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# Global solution for a system of semilinear diffusion-reaction equations with distinct diffusion coefficients

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**Abstract.** In this paper, we show the existence of solution for a relatively general system of semilinear parabolic equations with nonlinear reaction rate terms and inflow-outflow boundary conditions. Generally, to show the existence of global solution, it has been seen in the literature that either mass conservation or some growth condition on the source term is needed. Also, in several recent works only the nonlinearity up to certain order or of certain structure is allowed. However, our work considerably weakens the ones previously made by several authors on the coefficients of the elliptic operator, on the source (reaction rate) terms as well as on the boundary conditions. Our proof is also rather small and uses an argument based on implicit function theorem.

**Keywords:** semilinear parabolic equations, system of PDEs, equilibrium reaction rates, non-identical diffusion coefficients, global solution, mass action kinetics.

**2020 Mathematics Subject Classification:** 35K57, 35K55, 76S05, 47J35.

## 1 Introduction

Even a very simple semilinear parabolic equation (e.g., diffusion-reaction equation) can have merely local solutions, i.e. solutions in some perhaps small neighbourhood of the initial time $t_0$. The same applies to coupled systems of more than one variable. Therefore, it takes some special structure of the nonlinearities to guarantee the existence of the *global* solutions, i.e. solutions on any given time interval $[0, T)$, $T < \infty$. In this regard, nonlinearities which are obtained by modelling *equilibrium (reversible)* reactions amongst chemical species via mass-action kinetics bear some potential for producing global solutions since positive productions are accompanied by negative ones. Kräutle [11] and Mahato et al. [12] showed that the production rates of a large class of $J$ number of equilibrium (reversible) reactions of $I$ number of chemical species can be reduced to the following setting: let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^2$-boundary $\partial\Omega$, $S := [0, T)$ be the time interval for some $T < \infty$ and we denote by

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$\mathcal{S}$ the stoichiometric matrix such that $\mathcal{S} = (s_{ij}) \in M_{I\times J}$, the set of all $I \times J$ matrices, with $\operatorname{rank}(\mathcal{S}) = J$ and $|s_{ij}| \in \{0\} \cup [1,\infty)$. The entries $s_{ij} = \nu_{ij} - \tau_{ij}$ for $1 \le i \le I$ and $1 \le j \le J$, where $-\tau_{ij} \in \mathbb{Z}_0^-$ and $\nu_{ij} \in \mathbb{Z}_0^+$ are the stoichiometric coefficients for reversible reactions given by

$$\tau_{1j}X_1 + \tau_{2j}X_2 + \cdots + \tau_{Ij}X_I \rightleftharpoons \nu_{1j}X_1 + \nu_{2j}X_2 + \cdots + \nu_{Ij}X_I, \tag{1.1}$$

where $X_i$, $1 \le i \le I$, denotes the chemical species involved in $J$ reactions. Let $u = (u_1,\dots,u_I)$ be the unknown concentration vector of $I$ chemical species. For $k_j^f, k_j^b > 0$ and $j = 1,2,\dots,J$, we set

$$R_j^f(u) = k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}}, R_j^b(u) = k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}} \tag{1.2a}$$

$$R_j(u) = \left(R_j^f(u) - R_j^b(u)\right), \quad R = (R_1,\dots,R_J)^T, \quad \hat{f}(u) = \mathcal{S}R(u), \tag{1.2b}$$

where $s_{mj}^-$ and $s_{mj}^+$ denote the negative and positive parts of $s_{mj}$, respectively, such that $s_{mj} = s_{mj}^+ - s_{mj}^-$. For the $i-th$ species,

$$(\mathcal{S}R(u))_i = \sum_{j=1}^{J} s_{ij}R_j(u). \tag{1.3}$$

Furthermore, let $\mathbb{D} = (D_{0_1},\dots,D_{0_I})$, where $D_{0_i}$ denotes the symmetric $N\times N$ diffusive matrix of the $i$-th chemical species such that $D_{0_i} \in L^\infty(\Omega;\mathrm{Sym}(N))$. Let $\boldsymbol{q} : S\times\Omega \to \mathbb{R}^N$ denote the velocity vector such that $\nabla\cdot\boldsymbol{q} = 0$. For $i = 1,2,\dots,I$, we now define the following operators:

$$A := \operatorname{diag}(A_1,\dots,A_I), \quad B := \operatorname{diag}(B_1,\dots,B_I), \quad A_i(\mathbb{D},u) := A_i := \operatorname{div}(\boldsymbol{j}_i) \tag{1.4a}$$

$$\boldsymbol{j}_i := -a_i D_{0_i}\nabla u_i + \mathbf{q}u_i \quad (= \text{diffusive + advective flux}) \tag{1.4b}$$

$$B_i(\mathbb{D},u) := B_i := -(a_i D_{0i}\nabla u_i - \mathbf{q}u_i)\cdot\vec{n}, \tag{1.4c}$$

where $\vec{n} = \vec{n}(x)$ is the unit outward normal on $\partial\Omega$. The coefficients $a_i = a_i(x) \in \{0,1\}$ and $\theta \in (0,1]$ are explained below. The semilinear problem is: let $g$ and $h$ be given, then find a $u : \overline{S}\times\overline{\Omega} \to \mathbb{R}^I$ such that

$$\theta\frac{\partial u}{\partial t} + A(\mathbb{D},u) = \hat{f}(u) \ \text{ in } S\times\Omega, \quad u(0,\cdot) = g \quad \text{in } \Omega \text{ and} \tag{1.5}$$

$$B(\mathbb{D},u) = h \quad \text{on } S\times\partial\Omega. \tag{1.6}$$

For identical diffusion coefficients and $p > N+1$, in [11] Kräutle's showed that the solution $u(t,\cdot)$ belongs in $H^{2,p}(\Omega)^I$ for a.a. $t$. Since he deals with diffusion and reaction in porous media, in his setting the porosity $\theta$ might be different from one whereas, Mahato et. al. in [12] considers a free flow and thus in his setting $\theta = 1$. Kräutle in [11] splits $\partial\Omega$ in two disjoint parts $\Gamma_{\text{in}}$ and $\Gamma_{\text{out}}$, the inflow and outflow boundary parts, respectively, and specifies (1.5) as

$$-\mathbf{q}\cdot\vec{n} = 0 \quad \text{on } S\times\Gamma_{\text{in}}, \qquad -\mathbf{q}\cdot\vec{n} \le 0 \quad \text{on } S\times\Gamma_{\text{out}} \quad \text{and} \tag{1.7}$$

$$-D_{0i}\frac{\partial u_i}{\partial\vec{n}} = 0 \quad \text{on } S\times\Gamma_{\text{out}}, \qquad h \le 0 \quad \text{on } S\times\Gamma_{\text{in}} \quad \text{and} \quad h \ge 0 \quad \text{on } S\times\Gamma_{\text{out}}, \tag{1.8}$$

i.e. the outflow is entirely advective (cf. [11]). Let us denote the problem (1.5)–(1.6) by $(P)$. In order to model the conditions in (1.6) and (1.7), we choose $a_i(x) := 1$ on (1.5)–(1.7) and $a_i(x) := 0$ on $\Gamma_{\text{out}}$ whereas in $\Omega$, $a_i(x) = 1$. The existence of solution of (1.5)–(1.7) in [11] is based on $L^\infty$-estimates obtained via a Lyapunov functional, a fixed-point argument and classical $H^{2,p}$-theory for linear parabolic systems. An essential drawback of their approaches is that the diffusion coefficients need to be the same for all species. For a diffusion setting in a porous medium this can be justified by the observation that, usually, the advective flux dominates the diffusive one by order of magnitudes. In this note we address the issue of non-identical diffusion coefficients and show that the unique existence of weak solutions can still be guaranteed under certain assumptions which are given in next section.

### 1.1 Literature survey

In regards to the global in time solution, the authors in [18] showed that only mass control and positivity of the solutions are not sufficient to prevent the blowup in the solution and therefore we need a growth control condition. In the survey paper [16], the author has summarized the conditions (and limitations) under which the global solution can be guaranteed. The existence of weak solutions for $(P)$ is shown in [17] under the assumption that the nonlinearities belong to $L^1(S \times \Omega)$. For quadratic nonlinearities, the existence of weak solutions is shown in [4] via a duality method. The authors in [8,20] showed the existence of a global renormalized solution if nonlinearities satisfy the entropy condition: $\sum_{i=1}^{I} \hat{f}_i(u)(\log u_i + \alpha_i) \leq 0$ for all $u \in (0,\infty)^I$ for some $\alpha_1, \alpha_2, \ldots, \alpha_I \in \mathbb{R}$. In [9], the global classical solutions for $(P)$ with homogeneous Neumann boundary condition is shown under the assumption of mass conservation and entropy condition. In [9], the authors have assumed that $\hat{f}_i(u)$ has the cubic growth for $n = 1$ and $\hat{f}_i(u)$ has the quadratic growth for $n = 2$. The result later improved in [24] by utilizing a modified Gagliardo–Nirenbarg inequality. For higher-order nonlinearities in any space dimension if the diffusion coefficients are close to each other, i.e. if they are quasi-uniform, then the global existence of classical solutions is proved in [3, 5]. In [1], the authors have shown that under the polynomial growth condition, the $L^\infty$-norm of the classical solution can be obtained, however, later on this growth condition is removed in [3]. The authors in [19] proved the global existence and uniform boundedness for quadratic growth and dimension $n = 2$ by relaxing the mass conservation to mass dissipation. This result is improved in [15] by replacing the mass dissipation assumption with a weaker intermediate sum condition. In higher dimensions, the existence of global classical solutions for nonlinearities with quadratic growth has been proved in [2,6,23] and for the case of $\Omega = \mathbb{R}^n$ is deduced in [10]. The work in [2] is based on mass conservation assumption together with the entropy condition, whereas in [23] the mass conservation condition is replaced by the mass dissipation assumption. A more general work is done in [6] under the mass control assumption. The uniform in time bound for the solutions is shown in [7]. Thus, in the previous works the global classical solutions in any space dimensions and for the higher order nonlinearities is shown under the restriction that the diffusion coefficients are close to each other and on some particular structure of the nonlinearities. Our work shows the existence of weak solution in a $H^{1,p}$ setting and far less assumptions on the nonlinearities. Our argument to prove the existence of global solution is rather small and involves less calculations. We have also incorporated inflow-outflow boundary conditions which in turn do not disturb the global existence of the solution.

# 2 Mathematical preliminaries

## 2.1 Function spaces

Let $p, q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in [0, 1]$, $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. $\mathrm{Sym}(N)$ is the set of all real symmetric matrices $A = (A_{ij})$ normed by $|A|_{\mathrm{Sym}(N)} := \max_{i,j=1,\ldots,N} |A_{ij}|$. $(\cdot,\cdot)_{p,\lambda}$ and $[\cdot,\cdot]_\lambda$ stand for the real- and complex-interpolation functor, respectively (cf. [25]). Likewise $L^p(\Omega)$, $H^{\alpha,p}(\Omega)$, $\alpha \in \mathbb{N}$ and $C^\lambda(\overline{\Omega})$ denote the Lebesgue, Sobolev and Hölder spaces, respectively, with their usual norms (cf. [25]). $'' \hookrightarrow ''$ denotes a continuous imbedding. For a normed space $Y$, $L^p(S;Y)$ and $H^{1,p}(S;Y)$ are the (standard) Bochner and Sobolev–Bochner spaces (cf. [25]). $Y^*$ stands for the dual of $Y$. $\langle\cdot,\cdot\rangle_I$ denotes the inner product on $\mathbb{R}^I$ and $|\cdot|_I$ be the corresponding norm. If $X$ and $Y$ are normed spaces, then $\mathfrak{L}(X,Y)$ represents the set of all bounded linear operators from $X$ to $Y$ and $\mathrm{Iso}(X;Y)$ stands for the set of all linear isomorphisms of $\mathfrak{L}(X;Y)$. From here on we assume $p > N+2$ is fixed. The imbedding $L^p(\Omega)^I \hookrightarrow (H^{1,q}(\Omega)^*)^I$ is given by $L^p(\Omega)^I \ni h_0 \mapsto L_h : \langle L_h, w\rangle := \sum_{i=1}^I \int_\Omega h_{0i}(x) w_i(x)dx,\ w \in (H^{1,q}(\Omega))^I$. We set $F_p := F_p(S,\Omega) := L^p(S; H^{1,p}(\Omega)) \cap H^{1,p}(S; H^{1,q}(\Omega)^*)$ normed by

$$\|\psi\|_{F_p} := \|\psi\|_{L^p(S;H^{1,p}(\Omega))} + \|\psi'\|_{L^p(S;H^{1,q}(\Omega)^*)}, \tag{2.1}$$

where $u'$ is the distributional derivative of $u$. The solution space of the system under consideration is $F_p^I = \underbrace{F_p \times F_p \times \cdots \times F_p}_{I\text{-times}}$ and its norm is defined by $\|u\|_{F_p^I} := \left[\sum_{i=1}^I \|u_i\|_{F_p}^p\right]^{\frac{1}{p}}$. Note that for $p > N+2$, $F_p \subset C(\overline{S}; (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p})$ and $F_p \subset C(\overline{S}; C(\overline{\Omega}))$ (cf. [12]). For abbreviation, we set

$$V := H^{1,p}(\Omega)^I,\ W := H^{1,q}(\Omega)^I,\ W^* := [H^{1,q}(\Omega)^*]^I,\ V_0 := (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))^I_{1-\frac{1}{p},p}, \tag{2.2a}$$

$$V_{\partial\Omega} := L^p(\partial\Omega)^I,\ P_0 = L^p(S;V_0),\ P_1 = F_p^I,\ P_2 := L^p(S; H^{1,q}(\Omega)^*)^I,\ P := P_2 \times V_0, \tag{2.2b}$$

$$Q_1 := C(S; C(\overline{\Omega}))^I,\ Q_2 := L^p(S; C(\overline{\Omega}))^I,\ E := Q_0 \times P_1 \text{ and } Q_0 := L^\infty(\Omega; \mathrm{Sym}(N))^I. \tag{2.2c}$$

Then, for $p > N+2$, a simple embedding result from [12] yields

$$L^p(S;V) \cap H^{1,p}(S;W^*) \hookrightarrow C(\overline{S};V_0) \hookrightarrow C(\bar{S}; C(\bar{\Omega}))^I \hookrightarrow C(\bar{S}; L^\infty(\Omega))^I. \tag{2.3}$$

**Remark 2.1.** If $\mathbb{D} = (D_{0_1}, D_{0_2}, \ldots, D_{0_I}) \in L^\infty(\Omega; \mathrm{Sym}(N))^I$, i.e. $\forall i\ D_{0_i} \in L^\infty(\Omega; \mathrm{Sym}(N))$, then $|||\mathbb{D}|||_{L^\infty(\Omega;\mathrm{Sym}(N))^I} := \max_{i,j,k} \|D_{0_{ikj}}\|_{L^\infty(\Omega)} = \max_{i,j,k} \operatorname{ess\,sup}_{x\in\Omega} |D_{0_{ikj}}(x)|_{L^\infty(\Omega)}$. However, in our case, we only have $D_{0_{ikj}}(x) = D_0 =$ constant $\forall i,j,k$ and $x \in \Omega$, then $|||\mathbb{D}|||_{L^\infty(\Omega;Sym(N))^I} := \max_{i,j,k} \|D_{0_{ikj}}\|_{L^\infty(\Omega)} = \max_{i,j,k} \operatorname{ess\,sup}_{x\in\Omega} |D_{0_{ikj}}(x)| = D_0$.

To state the main theorem of the paper, we would require the following assumptions:

**A1.** let $D_0$ be a positive constant. For each $i = 1, 2, \ldots, I$, $D_{0_i} := \mathrm{diag}(D_0, \ldots, D_0) \in \mathrm{Sym}(N) \subset \mathbb{R}^{N\times N}$ be a diagonal matrix such that $\mathbb{D} := (D_{0_1}, \ldots, D_{0_I}) \in \mathrm{Sym}(N)^I$, where $I \in \mathbb{N}$.

**A2.** let $h \in L^p(\partial\Omega)^I$ and let $g \in V_0$ such that $g_i \geq 0$ for each $i$.

**A3.** let (1.6) and (1.7) hold true and $\vec{q} \in L^\infty(S\times\Omega)$ be such that $Q := \|\vec{q}\|_{L^\infty(S\times\Omega)} < \infty$ and $\vec{q}\cdot\vec{n} \in L^\infty(S \times \Gamma_{\mathrm{out}})$.

**Definition 2.2 (Weak formulation).** Let the assumptions A1–A3 hold true. Then, a vector $u \in F_p^I$ is said to be a weak solution of the problem $(P)$ if $u(0) = g$ and

$$\int_S \langle \partial_t u_i, \phi_i \rangle_{W^* \times W}\, dt + \int_S \int_\Omega (D_{0_i} \nabla u_i - \vec{q} u_i) \nabla \phi_i \, dx\, dt - \int_{S \times \Gamma_i n} h \phi_i \, ds\, dt$$
$$+ \int_{S \times \Gamma_o ut} \vec{q} \cdot \vec{n} \phi_i \, ds\, dt = \int_S \langle \hat{f}_i, \phi_i \rangle_{W^* \times W}\, dt, \quad \forall \phi \in L^q(S; W). \quad (2.4)$$

Next, we state the existence theorem from [13] which addresses the question of same diffusion coefficients in the system.

**Theorem 2.3.** *Let the assumptions A1–A3 hold true. Then, there exists a unique global positive weak solution (in the sense of Definition 2.2) $u \in F_p^I$ of the problem $(P)$.*

For Theorem 2.3, the global existence of solution follows from the construction of a particular type of Lyapunov function and exploiting the dissipative property of the reaction rate term and an application of Schaefer's fixed point theorem. The uniqueness and positivity follow from Gronwall's inequality.

Now, we shall state the main theorem of this paper which is existence of solution for different diffusion coefficients.

**Theorem 2.4.** *Let the assumptions A1–A3 hold true. Then, there is a neighborhood $U = U(D_0)$ in $L^\infty(\Omega; \mathrm{Sym}(N))^I$ such that (1.5)–(1.7) is solvable for all $\mathbb{D} \in U$. Moreover, the components of the solutions are non-negative.*

**Remark 2.5.** The proof of the implicit function theorem provides estimates for the size of $U(D_0)$. Here we do not yet go into detail, however, in [5,6] it has been shown that the if the diffusion co-efficients are very close to one and another, then there exists a classical solution.

**Remark 2.6.** In this note we do not directly employ the particular structure (1.5) of the reaction rates incorpoated into $\hat{f}(u)$, rather we use $\hat{f}$ is locally Lipschitz, then by Rademacher's theorem, we have $\hat{f} \in C^1(\mathbb{R}^I)^I$ in Theorem 2.4.

## 2.2 Operators

Let $U, V$ be two Banach spaces and $x \in U$, then a continuous linear operator $\Psi : U \to V$ is called the Fréchet derivative of the operator $T : U \to V$ at $x$ if $T(x + \theta) - T(x) = \Psi(\theta) + \phi(x, \theta)$ and $\lim_{\|\theta\|_U \to 0} \frac{\|\phi(x,\theta)\|_V}{\|\theta\|_U} = 0$ or, equivalently $\lim_{\|\theta\|_U \to 0} \frac{\|T(x+\theta) - T(x) - \Psi(\theta)\|_V}{\|\theta\|_U} = 0$. For a function $v = v(t, x)$, $t \in \overline{S}, x \in \overline{\Omega}$, we set $v(t) := v(t, \cdot)$. Let $H : U \to V$ and $G : \prod_{i=1}^n U_i \to V$, where $U, U_i$ and $V$ are Banach spaces. For functions $\xi \in U, \xi_i \in U_i\ \forall i$, $\mathcal{D}_{\xi^*} H(\xi)$ is the Fréchet derivative of $H = H(\xi)$ at $\xi^*$ and $\partial_i G := \partial_{\xi_i} G := \frac{\partial G}{\partial \xi_i}$ is the partial Fréchet derivative of $G = G(\xi_1, \ldots, \xi_n)$.

We will now define the following operators:

By Remark 2.6, $\widehat{f} \in C^1(\mathbb{R}^I, \mathbb{R}^I)$ (production-rate vector). Clearly, $\widehat{f} : L^p(S; V) \to L^p(S; W^*)$ and we define $\left\langle \widehat{f}(u), v \right\rangle := \int_\Omega \widehat{f}(u(x))^T v(x) dx$ (the $V$-realisation of $\widehat{f}$). We then introduce the operator $F : F_p^I \to L^p(S; W^*)$ via

$$\langle \mathcal{F}(u)(t), v \rangle := \int_\Omega \langle \widehat{f}(u(t,x)), v(x) \rangle_I dx \quad a.e.\ t \in S, v \in W. \quad (2.5)$$

We further define $A(\mathbb{D},u) := A_1(\mathbb{D},u) + A_2(\mathbb{D},u)$, where $A_1(\mathbb{D},u) := -(\operatorname{div}(D_{0_1}\nabla u_1), \ldots, \operatorname{div}(D_{0_I}\nabla u_I))^T$, $A_2(\mathbb{D},u) := (\operatorname{div} u_1\mathbf{q}, \ldots, \operatorname{div} u_I\mathbf{q})^T$, where

$$\begin{aligned}
A(\mathbb{D},u) &:= A_1(\mathbb{D},u) + A_2(\mathbb{D},u),\\
\langle A_1(\mathbb{D},u), v\rangle &:= \textstyle\sum_{i=1}^I \int_\Omega D_{0_i}\nabla u_i \cdot \nabla v_k\, dx, \quad u \in V, v \in W,\\
\langle A_2(\mathbb{D},u), v\rangle &:= \textstyle\sum_{i=1}^I \int_\Omega u_i\mathbf{q} \cdot \nabla v_i\, dx, \quad u \in V, v \in W \quad \text{and}\\
\langle B(h), v\rangle &:= \textstyle\sum_{i=1}^I \int_{\partial\Omega} h_i v_i\, d\sigma, v \in V.
\end{aligned}$$

We note that $A : E \to P_2$. The corresponding extensions for time dependent $u = u(t)$ is, with the same notation, given by $A(\mathbb{D},u)(t) := A(\mathbb{D},u(t))$. Similarly, we proceed with $A_1, A_2$ and $B$ and obtain

$$A, A_1, A_2 : E \to P_2, \quad B : L^p(S; V_{\partial\Omega}) \to P_2.$$

Also, note that $h \neq 0$ corresponds to non-homogeneous flux boundary conditions. Finally, we set

$$G_1(\mathbb{D},u) := u' + A(\mathbb{D},u) + B(h) - \mathcal{F}(u), \tag{2.6}$$
$$G_2(\mathbb{D},u) := u(0) - g, \tag{2.7}$$
$$G(\mathbb{D},u) := (G_1(\mathbb{D},u), G_2(\mathbb{D},u))^T. \tag{2.8}$$

Therefore, the problem (1.5)–(1.7) has now been formulated into an abstract evolution equation (2.6)–(2.8) and its weak formulation can be given by

**Definition 2.7.** Let the assumptions **A1–A3** be true. A function $u \in F_p^I$ is called a weak solution of problem (2.6)–(2.8), if $u(0) = g$ and $u'(t) + A(\mathbb{D}, u(t)) + B(h(t)) = \mathcal{F}(u(t))$ in $P_2$. Alternatively, a function $u \in F_p^I$ is a weak solution of (2.6)–(2.8) if $G(\mathbb{D}), u) = 0$ in $P$.

In order to prove Theorem 2.4, we will first look in to following lemmas:

**Lemma 2.8.** *Let $p > N+2$. For $k = 1,2$, let $P_k$ and $Q_k$ be the normed spaces, defined as in (2.2a)–(2.2c), such that $P_1 \hookrightarrow Q_1$, $Q_2 \hookrightarrow P_2$. Assume further that $M : Q_1 \to Q_2$ be a Fréchet differentiable operator and set $\overline{M} := M|_{P_1}$ = restriction of $M$ on $P_1$. Then,*

$$\mathfrak{L}(Q_1; Q_2) \hookrightarrow \mathfrak{L}(P_1; P_2), \quad \mathcal{D}_u\overline{M} = \mathcal{D}_u M|_{P_1} \in \mathfrak{L}(P_1; P_2). \tag{2.9}$$

*Proof.* Let $v \in Q_2$, then $\|v\|_{Q_2} \le C\|v\|_{Q_1}$. Since $P_1 \hookrightarrow Q_1$ and $Q_2 \hookrightarrow P_2$ , $\|v\|_{P_2} \le C\|v\|_{Q_2} \le C\|v\|_{Q_1} \le C\|v\|_{P_1}$. This concludes (2.9). Now, we shall prove (2.9). We note that $M : Q_1 \to Q_2$ is a Fréchet differentiable operator, i.e. $Q_1 \ni l \mapsto \mathcal{D}_l M(l) \in Q_2$ is a bounded linear operator from $Q_1$ to $Q_2$. Then, by the definition of Fréchet derivative, we have for $\varepsilon > 0$

$$\|M(u+l) - M(u) - \mathcal{D}_l M(l)\|_{Q_2} < \varepsilon\|l\|_{Q_1}.$$

Then, by $P_1 \hookrightarrow Q_1$, $Q_2 \hookrightarrow P_2$, we obtain

$$\|M(u+l) - M(u) - \mathcal{D}_l M(l)\|_{P_2} < \varepsilon\|l\|_{P_1}. \tag{2.10}$$

(2.10) implies that $\mathcal{D}_l M : P_1 \to P_2$ is a bounded linear operator, i.e. $M : P_1 \to P_2$ is a Fréchet differentiable operator. Now, for $u, l \in P_1$, $\overline{M}(u+l) = M(u+l)$, $\overline{M}(u) = M(u)$, therefore from (2.10), we have $\|\overline{M}(u+l) - \overline{M}(u) - \mathcal{D}_l\overline{M}(l)\|_{P_2} = \|M(u+l) - M(u) - \mathcal{D}_l M(l)\|_{P_2} < \varepsilon\|l\|_{P_1}$ which implies $\mathcal{D}_l\overline{M}|_{P_1} \in \mathfrak{L}(P_1; P_2)$. $\square$

**Lemma 2.9** (Implicit Function Theorem, cf. [22])**.** *Suppose that $X, Y, Z$ are Banach spaces, $C$ is an open subset of $X \times Y$ and $T : C \to Z$ is continuous operator. Suppose further that for some $(x_0, y_0) \in C$,*

*(i)* $T(x_0, y_0) = 0$,

*(ii) The Fréchet derivative of $T(\cdot,\cdot)$, when $x$ is fixed is denoted by $T_y(x, y)$, is called the partial Fréchet derivative w.r.t. $y$ which exists at each point $(x, y)$ in a neighbourhood of the point $(x_0, y_0)$ and is continuous at $(x, y)$.*

*(iii)* $[T_y(x_0, y_0)]^{-1} \in \mathfrak{L}(Z, Y)$.

*Then there is an open subset $U$ of $X$ containing $x_0$ and a unique continuous mapping $y : U \to Y$ such that $T(x, y(x)) = 0$ and $y(x_0) = y_0$.*

## 3 Proof of Theorem 2.4

Before we prove Theorem 2.4, we recall Theorem 2.4 since it deals with the system of semilinear parabolic PDEs with identical diffusion coefficients. It states that under the assumptions A1–A3, there exists a unique positive global weak solution $u \in P_1$ of the problem (1.5)–(1.7).

In other words, for a fixed $\mathbb{D} \in Q_0$ there exists a unique $u^* \in P_1$ such that we have

$$G(\mathbb{D}, u^*) = 0. \tag{3.1}$$

Now, in the spirit of Lemma 2.9, we denote $T = G, X = Q_0, Y = P_1$ and $Z = P$. Next, we will show that the following equations holds true:

$$\langle \mathcal{D}_{u^*}\mathcal{F}(u), v\rangle = \int_\Omega \langle \mathcal{D}_{u^*}\hat{f}(u(t,x)), v(x)\rangle_I dx \text{ for a.a. } t \in S,\ u \in P_1,\ v \in W, \tag{3.2}$$

$$\langle \mathcal{D}_{u^*}G_1(\mathbb{D}, u), v\rangle = \langle \partial_t u^* + A(\mathbb{D}, u^*) - \mathcal{D}_{u^*}F(u), v\rangle \text{ for } u \in P_1, \text{ and } v \in W, \tag{3.3}$$

$$\langle \mathcal{D}_{u^*}G_2(\mathbb{D}, u), v\rangle = \langle u^*(0) - g, v\rangle, \tag{3.4}$$

$$G_1 \in P_2 \text{ is Fréchet differentiable on } Q_0 \times P_1, \tag{3.5}$$

For fixed $(\mathbb{D}, u^*)$,

$$L := \mathcal{D}_{u^*}G(\mathbb{D}, u) \in \mathfrak{L}(P, P_1), \text{ i.e. } L = (\mathcal{D}_{u^*}G_1(\mathbb{D}, u), \mathcal{D}_{u^*}G_2(\mathbb{D}, u)) \in \mathrm{Iso}(P_1, P). \tag{3.6}$$

We shall prove (3.2)–(3.6) in several steps.

Step 1: At first, we show that $G : Q_0 \times P_1 \to P_2 \times P_0$ is a continuous operator. We note that $G_1(\mathbb{D}, u) := u' + A_0(\mathbb{D}, u) + B(h) - F(u), G_2(\mathbb{D}, u) = u(0) - g$, Then, for a $\phi \in \mathfrak{D} := C_0^\infty(S \times \Omega)^I$, we have

$$\begin{aligned}\langle G_1(\mathbb{D}, u), \phi\rangle &= \langle u' + A(\mathbb{D}, u) + B(h) - \mathcal{F}(u), \phi\rangle \\ &= -\langle u, \partial_t \phi\rangle + \int_{S\times\Omega} \langle \mathbb{D}\nabla u, \nabla v\rangle_I + \int_{S\times\Omega} \langle u\mathbf{q}, \nabla\phi\rangle_I + \int_{S\times\partial\Omega} \langle h, v\rangle_I - \int_{S\times\Omega} \langle \hat{f}, \phi\rangle_I.\end{aligned}$$

By Hölder's inequality and $\mathfrak{D} \hookrightarrow P_1 \hookrightarrow P_0 \hookrightarrow P_2$, it follows that $\|G_1(\mathbb{D}, u)\|_{P_2} < \infty$. We also note that $\|G_2(\mathbb{D}, u)\|_{P_0} < \infty$. Altogether these two estimates imply that $\|G(\mathbb{D}, u)\|_{P_2\times P_0} < \infty$, i.e. $G : Q_0 \times P_1 \to P_2 \times P_0$ is a continuous operator. Since $\hat{f} \in C^1(\mathbb{R}^I)^I$, then by definition of $\mathcal{F}(u)$ and Fréchet derivative, we obtain

$$\begin{aligned}\langle \mathcal{D}_{u^*}F(u), v\rangle &= \int_\Omega \Big\langle \Big[\hat{f}(u^*(t,x)+\theta(t,x)) - \hat{f}(u^*(t,x)) - \Psi(u^*(t,x),\theta(t,x))\Big], v(x)\Big\rangle_I \, dx \\ &= \int_\Omega \langle \mathcal{D}_{u^*}\hat{f}(u(t,x)), v(x)\rangle_I \, dx, \end{aligned} \tag{3.7}$$

for a.e. $t \in S$, $u, \theta \in V$ and $v \in W$. By (3.7), it follows that $\mathcal{D}_u F \in \mathfrak{L}(V; W)$ exists as $C(S) \hookrightarrow L^p(S)$. Now, since $H^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \hookrightarrow H^{1,q}(\Omega)^*$, the restriction $F|_{P_1}$ is Fréchet differentiable with $\mathcal{D}_u F|_{P_1} \in \mathfrak{L}(P_1; P_2)$ for all $u \in P_1$. Therefore, $\langle \mathcal{D}_u F, v\rangle$ exists in (3.2).

Step 2: We note from definition 2.1.1 that the Fréchet derivative of a linear operator $T$ is $T$ itself. Since, $\partial_t, A : F_p^I \to P_2$ are linear and $B$ can be treated as a constant w.r.t. $u \in F_p^I$, therefore the Fréchet derivative of $G_1$ will yield

$$\mathcal{D}_{u^*}G_1(\mathbb{D}, u) = \partial_t u^* + A(\mathbb{D}, u^*) - \mathcal{D}_{u^*}\mathcal{F}(u) \in P_2.$$

This concludes (3.3). Likewise, (3.4) follows with similar arguments. Furthermore, by step 1, we know $\mathcal{D}_u F|_{P_1} \in \mathfrak{L}(P_1; P_2)$. This implies $\mathcal{D}_u G_1 \in \mathfrak{L}(Q_0 \times P_1; P_2)$, i.e. $\mathcal{D}_u G_1 : Q_0 \times P_1 \to P_2$ exists, i.e. $G_1(\mathbb{D}, u^*) \in P_2$ is Fréchet differentiable on $Q_0 \times P_1$.

Step 3.: We have obtained the continuity of one of the partial derivative $(\mathcal{D}_u G_1(\mathbb{D}, .))$ of a total of two and the existence of $\mathcal{D}_u G_2(\mathbb{D}, u^*)$, we obtain the existence of $\mathcal{D}G(\cdot,\cdot)$. Now the estimate

$$\|\mathcal{D}_u G_2(\mathbb{D}, u^*)\|_{P_2} = \|u_0^*\|_{P_2} \le C\|u_0^*\|_{L^p(S;V_0)} \le C\|u_0^*\|_{P_1} < \infty,$$

by the definition of function spaces and a straightforward imbedding of $P_1 \hookrightarrow P_0 \hookrightarrow P_2$. This implies the continuity of $\mathcal{D}_u G_2$. Hence, the continuity of both $\mathcal{D}_u G_1$ and $\mathcal{D}_u G_2$ imply the continuity of $\mathcal{D}G$.

Step 4: Let $(f, g) \in P_2 \times V_0$. In order to verify (3.6) it remains to show that the problem:

$$\text{Find } (\mathbb{D}, u^*) \in Q_0 \times P_1 \text{ with} \tag{3.8}$$

$$\partial_t u^* + A(\mathbb{D}, u^*) = \mathcal{D}_{u^*}\mathcal{F}(u), \quad u^*(0) = g, \tag{3.9}$$

has a unique solution. The operator $A(\mathbb{D}, u^*)$ possesses the maximal parabolic regularity property on $P_1$ in the $L^p$-sense (cf. [21]). From [21], it follows that (3.8)–(3.9) is uniquely solvable. Upon combining the steps 1 to 4, all the three conditions of implicit function theorem (Lemma 2.9) satisfied. Therefore, there exists an open neighbourhood of $\mathbb{D}$, $U(\mathbb{D}) \subset Q_0$ such that $G(D, u) = 0$ for all $D \in U(\mathbb{D})$. Moreover, the size of the neighbourhood $U(\mathbb{D})$ can be estimated, however this will be addressed somewhere else. Now, for the positivity, we multiply the PDE (1.5) with $-u_i^-$ ($-1\times$ negative part of $u_i$) and integrate over $\Omega_i^-$ (support of $u_i^-$) for all $i = 1, 2, \ldots, I$ and for a.e. $t \in S$. We use the fact that $u(0) \ge 0$ which eventually yields the positivity of the solutions via Gronwall's inequality.

**Remark 3.1.** Although we did not show the size of this neighbourhood in which the diffusion coefficient must lie, in [6] an idea regarding that is mentioned and recently in [14] has been shown that this can be further refined and a rather general neighbourhood can be chosen.

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## References

[1] J. A. Cañizo, L. Desvillettes, K. Fellner, Improved duality estimates and applications to reaction-diffusion equations, *Comm. Partial Differential Equations* **39**(2014), No. 6, 1185–1204. https://doi.org/10.1080/03605302.2013.829500; MR3200093; Zbl 1295.35142

[2] M. C. Caputo, T. Goudon, A. Vasseur, Solutions of the 4-species quadratic reaction-diffusion system are bounded and $C^\infty$-smooth, in any space dimension, *Anal. PDE* **12**(2019), No. 7, 1773–1804. https://doi.org/10.2140/apde.2019.12.1773; MR3986541; Zbl 1423.35164

[3] B. P. Cupps, J. Morgan, B. Q. Tang, Uniform boundedness for reaction-diffusion systems with mass dissipation, *SIAM J. Math. Anal.* **53**(2021), No. 1, 323–350. https://doi.org/10.1137/19M1277977; MR4201440; Zbl 1456.35105

[4] L. Desvillettes, K. Fellner, M. Pierre, J. Vovelle, Global existence for quadratic systems of reaction-diffusion, *Adv. Nonlinear Stud.* **7**(2007), No. 3, 491–511. https://doi.org/10.1515/ans-2007-0309; MR2340282; Zbl 1330.35211

[5] K. Fellner, E. Latos, T. Suzuki, Global classical solutions for mass-conserving, (super)-quadratic reaction-diffusion systems in three and higher space dimensions, *Discrete Contin. Dyn. Syst. Ser. B* **21**(2016), No. 10, 3441–3462. https://doi.org/10.3934/dcdsb.2016106; MR3593433; Zbl 1361.35090

[6] K. Fellner, J. Morgan, B. Q. Tang, Global classical solutions to quadratic systems with mass control in arbitrary dimensions, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **37**(2020), 281–307. https://doi.org/10.1016/j.anihpc.2019.10.003; MR4072804; Zbl 1433.35167

[7] K. Fellner, J. Morgan, B. Q. Tang, Uniform-in-time bounds for quadratic reaction-diffusion systems with mass dissipation in higher dimensions, *Discrete Contin. Dyn. Syst. Ser. S* **14**(2021), No. 2, 635. https://doi.org/10.3934/dcdss.2020334; MR4201861; Zbl 1459.35236

[8] J. Fischer, Global existence of renormalized solutions to entropy-dissipating reaction–diffusion systems, *Arch. Ration. Mech. Anal.* **218**(2015), No. 1, 553–587. https://doi.org/10.1007/s00205-015-0866-x; MR3360745; Zbl 1323.35175

[9] T. Goudon, A. Vasseur, Regularity analysis for systems of reaction-diffusion equations, *Ann. Sci. Éc. Norm. Supér. (4)* **43**(2010), 117–142. https://doi.org/10.24033/asens.2117; MR2583266; Zbl 1191.35202

[10] Y. I. Kanel, Solvability in the large of a system of reaction-diffusion equations with the balance condition, *Differentsial'nye Uravneniya* **26**(1990), No. 3, 448–458. MR1053771; Zbl 0715.35036

[11] S. Kräutle, General multispecies reactive transport problems in porous media, Habilitation Thesis, 2008.

[12] H. S. Mahato, M. Böhm, An existence result for a system of coupled semilinear diffusion-reaction equations with flux boundary conditions, *European J. Appl. Math.* **26**(2015), 121–142. https://doi.org/10.1017/S0956792514000369; MR3315050; Zbl 1387.35336

[13] H. S. Mahato, M. Böhm, Global existence and uniqueness for a system of nonlinear multi-species diffusion-reaction equations, *J. Appl. Anal. Comput.* **3**(2013), No. 4, 357–376. MR3146167; Zbl 1302.35212

[14] H. S. Mahato, N. Ghosh, Global existence and uniform boundedness of the classical solutions for a system of reaction-diffusion equations with mass control, available on arXiv, 2023, 16 pp. https://doi.org/10.48550/arXiv.2209.07456

[15] J. Morgan, B. Q. Tang, Boundedness for reaction–diffusion systems with Lyapunov functions and intermediate sum conditions, *Nonlinearity* **33**(2020), No. 7, 3105–3133. https://doi.org/10.1088/1361-6544/ab8772; MR4105387; Zbl 1448.35293

[16] M. Pierre, Global existence in reaction-diffusion systems with control of mass: a survey, *Milan J. Math.* **78**(2010), No. 2, 417–455. https://doi.org/10.1007/s00032-010-0133-4; MR2781847; Zbl 1222.35106

[17] M. Pierre, Weak solutions and supersolutions in $L^1$ for reaction-diffusion systems, *J. Evol. Equ.* **3**(2003), 153–168. https://doi.org/10.1007/978-3-0348-8016-6_9; MR1977432; Zbl 1026.35047

[18] M. Pierre, D. Schmitt, Blowup in reaction-diffusion systems with dissipation of mass, *SIAM Rev.* **42**(2000), No. 1, 93–106. https://doi.org/10.1137/S0036144599359735; MR1738101; Zbl 0942.35033

[19] M. Pierre, T. Suzuki, Y. Yamada, Dissipative reaction diffusion systems with quadratic growth, *Indiana Univ. Math. J.* **68**(2017), 291–322. https://doi.org/10.1512/iumj.2019.68.7447; MR3922046; Zbl 1418.35237

[20] M. Pierre, T. Suzuki, R. Zou, Asymptotic behavior of solutions to chemical reaction–diffusion systems, *J. Math. Anal. Appl.* **450**(2017), No. 1, 152–168. https://doi.org/10.1016/j.jmaa.2017.01.022; MR3606162; Zbl 1381.35089

[21] J. Prüss, R. Schnaubelt, Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time, *J. Math. Anal. Appl.* **256**(2001), 405–430. https://doi.org/10.1006/jmaa.2000.7247; MR1821747; Zbl 0994.35076

[22] A. H. Siddiqi, *Functional analysis and applications*, Springer, Singapore, 2018. https://doi.org/10.1007/978-981-10-3725-2; MR3791838; Zbl 1398.46002

[23] P. Souplet, Global existence for reaction–diffusion systems with dissipation of mass and quadratic growth, *J. Evol. Equ.* **18**(2018), No. 4, 1713–1720. https://doi.org/10.1007/s00028-018-0458-y; MR3884766; Zbl 1516.35235

[24] B. Q. Tang, Global classical solutions to reaction-diffusion systems in one and two dimensions, *Commun. Math. Sci.* **16**(2018), No. 2, 411–423. https://doi.org/10.4310/CMS.2018.v16.n2.a5; MR3805025; Zbl 1390.35162

[25] H. Triebel, *Interpolation theory, function spaces, differential operators*, Johann Ambrosius Barth, Heidelberg, 1995. MR1328645; Zbl 0830.46028