Fully nonlinear degenerate equations with applications to Grad equations

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Abstract. We consider a class of degenerate elliptic fully nonlinear equations with applications to Grad equations:

$$\begin{cases} |Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}\big(D^2 u(x)\big) = f\big(|u \geq u(x)|\big) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\gamma \geq 1$ is a constant, Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary. We prove the existence of a $W^{2,p}$-viscosity solution to the above equation, which degenerates when the gradient of the solution vanishes.

Keywords: fully nonlinear degenerate elliptic equations, viscosity solution, Pucci's extremal operator, Dirichlet boundary value problem.

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1 Introduction

We investigate the following degenerate problem:

$$\begin{cases} |Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}\big(D^2 u(x)\big) = f\big(|u \geq u(x)|\big) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\gamma \geq 1$ is a constant, Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$, $f : [0, |\Omega|] \longrightarrow \mathbb{R}$ is a non-decreasing, non-negative continuous function and $u : \Omega \longrightarrow \mathbb{R}$. Here, $\mathcal{M}^+_{\lambda,\Lambda}$ is the Pucci's extremal operator. In our setting, by $u \geq u(x)$, we mean,

$$\{\omega \in \Omega : u(\omega) \geq u(x)\}$$

called the superlevel sets of u. We establish the existence of a $W^{2,p}$-viscosity solution (also known as L^p-viscosity solution) to (1.1). It is worth mentioning that the notion of $W^{2,p}$-viscosity solution was defined by Caffarelli et al. [7]. In the case when $\gamma = 0$ in (1.1), the existence of a $W^{2,p}$-viscosity solution is proven by L. Caffarelli and I. Tomasetti [8].

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The pioneer contribution in this direction was due to H. Grad [12], who introduced such equations, which appear in plasma physics, called "Grad equations". In their seminal work, Grad examined the following equation in three-dimension:

$$\Delta \Psi = F(V, \Psi, \Psi', \Psi''),$$

where the right hand side (R.H.S.) represents a second-order differential operator acting on $\Psi(V)$ for a surface defined by $\Psi = $ constant. Here, $\Psi'(V)$ represents the derivative with respect to volume and $\Psi(V)$ stands for the inverse function to $V(\Psi)$, denoting the volume enclosed by Ψ. Furthermore, they pointed out the potential for simplifying plasma equations by introducing u^* defined as:

$$u^*(t) := \inf \left\{ s : |u < s| \geq t \right\}.$$

These equations, also known as Queer Differential Equations in the literature, have a wide range of applications across various fields. One notable application is their appearance in plasma modeling, specifically in analyzing plasma confined within toroidal containers. We refer to [12] for the details. Moreover, these equations exhibit connections in financial mathematics, see [23]. R. Temam [22] pioneered the investigation of problems akin to (1.1) concerning the Laplacian, a direction extensively examined by several researchers. They notably established the existence of solutions to equations having the model structure:

$$\Delta u = g\big(|u < u(x)|, u(x)\big) + f(x),$$

by exploiting the properties of directional derivatives of u^*. For further insights into this topic, we refer the interested readers to the works of J. Mossino and R. Temam [17], as well as those by P. Laurence and E. Stredulinsky [15, 16], along with the related references therein.

In all the aforementioned research works, the problem was studied using variational methods. However, in a recent work, L. Caffarelli and I. Tomasetti [8] studied the equation similar to J. Mossino and R. Temam [17] for fully nonlinear uniformly elliptic operators using the viscosity approach. Specifically, they addressed the following problem:

$$\begin{cases} F\big(D^2 u(x)\big) = f\big(|u \geq u(x)|\big) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where F represents a convex, uniformly elliptic operator. They established the existence of a $W^{2,p}$-viscosity solution u to this problem, satisfying the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} \leq C \big[\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f\big(|u \geq u(x)|\big)\|_{p,\Omega}\big].$$

For further insights into the existence and qualitative questions pertaining to extremal Pucci's equations, we refer to [10, 11, 18, 20, 21, 24–26] .

Concurrently, equations involving gradient degenerate fully nonlinear elliptic operators have been widely investigated over the past decade. Pioneering works in this direction are attributed to I. Birindelli and F. Demengel. They proved several important results for these operators in a series of papers. These contributions involve comparison principle and Liouville-type results [3], regularity and uniqueness of eigenvalues and eigenfunctions [4, 5], $C^{1,\alpha}$ regularity in the radial case [6]. Furthermore, the equations of the form:

$$|Du|^\gamma F(D^2 u) = f \quad \text{in } B_1, \tag{1.2}$$

when $\gamma \geq 0$ is a constant and $f \in L^\infty(B_1, \mathbb{R})$, were investigated by C. Imbert and L. Silvestre [13]. In particular, they established the interior $C^{1,\alpha}$ regularity of solutions for equations of the form (1.2). One may also see [19] for insights into variable exponent degenerate mixed fully nonlinear local and nonlocal equations.

Motivated by the above works and recently by the work of L. Caffarelli and I. Tomasetti [8], it is natural to ask the following question:

Question: *Do we have the existence of a viscosity solution to* (1.1)*?*

The aim of this paper is to answer this question affirmatively. The crucial difference to our problem from [8] is due to the fact that $|Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2u)$ degenerates along the set of critical points,

$$\mathcal{C} := \{x : Du(x) = 0\}.$$

The problem is challenging due to the following reasons:

(C1) The R.H.S. of (1.1) is a function of measure of superlevel sets. This makes the problem nonlocal.

(C2) The L.H.S. of (1.1) is degenerate. The fundamental theory of L^p-viscosity solutions does not work directly here since it requires the uniform ellipticity of the operator. Also, when $f \in C(\Omega)$, the problem can be discussed in the C-viscosity sense but in the case of discontinuous data, when $f \in L^p(\Omega)$, the problem needs to be treated in the L^p-viscosity sense. We point out that this situation occurs while approximating the R.H.S. of (1.1).

We use the L^p-viscosity solution approach for Monge–Ampère equation as in [1, 8] to (1.1). To handle the above mentioned challenges, we first consider the following approximate problem:

$$\begin{cases} |Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2u(x)) + \varepsilon \Delta u = f(|u \geq u(x)|) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for $\varepsilon > 0$. Further, using the approximations in the R.H.S. of the equation and exploiting the results available for uniform elliptic operators, for instance, Theorem 2.5 and Theorem 2.7 (see next), we establish the existence of a viscosity solution to the approximate problem (1.3). This yields the existence of a viscosity solution to (1.1). More precisely, using the idea of Amadori et al. [1], we first get the existence of a $W^{2,p}$-viscosity solution to the approximate problem (1.3) by invoking Theorem 2.1 [8]. We recall that the estimate established in [8] is not adequate to claim the uniform bound on the $W^{2,p}$-viscosity solution of (1.3). To show the existence of a solution to the original problem (1.1), we seek the uniform bound on the solutions of (1.3), which is crucial in approaching $\varepsilon \longrightarrow 0^+$. We invoke the Alexandroff–Bakelman–Pucci (ABP) estimates from Caffarelli et al. [7] to sort this issue. These estimates play a crucial role in obtaining uniform bounds on the $W^{2,p}$-viscosity solutions to (1.3).

Throughout the paper, we consider Ω to be a bounded $C^{1,1}$ domain in $\mathbb{R}^N$, $N \geq 2$.

The main result of this paper is the following:

Theorem 1.1. *Let* $\gamma \geq 1$ *be a constant. Let* $\Omega \subset \mathbb{R}^N$ *be a bounded* $C^{1,1}$ *domain. Let* $f \in C([0, |\Omega|], \mathbb{R})$ *be a non-decreasing, non-negative function and* $g \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$, $p > N$. *Consider the problem*

$$\begin{cases} |Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2u(x)) = f(|u \geq u(x)|) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Then, there exists a $W^{2,p}$-viscosity solution of (1.4)*. Moreover, the solution satisfies the following estimate:*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(|u \geq u(x)|)\|_{p,\Omega}\right),$$

where $C > 0$ is a constant.

Remark 1.2. By Sobolev's embedding theorem we have that the solution is $C^{1,\alpha}(\overline{\Omega})$ regular for any $0 < \alpha < 1$.

The organization of the paper is as follows. In Section 2, we recall the basic definitions and several key results used in the ensuing sections of the paper. Section 3 is devoted to the proof of our main result. Here, we sketch the plan of our proof:

(i) Perturb the left-hand side (L.H.S.), i.e., the operator $|Du|^\gamma \mathcal{M}_{\lambda,\Lambda}^+(D^2u)$ by adding $\varepsilon\Delta u$, for $\varepsilon > 0$. (This makes the problem uniformly elliptic.)

(ii) Fix a Lipschitz function v in the R.H.S. of (1.3).

(iii) Construct a sequence of L^p-functions converging to R.H.S. (for fixed Lipschitz function v) and obtain a sequence of solutions.

(iv) Obtain the existence of solution to equation pertaining $|Du|^\gamma \mathcal{M}_{\lambda,\Lambda}^+(D^2u) + \varepsilon\Delta u$ for fixed Lipschitz function v in the R.H.S.

(v) Use Theorem 2.1 [8] (an application of Schaefer fixed point theorem) to show the existence of a solution to (1.3).

(vi) Establish the existence of a $W^{2,p}$-viscosity solution to (1.4).

2 Preliminaries

We recall that a continuous mapping $F : \mathcal{S}^N \longrightarrow \mathbb{R}$ is *uniformly elliptic* if:
For any $A \in \mathcal{S}^N$, where $\mathcal{S}^N$ is the set of all $N \times N$ real symmetric matrices, there exist two positive constants $\Lambda \geq \lambda > 0$ s.t.

$$\lambda\|B\| \leq F(A+B) - F(A) \leq N\Lambda\|B\| \quad \text{for all positive semi-definite } B \in \mathcal{S}^N,$$

and $\|B\|$ is the largest eigenvalue of B. Here, we have the usual partial ordering: $A \leq B$ in $\mathcal{S}^N$ means that $\langle A\xi, \xi \rangle \leq \langle B\xi, \xi \rangle$ for any $\xi \in \mathbb{R}^N$. In other words, $B - A$ is positive semidefinite.

Let $S \in \mathcal{S}^N$, then for the given two parameters $\Lambda \geq \lambda > 0$, Pucci's maximal operator is defined as follows:

$$\mathcal{M}_{\lambda,\Lambda}^+(S) := \Lambda \sum_{e_i \geq 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $\{e_i\}_{i=1}^N$ are the eigenvalues of S. This operator is uniformly elliptic and subadditive, that is

$$\mathcal{M}_{\lambda,\Lambda}^+(A+B) \leq \mathcal{M}_{\lambda,\Lambda}^+(A) + \mathcal{M}_{\lambda,\Lambda}^+(B),$$

for $A, B \in \mathcal{S}^N$. Clearly, for $\lambda = \Lambda = 1$, $\mathcal{M}_{\lambda,\Lambda}^+ \equiv \Delta$, the classical Laplace operator.

Next, we recall the notion of a viscosity solution. M. G. Crandall and P.-L. Lions [9] were the first to introduce the concept of a viscosity solution. Now, we recall the definition of *continuous viscosity solution* to the following equation:

$$|Du|^\gamma F(D^2u(x)) = f \quad \text{in } \Omega, \tag{2.1}$$

for $f \in C(\Omega)$.

Definition 2.1 ([3]). Let $u : \overline{\Omega} \longrightarrow \mathbb{R}$ be an upper semicontinuous (USC) function in Ω. Then, u is called a *viscosity subsolution* of (2.1) if

$$|D\varphi(x)|^{\gamma} F(D^2\varphi(x)) \geq f(x),$$

whenever $\varphi \in C^2(\Omega)$ and $x \in \Omega$ is a local maximizer of $u - \varphi$ with $D\varphi \neq \mathbf{0} \in \mathbb{R}^N$.

Definition 2.2 ([3]). Let $u : \overline{\Omega} \longrightarrow \mathbb{R}$ be a lower semicontinuous (LSC) function in Ω. Then, u is called a *viscosity supersolution* of (2.1) if

$$|D\psi(x)|^{\gamma} F(D^2\psi(x)) \leq f(x),$$

whenever $\psi \in C^2(\Omega)$ and $x \in \Omega$ is a local minimizer of $u - \psi$ with $D\psi \neq \mathbf{0} \in \mathbb{R}^N$.

Definition 2.3 ([3]). A continuous function u is said to be a *viscosity solution* to (2.1) if it is a supersolution as well as subsolution to (2.1).

Let $h \in L^p(\Omega)$, $g \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$ for $p > N$. Let us consider the problem

$$\begin{cases} |Du|^{\gamma} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) = h & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We mention that the classical definition of $W^{2,p}$-viscosity solution can not be applied for (2.2), due to the lack of uniform ellipticity. Consider the problem:

$$\begin{cases} |Du|^{\gamma} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mathcal{E}\Delta u = h & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

for $p \in \mathbb{R}^N$. Motivated by Caffarelli et al. [7] and Ishii et al. [14], we define the L^p-viscosity subsolution (supersolution) to (2.3) as follows.

Definition 2.4. Let u be an USC (respectively, LSC) function on $\overline{\Omega}$. We say that u is an L^p-viscosity subsolution (respectively, supersolution) to (2.3) if $u \leq g$ (resp., $u \geq g$) on $\partial\Omega$ and for all $\phi \in W^{2,p}(\Omega)$,

$$\text{ess} \liminf_{x \longrightarrow y} \left(|D\phi(x)|^{\gamma} \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi(x)) + \mathcal{E}\Delta\phi(x) - h(x)\right) \geq 0$$

$$\left(\text{resp., ess} \limsup_{x \longrightarrow y} \left(|D\phi(x)|^{\gamma} \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi(x)) + \mathcal{E}\Delta\phi(x) - h(x)\right) \leq 0\right),$$

for $y \in \Omega$, the point of local maxima (respectively, minima) to $u - \phi$.

We say that any continuous function u is an L^p-viscosity solution to (2.3) if it is both L^p-viscosity subsolution and supersolution to (2.3). Now, we state a result concerning the existence and uniqueness of $W^{2,p}$-viscosity solution to the operator F under certain hypotheses. The following result is due to N. Winter [27].

Theorem 2.5 ([27, Theorem 4.6]). *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^N$. Let $F(p, M)$ be a uniformly elliptic operator and convex in M-variable. Also, let $F(0,0) \equiv 0$ in Ω, $f \in L^p(\Omega)$ and $g \in W^{2,p}(\Omega)$ for $p > N$. Then, there exists a unique $W^{2,p}$-viscosity solution to*

$$\begin{cases} F(Du, D^2u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Moreover, $u \in W^{2,p}(\Omega)$ and

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f\|_{p,\Omega} \right),$$

for some positive constant C.

Theorem 2.6 ([2, Theorem 1.1]). *Let Ω be a bounded domain with C^2-boundary. Let $\gamma \geq 0$ and F be a uniformly elliptic operator and $f \in C(\overline{\Omega})$, $g \in C^{1,\beta}(\partial\Omega)$ for some $\beta \in (0,1)$. Then, any viscosity solution u of*

$$\begin{cases} |Du|^\gamma F(D^2u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is in $C^{1,\alpha}$ for some $\alpha = \alpha(\lambda, \Lambda, \|f\|_{\infty,\Omega}, N, \Omega,, \beta)$. Moreover, u satisfies the following estimate

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\left(\|g\|_{C^{1,\beta}(\partial\Omega)} + \|u\|_{\infty,\Omega} + \|f\|_{\infty,\Omega}^{\frac{1}{1+\gamma}} \right),$$

for some positive constant $C = C(\alpha)$.

The following result plays an important role in Step 5 of the proof of our main result.

Theorem 2.7 ([7, Theorem 3.8]). *Let F_i, F be uniformly elliptic and $p > N$. Let $f, f_i \in L^p(\Omega)$. Let $u_i \in C(\Omega)$ be $W^{2,p}$-viscosity subsolutions (supersolutions) to*

$$F_i(D^2u_i) = f_i \quad \text{in } \Omega,$$

for $i = 1, 2, \ldots$ Assume that $u_i \longrightarrow u$ locally uniformly in Ω. Also, assume that if for each $B(x_0, r) \subset \Omega$ and $g \in W^{2,p}(B(x_0, r))$, we have

$$\left\| \left(F_i(D^2u_i) - f_i(x) - F(D^2(u)) + f(x) \right)^+ \right\|_{p, B(x_0,r)} \longrightarrow 0,$$

$$\left(\left\| \left(F_i(D^2u_i) - f_i(x) - F(D^2(u)) + f(x) \right)^- \right\|_{p, B(x_0,r)} \longrightarrow 0 \right).$$

Then, u is a $W^{2,p}$-viscosity subsolution (supersolution) to

$$F(D^2u) = f \quad \text{in } \Omega.$$

3   Proof of our main result

Proof of Theorem 1.1. The original problem is

$$\begin{cases} |Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2u(x)) = f(|u \geq u(x)|) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Step 1: Perturbing the L.H.S. by adding $\varepsilon \Delta u$. Consider the approximate problem:

$$\begin{cases} |Du|^{\gamma} \mathcal{M}^+_{\lambda, \Lambda}(D^2 u) + \varepsilon \Delta u = f(|u \geq u(x)|) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

for $\varepsilon > 0$. Since, $Gu := |Du|^{\gamma} \mathcal{M}^+_{\lambda, \Lambda}(D^2 u) + \varepsilon \Delta u$ is uniformly elliptic, so by Theorem 2.1 [8], we immediately have the existence of a $W^{2,p}$-viscosity solution (say u_ε) to (3.2) satisfying the following estimate:

$$\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C\left(\|u_\varepsilon\|_{\infty, \Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(|u_\varepsilon \geq u_\varepsilon(x)|)\|_{p, \Omega} \right).$$

By the above estimate, one can not directly claim the uniform bound on u_ε, which is crucial in order to pass the limit $\varepsilon \longrightarrow 0$ to establish the existence of $W^{2,p}$-viscosity solution to (3.1). To overcome this difficulty, we further approximate problem (3.2).

Step 2: Freeze a Lipschitz function v for the R.H.S.. Next, following the arguments similar to [8], we fix a Lipschitz function v in Ω, and consider $h_v(x) := f(|v \geq v(x)|)$ and reduce to the following problem:

$$\begin{cases} |Du|^{\gamma} \mathcal{M}^+_{\lambda, \Lambda}(D^2 u) + \varepsilon \Delta u = h_v & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Step 3: Building a sequence of functions in R.H.S. We consider a sequence of functions $\{h_v^i\}_{i=1}^{\infty}$ defined as

$$h_v^i(x) := f\left(i \int_0^{\frac{1}{i}} |v \geq v(x) - t| dt \right).$$

We approximate the function

$$h_v(x) \left(= f(|v \geq v(x)|) \right)$$

in the R.H.S. of (3.3) by the sequence of functions $\{h_v^i\}_{i=1}^{\infty}$. Hence, we have the following approximate problem:

$$\begin{cases} |Du|^{\gamma} \mathcal{M}^+_{\lambda, \Lambda}(D^2 u) + \varepsilon \Delta u = h_v^i & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

for $i \geq 1$. Since $\{h_v^i\} \in L^p(\Omega)$. For each i, by Theorem 2.5, we have the existence of a unique $W^{2,p}$-viscosity solution to (3.4).

Lemma 3.1. *There exists a unique $W^{2,p}$-viscosity solution to (3.4). Moreover, it satisfies the following estimate:*

$$\|u_\varepsilon^i\|_{W^{2,p}(\Omega)} \leq C\left(\max_{\partial\Omega} g + \|g\|_{W^{2,p}(\Omega)} + f(|\Omega|)|\Omega|^{\frac{1}{p}} \right).$$

Proof. By Theorem 2.5, we have the existence of a unique $W^{2,p}$-viscosity solution u_ε^i to (3.4) satisfying the following estimate:

$$\|u_\varepsilon^i\|_{W^{2,p}(\Omega)} \leq C\left(\|u_\varepsilon^i\|_{\infty, \Omega} + \|g\|_{W^{2,p}(\Omega)} + \|h_v^i\|_{p, \Omega} \right),$$

Also, it is easy to observe that

$$\|h_v^i\|_{\infty, \Omega} \leq f(|\Omega|), \tag{3.5}$$

and

$$\begin{aligned}
\|h_v^i\|_{p,\Omega} &= \left(\int_\Omega |h_v^i(x)|^p dx \right)^{\frac{1}{p}} \\
&\leq \|h_v^i\|_{\infty,\Omega} |\Omega|^{\frac{1}{p}} \\
&\leq f(|\Omega|)|\Omega|^{\frac{1}{p}},
\end{aligned} \tag{3.6}$$

for each $i \geq 1$. Thus the sequence of functions h_v^i is uniformly bounded. Now, by ABP estimates established in [7], we have

$$\sup_\Omega u_\varepsilon^i \leq \sup_{\partial\Omega} u_\varepsilon^i + C\|h_v^i\|_{p,\Omega},$$

and similarly for the $\inf_\Omega u_\varepsilon^i$. For more details, see Proposition 3.3 [7]. Using this along with the estimates (3.5) and (3.6), we have the following:

$$\|u_\varepsilon^i\|_{W^{2,p}(\Omega)} \leq \widetilde{C},$$

where $\widetilde{C}$ is a positive constant independent of i and ε. $\qquad\square$

Step 4: Establish the existence of solution to (3.3). It further gives that $\{u_\varepsilon^i\}$ is uniformly bounded in $W^{2,p}(\Omega)$ (with respect to i). Now, by reflexivity of $W^{2,p}(\Omega)$, u_ε^i converges weakly in $W^{2,p}(\Omega)$. Moreover, since $p > N/2$. Using the similar arguments as above, we have the existence of a subsequence such that $u_\varepsilon^i \longrightarrow u_{\varepsilon,v}$ in the Lipschitz norm. As a consequence of Theorem 2.7, $u_{\varepsilon,v}$ is a $W^{2,p}$-viscosity solution to (3.3). Moreover, $u_{\varepsilon,v}$ satisfies the following estimate:

$$\|u_{\varepsilon,v}\|_{W^{2,p}(\Omega)} \leq C\left(\max_{\partial\Omega} |g| + \|g\|_{W^{2,p}(\Omega)} + \|f(|v \geq v(x)|)\|_{p,\Omega} \right).$$

Step 5: Establish the existence of solution to (3.2). Further, using Theorem 2.1 [8] (an application of Schaefer fixed point theorem), we have the existence of a $W^{2,p}$-viscosity solution to (3.2) for each $0 < \varepsilon < 1$, say u_ε (a Lipschitz fixed point). Moreover, u_ε satisfies the following estimate:

$$\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C\left(\max_{\partial\Omega} |g| + \|g\|_{W^{2,p}(\Omega)} + \|f(|u_\varepsilon \geq u_\varepsilon(x)|)\|_{p,\Omega} \right). \tag{3.7}$$

Step 6: Establish the existence of solution to (3.1) **on** $\varepsilon \longrightarrow 0$. Since u_ε is uniformly bounded in $W^{2,p}(\Omega)$ (with respect to ε) so we have that along some subsequence, u_ε converges weakly in $W^{2,p}(\Omega)$. Moreover, by the Rellich–Kondrasov theorem, along some subsequence $u_\varepsilon \longrightarrow u$ in $C(\overline{\Omega})$ (since $p > N$) to a Lipschitz function u. We further claim that u is an L^p-viscosity solution to (3.1). We use the idea of [1]. We just check the supersolution part. Further, one can check for the subsolution part using the similar arguments. Let, if possible, assume that u is not an L^p-viscosity supersolution to (3.1). Then by definition, there exists a point $x_0 \in \Omega$ and a function $\phi \in W^{2,p}(\Omega)$ with $D\phi \neq 0$ such that $u - \phi$ has local minimum at x_0 and

$$|D\phi|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) - f(|u \geq u(x_0)|) \geq \alpha \text{ a.e. in some ball } B(x_0, r), \tag{3.8}$$

for some constant $\alpha > 0$. In other words, $u - \phi$ restricted to $\overline{B(x_0, r)}$ has a global strict minima at x_0. Next, using the above information, we get a contradiction by constructing a function $\phi_\varepsilon = \phi - \psi_\varepsilon$ corresponding to u_ε such that

$$|D\phi_\varepsilon|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi_\varepsilon) + \varepsilon\Delta\phi_\varepsilon - f(|u \geq u(x_0)|) \geq \alpha \quad \text{a.e. in } B(x_0, r) \tag{3.9}$$

for small enough $\varepsilon > 0$ and

$$\phi_\varepsilon \longrightarrow \phi \quad \text{uniformly.}$$

Now, since u_ε is an L^p-viscosity solution to (3.2) so (3.9) implies that $u_\varepsilon - \phi_\varepsilon$ can not attain minimum in the ball $B(x_0, r)$. However, since $u_\varepsilon - \phi_\varepsilon$ is continuous and $B(x_0, r)$ is compact. Therefore, $u_\varepsilon - \phi_\varepsilon$ attains minimum in $\overline{B(x_0, r)}$. Let it be x_ε. It gives that $x_\varepsilon \longrightarrow x_0$ along some subsequence. It further implies that $x_\varepsilon \in B(x_0, r)$ for small enough ε, which is a contradiction. Thus such a function ϕ constructed in (3.8) does not exist, which proves our claim that u is an L^p-supersolution to (3.1). Similarly, one can check the subsolution part.

Next, we show that u is the limit function of the sequence of functions u_ε as $\varepsilon \longrightarrow 0$. Let if possible, ε_i and $\widetilde{\varepsilon}_i$ be two sequences approaching 0 with u and $\widetilde{u}$ being the corresponding limit functions to the sequences, respectively. Up to subsequences, we may assume that

$$\cdots \leq \widetilde{\varepsilon}_{i+1} \leq \varepsilon_i \leq \widetilde{\varepsilon}_i \leq \varepsilon_{i-1} \leq \cdots .$$

Our aim is to show that $w = u_{\varepsilon_i} - u_{\widetilde{\varepsilon}_{i+1}} \leq 0$. If we show that

$$|Dw|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + \mathcal{E}\Delta w \geq 0 \quad \text{in } \Omega \text{ (in } C\text{-viscosity sense)},$$

we are done. As by comparison principle, we would immediately get $w \leq 0$. Therefore, $u_{\varepsilon_i} \leq u_{\widetilde{\varepsilon}_{i+1}}$. Thus, in order to show that

$$w = u_{\varepsilon_i} - u_{\widetilde{\varepsilon}_{i+1}} \leq 0,$$

we only need to show that

$$|Dw|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + \mathcal{E}_i \Delta w \geq 0.$$

As shown above, it immediately gives $w \leq 0$. Let us assume the contrary, i.e., there exists some point $x_0 \in \Omega$ such that for some $\varphi \in C^2(\Omega)$, $w - \varphi$ attains local maxima at x_0, i.e., there exists a ball $B(x_0, r)$ such that

$$|D\varphi|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2 \varphi) + \mathcal{E}_i \Delta \varphi \leq -\alpha \quad \text{in } B(x_0, r),$$

for some $\alpha > 0$ and $w - \varphi = \left(u_{\varepsilon_i} - u_{\widetilde{\varepsilon}_{i+1}}\right) - \varphi = u_{\varepsilon_i} - \left(u_{\widetilde{\varepsilon}_{i+1}} + \varphi\right)$ has a global strict maximum at x_0 in $B(x_0, r)$. Now, consider a function

$$\Psi := \varphi + u_{\widetilde{\varepsilon}_{i+1}}.$$

Clearly, $\Psi \in W^{2,p}(\Omega)$ and touches u_{ε_i} from above at x_0. Also, consider a test function, Φ for

$u_{\widetilde{\varepsilon}_{i+1}}$ touching from below with $|D\Phi(x_0)|$ sufficiently larger than $|D\varphi(x_0)|$. We have

$$
\begin{aligned}
|D\Psi(x_0)|^\gamma & \mathcal{M}^+_{\lambda,\Lambda}\big(D^2\Psi(x_0)\big) + \mathcal{E}_i\Delta\Psi(x_0) - f\big(|u \geq u(x_0)|\big) + \alpha \\
& \leq |D\Psi(x_0)|^\gamma\big(\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi(x_0)) + \mathcal{M}^+_{\lambda,\Lambda}(D^2\Phi(x_0))\big) + \mathcal{E}_i\Delta\varphi(x_0) \\
& \quad + \mathcal{E}_i\Delta\Phi(x_0) - f\big(|u \geq u(x_0)|\big) + \alpha \\
& = |D\Psi(x_0)|^\gamma\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi(x_0)) + \mathcal{E}_i\Delta\varphi(x_0) + |D\Psi(x_0)|^\gamma\mathcal{M}^+_{\lambda,\Lambda}(D^2\Phi(x_0)) \\
& \quad + \mathcal{E}_i\Delta\Phi(x_0) - f\big(|u \geq u(x_0)|\big) + \alpha \\
& = |D\varphi(x_0) + D\Phi(x_0)|^\gamma\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi(x_0)) + \mathcal{E}_i\Delta\varphi(x_0) + \alpha \\
& \quad + |D\varphi(x_0) + D\Phi(x_0)|^\gamma\mathcal{M}^+_{\lambda,\Lambda}(D^2\Phi(x_0)) + \mathcal{E}_i\Delta\Phi(x_0) - f\big(|u \geq u(x_0)|\big) \\
& \leq \frac{|D\varphi(x_0) + D\Phi(x_0)|^\gamma}{|D\varphi(x_0)|^\gamma}\big(-\alpha - \mathcal{E}_i\Delta\varphi(x_0)\big) + \mathcal{E}_i\Delta\varphi(x_0) + \alpha \\
& \quad + \frac{|D\varphi(x_0) + D\Phi(x_0)|^\gamma}{|D\Phi(x_0)|^\gamma}\big(f(|u \geq u(x_0)|) - \mathcal{E}_i\Delta\Phi(x_0)\big) + \mathcal{E}_i\Delta\Phi(x_0) \\
& \quad - f\big(|u \geq u(x_0)|\big) \\
& \leq \frac{|D\varphi(x_0) + D\Phi(x_0)|^\gamma}{|D\varphi(x_0)|^\gamma}\big(-\alpha - \mathcal{E}_i\Delta\varphi(x_0)\big) + \mathcal{E}_i\Delta\varphi(x_0) + \alpha \\
& \quad + 2^{\gamma-1}\frac{|D\varphi(x_0)|^\gamma + |D\Phi(x_0)|^\gamma}{|D\Phi(x_0)|^\gamma}\big(f(|u \geq u(x_0)|) - \mathcal{E}_i\Delta\Phi(x_0)\big) + \mathcal{E}_i\Delta\Phi(x_0) \\
& \quad - f\big(|u \geq u(x_0)|\big) \\
& \leq \big(-\alpha - \mathcal{E}_i\Delta\varphi(x_0)\big)\left(\frac{|D\varphi(x_0) + D\Phi(x_0)|^\gamma}{|D\varphi(x_0)|^\gamma} - 1\right) \\
& \quad + \big(f(|u \geq u(x_0)|) - \mathcal{E}_i\Delta\Phi(x_0)\big)\left(2^{\gamma-1}\frac{(|D\varphi(x_0)|^\gamma + |D\Phi(x_0)|^\gamma)}{|D\Phi(x_0)|^\gamma} - 1\right),
\end{aligned}
\tag{3.10}
$$

for all large enough $i \in \mathbb{N}$. Note that in the second last step we used the fact that for any positive real numbers a, b and $r \geq 1$, we have

$$
|a + b|^r \leq 2^{r-1}\big(|a|^r + |b|^r\big).
$$

Further, by the choice of test function Φ made before (3.10), we have

$$
\mathcal{M}^+_{\lambda,\Lambda}(D^2\Psi) + \mathcal{E}_i\Delta\Psi - f\big(|u \geq u(x_0)|\big) \leq -\alpha < 0,
$$

which contradicts the fact that u_{ε_i} is an L^p-viscosity solution to (3.2). Thus we have that $u_{\varepsilon_i} \leq u_{\widetilde{\varepsilon}_{i+1}}$. Letting $i \longrightarrow \infty$, we get $u \leq \widetilde{u}$. Also, following the similar arguments, one can show that $u_{\widetilde{\varepsilon}_{i+1}} \leq u_{\varepsilon_i}$. Thus, we have $\widetilde{u} \leq u$ and hence $u = \widetilde{u}$.

Therefore, we have the existence of a $W^{2,p}$-viscosity solution, u to (3.1). Moreover, by (3.7), we have the following estimate:

$$
\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(|u \geq u(x)|)\|_{p,\Omega}\right). \qquad \square
$$

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References

[1] A. L. AMADORI, B. BRANDOLINI, C. TROMBETTI, Viscosity solutions of the Monge–Ampére equation with the right hand side in L^p, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **18**(2007), No. 3, 221–233. https://doi.org/10.4171/rlm/491; MR2318817; Zbl 1223.35167

[2] I. BIRINDELLI, F. DEMENGEL, $C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations, *ESAIM Control Optim. Calc. Var.* **20**(2014), 1009–1024. https://doi.org/10.1051/cocv/2014005; MR3264232; Zbl 1315.35108

[3] I. BIRINDELLI, F. DEMENGEL, Comparison principle and Liouville type results for singular fully nonlinear operators, *Ann. Fac. Sci. Toulouse Math.* **13**(2004), No. 2, 261–287. https://doi.org/10.5802/afst.1070; MR2126744; Zbl 1129.35369

[4] I. BIRINDELLI, F. DEMENGEL, Eigenvalue, maximum principle and regularity for fully nonlinear homogeneous operators, *Commun. Pure Appl. Anal.* **6**(2007), No. 2, 335–366. https://doi.org/10.3934/cpaa.2007.6.335; MR2289825; Zbl 1132.35032

[5] I. BIRINDELLI, F. DEMENGEL, Regularity and uniqueness of the first eigenfunction for singular, fully nonlinear elliptic operators, *J. Differential Equations* **249**(2010), No. 5, 1089–1110. https://doi.org/10.1016/j.jde.2010.03.015; MR2652165; Zbl 1193.35115

[6] I. BIRINDELLI, F. DEMENGEL, Regularity for radial solutions of degenerate fully nonlinear equations, *Nonlinear Anal.* **75**(2012), No. 17, 6237–6249. https://doi.org/10.1016/j.na.2012.06.028; MR2959802; Zbl 1254.35099

[7] L. CAFFARELLI, M. G. CRANDALL, M. KOCAN, A. SWIECH, On viscosity solutions of fully nonlinear equations with measurable ingredients, *Comm. Pure Appl. Math.* **49**(1993), No. 4, 365–398. https://doi.org/10.1002/(SICI)1097-0312(199604)49:4<365::AID-CPA3>3.0.CO;2-A; MR1376656; Zbl 0854.35032

[8] L. CAFFARELLI, I. TOMASETTI, Fully nonlinear equations with application to Grad equations in plasma physics, *Comm. Pure Appl. Math.* **76**(2023), No. 3, 604–615. https://doi.org/10.1002/cpa.22026; MR4544806; Zbl 07748336

[9] M. G. CRANDALL, P.-L. LIONS, Viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.* **277**(1983), No. 1, 1–42. https://doi.org/10.1090/S0002-9947-1983-0690039-8; MR0690039; Zbl 0599.35024

[10] P. FELMER, A. QUAAS, On critical exponents for the Pucci's extremal operators, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20**(2003), No. 5, 843–865. https://doi.org/10.1016/S0294-1449(03)00011-8; MR1995504; Zbl 1274.35115

[11] P. Felmer, A. Quaas, Positive radial solutions to a "semilinear" equation involving the Pucci's operator, *J. Differential Equations* **199**(2004), No. 2, 376–393. `https://doi.org/10.1016/j.jde.2004.01.001`; MR2047915; Zbl 1070.34032

[12] H. Grad, *Alternating dimension plasma transport in three dimensions*, Technical report, New York Univ. NY (USA), Courant Inst. of Mathematical Sciences, 1979. MR0584040; Zbl 0466.76125

[13] C. Imbert, L. Silvestre, $C^{1,\alpha}$ regularity of solutions of some degenerate, fully nonlinear elliptic equations, *Adv. Math.* **233**(2013), 196–206. `https://doi.org/10.1016/j.aim.2012.07.033`; MR2995669; Zbl 1262.35065

[14] H. Ishii, P.-L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, *J. Differential Equations* **83**(1990), 26–78. `https://doi.org/10.1016/0022-0396(90)90068-Z`; MR1031377; Zbl 0708.35031

[15] P. Laurence, E. Stredulinsky, A new approach to queer differential equations, *Comm. Pure Appl. Math.* **38**(1985), No. 3, 333–355. `https://doi.org/10.1002/cpa.3160380306`; MR0784478; Zbl 0818.35145

[16] P. Laurence, E. Stredulinsky, A bootstrap argument for grad generalized differential equations, *Indiana Univ. Math. J.* **38**(1989), No. 2, 377–415. `https://doi.org/10.1512/iumj.1989.38.38019`; MR0997389; Zbl 0712.35102

[17] J. Mossino, R. Temam, Directional derivative of the increasing rearrangement mapping and application to a queer differential equation in plasma physics, *Duke Math. J.* **48**(1981), No. 3, 475–495. `https://doi.org/10.1215/S0012-7094-81-04827-4`; MR0630581; Zbl 0476.35031

[18] P. Oza, J. Tyagi, Probabilistic method to solution representation formula & Hopf lemma to Pucci's equation, under review, 2023, 13 pp.

[19] P. Oza, J. Tyagi, Regularity of solutions to variable-exponent degenerate mixed fully nonlinear local and nonlocal equations, under review, 2023, 26 pp. `https://arxiv.org/abs/2302.06046`

[20] A. Quaas, Existence of a positive solution to a "semilinear" equation involving Pucci's operator in a convex domain, *Differential Integral Equations* **17**(2004), No. 5–6, 481–494. `https://doi.org/10.57262/die/1356060343`; MR2054930; Zbl 1174.35373

[21] A. Quaas, B. Sirakov, Existence results for nonproper elliptic equations involving the Pucci operator, *Comm. Partial Differential Equations* **31**(2006), No. 7–9, 987–1003. `https://doi.org/10.1080/03605300500394421`; MR2254600; Zbl 1237.35056

[22] R. Temam, Monotone rearrangement of a function and the Grad–Mercier equation of plasma physics, in: *Recent methods in nonlinear analysis, Proc. Int. Meet., Rome 1978*, Pitagora, Bologna, 1979, pp. 83–98. MR0533163; Zbl 0405.35066

[23] I. Tomasetti, *Fully nonlinear equations with applications to Grad equations in Plasma physics; Interaction between a one phase free boundary problem and an obstacle problem; Optimal trace Sobolev inequalities*, UT Electronic Theses and Dissertations, 2022.

[24] J. TYAGI, R. B. VERMA, Lyapunov-type inequality for extremal Pucci's equations, *J. Aust. Math. Soc.* **109**(2020), No. 3, 416–430. https://doi.org/10.1017/S1446788719000569; MR4190087; Zbl 1454.35142

[25] J. TYAGI, R. B. VERMA, Positive solution of extremal Pucci's equations with singular and sublinear nonlinearity, *Mediterr. J. Math.* **14**(2017), Article no. 148. https://doi.org/10.1007/s00009-017-0950-6; MR3661607; Zbl 1379.35125

[26] J. TYAGI, R. B. VERMA, Positive solution to extremal Pucci's equations with singular and gradient nonlinearity, *Discrete Contin. Dyn. Syst.* **39**(2019), No. 5, 2637–2659. https://doi.org/10.3934/dcds.2019110; MR3927527; Zbl 1421.35120

[27] N. WINTER, $W^{2,p}$- and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations, *Z. Anal. Anwend.* **28**(2009), No. 2, 129–164. https://doi.org/10.4171/ZAA/1377; MR2486925; Zbl 1206.35116