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# Well-posedness and controllability of a nonlinear system for surface waves

**Alex M. Montes**[1] and **Ricardo Córdoba**✉ [2]

[1]University of Cauca, Popayán, Colombia
[2]University of Nariño, San Juan de Pasto, Colombia

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**Abstract.** In this paper we study the well-posedness for the periodic Cauchy problem and the internal controllability of a one-dimensional system that describes the propagation of long water waves with small amplitude in the presence of surface tension. The well-posedness is proved by using the Fourier transform restriction method and the controllability is proved by using the moment method.

**Keywords:** nonlinear system, water waves, well-posedness, Bourgain spaces, spectral analysis, internal control.

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## 1 Introduction

In the present work we consider the periodic Cauchy problem and the internal controllability of the following one-dimensional system

$$\begin{cases} \eta_t + \partial_x^2 \Phi - \partial_x^4 \Phi + \partial_x (\eta \partial_x \Phi) = 0, \\ \Phi_t + \eta - \partial_x^2 \eta + \dfrac{1}{2} (\partial_x \Phi)^2 = 0, \end{cases} \tag{1.1}$$

which is a rescaled version of the system derived in [14] from the evolution of long water waves with small amplitude in the presence of surface tension, where $\Phi = \Phi(x,t)$ represents the nondimensional velocity potential on the bottom $z = 0$ and the variable $\eta = \eta(x,t)$ corresponds the free surface elevation.

As happens in water wave models, there is a Hamiltonian type structure which is clever to characterize the space for the study of the Cauchy problem. In our particular system (1.1), the Hamiltonian functional $\mathcal{H} = \mathcal{H}(t)$ is defined as

$$\mathcal{H}\begin{pmatrix} \eta \\ \Phi \end{pmatrix} = \frac{1}{2} \int_{\mathbb{R}} \left( \eta^2 + (\partial_x \eta)^2 + (\partial_x \Phi)^2 + (\partial_x^2 \Phi)^2 + \eta (\partial_x \Phi)^2 \right) dx,$$

✉Corresponding author. Email: rcordoba@udenar.edu.co

and the Hamiltonian type structure is given by

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix} = \mathcal{J}\mathcal{H}'\begin{pmatrix}\eta\\ \Phi\end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We see directly that the functional $\mathcal{H}$ is well defined when

$$\eta(\cdot,t), \Phi_x(\cdot,t) \in H^1(\mathbb{R}),$$

for $t$ in some interval. These conditions already characterize the natural space for the study of solutions of the system (1.1). Certainly, J. Quintero and A. Montes in [16] showed for the model (1.1) the existence of solitary wave solutions which propagate with speed of wave $\theta > 0$, i.e. solutions of the form

$$\eta(x,t) = u(x-\theta t), \quad \Phi(x,t) = v(x-\theta t),$$

in the energy space $H^1(\mathbb{R}) \times \mathcal{V}^2(\mathbb{R})$, where $H^1(\mathbb{R})$ is the usual Sobolev space of order 1 and the space $\mathcal{V}^2(\mathbb{R})$ is defined with respect to the norm given by

$$\|w\|^2_{\mathcal{V}^2(\mathbb{R})} = \|w'\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \left((w')^2 + (w'')^2\right) dx.$$

They also showed, using the estimates of the Kato's commutator, the local well-posedness for the Cauchy problem associated to the system (1.1) in the Sobolev type space $H^s(\mathbb{R}) \times \mathcal{V}^{s+1}(\mathbb{R})$, with $s > 3/2$, where $H^s(\mathbb{R})$ is the usual Sobolev space of order $s$ defined as the completion of the Schwartz class with respect to the norm

$$\|w\|_{H^s(\mathbb{R})} = \|\,(1+|\xi|)^s\,\widehat{w}(\xi)\|_{L^2_\xi}$$

and $\mathcal{V}^{s+1}(\mathbb{R})$ denotes the completion of the Schwartz class with respect to the norm

$$\|w\|_{\mathcal{V}^{s+1}(\mathbb{R})} = \|\,(1+|\xi|)^s\,|\xi|\widehat{w}(\xi)\|_{L^2_\xi},$$

where $\widehat{w}$ is the Fourier transform of $w$ in the space variable $x$ and $\xi$ is the variable in the frequency space related to the variable $x$. Using Bourgain type spaces, in work [13] the authors showed that the Cauchy problem associated to the system (1.1) is locally well-posedness in the space $H^s(\mathbb{R}) \times \mathcal{V}^{s+1}(\mathbb{R})$ for $s \geq 0$.

On the case of the periodic domain $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (the one-dimensional torus), it was proved in [15] the local well-posedness of the Cauchy problem associated to system (1.1) in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, for $s > 3/2$, where the periodic Sobolev space $H^s(\mathbb{T})$ is defined by

$$H^s(\mathbb{T}) = \left\{ w = \sum_{k\in\mathbb{Z}} w_k e^{ikx} \;:\; \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |w_k|^2 < +\infty \right\}$$

and the space $\mathcal{V}^{s+1}(\mathbb{T})$ is defined by the norm

$$\|w\|_{\mathcal{V}^{s+1}(\mathbb{T})} = \left[\sum_{k\in\mathbb{Z}} (1+|k|^2)^s |k|^2 |w_k|^2\right]^{1/2},$$

where $w_k = \widehat{w}(k)$ denotes the $k$-Fourier coefficient with respect to the spatial variable $x$. In this paper, we prove that the Cauchy problem associated to system (1.1) with the initial condition

$$\eta(x,0) = \eta_0(x), \quad \Phi(x,0) = \Phi_0(x) \tag{1.2}$$

is locally well-posed in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ for $s \geq 0$. Hence we improve the result found in [15].

To study the Cauchy problem (1.1)–(1.2) we use its integral equivalent formulation,

$$(\eta(t), \Phi(t)) = S(t)(\eta_0, \Phi_0) - \int_0^t S(t-t')\Big(\partial_x(\eta\partial_x\Phi), \frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt', \tag{1.3}$$

where $S(t)(\eta_0, \Phi_0)$ is the solution of the linear problem, that is

$$S(t)(\eta_0, \Phi_0) = (S_1(t)(\eta_0, \Phi_0), S_2(t)(\eta_0, \Phi_0))$$

with

$$S_1(t)(\eta_0, \Phi_0) = \sum_{k\in\mathbb{Z}} e^{ikx}\left[\cos(\phi(k)t)\widehat{\eta}_0(k) + |k|\sin(\phi(k)t)\widehat{\Phi}_0(k)\right],$$

$$S_2(t)(\eta_0, \Phi_0) = \sum_{k\in\mathbb{Z}} e^{ikx}\left[-\frac{\sin(\phi(k)t)\widehat{\eta}_0(k)}{|k|} + \cos(\phi(k)t)\widehat{\Phi}_0(k)\right],$$

and the function $\phi$ defined as

$$\phi(k) = |k|^3 + |k|$$

is the Fourier symbol associated to the spacial linear part of the system (1.1).

The method of proof will be the application of the contraction mapping principle in a suitable Banach function space $C\left([0,T] : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right) \cap \mathcal{Z}^s$, where the appropriated space-time weight norm for $\mathcal{Z}^s$ is determined by the knowledge of certain estimates for the solutions of the linear part. This method, introduced by J. Bourgain in [2]-[3] and simplified by C. Kenig, G. Ponce and L. Vega in [8]-[9], not only benefits of the above mentioned space-time estimates, but also exploits structural properties of the nonlinearity.

As usual when dealing with dispersive models in Bourgain spaces, we slightly modify the terms in the right-hand of (1.3) by means of a cut off function. In the following, let $\psi \in C_0^\infty(\mathbb{R})$ with support in $(-2, 2)$, such that $0 \leq \psi \leq 1$, and $\psi \equiv 1$ in $[-1, 1]$. Thus, for $0 < T < 1$ we consider the following modified version of (1.3),

$$(\eta(t), \Phi(t)) = \psi(t)S(t)(\eta_0, \Phi_0) - \psi(t)\int_0^t S(t-t')\psi(t')\Big(\partial_x(\eta\partial_x\Phi), \frac{1}{2}\left(\partial_x\Phi\right)^2\Big)(t')dt'. \tag{1.4}$$

We will show the existence of a solution of the integral problem (1.4) using the Banach fixed point theorem and appropriate linear and nonlinear estimates.

The second part of this paper is concerned with the internal control problem for the system (1.1) on the periodic domain $\mathbb{T}$: choose an appropriate internal control function

$$F = F(x,t) = (f_1(x,t), f_2(x,t))$$

to guide the model

$$\begin{cases} \eta_t + \partial_x^2\Phi - \partial_x^4\Phi + \partial_x(\eta\partial_x\Phi) = f_1, & x \in \mathbb{T},\ t \geq 0, \\ \Phi_t + \eta - \partial_x^2\eta + \dfrac{1}{2}\left(\partial_x\Phi\right)^2 = f_2, & x \in \mathbb{T},\ t \geq 0, \end{cases} \tag{1.5}$$

during a time interval $[0, T]$, from a given initial state to another preassigned terminal state, in an appropriate function space of system states.

During the last years, there have been many contributions to the internal controllability for different dispersive wave models. For instance, in the case of the Korteweg–de Vries equation

D. Russell and B. Zhang in [18] showed that for $T > 0$ and functions $u_0, u_T \in H^s(\mathbb{T})$, $s \geq 0$, one can always find a control $f$ so that the Cauchy problem

$$u_t + uu_x + u_{xxx} = f, \quad u(x,0) = u_0(x),$$

has a solution $u \in C\left([0,T] : H^s(\mathbb{T})\right)$ satisfying

$$u(x,T) = u_T(x), \quad x \in \mathbb{T},$$

when the initial and terminal states are sufficiently small. A similar result was proved by B. Zhang in [20] for the Boussinesq model,

$$u_{tt} - u_{xx} + \left(u^2 + u_{xx}\right)_{xx} = f, \quad u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x),$$

with the condition

$$u(x,T) = u_T(x), \quad u_t(x,T) = v_T(x),$$

in the space $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ with $s \geq 2$. In the work [5], E. Cerpa and I. Rivas showed controllability for the Boussinesq equation in low regularity, this is, in the space $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ with $s > -\frac{1}{4}$.

For the Benjamin–Bona–Mahony equation, L. Rosier and B. Zhang in [17] proved that

$$u_t + u_x - u_{xxt} + uu_x = a(x+ct)h(x,t),$$

with a moving distributed control is controllable in $H^s(\mathbb{T})$ for any $s \geq 1$ in (sufficiently) large time. The control time is chosen in such a way that the support of the control, which is moving at the constant velocity $c$, can visit all the domain $\mathbb{T}$.

C. Laurent, F. Linares and L. Rosier in [11] and F. Linares and L. Rosier in [12] considered the control problem for the Benjamin–Ono equation,

$$u_t + \mathcal{H}(u_{xx}) + uu_x = f, \quad u(x,0) = u_0(x), \quad u(x,T) = u_T(x).$$

In the latter work, authors proved a controllability result in $L^2(\mathbb{T})$ that allows to prove the global controllability in large time.

Our main result in Theorem 5.4 gives a positive answer to the internal controllability for the system (1.5) in a local sense. We will show that for $T > 0$ and initial an terminal states

$$(\eta_0, \Phi_0), \quad (\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}), \quad s \geq 0,$$

sufficiently small, there exists a control function $F = (f_1, f_2)$ such that the Cauchy problem associated to the system (1.5) with the initial condition

$$\eta(x,0) = \eta_0(x), \quad \Phi_0(x,0) = \Phi_0(x), \quad x \in \mathbb{T}, \tag{1.6}$$

has a solution $(\eta, \Phi) \in C\left([0,T] : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right)$ satisfying the condition

$$\eta(x,T) = \eta_T(x), \quad \Phi(x,T) = \Phi_T(x), \quad x \in \mathbb{T}.$$

Following the same approach used in the case of the KdV equation and Boussinesq equation, we restrict our attention to a control of the form

$$F(x,t) = (f_1(x,t), f_2(x,t)) = (\rho_1 h_1(x,t), \rho_2 h_2(x,t)),$$

with $\rho_i$ being a smooth function defined on $\mathbb{T}$. Thus

$$H(x,t) = (h_1(x,t), h_2(x,t))$$

is a new control input. Here, the function $\rho_i$ can have a support strictly contained on the torus; thus, it can represent a localization of the control $h_i(x,t)$, which would be only able to act on a part of the domain. First, we perform a spectral analysis for the operator

$$M = \begin{pmatrix} 0 & -(I-\partial_x^2)\partial_x^2 \\ -(I-\partial_x^2) & 0 \end{pmatrix},$$

defined in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$. Using that the $k$-Fourier symbol for the operator $M$ is given by

$$M_k = \begin{pmatrix} 0 & (1+k^2)k^2 \\ -(1+k^2) & 0 \end{pmatrix},$$

we prove for $M$ the existence of a discrete spectral decomposition since the eigenvectors form a Riesz basis of the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$. Next, using this spectral analysis and the moment method we establish that the linear system associated with (1.5),

$$\begin{cases} \eta_t + \partial_x^2\Phi - \partial_x^4\Phi = f_1, \\ \Phi_t + \eta - \partial_x^2\eta = f_2, \end{cases} \tag{1.7}$$

is exactly controllable in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, with the conditions

$$\eta(0) = \eta_0, \quad \eta(T) = \eta_T, \quad \Phi(0) = \Phi_0, \quad \Phi(T) = \Phi_T. \tag{1.8}$$

Finally, the nonlinear problem is treated as a perturbation by fixed point theory.

The paper is organized as follows. In Section 2, first we define the Bourgain spaces related to our problem and next we establish all the linear estimates needed to proved the result of well-posedness. In Section 3 we estimate the bilinear forms $\partial_x(\eta\partial_x\Phi)$ and $(\partial_x\Phi)(\partial_x\Phi_1)$ associated to the nonlinear part of the system. The Section 4 will be dedicated to establish the result of local well-posedness, via a standard fixed point argument. In Section 5.1, we perform the spectral analysis for the operator $M$ defined in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, for $s \geq 0$. In Section 5.2, by solving a moment problem we found the characterization of the internal control $F = (f_1, f_2)$ for the linear problem (1.7)–(1.8). In Section 5.3, we prove the exact controllability result for the nonlinear problem, by imposing smallness of the initial and terminal states. The proof of this result is mainly based on the linear controllability and the Banach Fixed Point Theorem.

## 2 Bourgain spaces and linear estimates

We star with the definition of the Bourgain type spaces. We consider the space $\mathcal{Y}$ of functions $w$ such that

(i) $w : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, (ii) $w(x,\cdot) \in \mathcal{S}(\mathbb{R})$ for all $x \in \mathbb{T}$,

(iii) $x \to w(x,\cdot) \in C^\infty(\mathbb{R})$, (iv) $\widehat{w}(0,t) = 0$ for all $t \in \mathbb{R}$,

**Definition 2.1.** For $s, \beta \in \mathbb{R}$ we define the Bourgain spaces $X^{s,\beta}$ to be the completion of the space $\mathcal{Y}$ with respect to the norm

$$\|w\|_{X^{s,\beta}} = \|\langle k\rangle^s \langle |\tau| - \phi(k)\rangle^\beta \widetilde{w}\|_{\ell^2_k L^2_\tau},$$

where $\langle a\rangle = 1 + |a|$; $\tilde{w}$ denotes the time-space Fourier transform of $w$,

$$\widetilde{w}(k,\tau) = \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-ixk-it\tau} w(x,t)dxdt;$$

and the function $\phi$ is defined as

$$\phi(k) = |k|^3 + |k|.$$

The spaces $Y^{s+1,\beta}$ to be the completion of the Schwartz class $\mathcal{S}_{per,2\pi} = \mathcal{S}(\mathbb{T}\times\mathbb{R})$ with respect to the norm

$$\|w\|_{Y^{s+1,\beta}} = \||k|\langle |\tau| - \phi(k)\rangle^\beta \langle k\rangle^s \widetilde{w}\|_{\ell^2_k L^2_\tau}.$$

We similarly introduce the spaces $Z^s, W^{s+1}$, $s \in \mathbb{R}$, with the norms

$$\|w\|_{Z^s} = \|w\|_{X^{s,-1/2}} + \left\|\frac{\langle k\rangle^s \widetilde{w}(k,\tau)}{\langle |\tau| - \phi(k)\rangle}\right\|_{\ell^2_k L^1_\tau},$$

and

$$\|w\|_{W^{s+1}} = \|w\|_{Y^{s+1,-1/2}} + \left\|\frac{|k|\langle k\rangle^s \widetilde{w}(k,\tau)}{\langle |\tau| - \phi(k)\rangle}\right\|_{\ell^2_k L^1_\tau}.$$

Also, we consider the spaces $U^s, V^{s+1}, s \in \mathbb{R}$, where $U^s$ denotes the completion of the Schwartz class $\mathcal{S}_{per,2\pi}$ with respect to the norm

$$\|w\|_{U^s} = \|w\|_{X^{s,1/2}} + \|\langle k\rangle^s \widetilde{w}(k,\tau)\|_{\ell^2_k L^1_\tau}$$

and $V^{s+1}$ denotes the completion of the Schwartz class $\mathcal{S}_{per,2\pi}$ with respect to the norm

$$\|w\|_{V^{s+1}} = \|w\|_{Y^{s+1,1/2}} + \||k|\langle k\rangle^s \widetilde{w}(k,\tau)\|_{\ell^2_k L^1_\tau}.$$

For $T > 0$ we denote by $U^s_T$ the space space of the restrictions to the interval $[0,T]$ of the elements $w \in U^s$ with norm defined by

$$\|\eta\|_{U^s_T} = \inf_{w\in U^s}\{\|w\|_{U^s} : \eta(t) = w(t) \quad \text{on } [0,T]\}.$$

and by $V^{s+1}_T$ the space space of the restrictions to the interval $[0,T]$ of the elements $w \in V^{s+1}$ with norm defined by

$$\|\Phi\|_{V^{s+1}_T} = \inf_{w\in V^{s+1}}\{\|w\|_{V^{s+1}} : \Phi(t) = w(t) \quad \text{on } [0,T]\}.$$

Next we look at some basic results.

**Lemma 2.2.** *Let $s \in \mathbb{R}$, then there exists $C > 0$ such that*

*(i)* $\|\psi\eta\|_{X^{s,-1/2}} \le C\|\eta\|_{X^{s,-1/2}}$,

*(ii)* $\|\psi\Phi\|_{Y^{s+1,-1/2}} \le C\|\Phi\|_{Y^{s+1,-1/2}}$.

*Proof.* We will use the notation $\widehat{w}^{(t)}$ for the Fourier transform of $w$ in the time variable $t$. Note that

$$\widetilde{\psi\eta}(k,\tau) = \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-ixk}e^{-it\tau}\left(\int_{\mathbb{R}} e^{it\lambda}\widehat{\psi}^{(t)}(\lambda)d\lambda\right)\eta(x,t)dxdt$$
$$= \int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda$$

and also

$$\|\psi\eta\|^2_{X^{s,-1/2}} = \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle^{-1}\Big|\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\Big|^2 d\tau.$$

Moreover

$$\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{-\infty}^{0}\langle|\tau|-\phi(k)\rangle^{-1}\Big|\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\Big|^2 d\tau$$
$$\leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle\tau+\phi(k)\rangle^{-1}\Big|\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\Big|^2 d\tau$$
$$= \Big\|\langle\tau+\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\Big\|^2_{\ell^2_kL^2_\tau}$$
$$\leq \int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\|\langle\tau+\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\widetilde{\eta}(k,\tau-\lambda)\|^2_{\ell^2_kL^2_\tau}d\lambda$$

and

$$\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{0}^{+\infty}\langle|\tau|-\phi(k)\rangle^{-1}\Big|\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\Big|^2 d\tau$$
$$\leq \Big\|\langle\tau-\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\Big\|^2_{\ell^2_kL^2_\tau}$$
$$\leq \int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\|\langle\tau-\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\widetilde{\eta}(k,\tau-\lambda)\|^2_{\ell^2_kL^2_\tau}d\lambda.$$

Next, using the inequality

$$\big||\tau|-\phi(k)\big| \leq \min\{|\tau-\phi(k)|,|\tau+\phi(k)|\},$$

we have for all $\lambda\in\mathbb{R}$, $\langle|\tau|-\phi(k)\rangle \leq \langle\tau\pm\phi(k)\rangle \leq \langle\tau+\lambda\pm\phi(k)\rangle\langle\lambda\rangle$, and then

$$\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\|\langle\tau\pm\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\widetilde{\eta}(k,\tau-\lambda)\|^2_{\ell^2_kL^2_\tau}d\lambda$$
$$= \int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\Big(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle\tau+\lambda\pm\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|^2d\tau\Big)d\lambda$$
$$\leq \int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\Big(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle\lambda\rangle\langle|\tau|-\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|^2d\tau\Big)d\lambda$$
$$= \|\langle|\tau|-\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\widetilde{\eta}\|^2_{\ell^2_kL^2_\tau}\int_{\mathbb{R}}\langle\lambda\rangle|\widehat{\psi}^{(t)}(\lambda)|^2d\lambda$$
$$\leq C\|\eta\|^2_{X^{s,-1/2}}.$$

Thus, we conclude that

$$\|\psi\eta\|_{X^{s,-1/2}} \leq C\|\eta\|_{X^{s,-1/2}}.$$

In a similar fashion we have that

$$\int_{\mathbb{R}} |\widehat{\psi}(\lambda)^{(t)}|^2 \|\langle \tau \pm \phi(k)\rangle^{-1/2}|k|\langle k\rangle^s \widetilde{\Phi}(k,\tau-\lambda)\|^2_{\ell^2_k L^2_\tau} d\lambda$$
$$\leq \|\langle |\tau|-\phi(k)\rangle^{-1/2}|k|\langle k\rangle^s \widetilde{\Phi}\|^2_{\ell^2_k L^2_\tau} \int_{\mathbb{R}} \langle \lambda\rangle |\widehat{\psi}^{(t)}(\lambda)|^2 d\lambda \leq C\|\Phi\|^2_{Y^{s+1,-1/2}}.$$

Therefore

$$\|\psi\Phi\|_{Y^{s+1,-1/2}} \leq C\|\Phi\|_{Y^{s+1,-1/2}}. \qquad \square$$

Similarly, we have also the following lemma.

**Lemma 2.3.** *Let $s\in\mathbb{R}$, then there exists $C>0$ such that*

*(i)* $\left\| \frac{\langle k\rangle^s \widetilde{\psi\eta}(k,\tau)}{\langle |\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \leq C \left\| \frac{\langle k\rangle^s \widetilde{\eta}(k,\tau)}{\langle |\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau}$,

*(ii)* $\left\| \frac{|k|\langle k\rangle^s \widetilde{\psi\Phi}(k,\tau)}{\langle |\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \leq C \left\| \frac{|k|\langle k\rangle^s \widetilde{\Phi}(k,\tau)}{\langle |\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau}$.

In the following lemmas we establish estimations related with the semigroup $S(t)$.

**Lemma 2.4.** *Let $s\in\mathbb{R}$, then exists $C_1>0$ such that*

$$\|\psi(t)S_1(t)(\eta_0,\Phi_0)\|_{U^s} \leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})},$$

$$\|\psi(t)S_2(t)(\eta_0,\Phi_0)\|_{V^{s+1}} \leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}.$$

*Proof.* We see that

$$\Big[\psi(t)\sum_{k\in\mathbb{Z}} e^{ikx}e^{\pm i\phi(k)t}\widehat{\eta}_0(k)\Big]^{\sim}(k,\tau) = \widehat{\eta}_0(k)\widehat{\psi}^{(t)}(\tau\mp\phi(k)).$$

So that

$$\Big\|\psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i\phi(k)t}\widehat{\eta}_0(k)\Big\|^2_{X^{s,1/2}} = \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\eta}_0(k)|^2\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|^2d\tau$$
$$\leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\eta}_0(k)|^2\int_{\mathbb{R}}\langle\tau\rangle|\widehat{\psi}^{(t)}(\tau)|^2d\tau \leq C\|\eta_0\|^2_{H^s(\mathbb{T})}.$$

Also, we note that

$$\Big\|\langle k\rangle^s\Big[\psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i\phi(k)t}\widehat{\eta}_0(k)\Big]^{\sim}\Big\|^2_{\ell^2_k L^1_\tau} = \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\eta}_0(k)|^2\left(\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|d\tau\right)^2$$
$$\leq C\|\eta_0\|^2_{H^s(\mathbb{T})}.$$

In a similar fashion,

$$\Big\|\psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i\phi(k)t}|k|\widehat{\Phi}_0(k)\Big\|^2_{X^{s,1/2}}$$
$$= \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|k|^2|\widehat{\Phi}_0(k)|^2\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|^2d\tau \leq C\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}$$

and

$$\Big\|\langle k\rangle^s\Big[\psi(t)\sum_{k\in\mathbb{Z}}e^{ixk}e^{\pm i\phi(k)t}|k|\widehat{\Phi}_0(k)\Big]^{\sim}\Big\|^2_{\ell^2_kL^1_\tau}=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|k|^2|\widehat{\Phi}_0(k)|^2\left(\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|d\tau\right)^2$$
$$\leq C\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}.$$

Thus, from the previous estimates we obtain that

$$\|\psi(t)S_1(t)(\eta_0,\Phi_0)\|_{U^s}\leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}.$$

Similarly we have that

$$\Big\|\psi(t)\sum_{k\in\mathbb{Z}}\frac{e^{ikx}e^{\pm i\phi(k)t}\widehat{\eta}_0(k)}{|k|}\Big\|_{Y^{s+1,1/2}}=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\eta}_0(k)|^2\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|^2d\tau$$
$$\leq C\|\eta_0\|^2_{H^s(\mathbb{T})}$$

and

$$\Big\||k|\langle k\rangle^s\Big[\psi(t)\sum_{k\in\mathbb{Z}}\frac{e^{ikx}e^{\pm i\phi(k)t}\widehat{\eta}_0(k)}{|k|}\Big]^{\sim}\Big\|_{\ell^2_kL^1_\tau}=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\eta}_0(k)|^2\Big(\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|d\tau\Big)^2$$
$$\leq C\|\eta_0\|^2_{H^s(\mathbb{T})}.$$

Also, we have that

$$\Big\|\psi(t)\sum_{k\in\mathbb{Z}}e^{ikx}e^{\pm i\phi(k)t}\widehat{\Phi}_0(k)\Big\|_{Y^{s+1,1/2}}=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|k|^2|\widehat{\Phi}_0(k)|^2\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|^2d\tau$$
$$\leq C\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}$$

and

$$\Big\||k|\langle k\rangle^s\Big[\psi(t)\sum_{k\in\mathbb{Z}}e^{ikx}e^{\pm i\phi(k)t}\widehat{\Phi}_0(k)\Big]^{\sim}\Big\|_{\ell^2_kL^1_\tau}=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|k|^2|\widehat{\Phi}_0(k)|^2\Big(\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|d\tau\Big)^2$$
$$\leq C\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}.$$

Then we conclude that

$$\|\psi(t)S_2(t)(\eta_0,\Phi_0)\|_{V^{s+1}}\leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}.$$ □

**Lemma 2.5.** *Let $s\in\mathbb{R}$, then there exists $C_2>0$ such that*

*(i)* $\Big\|\psi(t)\int_0^t f(t')dt'\Big\|_{H_t^{1/2}}\leq C_2\left(\|f\|_{H_t^{-1/2}}+\|\langle\tau\rangle^{-1}\widehat{f}^{(t)}\|_{L^1_\tau}\right).$

*(ii)* $\Big\|\psi(t)\int_0^t S_1(t-t')(\eta,\Phi)(t')\,dt'\Big\|_{U^s}\leq C_2\left(\|\eta\|_{Z^s}+\|\Phi\|_{W^{s+1}}\right),$

*(iii)* $\Big\|\psi(t)\int_0^t S_2(t-t')(\eta,\Phi)(t')\,dt'\Big\|_{V^{s+1}}\leq C_2\left(\|\eta\|_{Z^s}+\|\Phi\|_{W^{s+1}}\right).$

*Proof.* For the inequality (i) see Remark 3.13 of [4]. To prove the inequality (ii), first we note that

$$\begin{aligned}&\Big(\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big)^{\wedge}(k,t)\\&=\psi(t)\int_0^t e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\\&=e^{\pm i\phi(k)t}\psi(t)\int_0^t e^{\mp i\phi(k)t'}\widehat{\eta}(k,t')dt'=e^{\pm i\phi(k)t}\widehat{w}(k,t),\end{aligned}$$

where $w(x,t)=\psi(t)\int_0^t e^{\mp i\phi(k)t'}\eta(x,t')dt'$. Then we obtain that

$$\Big[\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big]^{\sim}(k,\tau)=\widetilde{w}(k,\tau\mp\phi(k)).$$

Using the fact that

$$\max\{\big||\tau+\phi(k)|-\phi(k)\big|,\big||\tau-\phi(k)|-\phi(k)\big|\}\leq|\tau|$$

we have that

$$\begin{aligned}\Big\|\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big\|^2_{X^{s,1/2}}&=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle|\tau\pm\phi(k)|-\phi(k)\rangle|\widetilde{w}(k,\tau)|^2d\tau\\&\leq\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle\tau\rangle|\widetilde{w}(k,\tau)|^2d\tau=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\widehat{w}\|^2_{H_t^{1/2}}.\end{aligned}$$

By using part (i) we have that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\widehat{w}\|^2_{H_t^{1/2}}&=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big\|\psi(t)\int_0^t e^{\mp i\phi(k)t'}\widehat{\eta}(k,t')dt'\Big\|^2_{H_t^{1/2}}\\&\leq C\Big(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big\|e^{\mp i\phi(k)t}\widehat{\eta}(k,t)\Big\|^2_{H_t^{-1/2}}\\&\quad+\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big\|\langle\tau\rangle^{-1}\Big[e^{\mp i\phi(k)t}\widehat{\eta}(k,t)\Big]^{\wedge(t)}\Big\|^2_{L^1_\tau}\Big)\\&\leq C\Big[\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|^2d\tau\\&\quad+\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big(\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|d\tau\Big)^2\Big].\end{aligned}$$

Thus

$$\Big\|\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big\|^2_{X^{s,1/2}}\leq C\|\eta\|^2_{Z^s}.$$

Let $\varrho$ a smooth cutoff function in the time variable, supported in $A=[-1,1]$. Then

$$\begin{aligned}\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'&=\psi(t)\sum_{k\in\mathbb{Z}}e^{ixk}\int_{\mathbb{R}}e^{\pm it\phi(k)}\widetilde{\eta}(k,\tau)\Big(\int_0^t e^{it'(\tau\mp\phi(k))}dt'\Big)d\tau\\&=\psi(t)\sum_{k\in\mathbb{Z}}e^{ixk}\int_{\mathbb{R}}\frac{e^{i\tau t}-e^{\pm it\phi(k)}}{i(\tau\mp\phi(k))}\widetilde{\eta}(k,\tau)d\tau=S_1+S_2-S_3,\end{aligned}$$

where

$$S_1 = \psi(t) \sum_{k\in\mathbb{Z}} e^{ixk} \int_{\mathbb{R}} \frac{e^{i\tau t} - e^{\pm it\phi(k)}}{i(\tau \mp \phi(k))} \varrho(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)d\tau,$$

$$S_2 = \psi(t) \sum_{k\in\mathbb{Z}} e^{ixk} \int_{\mathbb{R}} \frac{[1-\varrho(\tau \mp \phi(k))]}{i(\tau \mp \phi(k))} \widetilde{\eta}(k,\tau)e^{i\tau t}d\tau,$$

$$S_3 = \psi(t) \sum_{k\in\mathbb{Z}} e^{i(xk\pm t\phi(k))} \int_{\mathbb{R}} \frac{[1-\varrho(\tau \mp \phi(k))]}{i(\tau \mp \phi(k))} \widetilde{\eta}(k,\tau)d\tau.$$

Now,

$$\begin{aligned}
S_1 &= \psi(t) \sum_{k\in\mathbb{Z}} e^{ixk} \int_{\mathbb{R}} \frac{e^{\pm it\phi(k)}\left(e^{it(\tau\mp\phi(k))} - 1\right)}{i(\tau \mp \phi(k))} \varrho(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)d\tau \\
&= \psi(t) \sum_{k\in\mathbb{Z}} e^{i(xk\pm\phi(k))} \int_{\mathbb{R}} \sum_{n\geq 1} \frac{t^n[i(\tau \mp \phi(k))]^{n-1}}{n!} \varrho(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)d\tau \\
&= \psi(t) \sum_{n\geq 1} \frac{t^n i^{n-1}}{n!} \Big( \sum_{k\in\mathbb{Z}} e^{i(xk\pm t\phi(k))} \int_{\mathbb{R}} (\tau \mp \phi(k))^{n-1} \varrho(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)d\tau \Big).
\end{aligned}$$

Thus, using the notation

$$f_n(k) = \int_{\mathbb{R}} i^{n-1}(\tau \mp \phi(k))^{n-1} \varrho(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)d\tau, \quad \omega_n(t) = \psi(t)t^n,$$

we see that

$$\begin{aligned}
\widetilde{S_1}(k,\tau) &= \Big[ \sum_{n\geq 1} \frac{\omega_n(t)}{n!} \Big( \sum_{k\in\mathbb{Z}} e^{i(xk\pm t\phi(k))} f_n(k) \Big) \Big]^{\sim}(k,\tau) \\
&= \sum_{n\geq 1} \frac{1}{n!} f_n(k) \widehat{\omega}_n^{(t)}(\tau \mp \phi(k)).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\langle k\rangle^s \widetilde{S}_1\|^2_{\ell^2_k L^1_\tau} &\leq C \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \Big( \|\chi_A(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)\|_{L^1_\tau} \sum_{n\geq 1} \frac{1}{n!} \int_{\mathbb{R}} |\widehat{\omega}_n^{(t)}(\tau \mp \phi(k))| d\tau \Big)^2 \\
&\leq C \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \Big( \int_{\mathbb{R}} \langle \tau \mp \phi(k)\rangle^{-1} |\widetilde{\eta}(k,\tau)| d\tau \Big)^2 = C \|\langle k\rangle^s \langle |\tau| - \phi(k)\rangle^{-1} \widetilde{\eta}\|^2_{\ell^2_k L^1_\tau}.
\end{aligned}$$

Now, if we use the notation

$$g(k,\tau) = [i(\tau \mp \phi(k))]^{-1}[1-\varrho(\tau \mp \phi(k))]\widetilde{\eta}(k,\tau)$$

then

$$\widetilde{S}_2(k,\tau) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{T}} e^{-ixk} e^{-it\tau} \psi(t) g^{\sim^{-1}}(x,t)dxdt = \widehat{\psi}^{(t)}(\tau) * g(k,\tau).$$

So that, from Young's inequality,

$$\begin{aligned}
\|\langle k\rangle^s \widetilde{S}_2\|^2_{\ell^2_k L^1_\tau} &\leq \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \|\widehat{\psi}^{(t)}(\tau)\|^2_{L^1_\tau} \|g(k,\tau)\|^2_{L^1_\tau} \\
&\leq C \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left( \int_{\mathbb{R}} \langle \tau \mp \phi(k)\rangle^{-1} |\chi_B(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)| d\tau \right)^2 \\
&\leq C \|\langle k\rangle^s \langle |\tau| - \phi(k)\rangle^{-1} \widetilde{\eta}\|^2_{\ell^2_k L^1_\tau},
\end{aligned}$$

where $B = \{\tau : |\tau| \geq 1\}$. Next, let

$$\widehat{h}(k) = \int_{\mathbb{R}} [i(\tau \mp \phi(k))]^{-1}[1 - \varrho(\tau \mp \phi(k))]\widetilde{\eta}(k,\tau)d\tau.$$

Then

$$\begin{aligned}\|\langle k\rangle^s \widetilde{S}_3\|^2_{\ell^2_k L^1_\tau} &= \|\langle k\rangle^s \widehat{h}(k)\widehat{\psi}^{(t)}(\tau \mp \phi(k))\|^2_{\ell^2_k L^1_\tau}\\ &\leq C\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}\Big(\int_{\mathbb{R}} \langle \tau \mp \phi(k)\rangle^{-1}|\chi_B(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)|d\tau\Big)^2\\ &\leq C\|\langle k\rangle^s\langle |\tau| - \phi(k)\rangle^{-1}\widetilde{\eta}\|^2_{\ell^2_k L^1_\tau}.\end{aligned}$$

Hence, from the previous estimates we conclude that

$$\Big\|\langle k\rangle^s\Big[\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big]^\sim\Big\|_{\ell^2_k L^1_\tau} \leq C\|\langle k\rangle^s\langle |\tau| - \phi(k)\rangle^{-1}\widetilde{\eta}\|_{\ell^2_k L^1_\tau}.$$

In what follows we will use similar arguments. First

$$\Big[\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}|k|\widehat{\Phi}(k,t')dt'\Big]^\sim(k,\tau) = \widetilde{v}(k,\tau \mp \phi(k)).$$

where $v(x,t) = \psi(t)\int_0^t e^{\mp i\phi(k)t'}|k|\Phi(x,t')dt'$. Then we obtain that

$$\begin{aligned}\Big\|\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}|k|\widehat{\Phi}(k,t')dt'\Big\|^2_{X^{s,1/2}} &\leq \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}\|\widehat{v}\|^2_{H^{1/2}_t}\\ &\leq C\Big[\sum_{k\in\mathbb{Z}} |k|^2\langle k\rangle^{2s}\int_{\mathbb{R}} \langle \tau \mp \phi(k)\rangle^{-1}|\widetilde{\Phi}(k,\tau)|^2 d\tau\\ &\quad + \sum_{k\in\mathbb{Z}} |k|^2\langle k\rangle^{2s}\Big(\int_{\mathbb{R}} \langle \tau \mp \phi(k)\rangle^{-1}|\widetilde{\Phi}(k,\tau)|d\tau\Big)^2\Big]\\ &\leq C\|\Phi\|^2_{W^{s+1}}.\end{aligned}$$

Now,

$$\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}|k|\widehat{\Phi}(k,t')dt' = S_4 + S_5 - S_6,$$

where

$$\begin{aligned}S_4 &= \psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}\int_{\mathbb{R}} \frac{e^{i\tau t} - e^{\pm it\phi(k)}}{i(\tau \mp \phi(k))}\varrho(\tau \mp \phi(k))|k|\widetilde{\Phi}(k,\tau)d\tau,\\ S_5 &= \psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}\int_{\mathbb{R}} \frac{[1 - \varrho(\tau \mp \phi(k))]}{i(\tau \mp \phi(k))}|k|\widetilde{\Phi}(k,\tau)e^{i\tau t}d\tau,\\ S_6 &= \psi(t)\sum_{k\in\mathbb{Z}} e^{i(xk \pm t\phi(k))}\int_{\mathbb{R}} \frac{[1 - \varrho(\tau \mp \phi(k))]}{i(\tau \mp \phi(k))}|k|\widetilde{\Phi}(k,\tau)d\tau.\end{aligned}$$

We note that

$$S_4 = \psi(t)\sum_{n\geq 1} \frac{t^n i^{n-1}}{n!}\Big(\sum_{k\in\mathbb{Z}} e^{i(xk \pm t\phi(k))}\int_{\mathbb{R}} (\tau \mp \phi(k))^{n-1}\varrho(\tau \mp \phi(k))|k|\widetilde{\Phi}(k,\tau)d\tau\Big).$$

Thus, if we use the notation

$$\zeta_n(k) = \int_{\mathbb{R}} i^{n-1}(\tau \mp \phi(k))^{n-1}\varrho(\tau \mp \phi(k))|k|\widetilde{\Phi}(k,\tau)d\tau, \quad \omega_n(t) = \psi(t)t^n,$$

we obtain that

$$\begin{aligned}\widetilde{S}_4(k,\tau) &= \Big[\sum_{n\geq 1}\frac{\omega_n(t)}{n!}\Big(\sum_{k\in\mathbb{Z}} e^{i(xk\pm t\phi(k))}\zeta_n(k)\Big)\Big]^{\sim}(k,\tau)\\ &= \sum_{n\geq 1}\frac{1}{n!}\zeta_n(k)\widehat{\omega}_n^{(t)}(\tau\mp\phi(k)).\end{aligned}$$

Therefore

$$\begin{aligned}\|\langle k\rangle^s\widetilde{S}_4\|^2_{\ell^2_k L^1_\tau} &\leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big(\sum_{n\geq 1}\frac{1}{n!}|\zeta_n(k)|\int_{\mathbb{R}}|\widehat{\omega}_n^{(t)}(\tau\mp\phi(k))|d\tau\Big)^2\\ &\leq C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\chi_A(\tau\mp\phi(k))|k|\widetilde{\Phi}(k,\tau)\|^2_{L^1_\tau}\\ &= C\||k|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\Phi}\|^2_{\ell^2_k L^1_\tau}.\end{aligned}$$

Using the notation $g_1(k,\tau) = [i(\tau\mp\phi(k))]^{-1}[1-\varrho(\tau\mp\phi(k))]|k|\widetilde{\Phi}(k,\tau)$ we see that

$$\widetilde{S}_5(k,\tau) = \Big[\psi(t)\sum_{k\in\mathbb{Z}} e^{ikx}\int_{\mathbb{R}} e^{it\tau}g_1(k,\tau)d\tau\Big]^{\sim}(k,\tau) = \widehat{\psi}^{(t)}(\tau) * g_1(k,\tau).$$

Hence

$$\begin{aligned}\|\langle k\rangle^s\widetilde{S}_5\|^2_{\ell^2_k L^1_\tau} &\leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\widehat{\psi}^{(t)}(\tau)\|^2_{L^1_\tau}\|g_1(k,\tau)\|^2_{L^1_\tau}\\ &\leq C\||k|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\Phi}\|^2_{\ell^2_k L^1_\tau}.\end{aligned}$$

Now, let $\widehat{h}_1(k) = \int_{\mathbb{R}}[i(\tau\mp\phi(k))]^{-1}[1-\varrho(\tau\mp\phi(k))]|k|\widetilde{\Phi}(k,\tau)d\tau$. Then

$$\begin{aligned}\|\langle k\rangle^s\widetilde{S}_6\|^2_{\ell^2_k L^1_\tau} &= \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{h}_1(k)|^2\left(\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|d\tau\right)^2\\ &\leq C\||k|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\Phi}\|^2_{\ell^2_k L^1_\tau}.\end{aligned}$$

Consequently, from the previous estimates we have that

$$\Big\|\langle k\rangle^s\Big[\psi(t)\int_0^t\sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}|k|\widehat{\Phi}(k,t')dt'\Big]^{\sim}\Big\|_{\ell^2_k L^1_\tau} \leq C\||k|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\Phi}\|_{\ell^2_k L^1_\tau}.$$

Therefore, we conclude that

$$\Big\|\psi(t)\int_0^t S_1(t-t')(\eta,\Phi)(t')\,dt'\Big\|_{U^s} \leq C_2\Big(\|\eta\|_{Z^s}+\|\Phi\|_{W^{s+1}}\Big).$$

Similarly we obtain the other inequality in (iii). □

In the following lemma we show the continuous embedding of the space $U^s\times V^{s+1}$ in the class $C(\mathbb{R} : H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T}))$ for $s\in\mathbb{R}$.

**Lemma 2.6.** *Let $s \in \mathbb{R}$, then there exists $C > 0$ such that*

$$\|(\eta,\Phi)\|_{C(\mathbb{R}:H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T}))} \leq C\|(\eta,\Phi)\|_{U^s\times V^{s+1}}.$$

*Proof.* First we prove that $U^s \subseteq L^\infty(\mathbb{R} : H^s(\mathbb{T}))$. Since

$$\|\eta(t)\|_{H^s(\mathbb{T})} \leq \Big(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big(\int_{\mathbb{R}}|\widetilde{\eta}(k,\tau)|d\tau\Big)^2\Big)^{1/2} \leq \|\eta\|_{U^s},$$

we have that $\|\eta\|_{L^\infty(\mathbb{R}:H^s(\mathbb{T}))} \leq \|\eta\|_{U^s}$. Now,

$$\|\eta(t)-\eta(t')\|^2_{H^s(\mathbb{T})} \leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big(\int_{\mathbb{R}}|e^{it\tau}-e^{it'\tau}||\widetilde{\eta}(k,\tau)|d\tau\Big)^2.$$

Then, using the Dominated Convergence Theorem,

$$\|\eta(t)-\eta(t')\|_{H^s(\mathbb{T})} \to 0, \quad t \to t'.$$

Thus $\eta \in C(\mathbb{R} : H^s(\mathbb{T}))$ and moreover $\|\eta\|_{C(\mathbb{R}:H^s(\mathbb{T}))} \leq C\|\eta\|_{U^s}$. Finally,

$$\|\Phi(t)\|_{\mathcal{V}^{s+1}(\mathbb{T})} = \Big(\sum_{k\in\mathbb{Z}}|k|^2\langle k\rangle^{2s}\Big|\int_{\mathbb{R}}e^{it\tau}\widetilde{\Phi}(k,\tau)d\tau\Big|^2\Big)^{1/2} \leq \|\Phi\|_{V^{s+1}}.$$

Hence, as in the previous case, $\|\Phi\|_{C(\mathbb{R}:\mathcal{V}^{s+1}(\mathbb{T}))} \leq C\|\Phi\|_{V^{s+1}}$ and then

$$\|(\eta,\Phi)\|_{C(\mathbb{R}:H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T}))} \leq C\|(\eta,\Phi)\|_{U^s\times V^{s+1}}. \qquad \square$$

## 3 Bilinear estimates

Before proceed to the proof of the bilinear estimates, we state some elementary calculus inequalities that will be useful later, and whose proofs can be seen, respectively, in Lemma 5.3 of [10], Lemma 2.5 of [19], and Lemma 4.2 in [6].

**Lemma 3.1.** *If $\mu > 1/2$ and $\nu = \nu(k,\tau) > 0$, then*

$$\sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}}\sum_{k_1\in\mathbb{Z}}\frac{1}{(\nu+|k_1^2+\alpha_1k_1+\alpha_2|)^\mu} < +\infty,$$

*where $\alpha_1 = \alpha_1(k,\tau)$ and $\alpha_2 = \alpha_2(k,\tau)$.*

**Lemma 3.2.** *If $\mu > 1/3$ and $\nu = \nu(k,\tau) > 0$, then*

$$\sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}}\sum_{k_1\in\mathbb{Z}}\frac{1}{(\nu^3+|k_1^3+\alpha_1k_1^2+\alpha_2k_1+\alpha_3|)^\mu} < +\infty,$$

*where $\alpha_1 = \alpha_1(k,\tau), \alpha_2 = \alpha_2(k,\tau)$ and $\alpha_3 = \alpha_3(k,\tau)$.*

**Lemma 3.3.** *For $p,q > 0$ and $r = \min\{p,q,p+q-1\}$ with $p+q > 1$, we have that*

$$\int_{\mathbb{R}}\frac{dx}{\langle x-\lambda\rangle^p\langle x-\mu\rangle^q} \leq \frac{C}{\langle\lambda-\mu\rangle^r}. \tag{3.1}$$

The following nonlinear estimates constitute an important result for this work. We prove these estimates using a method originally due to Bourgain (see [2,3]) and considerably improved by Kenig, Ponce and Vega (see [8,9]).

**Lemma 3.4.** *Let $s \geq 0$, then exists $C_3 > 0$ such that*

*(i)* $\|\partial_x(\eta\partial_x\Phi)\|_{X^{s,-1/2}} \leq C_3\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}}$,

*(ii)* $\|(\partial_x\Phi)(\partial_x\Phi_1)\|_{Y^{s+1,-1/2}} \leq C_3\|\Phi\|_{Y^{s+1,1/2}}\|\Phi_1\|_{Y^{s+1,1/2}}$.

*Proof.* First we note that

$$\begin{aligned}&\|\partial_x(\eta\partial_x\Phi)\|_{X^{s,-1/2}}\\ &\quad= \|\langle|\tau|-\phi(k)\rangle^{-1/2}k\langle k\rangle^s(\widetilde{\eta}*\widetilde{\partial_x\Phi})(k,\tau)\|_{\ell^2_kL^2_\tau}\\ &\quad= \sup_{\|h\|_{\ell^2_kL^2_\tau}=1}\left|\sum_{k,k_1\in\mathbb{Z}}\int_{\mathbb{R}^2}k\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1/2}\widetilde{\eta}(k-k_1,\tau-\tau_1)k_1\widetilde{\Phi}(k_1,\tau_1)h(k,\tau)\,d\tau d\tau_1\right|.\end{aligned}$$

Thus, by letting

$$f(k,\tau) = \langle|\tau|-\phi(k)\rangle^{1/2}\langle k\rangle^s\widetilde{\eta}(k,\tau), \quad g(k,\tau) = \langle|\tau|-\phi(k)\rangle^{1/2}\langle k\rangle^s k\widetilde{\Phi}(k,\tau),$$

we have that (i) is equivalent to

$$|J(f,g,h)| \leq C\|f\|_{\ell^2_kL^2_\tau}\|g\|_{\ell^2_kL^2_\tau}\|h\|_{\ell^2_kL^2_\tau}, \tag{3.2}$$

where

$$J(f,g,h) = \sum_{k,k_1\in\mathbb{Z}}\int_{\mathbb{R}^2}\frac{k\langle k\rangle^s}{\langle k_1\rangle^s\langle k-k_1\rangle^s}\frac{f(k-k_1,\tau-\tau_1)g(k_1,\tau_1)h(k,\tau)d\tau d\tau_1}{\langle|\tau|-\phi(k)\rangle^{1/2}\langle|\tau_1|-\phi(k_1)\rangle^{1/2}\langle|\tau-\tau_1|-\phi(k-k_1)\rangle^{1/2}}.$$

For to perform the inequality (3.2), we analyse all possible cases for the sign of $\tau, \tau_1$ and $\tau-\tau_1$. To do this we split $\mathbb{Z}^2\times\mathbb{R}^2$ into the following regions

$$\begin{aligned}\Gamma_1 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1<0, \tau-\tau_1<0\},\\ \Gamma_2 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1\geq 0, \tau-\tau_1<0, \tau\geq 0\},\\ \Gamma_3 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1\geq 0, \tau-\tau_1<0, \tau<0\},\\ \Gamma_4 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1<0, \tau-\tau_1\geq 0, \tau\geq 0\},\\ \Gamma_5 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1<0, \tau-\tau_1\geq 0, \tau<0\},\\ \Gamma_6 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1\geq 0, \tau-\tau_1\geq 0\}.\end{aligned}$$

We note that $\tau_1 < 0$ and $\tau-\tau_1 < 0$ implies $\tau < 0$, and $\tau_1 \geq 0$ and $\tau-\tau_1 \geq 0$ implies $\tau \geq 0$. Then the cases $\tau_1 < 0$, $\tau-\tau_1 < 0$, $\tau \geq 0$ and $\tau_1 \geq 0$, $\tau-\tau_1 \geq 0$, $\tau < 0$ cannot occur. Now, since

$$1+|k| \leq (1+|k_1|)(1+|k-k_1|),$$

then for $s \geq 0$ we see that

$$\frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{2s}\langle k-k_1\rangle^{2s}} \leq 1. \tag{3.3}$$

So, we will prove the inequality (3.2) with $Z(f,g,h)$ instead of $J(f,g,h)$, where

$$Z(f,g,h) = \sum_{k,k_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{kf(k_2,\tau_2)g(k_1,\tau_1)h(k,\tau)\, d\tau d\tau_1}{\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}$$

with $k_2 = k-k_1, \tau_2 = \tau-\tau_1$ and $\sigma,\sigma_1,\sigma_2$ belonging to one of the following cases

$$\begin{aligned}
&(C_1)\, \sigma = \tau + |k|^3 + |k|,\ \sigma_1 = \tau_1 + |k_1|^3 + |k_1|,\ \sigma_2 = \tau_2 + |k_2|^3 + |k_2|,\\
&(C_2)\, \sigma = \tau - |k|^3 - |k|,\ \sigma_1 = \tau_1 - |k_1|^3 - |k_1|,\ \sigma_2 = \tau_2 + |k_2|^3 + |k_2|,\\
&(C_3)\, \sigma = \tau + |k|^3 + |k|,\ \sigma_1 = \tau_1 - |k_1|^3 - |k_1|,\ \sigma_2 = \tau_2 + |k_2|^3 + |k_2|,\\
&(C_4)\, \sigma = \tau - |k|^3 - |k|,\ \sigma_1 = \tau_1 + |k_1|^3 + |k_1|,\ \sigma_2 = \tau_2 - |k_2|^3 - |k_2|,\\
&(C_5)\, \sigma = \tau + |k|^3 + |k|,\ \sigma_1 = \tau_1 + |k_1|^3 + |k_1|,\ \sigma_2 = \tau_2 - |k_2|^3 - |k_2|,\\
&(C_6)\, \sigma = \tau - |k|^3 - |k|,\ \sigma_1 = \tau_1 - |k_1|^3 - |k_1|,\ \sigma_2 = \tau_2 - |k_2|^3 - |k_2|.
\end{aligned}$$

By hypotheses we have that $\widehat{\eta}(0,t) = 0$, for all $t \in \mathbb{R}$. Thus, if $k = k_1$ then $f(k_2,\tau_2) = 0$. Similarly if $k_1 = 0$ then $g(k_1,\tau_1) = 0$. Then, we will estimate $Z(f,g,h)$ when $k \neq 0$, $k_1 \neq 0$ and $k - k_1 \neq 0$.

By symmetry it is sufficient to estimate $Z(f,g,h)$ into the following set

$$R = \{(k,k_1,\tau,\tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 \,:\, |\sigma_2| \leq |\sigma_1|\}.$$

Now, we write $Z(f,g,h)$ as the sum $S_1 + S_2$, where

$$S_j = \sum_k \sum_{k_1} \iint_{R_j} \frac{kf(k_2,\tau_2)g(k_1,\tau_1)h(k,\tau)\chi_{R_j}\, d\tau d\tau_1}{\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}, \quad j = 1,2,$$

and the sets $R_1$, $R_2$ are defined by

$$R_1 = \{(k,k_1,\tau,\tau_1) \in R \,:\, |\sigma_1| \leq |\sigma|\}, \quad R_2 = \{(k,k_1,\tau,\tau_1) \in R \,:\, |\sigma| \leq |\sigma_1|\}.$$

We first consider $\sigma,\sigma_1,\sigma_2$ as the case $(C_1)$. We will use the notations $\sum_k F_1(k)$, $\int F_2(x)dx$ to indicate that the sum or the integral are calculated, respectively, at some subset of $\mathbb{Z}$ or $\mathbb{R}$. Using the Cauchy–Schwarz inequality,

$$|S_1|^2 \leq \|h\|^2_{\ell^2_k L^2_\tau} \sum_k \int \left( \sum_{k_1} \int |f(k_2,\tau_2)g(k_1,\tau_1)|^2 d\tau_1 \right) \left( \sum_{k_1} \int \frac{\chi^2_{R_1}|k|^2 d\tau_1}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle} \right) d\tau.$$

We will prove that the expression

$$\sum_{k_1} \int \frac{\chi^2_{R_1}|k|^2 d\tau_1}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle} = \frac{|k|^2}{\langle\sigma\rangle} \sum_{k_1} \int \frac{\chi^2_{R_1} d\tau_1}{\langle\sigma_1\rangle\langle\sigma_2\rangle}$$

is bounded. But, by using inequality (3.1) in Lemma 3.3 we have that

$$\int_{\mathbb{R}} \frac{d\tau_1}{\langle\tau_1 + |k_1|^3 + |k_1|\rangle\langle\tau - \tau_1 + |k_2|^3 + |k_2|\rangle} \leq \frac{C}{\langle\tau + |k_1|^3 + |k_1| + |k_2|^3 + |k_2|\rangle}.$$

Then we will prove that there exists $C > 0$ such that

$$\frac{|k|^2}{\langle\tau + |k|^3 + |k|\rangle} \sum_{k_1} \frac{1}{\langle\tau + |k_1|^3 + |k_1| + |k - k_1|^3 + |k - k_1|\rangle} \leq C, \quad \text{on } R_1.$$

Since for $k \neq 0$, $k_1 \neq 0$ and $k \neq k_1$, $|k| = |k_1 + (k-k_1)| \leq |k_1| + |k-k_1| \leq 2|k_1(k-k_1)|$. Then

$$\frac{k^2}{2} \leq |kk_1(k-k_1)|. \tag{3.4}$$

Moreover, we observe the relation

$$\begin{aligned} \tau + |k|^3 + |k| - \left[\tau_1 + |k_1|^3 + |k_1| + \tau_2 + |k_2|^3 + |k_2|\right] \\ = |k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|. \end{aligned} \tag{3.5}$$

If $(k, k_1, \tau, \tau_1) \in R_1$ then

$$|\tau_2 + |k_2|^3 + |k_2|| \leq |\tau_1 + |k_1|^3 + |k_1|| \leq |\tau + |k|^3 + |k||.$$

Hence, using the triangle inequality in (3.5) we conclude that

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| \leq 3|\tau + |k|^3 + |k||. \tag{3.6}$$

**Assume $k_1 > 0$ and $k - k_1 > 0$.** Then $k > k_1 > 0$ and, using Lemma 3.1, we obtain that

$$\begin{aligned} \sum_{k_1} \frac{1}{\langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1|\rangle} &\leq \sum_{k_1 \in \mathbb{Z}} \frac{1}{\langle \tau + k^3 + k - 3k^2 k_1 + 3kk_1^2\rangle} \\ &\leq \sup_{(k,\tau)\in \mathbb{Z}^* \times \mathbb{R}} \sum_{k_1 \in \mathbb{Z}} \frac{1}{\left(\frac{1}{3|k|} + \left|k_1^2 - kk_1 + \frac{\tau}{3k} + \frac{k^2}{3} + \frac{1}{3}\right|\right)} \\ &\leq C, \end{aligned}$$

where $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. Moreover,

$$|k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2| = k^3 + k - k_1^3 - k_1 - k_2^3 - k_2 = 3kk_1(k-k_1) > 0.$$

So, from (3.6) and inequality (3.4) we see that

$$|\tau + k^3 + k| \geq |kk_1(k-k_1)| \geq \frac{|k|^2}{2}.$$

Thus

$$\frac{|k|^2}{\langle \tau + |k|^3 + |k|\rangle} \leq C.$$

**Assume $k_1 < 0$ and $k - k_1 < 0$.** Then $k < k_1 < 0$ and, using Lemma 3.1, we see that

$$\begin{aligned} \sum_{k_1} \frac{1}{\langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1|\rangle} &\leq \sum_{k_1 \in \mathbb{Z}} \frac{1}{\langle \tau - k^3 - k + 3k^2 k_1 - 3kk_1^2\rangle} \\ &\leq \sup_{(k,\tau)\in \mathbb{Z}^* \times \mathbb{R}} \sum_{k_1 \in \mathbb{Z}} \frac{1}{\left(\frac{1}{3|k|} + \left|k_1^2 - kk_1 - \frac{\tau}{3k} + \frac{k^2}{3} + \frac{1}{3}\right|\right)} \\ &\leq C. \end{aligned}$$

Moreover

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = \big| - k^3 - k + k_1^3 + k_1 + k_2^3 + k_2\big| = 3|kk_1(k-k_1)|.$$

Hence from (3.6) and (3.4) we obtain that

$$|\tau - k^3 - k| \geq |kk_1(k-k_1)| \geq \frac{|k|^2}{2} \quad \text{and} \quad \frac{|k|^2}{\langle \tau - k^3 - k\rangle} \leq C.$$

**Assume $k_1 > 0$ and $k - k_1 < 0$.** Using Lemma 3.2,

$$\begin{aligned}
&\sum_{k_1} \frac{1}{\langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1|\rangle} \\
&\qquad \leq \sum_{k_1 \in \mathbb{Z}} \frac{1}{\langle \tau + 2k_1^3 + 2k_1 - k^3 - k + 3k^2k_1 - 3kk_1^2\rangle} \\
&\qquad \leq \sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}} \sum_{k_1\in\mathbb{Z}} \frac{1}{\left(\frac{1}{2} + \left|k_1^3 - \frac{3}{2}kk_1^2 + k_1 + \frac{3}{2}k^2k_1 + \frac{\tau}{2} - \frac{k^3}{2} - \frac{k}{2}\right|\right)} \leq C.
\end{aligned}$$

Moreover, if $k > 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k-k_1| \left[\left(k_1 - \frac{k}{4}\right)^2 + \frac{15k^2}{16} + 1\right] \geq \frac{15}{8}k^2.$$

If $k < 0$,

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k_1| \left[\left(k_1 - \frac{3k}{4}\right)^2 + \frac{15k^2}{16} + 1\right] \geq \frac{15k^2}{8}.$$

Thus, from inequality (3.6) we have that

$$|\tau + |k|^3 + |k|| \geq \frac{5k^2}{8} \quad \text{and} \quad \frac{|k|^2}{\langle \tau + |k|^3 + |k|\rangle} \leq C.$$

**Assume $k_1 < 0$ and $k - k_1 > 0$.** Using Lemma 3.2,

$$\begin{aligned}
&\sum_{k_1} \frac{1}{\langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1|\rangle} \\
&\qquad \leq \sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}} \sum_{k_1\in\mathbb{Z}} \frac{1}{\left(\frac{1}{2} + \left|k_1^3 - \frac{3}{2}kk_1^2 + k_1 + \frac{3}{2}k^2k_1 - \frac{\tau}{2} - \frac{k^3}{2} - \frac{k}{2}\right|\right)} \leq C.
\end{aligned}$$

Now, if $k > 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k_1| \left[\left(k_1 - \frac{3k}{4}\right)^2 + \frac{15k^2}{16} + 1\right] \geq \frac{15k^2}{8}.$$

and if $k < 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k-k_1| \left[\left(k_1 - \frac{k}{4}\right)^2 + \frac{15k^2}{16} + 1\right] \geq \frac{15}{8}k^2.$$

Thus, by using (3.6),

$$|\tau + |k|^3 + |k|| \geq \frac{5k^2}{8} \quad \text{and} \quad \frac{|k|^2}{\langle \tau + |k|^3 + |k|\rangle} \leq C.$$

Consequently, from the previous estimates there exists $C > 0$ such that

$$\frac{|k|^2}{\langle\sigma\rangle}\sum_{k_1}\int \frac{d\tau_1}{\langle\sigma_1\rangle\langle\sigma_2\rangle} \leq C, \quad \text{on} \quad R_1.$$

Therefore

$$\begin{aligned}|S_1|^2 &\leq C\|h\|^2_{\ell^2_k L^2_\tau}\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}|g(k_1,\tau_1)|^2\Big(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|f(k_2,\tau_2)|^2 d\tau\Big)d\tau_1\\ &\leq C\|f\|^2_{\ell^2_k L^2_\tau}\|g\|^2_{\ell^2_k L^2_\tau}\|h\|^2_{\ell^2_k L^2_\tau}.\end{aligned}$$

In a similar fashion,

$$|S_2|^2 \leq \|g\|^2_{\ell^2_k L^2_\tau}\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}\left(\sum_k\int|f(k_2,\tau_2)h(k,\tau)|^2 d\tau\right)\left(\sum_k\int\frac{\chi^2_{R_2}|k|^2 d\tau}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle}\right)d\tau_1.$$

We will prove that the expression

$$\sum_k\int\frac{\chi^2_{R_2}|k|^2 d\tau}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle} = \frac{1}{\langle\sigma_1\rangle}\sum_k\int\frac{\chi^2_{R_2}|k|^2 d\tau}{\langle\sigma\rangle\langle\sigma_2\rangle}$$

is bounded. Using inequality (3.1) in Lemma 3.3 we have that

$$\int_{\mathbb{R}}\frac{d\tau}{\langle\tau+|k|^3+|k|\rangle\langle\tau-\tau_1+|k_2|^3+|k_2|\rangle} \leq \frac{C}{\langle\tau_1+|k|^3+|k|-|k_2|^3-|k_2|\rangle}.$$

Thus, we will show that there exists $C > 0$ such that

$$\frac{1}{\langle\tau_1+|k_1|^3+|k_1|\rangle}\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle} \leq C, \quad \text{on } R_2.$$

We note that if $(k,k_1,\tau,\tau_1)\in R_2$,

$$|\tau+|k|^3+|k|| \leq |\tau_1+|k_1|^3+|k_1||, \qquad |\tau_2+|k_2|^3+|k_2|| \leq |\tau_1+|k_1|^3+|k_1||.$$

So, using the triangle inequality in (3.5) we see that

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big| \leq 3|\tau_1+|k_1|^3+|k_1||. \tag{3.7}$$

**Assume $k > 0$ and $k - k_1 > 0$.** Thus

$$\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k_2|^3-|k_2|\rangle} \leq \sum_{k\in\mathbb{Z}}\frac{|k|^2}{\langle\tau_1+k_1^3+k_1+3k^2k_1-3kk_1^2\rangle} =: J_1.$$

Moreover, if $k_1 > 0$ then, using inequality (3.4),

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big| = 3|kk_1(k-k_1)| \geq \frac{3k^2}{2}$$

and if $k_1 < 0$ then

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big| = 2|k_1|\left[\left(k_1-\frac{3k}{4}\right)^2+\frac{15k^2}{16}+1\right] \geq \frac{15k^2}{8}.$$

Hence, from (3.7) there exists $C>0$ such that

$$|\tau_1+|k_1|^3+|k_1||\geq Ck^2$$

and consequently, using Lemma 3.1, we obtain

$$\frac{1}{\langle\tau_1+|k_1|^3+|k_1|\rangle}J_1\leq\ C\sup_{(k_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{\left(\frac{1}{3|k_1|}+\left|k^2-kk_1+\frac{\tau_1}{3k_1}+\frac{k_1^2}{3}+\frac{1}{3}\right|\right)}\leq C.$$

**Assume $k<0$ and $k-k_1<0$.** Thus

$$\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle}\leq\sum_{k\in\mathbb{Z}}\frac{|k|^2}{\langle\tau_1-k_1^3-k_1-3k^2k_1+3kk_1^2\rangle}=:J_2.$$

Moreover, if $k_1>0$ then

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big|=2|k_1|\left[\left(k_1-\frac{3k}{4}\right)^2+\frac{15k^2}{16}+1\right]\geq\frac{15k^2}{8}.$$

If $k_1<0$ then, using (3.4),

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big|=3|kk_1(k-k_1)|\geq\frac{3k^2}{2}.$$

Thus, from (3.7) there exists $C>0$ such that

$$|\tau_1+|k_1|^3+|k_1||\geq Ck^2$$

and so

$$\frac{1}{\langle\tau_1+k_1^3+k_1\rangle}J_2\leq\ C\sup_{(k_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{\langle\tau_1-k_1^3-k_1-3k^2k_1+3kk_1^2\rangle}\leq C.$$

**Assume $k>0$ and $k-k_1<0$.** Then $k_1>k>0$ and

$$\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle}\leq\sum_{k\in\mathbb{Z}}\frac{|k|^2}{\langle\tau_1+2k^3+2k-3k^2k_1+3kk_1^2-k_1^3-k_1\rangle}=:J_3.$$

Moreover, we see that

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big|\geq\frac{15k^2}{8}\quad\text{and}\quad|\tau_1+k_1^3+k_1|\geq\frac{5k^2}{8}.$$

Consequently

$$\begin{aligned}\frac{1}{\langle\tau_1+k_1^3+k_1\rangle}J_3&\leq C\sup_{(k_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{\langle\tau_1+2k^3+2k-3k^2k_1+3kk_1^2-k_1^3-k_1\rangle}\\&\leq\sup_{(k_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{\left(\frac{1}{2}+\left|k^3-\frac{3}{2}k^2k_1+k+\frac{3}{2}kk_1^2+\frac{\tau_1}{2}-\frac{k_1^3}{2}-\frac{k_1}{2}\right|\right)}\leq C.\end{aligned}$$

**Assume $k < 0$ and $k - k_1 > 0$.** Then

$$\sum_k \frac{|k|^2}{\langle \tau_1 + |k|^3 + |k| - |k-k_1|^3 - |k-k_1| \rangle}$$
$$\leq \sum_{k\in\mathbb{Z}} \frac{|k|^2}{\langle \tau_1 - 2k^3 - 2k + k_1^3 + k_1 + 3k^2k_1 - 3kk_1^2 \rangle} =: J_4.$$

Also we see that

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| \geq \frac{15k^2}{8} \quad \text{and} \quad |\tau_1 + k_1^3 + k_1| \geq \frac{5k^2}{8}.$$

Thus, using (3.7),

$$|\tau_1 + k_1^3 + k_1| \geq \frac{5k^2}{8}.$$

So that

$$\frac{1}{\langle \tau_1 + k_1^3 + k_1 \rangle} J_4 \leq C \sup_{(k_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{\langle \tau_1 - 2k^3 - 2k + 3k^2k_1 - 3kk_1^2 + k_1^3 + k_1 \rangle}$$
$$\leq C \sup_{(k_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{\left(\frac{1}{2} + \left|k^3 - \frac{3}{2}k^2k_1 + k + \frac{3}{2}kk_1^2 - \frac{\tau_1}{2} - \frac{k_1^3}{2} - \frac{k_1}{2}\right|\right)} \leq C.$$

Hence, from previous estimates we see that there exists $C > 0$ such that

$$\frac{1}{\langle \sigma_1 \rangle} \sum_k \int \frac{|k|^2 d\tau}{\langle \sigma \rangle \langle \sigma_2 \rangle} \leq C, \quad \text{on } R_2.$$

Therefore

$$|S_2|^2 \leq C\|f\|^2_{\ell^2_k L^2_\tau} \|g\|^2_{\ell^2_k L^2_\tau} \|h\|^2_{\ell^2_k L^2_\tau}.$$

The proof of the others is similar to case $(C_1)$. Finally, note that

$$\|(\partial_x \Phi)(\partial_x \Phi_1)\|_{Y^{s+1,-1/2}}$$
$$= \sup_{\|h\|_{\ell^2_k L^2_\tau}=1} \left| \sum_{k,k_1\in\mathbb{Z}} \int_{\mathbb{R}^2} k\langle k \rangle^s \langle |\tau| - \phi(k) \rangle^{-1/2} (k-k_1) \widetilde{\Phi}(k-k_1, \tau-\tau_1) k_1 \widetilde{\Phi}_1(k_1,\tau_1) h(k,\tau)\, d\tau d\tau_1 \right|.$$

Then, by letting

$$f(k,\tau) = \langle |\tau| - \phi(k) \rangle^{1/2} \langle k \rangle^s k \widetilde{\Phi}(k,\tau), \quad f_1(k,\tau) = \langle |\tau| - \phi(k) \rangle^{1/2} \langle k \rangle^s k \widetilde{\Phi}_1(k,\tau)$$

we have that (ii) is equivalent to

$$|K(f, f_1, h)| \leq C\|f\|_{\ell^2_k L^2_\tau} \|f_1\|_{\ell^2_k L^2_\tau} \|h\|_{\ell^2_k L^2_\tau}, \tag{3.8}$$

where

$$K(f, f_1, h) = \sum_{k,k_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{k\langle k \rangle^s}{\langle k_1 \rangle^s \langle k-k_1 \rangle^s} \frac{f(k-k_1, \tau-\tau_1) f_1(k_1,\tau_1) h(k,\tau)\, d\tau d\tau_1}{\langle |\tau| - \phi(k) \rangle^{1/2} \langle |\tau_1| - \phi(k_1) \rangle^{1/2} \langle |\tau-\tau_1| - \phi(k-k_1) \rangle^{1/2}}.$$

The proof of (3.8) is analogous to the proof of (3.2). □

The proof of the following estimates is analogous to the proof of Lemma 3.4.

**Lemma 3.5.** *Let $s \geq 0$, then there exists $C_4 > 0$ such that*

*(i)* $\left\| \frac{\langle k\rangle^s [\partial_x(\eta\partial_x\Phi)]^{\sim}(k,\tau)}{\langle |\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \leq C_4 \|\eta\|_{X^{s,1/2}} \|\Phi\|_{Y^{s+1,1/2}}$,

*(ii)* $\left\| \frac{|k|\langle k\rangle^s [(\partial_x\Phi)(\partial_x\Phi_1)]^{\sim}(k,\tau)}{\langle |\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \leq C_4 \|\Phi\|_{Y^{s+1,1/2}} \|\Phi_1\|_{Y^{s+1,1/2}}$.

*Proof.* First, notice that

$$
\begin{aligned}
&\left\| \frac{\langle k\rangle^s [\partial_x(\eta\partial_x\Phi)]^{\sim}(k,\tau)}{\langle |\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \\
&\quad = \left\| \frac{k\langle k\rangle^s}{\langle |\tau|-\phi(k)\rangle} \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \frac{f(k-k_1,\tau-\tau_1)g(k_1,\tau_1)d\tau_1}{\langle k_1\rangle^s \langle k-k_1\rangle^s \langle |\tau_1|-\phi(k_1)\rangle^{1/2} \langle |\tau-\tau_1|-\phi(k-k_1)\rangle^{1/2}} \right\|_{\ell^2_k L^1_\tau} \\
&\quad =: J(f,g),
\end{aligned}
$$

where

$$
f(k,\tau) = \langle |\tau|-\phi(k)\rangle^{1/2} \langle k\rangle^s \widetilde{\eta}(k,\tau), \quad g(k,\tau) = \langle |\tau|-\phi(k)\rangle^{1/2} \langle k\rangle^s k \widetilde{\Phi}(k,\tau).
$$

In view of inequality (3.3) we will prove inequality in (i) with $Z(f,g)$ instead of $J(f,g)$ where

$$
Z(f,g) = \left\| \frac{k}{\langle\sigma\rangle} \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \frac{f(k_2,\tau_2)g(k_1,\tau_1)d\tau_1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right\|_{\ell^2_k L^1_\tau}.
$$

More exactly, we will study the expression

$$
Z_j(f,g) = \left\| \frac{k}{\langle\sigma\rangle} \sum_{k_1} \int \frac{f(k_2,\tau_2)g(k_1,\tau_1)\chi_{R_j} d\tau_1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right\|_{\ell^2_k L^1_\tau}, \quad j = 1,2,
$$

with $k_2 = k - k_1, \tau_2 = \tau - \tau_1$; $\sigma, \sigma_1, \sigma_2$ belonging to one of the cases $(C_1)$-$(C_6)$; and the sets $R$ $R_1$, $R_2$ are defined by

$$
R = \{(k,k_1,\tau,\tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |\sigma_2| \leq |\sigma_1|\},
$$

$$
R_1 = \{(k,k_1,\tau,\tau_1) \in R : |\sigma_1| \leq |\sigma|\} \quad \text{and} \quad R_2 = \{(k,k_1,\tau,\tau_1) \in R : |\sigma| \leq |\sigma_1|\}.
$$

Using a duality argument we see that

$$
Z_1(f,g) = \sup_{\|h\|_{\ell^2_k}=1} \left| \sum_k \sum_{k_1} \int\int \frac{kf(k_2,\tau_2)g(k_1,\tau_1)h(k)\chi_{R_1} d\tau d\tau_1}{\langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right|.
$$

Now, consider $\sigma, \sigma_1, \sigma_2$ as in the case $(C_1)$ and note that

$$
\begin{aligned}
&\left[ \sum_k \sum_{k_1} \int\int \frac{|kf(k_2,\tau_2)g(k_1,\tau_1)h(k)|\chi_{R_1} d\tau d\tau_1}{\langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right]^2 \\
&\quad \leq \|g\|^2_{\ell^2_k L^2_\tau} \sum_{k_1} \left\| \sum_{k\in\mathbb{Z}} |h(k)|^2 \int_{\mathbb{R}} |f(k_2,\tau_2)|^2 d\tau \right\|_{L^\infty_{\tau_1}} \sum_k \int\int \frac{\chi^2_{R_1}|k|^2 d\tau d\tau_1}{\langle\sigma\rangle^2\langle\sigma_1\rangle\langle\sigma_2\rangle}.
\end{aligned}
$$

Then we will prove that the expression

$$\sum_k |k|^2 \int \frac{1}{\langle\sigma\rangle^2}\Big(\int \frac{\chi^2_{R_1} d\tau_1}{\langle\sigma_1\rangle\langle\sigma_2\rangle}\Big)d\tau$$

is bounded. In fact, if $(k, k_1, \tau, \tau_1) \in R_1$,

$$|\tau_2 + |k_2|^3 + |k_2|| \le |\tau_1 + |k_1|^3 + |k_1|| \le |\tau + |k|^3 + |k||. \tag{3.9}$$

Hence, using inequality (3.1) in Lemma 3.3, we have for $0 < r < 1/4$ that

$$\begin{aligned}
&\frac{1}{\langle\tau + |k|^3 + |k|\rangle^2}\int \frac{d\tau_1}{\langle\tau + |k_1|^3 + |k_1|\rangle\langle\tau_2 + |k_2|^3 + |k_2|\rangle}\\
&\quad\le \frac{1}{\langle\tau + |k|^3 + |k|\rangle^{2(1-r)}}\int_{\mathbb{R}} \frac{d\tau_1}{\langle\tau + |k_1|^3 + |k_1|\rangle^{1+r}\langle\tau_2 + |k_2|^3 + |k_2|\rangle^{1+r}}\\
&\quad\le \frac{C}{\langle\tau + |k|^3 + |k|\rangle^{2(1-r)}\langle\tau + |k_1|^3 + |k_1| + |k_2|^3 + |k_2|\rangle^{1+r}}.
\end{aligned}$$

So, for $0 < r < 1/4$, we will prove that there exists $C > 0$ such that

$$\sum_k |k|^2 \int \frac{d\tau}{\langle\tau + |k|^3 + |k|\rangle^{2(1-r)}\langle\tau + |k_1|^3 + |k_1| + |k_2|^3 + |k_2|\rangle^{1+r}} \le C, \quad \text{on } R_1.$$

The importance of the choice of $r$ will be noted later. We have the relation

$$\begin{aligned}
\tau + |k|^3 + |k| - \big[\tau_1 + |k_1|^3 + |k_1| + \tau_2 + |k_2|^3 + |k_2|\big]&\\
= |k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|.&
\end{aligned} \tag{3.10}$$

Using the triangle inequality in (3.10) and inequality (3.9) we obtain that

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| \le 3|\tau + |k|^3 + |k||. \tag{3.11}$$

**Assume $k_1 > 0$ and $k - k_1 > 0$.** Then

$$|k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 + |k_2| = 3kk_1(k - k_1) > 0.$$

Thus, from (3.4) and (3.10) we see that

$$|\tau + k^3 + k| \ge |kk_1(k - k_1)| \ge \frac{|k|^2}{2},$$

and consequently, for $0 < r < 1/4$, we have that

$$\begin{aligned}
&\sum_k |k|^2 \int \frac{d\tau}{\langle\tau + |k|^3 + |k|\rangle^{2(1-r)}\langle\tau + |k_1|^3 + |k_1| + |k - k_1|^3 + |k - k_1|\rangle^{1+r}}\\
&\quad= \sum_k |k|^2 \int \frac{d\tau}{\langle\tau + k^3 + k\rangle^{2(1-r)}\langle\tau + k_1^3 + k_1 + (k - k_1)^3 + (k - k_1)\rangle^{1+r}}\\
&\quad\le C\sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^{2-4r}}\int_{\mathbb{R}} \frac{d\tau}{\langle\tau + k^3 + k - 3k^2k_1 + 3kk_1^2\rangle^{1+r}}.
\end{aligned}$$

**Assume $k_1 < 0$ and $k - k_1 < 0$.** Then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 + |k_2|\big| = 3|kk_1(k - k_1)|.$$

So, using (3.4) and (3.11) we see that

$$|\tau - k^3 - k| \geq |kk_1(k-k_1)| \geq \frac{|k|^2}{2}.$$

Hence, for $0 < r < 1/4$,

$$\sum_k |k|^2 \int \frac{d\tau}{\langle \tau + |k|^3 + |k| \rangle^{2(1-r)} \langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle^{1+r}}$$
$$\leq C \sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^{2-4r}} \int_{\mathbb{R}} \frac{d\tau}{\langle \tau - k^3 + 3k^2k_1 - 3kk_1^2 - k\rangle^{1+r}}.$$

**Assume $k_1 > 0$ and $k - k_1 < 0$.** Hence, if $k > 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k-k_1|\Big[\Big(k_1 - \frac{k}{4}\Big)^2 + \frac{15k^2}{16} + 1\Big] \geq \frac{15}{8}k^2$$

and if $k < 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k_1|\Big[\Big(k_1 - \frac{3k}{4}\Big)^2 + \frac{15k^2}{16} + 1\Big] \geq \frac{15}{8}k^2.$$

So, from inequality (3.11) we conclude that

$$|\tau + |k|^3 + |k|| \geq \frac{5}{8}k^2,$$

and

$$\sum_k |k|^2 \int \frac{d\tau}{\langle \tau + |k|^3 + |k| \rangle^{2(1-r)} \langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle^{1+r}}$$
$$\leq C \sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^{2-4r}} \int_{\mathbb{R}} \frac{d\tau}{\langle \tau + 2k_1^3 + 2k_1 - k^3 + 3k^2k_1 - 3kk_1^2 - k\rangle^{1+r}}.$$

**Assume $k_1 < 0$ and $k - k_1 > 0$.** Hence, if $k > 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k_1|\Big[\Big(k_1 - \frac{3k}{4}\Big)^2 + \frac{15k^2}{16} + 1\Big] \geq \frac{15k^2}{8}$$

and if $k < 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k-k_1|\Big[\Big(k_1 - \frac{k}{4}\Big)^2 + \frac{15k^2}{16} + 1\Big] \geq \frac{15}{8}k^2.$$

Thus, using (3.11),

$$|\tau + |k|^3 + |k|| \geq \frac{5}{8}k^2$$

and so

$$\sum_k |k|^2 \int \frac{d\tau}{\langle \tau + |k|^3 + |k| \rangle^{2(1-r)} \langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle^{1+r}}$$
$$\leq C \sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^{2-4r}} \int_{\mathbb{R}} \frac{d\tau}{\langle \tau - 2k_1^3 - 2k_1 + k^3 - 3k^2k_1 + 3kk_1^2 + k\rangle^{1+r}}.$$

Therefore, for any $h \in \ell^2_k$ we have that

$$|Z_1|^2 \leq C\|g\|^2_{\ell^2_k L^2_\tau}\Big(\sum_{k\in\mathbb{Z}}|h(k)|^2\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}|f(k_2,\tau)|^2 d\tau\Big)$$
$$\leq C\|f\|^2_{\ell^2_k L^2_\tau}\|g\|^2_{\ell^2_k L^2_\tau}\|h\|^2_{\ell^2_k}.$$

Next, choosing $1/2 < r < 3/4$ we see that

$$|Z_2| \leq C\Big\|\frac{|k|}{\langle\sigma\rangle^{1-r}}\sum_{k_1}\int\frac{|f(k_2,\tau_2)g(k_1,\tau_1)|\chi_{R_2}d\tau_1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\Big\|_{\ell^2_k L^2_\tau}$$
$$= C\sup_{\|h\|_{\ell^2_k L^2_\tau}=1}\Big|\sum_k\sum_{k_1}\int\int\frac{|kf(k_2,\tau_2)g(k_1,\tau_1)|h(k,\tau)\chi_{R_2}d\tau d\tau_1}{\langle\sigma\rangle^{1-r}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\Big|.$$

As before, for $1/2 < r < 3/4$ it is possible to prove that there exists $C > 0$ such that

$$\frac{1}{\langle\tau_1+|k_1|^3+|k_1|\rangle}\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle^{2(1-r)}} \leq C, \quad \text{on } R_2.$$

and, by using Lemma 3.3, we have that

$$\int_{\mathbb{R}}\frac{d\tau}{\langle\tau+|k|^3+|k|\rangle^{2(1-r)}\langle\tau_2+|k_2|^3+|k_2|\rangle} \leq \frac{C}{\langle\tau_1+|k|^3+|k|-|k_2|^3-|k_2|\rangle^{2(1-r)}},$$

then the expression

$$\frac{1}{\langle\sigma_1\rangle}\sum_k|k|^2\int\frac{\chi^2_{R_2}d\tau}{\langle\sigma\rangle^{2(1-r)}\langle\sigma_2\rangle}$$

is bounded. Therefore, for any $h \in \ell^2_k L^2_\tau$ we have that

$$|Z_2|^2 \leq C\|g\|^2_{\ell^2_k L^2_\tau}\sum_{k_1}\int\Big(\sum_k\int|f(k_2,\tau_2)h(k,\tau)|^2 d\tau\Big)\Big(\sum_k\int\frac{\chi^2_{R_2}|k|^2 d\tau}{\langle\sigma\rangle^{2(1-r)}\langle\sigma_1\rangle\langle\sigma_2\rangle}\Big)d\tau_1$$
$$\leq C\|f\|^2_{\ell^2_k L^2_\tau}\|g\|^2_{\ell^2_k L^2_\tau}\|h\|^2_{\ell^2_k L^2_\tau}.$$

In a similar way we have the rest of the proof. □

As a direct consequence of previous lemmas we have the following corollary.

**Corollary 3.6.** *Let $s \geq 0$, then there exists $C_5 > 0$ such that*

*(i)* $\|\psi\partial_x(\eta\partial_x\Phi)\|_{Z^s} \leq C_5\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}}$,

*(ii)* $\|\psi(\partial_x\eta)(\partial_x\Phi_1)\|_{W^{s+1}} \leq C_5\|\Phi\|_{Y^{s+1,1/2}}\|\Phi_1\|_{Y^{s+1,1/2}}$.

## 4 Well-posedness

In this section we establish the local well-posedness for the model (1.1) in the space $U^s \times V^s$.

**Theorem 4.1.** *Let* $s \geq 0$*, then for all* $(\eta_0, \Phi_0) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ *we have that there exist a time* $T = T(\|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}) > 0$ *and a unique solution* $(\eta, \Phi)$ *of the Cauchy problem* (1.1)-(1.2) *such that*

$$\eta \in C([0,T] : H^s(\mathbb{T})) \cap U^s_T \quad \text{and} \quad \Phi \in C([0,T] : \mathcal{V}^{s+1}(\mathbb{T})) \cap V^{s+1}_T.$$

*Moreover, for all* $0 < T' < T$ *there exists a neighborhood* $\mathbb{V}$ *of* $(\eta_0, \Phi_0)$ *in* $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ *such that the map data-solution is Lipschitz from* $\mathbb{V}$ *in the class*

$$C([0,T'] : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})) \cap (U^s_T \times V^{s+1}_T).$$

*Proof.* For $(\eta_0, \Phi_0) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ we consider the operator $\Gamma = (\Gamma_1, \Gamma_2)$ where

$$\Gamma_1(\eta,\Phi)(t) = \psi(t)S_1(t)(\eta_0,\Phi_0) - \psi(t)\int_0^t S_1(t-t')\psi(t')\Big(\partial_x(\eta\partial_x\Phi), \frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt'$$

and

$$\Gamma_2(\eta,\Phi)(t) = \psi(t)S_2(t)(\eta_0,\Phi_0) - \psi(t)\int_0^t S_2(t-t')\psi(t')\Big(\partial_x(\eta\partial_x\Phi), \frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt'.$$

Let $Z_M$ the closed ball of radius $M$ centered at the origin in $U^s \times V^{s+1}$,

$$Z_M = \{(\eta,\Phi) \in U^s \times V^{s+1} : \|(\eta,\Phi)\|_{U^s\times V^{s+1}} \leq M\}.$$

We will show that the correspondence $(\eta,\Phi) \mapsto \Gamma(\eta,\Phi)$ maps $Z_M$ into itself and defines a contraction if $M$ is well chosen. In fact, using Lemma 2.4, Lemma 2.5, and Corollary 3.6 we have that

$$\begin{aligned}
\|\Gamma_1(\eta,\Phi)\|_{U^s} &\leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} + C_2\Big(\|\psi\partial_x(\eta\partial_x\Phi)\|_{Z^s} + \|\psi(\partial_x\Phi)^2\|_{W^{s+1}}\Big)\\
&\leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} + C_2C_5\Big(\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}} + \|\Phi\|^2_{Y^{s+1,1/2}}\Big)\\
&\leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} + C_2C_5\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}}
\end{aligned}$$

and also that

$$\begin{aligned}
\|\Gamma_2(\eta,\Phi)\|_{V^{s+1}} &\leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} + C_2\Big(\|\psi\partial_x(\eta\partial_x\Phi)\|_{Z^s} + \|\psi(\partial_x\Phi)^2\|_{W^{s+1}}\Big)\\
&\leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} + C_2C_5\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}},
\end{aligned}$$

so that

$$\|\Gamma(\eta,\Phi)\|_{U^s\times V^{s+1}} \leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} + C_2C_5\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}}. \tag{4.1}$$

Choosing $M = 2C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}$ such that

$$K_1 = 4C_1C_2C_5\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} < 1,$$

we obtain that

$$\begin{aligned}
\|\Gamma(\eta,\Phi)\|_{U^s\times V^{s+1}} &\leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}(1 + 4C_1C_2C_5\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})})\\
&\leq 2C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} = M
\end{aligned}$$

and that $\Gamma$ maps $Z_M$ to itself. Now, let us prove that $\Gamma$ is a contraction. In fact, if $(\eta, \Phi)$, $(\eta_1, \Phi_1) \in Z_M$, using Lemma 2.5 and Corollary 3.6 we have that

$$
\begin{aligned}
&\|\Gamma_1(\eta,\Phi) - \Gamma_1(\eta_1,\Phi_1)\|_{U^s} \\
&\quad \leq C_2\Big(\|\psi\partial_x(\eta\partial_x\Phi - \eta_1\partial_x\Phi_1)\|_{Z^s} + \|\psi((\partial_x\Phi)^2 - (\partial_x\Phi_1)^2)\|_{W^{s+1}}\Big) \\
&\quad \leq C_2\Big(\|\partial_x(\eta\partial_x(\Phi-\Phi_1))\|_{Z^s} + \|\partial_x((\eta-\eta_1)\partial_x\Phi_1)\|_{Z^s} \\
&\qquad + \|\partial_x(\Phi-\Phi_1)\partial_x(\Phi+\Phi_1)\|_{W^{s+1}}\Big) \\
&\quad \leq C_2C_5\Big(\|\eta\|_{X^{s,1/2}}\|\Phi-\Phi_1\|_{Y^{s+1,1/2}} + \|\eta-\eta_1\|_{X^{s,1/2}}\|\Phi_1\|_{Y^{s+1,1/2}} \\
&\qquad + \|\Phi-\Phi_1\|_{Y^{s+1,1/2}}\|\Phi+\Phi_1\|_{Y^{s+1,1/2}}\Big) \\
&\quad \leq C_2C_5\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\left(\|(\eta,\Phi)\|_{U^s\times V^{s+1}} + \|(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\right).
\end{aligned}
$$

In a similar fashion we see that

$$
\begin{aligned}
&\|\Gamma_2(\eta,\Phi) - \Gamma_2(\eta_1,\Phi_1)\|_{V^{s+1}} \\
&\quad \leq C_2\Big(\|\psi\partial_x(\eta\partial_x\Phi - \eta_1\partial_x\Phi_1)\|_{Z^s} + \|\psi((\partial_x\Phi)^2 - (\partial_x\Phi_1)^2)\|_{W^{s+1}}\Big) \\
&\quad \leq C_2C_5\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\left(\|(\eta,\Phi)\|_{U^s\times V^{s+1}} + \|(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\right).
\end{aligned}
$$

Then, we conclude

$$
\begin{aligned}
&\|\Gamma(\eta,\Phi) - \Gamma(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}} \\
&\quad \leq C_2C_5\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\left(\|(\eta,\Phi)\|_{U^s\times V^{s+1}} + \|(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\right).
\end{aligned} \tag{4.2}
$$

So, if (4) holds we obtain that

$$
\|\Gamma(\eta,\Phi) - \Gamma(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}} \leq K_1\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}
$$

and then $\Gamma$ is a contraction in $Z_M$. Thus, the contraction mapping principle guarantees the existence of a unique fixed point $(\eta, \Phi)$ of $\Gamma$ in $Z_M$, which is solution of the truncated integral problem (1.4). Now, if $(\eta_1, \Phi_1)$ is a restriction of $(\eta, \Phi)$ on $[0, T]$, then using Lemma 2.6 we have that

$$
\eta_1 \in C([0,T] : H^s(\mathbb{T})) \cap U^s, \quad \Phi_1 \in C([0,T] : \mathcal{V}^{s+1}(\mathbb{T})) \cap V^{s+1}
$$

and $(\eta_1, \Phi_1)$ is a solution of the integral problem (1.3) on $[0, T]$.

By the fixed point argument used we have the uniqueness of the solution of the truncated integral problem (1.4) in the set $Z_M$. We will use an argument as in [1] to obtain the uniqueness of the integral problem (1.3) in the space $U^s_T \times V^{s+1}_T$.

Let $T > 0$ and $(\eta, \Phi) \in U^s \times V^{s+1}$ be the solution of the truncated integral problem (1.4) obtained above and $(\eta_1, \Phi_1) \in U^s_T \times V^{s+1}_T$ a solution of the integral problem (1.3) with the same initial data $(\eta_0, \Phi_0) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$. Fix an extension $(\eta_2, \Phi_2) \in U^s \times V^{s+1}$ of $(\eta_1, \Phi_1)$, then, for some $T^* < T < 1$ to be fixed later, we have that

$$
\eta_2(t) = \psi(t)S_1(t)(\eta_0,\Phi_0) - \psi(t)\int_0^t S_1(t-t')\psi(t')\Big(\partial_x(\eta_2\partial_x\Phi_2), \frac{1}{2}(\partial_x\Phi_1)^2\Big)(t')dt'
$$

and

$$
\Phi_2(t) = \psi(t)S_2(t)(\eta_0,\Phi_0) - \psi(t)\int_0^t S_2(t-t')\psi(t')\Big(\partial_x(\eta_2\partial_x\Phi_2), \frac{1}{2}(\partial_x\Phi_1)^2\Big)(t')dt',
$$

for all $t \in [0, T^*]$.

Now, by definition of $U^s_{T^*} \times V^{s+1}_{T^*}$ we have that for any $\epsilon > 0$, there exists $(\omega, \vartheta) \in U^s \times V^{s+1}$ such that for all $t \in [0, T^*]$,

$$\omega(t) = \eta(t) - \eta_2(t), \quad \vartheta(t) = \Phi(t) - \Phi_2(t)$$

and

$$\|\omega\|_{U^s} \le \|\eta - \eta_2\|_{U^s_{T^*}} + \epsilon, \quad \|\vartheta\|_{V^{s+1}} \le \|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} + \epsilon. \tag{4.3}$$

We define

$$\omega_1(t) = -\psi(t) \int_0^t S_1(t-t')\psi(t')\Big(\partial_x(\eta\partial_x\vartheta) + \partial_x(\omega\partial_x\Phi_2), \frac{1}{2}\partial_x\vartheta\partial_x(\Phi + \Phi_2)\Big)(t')dt',$$

$$\vartheta_1(t) = -\psi(t) \int_0^t S_2(t-t')\psi(t')\Big(\partial_x(\eta\partial_x\vartheta) + \partial_x(\omega\partial_x\Phi_2), \frac{1}{2}\partial_x\vartheta\partial_x(\Phi + \Phi_2)\Big)(t')dt'.$$

Then we have that $\omega_1(t) = \eta(t) - \eta_2(t)$ and $\vartheta_1(t) = \Phi(t) - \Phi_2(t)$ for all $t \in [0, T^*]$. Thus, from Lemma 2.5 and Corollary 3.6 we obtain that

$$\begin{aligned}\|\eta - \eta_2\|_{U^s_{T^*}} &\le \|\omega_1\|_{U^s} \\ &\le C_2C_5\|(\omega,\vartheta)\|_{U^s\times V^{s+1}}\Big(\|(\eta,\Phi)\|_{U^s\times V^{s+1}} + \|(\eta_2,\Phi_2)\|_{U^s\times V^{s+1}}\Big) \\ &\le 2C_2C_5N\|(\omega,\vartheta)\|_{U^s\times V^{s+1}}\end{aligned} \tag{4.4}$$

where we assume that

$$\max\{\|(\eta,\Phi)\|_{U^s\times V^{s+1}}, \ \|(\eta_2,\Phi_2)\|_{U^s\times V^{s+1}}\} \le N.$$

In a similar fashion we have that

$$\begin{aligned}\|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} \le \|\vartheta_1\|_{V^{s+1,1/2}} &\le C_2C_5\|(\omega,\vartheta)\|_{U^s\times V^{s+1}}\Big(\|(\eta,\Phi)\|_{U^s\times V^{s+1}} + \|(\eta_1,\Phi_2)\|_{U^s\times V^{s+1}}\Big) \\ &\le 2C_2C_5N\|(\omega,\vartheta)\|_{U^s\times V^{s+1}}.\end{aligned} \tag{4.5}$$

If $4C_2C_5N \le 1/2$, then we obtain, using (4.3), (4.4) and (4.5), that

$$\begin{aligned}\|\eta - \eta_2\|_{U^s_{T^*}} + \|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} &\le 4C_2C_5N\|(\omega,\vartheta)\|_{U^s\times V^{s+1}} \\ &\le \frac{1}{2}\Big(\|\eta - \eta_2\|_{U^s_{T^*}} + \epsilon + \|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} + \epsilon\Big).\end{aligned}$$

So, we see that

$$\|\eta - \eta_2\|_{U^s_{T^*}} + \|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} \le 2\epsilon.$$

Therefore $\eta = \eta_2$ and $\Phi = \Phi_2$ on $[0, T^*]$. Now, since the argument does not depend on the initial data, we can iterate this process a finite number of times to extend the uniqueness result in the whole existence interval $[0, T]$.

Combining an identical argument to the one used in the existence proof with Lemma 2.6, one can easily show that the map data-solution is locally Lipschitz. □

# 5 Internal controllability

## 5.1 Spectral analysis

In this section we perform the spectral analysis for the operator

$$M = \begin{pmatrix} 0 & -(I-\partial_x^2)\partial_x^2 \\ -(I-\partial_x^2) & 0 \end{pmatrix},$$

defined in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$. The result in this analysis will be the basis to transfer the internal controllability of the associated linear system to the nonlinear system. Let us define

$$E_{1,k} = \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}, \quad E_{2,k} = \begin{pmatrix} 0 \\ \frac{1}{k}e^{ikx} \end{pmatrix},$$

for $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. If we set

$$M_k = \begin{pmatrix} 0 & (1+k^2)k^2 \\ -(1+k^2) & 0 \end{pmatrix}, \qquad \Sigma_k = \begin{pmatrix} 0 & (1+k^2)k \\ -(1+k^2)k & 0 \end{pmatrix}, \quad k \in \mathbb{Z}^*$$

then we see directly that

$$M_k(E_{1,k}, E_{2,k}) = (E_{1,k}, E_{2,k})\Sigma_k, \quad k \in \mathbb{Z}^*.$$

Moreover, we have that the eigenvalues for $\Sigma_k$ are

$$\lambda_{1,k} = i\sqrt{(1+k^2)^2k^2}, \quad \lambda_{2,k} = -i\sqrt{(1+k^2)^2k^2}, \quad k \in \mathbb{Z}^*,$$

with corresponding eigenvectors

$$\tilde{e}_{1,k} = \begin{pmatrix} 1 \\ \frac{\lambda_{1,k}}{(1+k^2)k} \end{pmatrix}, \quad \tilde{e}_{2,k} = \begin{pmatrix} 1 \\ \frac{\lambda_{2,k}}{(1+k^2)k} \end{pmatrix}, \quad k \in \mathbb{Z}^*.$$

Thus, we have that

$$\begin{aligned} M(E_{1,k}, E_{2,k})(\tilde{e}_{1,k}, \tilde{e}_{2,k}) &= (E_{1,k}, E_{2,k})\Sigma_k(\tilde{e}_{1,k}, \tilde{e}_{2,k}) \\ &= (\lambda_{1,k}(E_{1,k}, E_{2,k})\tilde{e}_{1,k}, \lambda_{2,k}(E_{1,k}, E_{2,k})\tilde{e}_{2,k}), \; k \in \mathbb{Z}^*, \end{aligned}$$

meaning that $\lambda_{1,k}$ and $\lambda_{2,k}$ are the eigenvalues for the operator $M$ with corresponding eigenvectors

$$\eta_{j,k} = (E_{1,k}, E_{2,k})\tilde{e}_{j,k}, \quad j = 1, 2, \;\; k \in \mathbb{Z},$$

where

$$\lambda_{1,0} = \lambda_{2,0} = 0, \quad \tilde{e}_{1,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{e}_{2,0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E_{1,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E_{2,0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

On the other hand, we see that

$$\lim_{k\to\pm\infty} \frac{\lambda_{1,k}}{(1+k^2)k} = \pm i, \quad \lim_{k\to\pm\infty} \frac{\lambda_{2,k}}{(1+k^2)k} = \mp i.$$

Then

$$\lim_{k\to\infty} (\tilde{e}_{1,k}, \tilde{e}_{2,k}) = \begin{pmatrix} 1 & 1 \\ \pm i & \mp i \end{pmatrix} \quad \text{and} \quad \lim_{k\to\infty} \det(\tilde{e}_{1,k}, \tilde{e}_{2,k}) = \mp 2i \neq 0.$$

In other words, $\{\nu_{1,0}, \nu_{2,0}, \nu_{1,k}, \nu_{2,k} \ : \ k \in \mathbb{Z}^*\}$ forms a Riesz basis for $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ with

$$\nu_{j,k} = \frac{\eta_{j,k}}{\|\eta_{j,k}\|_{H^s \times \mathcal{V}^{s+1}}}.$$

Moreover, we also have for $j = 1, 2$ that $\nu_{j,k} = \vec{b}_{j,k} e^{ikx}$ with

$$0 < B_1 \leq \|\vec{b}_{j,k}\| \leq B_2, \quad k \in \mathbb{Z}, \quad s \geq 0. \tag{5.1}$$

From the above discussion we have the following result.

**Theorem 5.1.** *Let $\lambda_k$ and $\phi_{j,k}$, $j = 1, 2$ be given by*

$$\lambda_k = i \ \mathrm{sign}(k) \sqrt{(1 + k^2)^2 k^2}, \quad k \in \mathbb{Z},$$

$$\phi_{1,k} = \begin{cases} \nu_{1,k}, & k = 0, 1, 2, 3, \ldots \\ \nu_{2,k}, & k = -1, -2, -3, \ldots, \end{cases} \qquad \phi_{2,k} = \begin{cases} \nu_{1,-k}, & k = 1, 2, 3, \ldots \\ \nu_{2,-k}, & k = 0, -1, -2, -3, \ldots \end{cases}$$

*then we have that*

- *(i) The spectrum of the operator $M$ is $\sigma(M) = \{\lambda_k \ : \ k \in \mathbb{Z}\}$, in which each $\lambda_k$ is a double eigenvalue with eigenvectors $\phi_{1,k}$ and $\phi_{2,k}$.*
- *(ii) The set $\{\phi_{1,k}, \phi_{2,k} \ : \ k \in \mathbb{Z}\}$ forms an orthonormal basis for the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ such that any $(\eta, \Phi) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ has the following Fourier expansion*

$$(\eta, \Phi) = \sum_{k \in \mathbb{Z}} (\alpha_{1,k}\phi_{1,k} + \alpha_{2,k}\phi_{2,k}), \quad \alpha_{j,k} = \langle (\eta, \Phi), \phi_{j,k} \rangle_Q, \quad j = 1, 2, \ k \in \mathbb{Z},$$

*where $Q = L^2(\mathbb{T}) \times L^2(\mathbb{T})$.*

## 5.2 Linear controllability

In this section we consider the internal control problem for the linear system

$$\begin{cases} \eta_t + \partial_x^2 \Phi - \partial_x^4 \Phi = f_1, \\ \Phi_t + \eta - \partial_x^2 \eta = f_2, \end{cases} \tag{5.2}$$

with the initial condition

$$\eta(x, 0) = \eta_0(x), \quad \Phi(x, 0) = \Phi_0(x). \tag{5.3}$$

**Theorem 5.2.** *Suppose that $\rho = (\rho_1, \rho_2)$ is a non-zero smooth function defined on $\mathbb{T}$. Let $s \geq 0$ and $T > 0$, then for any $(\eta_0, \Phi_0)$, $(\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ there exists a function $H = (h_1, h_2) \in L^2\left(0, T; H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right)$ such that if*

$$F = (f_1(x, t), f_2(x, t)) = (\rho_1 h_1(x, t), \rho_2(x) h_2(x, t))$$

*we have that the problem (5.2)–(5.3) has a unique solution*

$$(\eta, \Phi) \in C\left([0, T] \ : \ H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right)$$

*satisfying*

$$\eta(x, T) = \eta_T(x), \quad \Phi(x, T) = \Phi_T(x).$$

*Moreover, there exists $C = C(T) > 0$ such that*

$$\|H\|_{L^2(0,T; H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}))} \leq C \left( \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} + \|(\eta_T, \Phi_T)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} \right).$$

*Proof.* For sake of simplicity in the proof we will consider $\rho_2 = \Phi_0 = \Phi_T = 0$. For any function $h = h(x,t)$, we define the control operator $L$ by

$$(Lh)(x,t) = \rho_1(x)h(x,t).$$

If $f_1 = Lh$ and $f_2 = 0$, then we rewrite the problem (5.2)–(5.3) as the following first order linear

$$\begin{pmatrix}\eta \\ \Phi\end{pmatrix}_t = M\begin{pmatrix}\eta \\ \Phi\end{pmatrix} + Bh, \tag{5.4}$$

with the initial condition

$$\eta(x,0) = \eta_0(x), \quad \Phi(x,0) = 0 \tag{5.5}$$

where

$$Bh = \begin{pmatrix}Lh \\ 0\end{pmatrix}.$$

In this case for $h \in L^2(0,T;H^s(\mathbb{T}))$, the solution of the linear problem (5.4)–(5.5) is given by

$$(\eta(t),\Phi(t)) = S(t)\,(\eta_0,0) + \int_0^t S(t-\tau)Bh(\tau)\,d\tau.$$

Now, using the spectral analysis on the operator $M$ we have that

$$\begin{aligned}(\eta(t),\Phi(t)) = &\sum_{n\in\mathbb{Z}} e^{\lambda_n t}(\alpha_{1,n}\phi_{1,n} + \alpha_{2,n}\phi_{2,n}) \\ &+ \sum_{n\in\mathbb{Z}}\int_0^t e^{\lambda_n(t-\tau)}\left(\beta_{1,n}(\tau)\phi_{1,n} + \beta_{2,n}(\tau)\phi_{2,n}\right)d\tau,\end{aligned} \tag{5.6}$$

where $\alpha_{j,n}$ and $\beta_{j,n}$, for $j = 1,2$, $n \in \mathbb{Z}$, are given by

$$\alpha_{j,n} = \left\langle(\eta_0,\Phi_0),\phi_{j,n}\right\rangle_Q, \quad \beta_{j,n}(t) = \left\langle Bh,\phi_{j,n}\right\rangle_Q. \tag{5.7}$$

We verify easily that $L$ is a self-adjoint operator in $L^2(\mathbb{T})$ such that

$$\langle Bh,(\eta,\Phi)\rangle_Q = \langle(Lh,0),(\eta,\Phi)\rangle_Q = \langle Lh,\eta\rangle_{L^2(\mathbb{T})} = \langle h,L\eta\rangle_{L^2(\mathbb{T})},$$

then we have that

$$\alpha_{j,n} = \left\langle\eta_0,\phi_{j,n}^{(1)}\right\rangle_{L^2(\mathbb{T})}, \quad \beta_{j,n}(t) = \left\langle h(\cdot,t),L(\phi_{j,n}^{(1)})\right\rangle_{L^2(\mathbb{T})},$$

where $\phi^{(m)}$ denoting the $m$ component of $\phi$.

The internal control problem consists of finding a $h \in L^2(0,T;H^s(\mathbb{T}))$ such that

$$\eta(x,T) = \eta_T(x), \quad \Phi(x,T) = 0.$$

Then, let $\eta_0$ and $\eta_T$ be having the following decompositions

$$\eta_0 = \sum_{n\in\mathbb{Z}}\left(\alpha_{1,n}\phi_{1,n}^{(1)} + \alpha_{2,n}\phi_{2,n}^{(1)}\right), \quad \eta_T = \sum_{n\in\mathbb{Z}}\left(\gamma_{1,n}\phi_{1,n}^{(1)} + \gamma_{2,n}\phi_{2,n}^{(1)}\right),$$

where $\gamma_{j,n} = \langle\eta_T,\phi_{j,n}^{(1)}\rangle_{L^2(\mathbb{T})}$. But, we know that

$$\begin{aligned}(\eta(x,T),\Phi(x,T)) = &\sum_{n\in\mathbb{Z}} e^{\lambda_n T}\left(\alpha_{1,n}\phi_{1,n} + \alpha_{2,n}\phi_{2,n}\right) \\ &+ \sum_{n\in\mathbb{Z}}\int_0^T e^{\lambda_n(T-\tau)}\left(\beta_{1,n}(\tau)\phi_{1,n} + \beta_{2,n}(\tau)\phi_{2,n}\right)\,d\tau.\end{aligned}$$

So, in each node, we have for $j = 1, 2$ and $n \in \mathbb{Z}$ that

$$\alpha_{j,n} + \int_0^T e^{-\lambda_n \tau} \beta_{j,n}(\tau)\, d\tau = \gamma_{j,n} e^{-\lambda_n T}.$$

Now, from [7] we have that $\mathcal{P} = \{e^{\lambda_k t} : k \in \mathbb{Z}\}$ is a Riesz basis for its closed span $\mathcal{P}_T = \overline{\text{gen}}\, \mathcal{P}$ generated in $L^2(0, T)$, with a unique dual Riesz basis given by $\mathcal{L} = \{q_k : k \in \mathbb{Z}\}$ satisfying that

$$\int_0^T q_l(t) \overline{e^{\lambda_k t}}\, dt = \delta_l^k, \quad l, k \in \mathbb{Z}.$$

We assume that $f_1$ has the form $f_1 = Lh$ with $h$ given by the expansion

$$h(x,t) = \sum_{l \in \mathbb{Z}} q_l(t) \left( c_{1,l} L(\phi_{1,l}^{(1)}) + c_{2,l} L(\phi_{2,l}^{(1)}) \right), \tag{5.8}$$

where the coefficients $c_{1,l}$ and $c_{2,l}$ are to be determined so that, among other things, the series (5.8) is appropriately convergent. In this case, for $j = 1, 2$ and $n \in \mathbb{Z}$ we have that

$$\begin{aligned}
\int_0^T e^{-\lambda_n \tau} \beta_{j,n}(\tau)\, d\tau &= \int_0^T e^{-\lambda_n \tau} \left\langle h(\cdot, \tau), L(\phi_{j,n}^{(1)}) \right\rangle_{L^2(\mathbb{T})} d\tau \\
&= \sum_{k \in \mathbb{Z}} \int_0^T e^{-\lambda_n \tau} \Big( \int_{\mathbb{T}} h(y, \tau) e^{iky}\, dy \Big) \overline{\left( L(\phi_{j,n}^{(1)}) \right)_k}\, d\tau \\
&= \sum_{k \in \mathbb{Z}} \overline{\left( L(\phi_{j,n}^{(1)}) \right)_k} \int_{\mathbb{T}} \left( \sum_{l \in \mathbb{Z}} \int_0^T q_l(\tau) \overline{e^{\lambda_n \tau}} d\tau \left( c_{1,l} L(\phi_{1,l}^{(1)}) + c_{2,l} L(\phi_{2,l}^{(1)}) \right) \right) e^{iky}\, dy \\
&= \sum_{k \in \mathbb{Z}} \overline{\left( L(\phi_{j,n}^{(1)}) \right)_k} \int_{\mathbb{T}} \left( c_{1,n} L(\phi_{1,n}^{(1)}) + c_{2,n} L(\phi_{2,n}^{(1)}) \right) e^{iky}\, dy \\
&= c_{1,n} \left\langle L(\phi_{1,n}^{(1)}), L(\phi_{j,n}^{(1)}) \right\rangle_{L^2(\mathbb{T})} + c_{2,n} \left\langle L(\phi_{2,n}^{(1)}), L(\phi_{j,n}^{(1)}) \right\rangle_{L^2(\mathbb{T})},
\end{aligned}$$

where $\left( L(\phi_{j,n}^{(1)}) \right)_k = \left( \widehat{L(\phi_{j,n}^{(1)})} \right)(k)$. Now, using the computations above, for $n \in \mathbb{Z}$, we have that $c_{1,n}$ and $c_{2,n}$ must satisfy the linear system

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} c_{1,n} \\ c_{2,n} \end{pmatrix} = \begin{pmatrix} -\alpha_{1,n} + \gamma_{1,n} e^{-\lambda_n T} \\ -\alpha_{2,n} + \gamma_{2,n} e^{-\lambda_n T} \end{pmatrix},$$

where $a_{jl} = \langle L(\phi_{j,n}^{(1)}), L(\phi_{l,n}^{(1)}) \rangle_{L^2(\mathbb{T})}$. Using the fact that $L(\phi_{1,n}^{(1)})$ and $L(\phi_{2,n}^{(1)})$ are linear independent, we obtain that

$$\Delta_n = \det \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \|L(\phi_{1,n}^{(1)})\|_{L^2(\mathbb{T})}^2 \|L(\phi_{2,n}^{(1)})\|_{L^2(\mathbb{T})}^2 - \left| \langle L(\phi_{1,n}^{(1)}), L(\phi_{2,n}^{(1)}) \rangle_{L^2(\mathbb{T})} \right|^2 \neq 0.$$

Moreover, using that $\nu_{j,n}^{(1)} = b_{j,n}^{(1)} e^{inx}$ and the estimate (5.1), we have that

$$\|L(\phi_{j,n}^{(1)})\|_{L^2(\mathbb{T})}^2 \sim |b_{j,n}^{(1)}|^2 \geq C > 0. \tag{5.9}$$

In addition (see the estimations by B. Zhang for the Boussinesq equation in [20]), it is not hard to prove that

$$\left\langle L(\phi_{1,n}^{(1)}), L(\phi_{2,n}^{(1)}) \right\rangle \to 0, \quad n \to \infty.$$

Hence there exists $\epsilon > 0$ such that $|\Delta_n| > \epsilon$. So, we conclude that $c_{1,n}$ and $c_{2,n}$ are uniquely determine by

$$c_{1,n} = \frac{\begin{vmatrix} -\alpha_{1,n} + \gamma_{1,n} e^{-\lambda_n T} & a_{21} \\ -\alpha_{2,n} + \gamma_{2,n} e^{-\lambda_n T} & a_{22} \end{vmatrix}}{\Delta_n}, \quad c_{2,n} = \frac{\begin{vmatrix} a_{11} & -\alpha_{1,n} + \gamma_{1,n} e^{-\lambda_n T} \\ a_{12} & -\alpha_{2,n} + \gamma_{2,n} e^{-\lambda_n T} \end{vmatrix}}{\Delta_n}. \tag{5.10}$$

It remains to show that $h$ defined by (5.8)–(5.10) belongs to the space $L^2(0,T;H^s(\mathbb{T}))$ provided that $\eta_0, \eta_T \in H^s(\mathbb{T})$. To this end, let us write

$$L(\phi_{j,l}^{(1)}) = \sum_{k\in\mathbb{Z}} a_{j,lk} e^{ikx}, \quad a_{j,lk} = \left( L(\phi_{j,l}^{(1)}) \right)_k, \quad l,k \in \mathbb{Z}, \ j = 1,2.$$

Thus

$$h(x,t) = h_1(x,t) + h_2(x,t),$$

where

$$h_j(x,t) = \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,l} a_{j,lk} q_l(t) e^{ikx}, \quad j = 1,2.$$

From this, we conclude that

$$\begin{aligned}
&\|h_j\|^2_{L^2(0,T;H^s(\mathbb{T}))} \\
&\quad = \int_0^T \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} | \left(h_j(\cdot,t)\right)_k |^2 \, dt = \int_0^T \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \Big| \sum_{l\in\mathbb{Z}} a_{j,lk} c_{j,l} q_l(t) \Big|^2 dt \\
&\quad = \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \int_0^T \Big| \sum_{l\in\mathbb{Z}} a_{j,lk} c_{j,l} q_l(t) \Big|^2 dt \le C \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \sum_{l\in\mathbb{Z}} |c_{j,l}|^2 |a_{j,lk}|^2 \\
&\quad = C \sum_{l\in\mathbb{Z}} |c_{j,l}|^2 \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |a_{j,lk}|^2,
\end{aligned}$$

where the constant $C > 0$ comes from the Riesz basis property of $\mathcal{L}$ in $\mathcal{P}_T$. Now, using (5.1), if $\rho_1 = \sum_{m\in\mathbb{Z}} \rho_m^1 e^{imx}$ we have that there exists $C > 0$ such that

$$\begin{aligned}
|a_{j,lk}| &= \left| \left\langle L(\phi_{j,l}^{(1)}), e^{ikx} \right\rangle_{L^2(\mathbb{T})} \right| = \left| \left\langle \rho_1 \phi_{j,l}^{(1)}, e^{ikx} \right\rangle_{L^2(\mathbb{T})} \right| \\
&= \left| \sum_{m\in\mathbb{Z}} \rho_m^1 \left\langle \phi_{j,l}^{(1)} e^{imx}, e^{ikx} \right\rangle_{L^2(\mathbb{T})} \right| = \left| \sum_{m\in\mathbb{Z}} \rho_m^1 \left\langle b_{j,l}^{(1)} e^{ilx} e^{imx}, e^{ikx} \right\rangle_{L^2(\mathbb{T})} \right| \\
&\le C \Big| \int_{\mathbb{T}} e^{-ix(k-l)} \sum_{m\in\mathbb{Z}} \rho_m^1 e^{imx} \, dx \Big| \le C \left| \rho_{k-l}^1 \right|.
\end{aligned}$$

Then we see that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |a_{j,lk}|^2 &\le C \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |\rho_{k-l}^1|^2 \le C \sum_{k\in\mathbb{Z}} (1+|k+l|)^{2s} |\rho_k^1|^2 \\
&\le C(1+|l|)^{2s} \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |\rho_k^1|^2 = (1+|l|)^{2s} \|\rho_1\|^2_{H^s(\mathbb{T})}.
\end{aligned}$$

On the other hand, we can to see that

$$\begin{aligned}
|c_{1,l}|^2 &\le C\left(|a_{22}|^2 + |a_{21}|^2\right)\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right) \\
&= C\left( \|L(\phi_{2,l}^{(1)})\|^4_{L^2(\mathbb{T})} + \left| \left\langle L(\phi_{2,l}^{(1)}), L(\phi_{1,l}^{(1)}) \right\rangle_{L^2(\mathbb{T})} \right|^2 \right) \left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right) \\
&\le C\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right).
\end{aligned}$$

Similarly,

$$|c_{2,l}|^2 \leq C\Big(\|L(\phi_{1,l}^{(1)})\|_{L^2(\mathbb{T})}^4 + \Big|\left\langle L(\phi_{1,l}^{(1)}), L(\phi_{2,l}^{(1)})\right\rangle_{L^2(\mathbb{T})}\Big|^2\Big) \times \left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right)$$
$$\leq C\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right).$$

From this, we conclude that

$$\|h\|_{L^2(0,T;H^s(\mathbb{T}))}^2 \leq C\|\rho_1\|_{H^s(\mathbb{T})}^2 \left(\sum_{l\in\mathbb{Z}}(1+|l|)^{2s}\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{1,l}|^2\right)\right)$$
$$\leq C\|\rho_1\|_{H^s(\mathbb{T})}^2 \left(\|\eta_0\|_{H^s(\mathbb{T})}^2 + \|\eta_T\|_{H^s(\mathbb{T})}^2\right).$$

Now we consider $\rho_1 = \eta_0 = \eta_T = 0$. Since the proof is similar to the previous case, we only present some ideas. The solution of the linear problem

$$\begin{pmatrix}\eta\\ \Phi\end{pmatrix}_t = M\begin{pmatrix}\eta\\ \Phi\end{pmatrix} + Bh, \quad (\eta(x,0),\Phi(x,0)) = (0,\Phi_0(x))$$

with

$$(Bh)(x,t) = \begin{pmatrix}0\\ (Lh)(x,t)\end{pmatrix} = \begin{pmatrix}0\\ \rho_2(x)h(x,t)\end{pmatrix}$$

is given by

$$(\eta(t),\Phi(t)) = \sum_{n\in\mathbb{Z}} e^{\lambda_n t}\left(\alpha_{1,n}\phi_{1,n} + \alpha_{2,n}\phi_{2,n}\right) + \sum_{n\in\mathbb{Z}}\int_0^t e^{\lambda_n(t-\tau)}\left(\beta_{1,n}(\tau)\phi_{1,n} + \beta_{2,n}(\tau)\phi_{2,n}\right)d\tau,$$

where $\alpha_{j,n}$ and $\beta_{j,n}$ for $j = 1,2$, $n \in \mathbb{Z}$ are given by

$$\alpha_{j,n} = \left\langle \Phi_0, \phi_{j,n}^{(2)}\right\rangle_{L^2(\mathbb{T})}, \quad \beta_{j,n}(t) = \left\langle h(\cdot,t), L(\phi_{j,n}^{(2)})\right\rangle_{L^2(\mathbb{T})}.$$

Then, using the decompositions

$$\Phi_0 = \sum_{n\in\mathbb{Z}}\left(\alpha_{1,n}\phi_{1,n}^{(2)} + \alpha_{2,n}\phi_{2,n}^{(2)}\right), \quad \Phi_T = \sum_{n\in\mathbb{Z}}\left(\gamma_{1,n}\phi_{1,n}^{(2)} + \gamma_{2,n}\phi_{2,n}^{(2)}\right),$$

where $\gamma_{j,n} = \langle \Phi_T, \phi_{j,n}^{(2)}\rangle_{L^2(\mathbb{T})}$, then we have for $j = 1,2$ and $n \in \mathbb{Z}$ that

$$\alpha_{j,n} + \int_0^T e^{-\lambda_n\tau}\beta_{j,n}(\tau)\,d\tau = \gamma_{j,n}e^{-\lambda_n T}.$$

Now, we take the control $h$ to have the form

$$h(x,t) = \sum_{l\in\mathbb{Z}} q_l(t)\left(c_{1,l}L(\phi_{1,l}^{(2)}) + c_{2,l}L(\phi_{2,l}^{(2)})\right)$$

with

$$\int_0^T q_l(t)\overline{e^{\lambda_k t}}\,dt = \delta_l^k, \quad l,k \in \mathbb{Z}.$$

Then, for $j = 1, 2$ and $n \in \mathbb{Z}$ we have that

$$\begin{aligned}\int_0^T e^{-\lambda_n \tau} \beta_{j,n}(\tau)\, d\tau &= \sum_{k\in\mathbb{Z}} \overline{\left(L\phi_{j,n}^{(2)}\right)_k} \int_0^T e^{-\lambda_n \tau}\Big( \int_{\mathbb{T}} h(y,\tau) e^{iky}\, dy\Big)\, d\tau \\ &= \sum_{k\in\mathbb{Z}} \overline{\left(L(\phi_{j,n}^{(2)})\right)_k} \int_{\mathbb{T}} \left(c_{1,n} L(\phi_{1,n}^{(2)}) + c_{2,n} L(\phi_{2,n}^{(2)})\right) e^{iky}\, dy \\ &= c_{1,n} \left\langle L(\phi_{1,n}^{(2)}), L(\phi_{j,n}^{(2)})\right\rangle_{L^2(\mathbb{T})} + c_{2,n} \left\langle L(\phi_{2,n}^{(2)}), L(\phi_{j,n}^{(2)})\right\rangle_{L^2(\mathbb{T})}.\end{aligned}$$

Thus, the coefficients $c_{1,n}$ and $c_{2,n}$ are uniquely determine by

$$c_{1,n} = \frac{\begin{vmatrix} -\alpha_{1,n} + \gamma_{1,n} e^{-\lambda_n T} & a_{21} \\ -\alpha_{2,n} + \gamma_{2,n} e^{-\lambda_n T} & a_{22} \end{vmatrix}}{\Delta_n}, \quad c_{2,n} = \frac{\begin{vmatrix} a_{11} & -\alpha_{1,n} + \gamma_{1,n} e^{-\lambda_n T} \\ a_{12} & -\alpha_{2,n} + \gamma_{2,n} e^{-\lambda_n T} \end{vmatrix}}{\Delta_n},$$

where

$$a_{jl} = \left\langle L(\phi_{j,n}^{(2)}), L(\phi_{l,n}^{(2)})\right\rangle_{L^2(\mathbb{T})}$$

and

$$\Delta_n = \|L(\phi_{1,n}^{(2)})\|^2_{L^2(\mathbb{T})} \|L(\phi_{2,n}^{(2)})\|^2_{L^2(\mathbb{T})} - \left|\langle L(\phi_{1,n}^{(2)}), L(\phi_{2,n}^{(2)})\rangle_{L^2(\mathbb{T})}\right|^2.$$

Finally, we write

$$\rho_2 = \sum_{k\in\mathbb{Z}} \rho_k^2 e^{ikx},\ L(\phi_{j,l}^{(2)}) = \sum_{k\in\mathbb{Z}} a_{j,lk} e^{ikx},\ a_{j,lk} = \left(L(\phi_{j,l}^{(2)})\right)_k,\ l,k \in \mathbb{Z},\ j = 1,2.$$

Then

$$h(x,t) = \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{1,l} a_{1,lk} q_l(t) e^{ikx} + \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{2,l} a_{2,lk} q_l(t) e^{ikx}.$$

Hence, we see that

$$\begin{aligned}\|h\|^2_{L^2(0,T;\mathcal{V}^{s+1}(\mathbb{T}))} &= \sum_{j=1}^{2} \sum_{k\in\mathbb{Z}} |k|^2 (1+|k|)^{2s} \int_0^T \Big|\sum_{l\in\mathbb{Z}} a_{j,lk} c_{j,l} q_l(t)\Big|^2 dt \\ &\leq C \sum_{j=1}^{2} \sum_{l\in\mathbb{Z}} |c_{j,l}|^2 \sum_{k\in\mathbb{Z}} |k|^2 (1+|k|)^{2s} |a_{j,lk}|^2.\end{aligned}$$

The fact $\left|a_{j,lk}\right| \leq C \left|\rho_{k-l}^2\right|$ implies

$$\sum_{k\in\mathbb{Z}} |k|^2 (1+|k|)^{2s} |a_{j,lk}|^2 \leq C \sum_{k\in\mathbb{Z}} |k+l|^2 (1+|k+l|)^{2s} |\rho_k^2|^2 \leq C|l|^2 \left(1+|l|\right)^{2s} \|\rho_2\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}.$$

Then, using

$$|c_{j,l}|^2 \leq C\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right),$$

we conclude that

$$\begin{aligned}\|h\|^2_{L^2(0,T;\mathcal{V}^{s+1}(\mathbb{T}))} &\leq C\|\rho_2\|^2_{\mathcal{V}^{s+1}(\mathbb{T})} \left(\sum_{l\in\mathbb{Z}} |l|^2 (1+|l|)^{2s} \left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{1,l}|^2\right)\right) \\ &\leq C\|\rho_2\|^2_{H^s(\mathbb{T})} \left(\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})} + \|\Phi_T\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}\right).\end{aligned}$$

□

**Remark 5.3.** If $T > 0$ and $(\eta_0, \Phi_0), (\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ with $s \geq 0$, the Theorem 5.2 implies that there is $F$ such that

$$S(T)(\eta_0, \Phi_0) + \int_0^T S(T-\tau)F(\tau)\, d\tau = (\eta_T, \Phi_T).$$

Moreover, we have that there exists $C = C(T) > 0$ such that

$$\|F\|_{L^1(0,T;H^s\times\mathcal{V}^{s+1})} \leq C\left(\|(\eta_0, \Phi_0)\|_{H^s\times\mathcal{V}^{s+1}} + \|(\eta_T, \Phi_T)\|_{H^s\times\mathcal{V}^{s+1}}\right).$$

## 5.3 Nonlinear controllability

Now we turn to the nonlinear system

$$\begin{cases} \eta_t + \partial_x^2\Phi - \partial_x^4\Phi + \partial_x(\eta\partial_x\Phi) = f_1, \\ \quad \Phi_t + \eta - \partial_x^2\eta + \dfrac{1}{2}(\partial_x\Phi)^2 = f_2, \end{cases} \tag{5.11}$$

with the initial condition

$$\eta(x,0) = \eta_0(x), \quad \Phi(x,0) = \Phi_0(x), \tag{5.12}$$

**Theorem 5.4.** *Let $s \geq 0$ and $T > 0$ be given, then there exists $\delta > 0$ such that for any $(\eta_0, \Phi_0)$, $(\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ satisfying*

$$\|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}, \quad \|(\eta_T, \Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} < \delta,$$

*there exists a control function $F = (f_1, f_2) \in L^1(0, T; H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}))$ such that the solution $(\eta, \Phi) \in C\left([0, T] : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right) \cap U_T^s \times V_T^{s+1}$ of the problem (5.11)–(5.12) satisfies*

$$\eta(x, T) = \eta_T(x), \quad \Phi(x, T) = \Phi_T(x).$$

*Proof.* We rewrite the Cauchy problem (5.11)–(5.12) in its equivalent form:

$$\begin{aligned} (\eta(t), \Phi(t)) = S(t)(\eta_0, \Phi_0) &+ \int_0^t S(t-t')F(t')\, dt' \\ &- \int_0^t S(t-t')\left(\partial_x(\eta\partial_x\Phi), \frac{1}{2}(\partial_x\Phi)^2\right)(t')dt'. \end{aligned} \tag{5.13}$$

Now, for any $(\eta, \Phi) = (\eta(x,t), \Phi(x,t))$ we define

$$w((\eta, \Phi), T) = \int_0^T S(T-t')\left(\partial_x(\eta\partial_x\Phi), \frac{1}{2}(\partial_x\Phi)^2\right)(t')\, dt'$$

According to Theorem 5.2, for given $(\eta_0, \Phi_0), (\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, if one chooses

$$F = F_{(\eta,\Phi)}$$

such that

$$S(T)(\eta_0, \Phi_0) + \int_0^T S(T-t')F_{(\eta,\Phi)}(t')\, dt' = (\eta_T, \Phi_T) + w((\eta, \Phi), T)$$

in the equation (5.13), then

$$(\eta(t),\Phi(t)) = S(t)(\eta_0,\Phi_0) + \int_0^t S(t-t')F_{(\eta,\Phi)}(t')\,dt' - \int_0^t S(t-t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt', \tag{5.14}$$

with $(\eta(0),\Phi(0)) = (\eta_0,\Phi_0)$ and

$$\begin{aligned}(\eta(T),\Phi(T)) &= S(T)(\eta_0,\Phi_0) + \int_0^T S(T-t')F_{(\eta,\Phi)}(t')dt' \\ &\quad - \int_0^T S(T-t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt' \\ &= (\eta_T,\Phi_T) + w((\eta,\Phi),T) - w((\eta,\Phi),T) = (\eta_T,\Phi_T).\end{aligned}$$

This suggests that we consider the map

$$\Gamma(\eta,\Phi) = S(t)(\eta_0,\Phi_0) + \int_0^t S(t-t')F_{(\eta,\Phi)}(t')dt' - \int_0^t S(t-t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt'.$$

If the map $\Gamma$ is shown to be a contraction in an appropriate space, then its fixed point $(\eta,\Phi)$ is a solution of (5.11)-(5.12) and satisfies $(\eta(x,T),\Phi(x,T)) = (\eta_T(x),\Phi_T(x))$. We show this is the case in the space $U^s\times V^{s+1}$.

As in the case of the KdV equation in [18], we modify the map $\Gamma=(\Gamma_1,\Gamma_2)$ as follow:

$$\begin{aligned}\Gamma_1(\eta,\Phi)(t) &= \psi_1(t)S_1(t)(\eta_0,\Phi_0) + \psi_1(t)\int_0^t S_1(t-t')\psi_2(t')F_{(\eta,\Phi)}(t')dt' \\ &\quad - \psi_1(t)\int_0^t S_1(t-t')\psi_2(t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt',\end{aligned}$$

and

$$\begin{aligned}\Gamma_2(\eta,\Phi)(t) &= \psi_1(t)S_2(t)(\eta_0,\Phi_0) + \psi_1(t)\int_0^t S_2(t-t')\psi_2(t')F_{(\eta,\Phi)}(t')dt' \\ &\quad - \psi_1(t)\int_0^t S_2(t-t')\psi_2(t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt',\end{aligned}$$

where $\psi_1$ is a smooth function with its support inside the interval $(T-1,T+1)$ and $\psi_1(t)=1$ for $t\in[-T,T]$, and $\psi_2$ is a nonnegative smooth function with $supp\,\psi_2\subset(-T-1,T+1)$ satisfying $\psi_2(t)=1$ for any $t$ in the support of $\psi_1$.

As in Theorem 4.1, let $Z_M$ be the closed ball of radius $M$ centered at the origin in $U^s\times V^{s+1}$. Using Remark 5.3 and slight modifications of results in Lemmas 2.2–2.6 we have that

$$\begin{aligned}\|w((\eta,\Phi),T)\|_{H^s\times\mathcal{V}^{s+1}} &= \Big\|\int_0^T S(T-t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')\,dt'\Big\|_{H^s\times\mathcal{V}^{s+1}} \\ &\le \sup_{t\in\mathbb{R}}\Big\|\psi_1(t)\int_0^t S(t-t')\psi_2(t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')\,dt'\Big\|_{H^s\times\mathcal{V}^{s+1}} \\ &\le C\Big\|\psi_1(t)\int_0^t S(t-t')\psi_2(t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')\,dt'\Big\|_{U^s\times V^{s+1}} \\ &\le C\Big(\|\partial_x(\eta\partial_x\Phi)\|_{Z^s} + \|(\partial_x\Phi)^2\|_{W^{s+1}}\Big) \\ &\le C\left(\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}} + \|\Phi\|^2_{Y^{s+1,1/2}}\right)\end{aligned}$$

and also that

$$\begin{aligned}\|\Gamma_1(\eta,\Phi)\|_{U^s} &\le C(T)\left(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|F_{(\eta,\Phi)}\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}\right)\\ &\quad + C_2C_5\left(\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}}+\|\Phi\|^2_{Y^{s+1,1/2}}\right)\\ &\le C(T)\big(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}}\big)\end{aligned}$$

and

$$\|\Gamma_2(\eta,\Phi)\|_{V^{s+1}} \le C(T)\big(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}}\big).$$

So that

$$\|\Gamma(\eta,\Phi)\|_{U^s\times V^{s+1}} \le C(T)\big(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}}\big).$$

Choosing $\delta > 0$ and

$$M = 2C(T)\left(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}\right)$$

in such a way that

$$2C^2(T)M < 1, \quad 2C(T)\delta \le M,$$

then we conclude for any $(\eta,\Phi)\in Z_M$ that

$$\begin{aligned}\|\Gamma(\eta,\Phi)\|_{U^s\times V^{s+1}} &\le C(T)\left(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}\right)\left(1+4C^2(T)M\right)\\ &\le 2C(T)\left(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}\right)\\ &\le 2C(T)\delta \le M\end{aligned}$$

provided that $\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} < \delta$. Now, using the same of computations, we have that $\Gamma$ is a contraction on $Z_M$, and so, the Banach Fixed Point Theorem guaranties the existence fixed point $(\eta,\Phi)$ of $\Gamma$ in $Z_M$. This fixed point $(\eta,\Phi)$ is a unique solution of the integral equation

$$\begin{aligned}(\eta(t),\Phi(t)) &= \psi_1(t)S(t)(\eta_0,\Phi_0)+\psi_1(t)\int_0^t S(t-t')\psi_2(t')F_{(\eta,\Phi)}(t')dt'\\ &\quad -\psi_1(t)\int_0^t S(t-t')\psi_2(t')\left(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\right)(t')dt'.\end{aligned}$$

In particular for $t\in[0,T]$,

$$\begin{aligned}(\eta(t),\Phi(t)) &= S(t)(\eta_0,\Phi_0)+\int_0^t S(t-t')(t')F_{(\eta,\Phi)}(t')dt'\\ &\quad -\int_0^t S(t-t')(t')\left(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\right)(t')dt'.\end{aligned}$$

That is to say, $(\eta,\Phi)\in C\left([0,T]:H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})\right)$ solves

$$\begin{cases}\eta_t+\partial_x^2\Phi-\partial_x^4\Phi+\partial_x(\eta\partial_x\Phi)=f_1,\\ \Phi_t+\eta-\partial_x^2\eta+\frac{1}{2}(\partial_x\Phi)^2=f_2,\end{cases}$$

with the conditions

$$\eta(x,0)=\eta_0(x),\quad \Phi(x,0)=\Phi_0(x),\quad \eta(x,T)=\eta_T(x),\quad \Phi(x,T)=\Phi_T(x) \qquad \square$$

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