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Existence results for singular nonlinear BVPs in the critical regime

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Abstract. We study the existence of solutions for a class of boundary value problems on the half line, associated to a third order ordinary differential equation of the type

$$\left(\Phi(k(t,u'(t))u''(t))\right)'(t) = f\left(t,u(t),u'(t),u''(t)\right), \quad \text{a.a. } t \in \mathbb{R}_0^+.$$

The prototype for the operator Φ is the Φ-*Laplacian*; the function k is assumed to be continuous and it may vanish in a subset of zero Lebesgue measure, so that the problem can be *singular*; finally, f is a Carathéodory function satisfying a weak growth condition of *Winter–Nagumo* type. The approach we follow is based on fixed point techniques combined with the upper and lower solutions method.

Keywords: boundary value problems on unbounded domains, heteroclinic solutions, Φ-Laplacian operator, singular equations, Wintner–Nagumo condition, third order ODEs.

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1 Introduction

In this paper we are concerned with the existence of solutions to boundary value problems (BVPs) on the half line, associated to strongly nonlinear third order ordinary differential equations, of the type

$$\begin{cases} \left(\Phi\left(k(\cdot,u'(\cdot))u''(\cdot)\right)\right)'(t) = f\left(t,u(t),u'(t),u''(t)\right), & \text{a.a. } t \in \mathbb{R}_0^+, \\ u(0) = u_0, \;\; u'(0) = \nu_1, \;\; u'(+\infty) = \nu_2. \end{cases} \tag{$\mathcal{P}$}$$

Here, the operator $\Phi : \mathbb{R} \to \mathbb{R}$ is the so-called Φ-*Laplacian* and $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function satisfying a weak growth condition of *Winter–Nagumo* type. Moreover, the function $k : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ is continuous and it may vanish in a subset of zero Lebesgue measure, so that problem ($\mathcal{P}$) is possibly *singular*. Finally, $u_0, \nu_1, \nu_2 \in \mathbb{R}$ are fixed real numbers.

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According to the existing literature, Φ-Laplacian type equations involve a strictly increasing homeomorphism

$$\Phi : (-a, a) \to (-b, b), \quad \text{with } 0 < a, b \leq +\infty,$$

such that $\Phi(0) = 0$. When $a = b = +\infty$ the main prototype for the Φ-Laplacian is the classical *r-Laplacian* $\Phi(s) = |s|^{r-2}s$, with $r > 1$. When $a = +\infty$ and $b < +\infty$, the map Φ is usually called *non-surjective* or *bounded* Φ-Laplacian and its main prototype is the mean curvature operator

$$\Phi(s) = \frac{s}{\sqrt{1+s^2}}, \quad s \in \mathbb{R},$$

cf. [4, 21]. When $a < +\infty$, the Φ-Laplacian is said to be *singular* and in this case the main prototype is the relativistic operator

$$\Phi(s) = \frac{s}{\sqrt{1-s^2}}, \quad s \in (-1, 1),$$

see [5–7, 15, 28]. Further details on *Φ-Laplacians* BVPs can be found also in [18, 22].

One of the main reasons to study problem ($\mathcal{P}$) is the great amount of applications of the Φ-Laplace operator in different fields of physics and applied mathematics, such as non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity, theory of capillary surfaces, see e.g. [20, 25], and, more recently, the modeling of glaciology, see for instance [12, 23, 29]. Moreover, problems like ($\mathcal{P}$) find many applications in fluid dynamics as generalizations of the Blasius problem modeling the flat plate problem in boundary layer theory for viscous fluids, cf. [16].

Due to the wide class of their applications, as well as for a more theoretical interest, several papers have been devoted to Φ-Laplacian type equations. Many contributions concern BVPs associated to a second order counterpart of ($\mathcal{P}$) involving equations of the type

$$\left(\Phi(k(\cdot, u(\cdot))u'(\cdot))\right)'(t) = f\left(t, u(t), u'(t)\right), \tag{1.1}$$

both in bounded and unbounded domains, under various assumptions for Φ and f, alongside with different types of boundary conditions, see [8,14,27]. We also mention [30] for third-order BVPs and [26] for higher-order BVPs in the half-line.

Recently, the multiplicity of solutions to differential equations with Φ-Laplacian has been largely investigated under periodic, Dirichlet or Neumann boundary conditions. In [17, 19] second order differential equations posed in bounded domains are considered by means of the fixed point index theory, while in [3] the authors find positive unbounded solutions for singular second-order BVPs set on the half-line.

We prove the solvability of ($\mathcal{P}$) assuming that f may have *critical* rate of decay -1 at infinity, that is $f(t, \cdot, \cdot, \cdot) \sim 1/t$ as $t \to +\infty$, cf. assumption (H_3) and Remark 3.7. Together with this assumption, we require a suitable form of the so-called *Nagumo–Wintner* condition on f, cf. (3.2) in assumption (H_2). The *Nagumo–Wintner* condition allows us to obtain a priori estimates on the derivatives of any solution u to ($\mathcal{P}$) on compact intervals of $\mathbb{R}$, see Lemma 3.11. We stress the fact that the version of the *Nagumo–Wintner* condition most frequently used in the literature for second order BVPs (see e.g. [10, 15]) is not useful in our case since it would not provide the desired estimates on the *higher* order derivative of the solution. Hence, a suitable adaptation of this condition turns out to be necessary in this context.

Our main result is Theorem 3.9 in which we prove the existence of a weak solution for problem ($\mathcal{P}$), in a sense that will be specified later. The proof is based on a fixed point technique, combined with the method of lower and upper solutions, named, respectively α and β; see assumption (H_1). More precisely, we start by proving the existence of a solution u_n to the auxiliary boundary value problem

$$\begin{cases} (\Phi(k(\cdot,u'(\cdot))u''(\cdot)))'(t) = f(t,u(t),u'(t),u''(t)), & \text{a.a. } t \in I_n, \\ u(0) = u_0,\ u'(0) = \alpha'(0),\ u'(n) = \beta'(n), \end{cases} \tag{$\mathcal{P}_n$}$$

where $I_n = [0,n]$, with $n \in \mathbb{N}$ sufficiently large. Then, by means of the a priori estimates provided by the *Nagumo–Wintner* condition, we show that a suitable sequence $(x_n)_n$, constructed by extending the functions u_n to the entire half-line, somehow converges to a solution of $(\mathcal{P})$.

Despite assumptions (H_1)–(H_3) seem rather technical, they are fulfilled by a wide class of functions, as we shall prove in Section 4. In particular, they hold for BVPs of the following type

$$\begin{cases} \left(\Phi\left(k(\cdot,u'(\cdot))u''(\cdot)\right)\right)'(t) = f_1\left(t,u(t),u'(t)\right) f_2(u''(t)), & \text{a.a. } t \in \mathbb{R}_0^+, \\ u(0) = u_0,\ \ u'(0) = \nu_1,\ \ u'(+\infty) = \nu_2, \end{cases}$$

where f_1 and f_2 satisfies suitable growth conditions and *either*

$$\begin{aligned} &k(t,y) = k_1(t)k_2(y), && \text{with } k_1 \geq 0,\ k_2 > 0 \text{ in } [\nu_1,\nu_2], \quad \textit{or} \\ &k(t,y) = k_1(t) + k_2(y), && \text{with } k_1,k_2 \geq 0. \end{aligned}$$

The *critical* rate for f, as well as the possibility of dealing with *singular* equations, have been yet considered in [11], where the authors prove the existence of heteroclinic solutions for BVPs associated to (1.1). A similar framework, with $k = k(t)$, can be found in the recent work [2], where BVPs associated to third order differential equations are studied in a compact domain, and in [1], where the author proves existence results for integro-differential BVPs in a non-critical regime and in the half-line. In this context, Theorem 3.9 throws a further light on the subject treating third order equations and it extends Theorem 3.3 of [2] since the function k is more general and solutions are obtained in the half-line. In particular, in [2] only solutions on compact domains are considered and the function k only depends on t. Concerning unbounded domains, a first contribution for the existence is contained in [1], where the sub-critical regime for the asymptotic behavior of the right-hand side f is investigated together with some non-existence results. The present work aims at providing a careful study of the critical regime for the asymptotic behavior of f at the same time generalizing the choice of the function k, which can also depend on the function v and not only on t.

By performing the change of variable $v(t) = u'(t)$, our results apply to integro-differential BVPs of the type

$$\begin{cases} \left(\Phi(k(\cdot,v(\cdot))v'(\cdot))\right)'(t) = f\left(t, \displaystyle\int_0^t v(s)\,ds, v(t), v'(t)\right), & \text{a.a. } t \in \mathbb{R}_0^+, \\ v(0) = \nu_1,\ \ v(+\infty) = \nu_2. \end{cases}$$

Hence, when $\nu_1 \neq \nu_2$, our analysis leads to the existence of heteroclinic solutions for such BVPs. These solutions are relevant in the study of biological, physical and chemical models since they represent a phase transition process in which the system evolves from an unstable equilibrium to a stable one; see [24,27,31] and the references therein.

Finally, we highlight that a straightforward adaptation of Theorem 3.9 to problem $(\mathcal{P})$ with $k = k(t,u(t),u'(t))$ directly follows in $\mathbb{R}_0^+$ considering an additional monotonicity assumption on the second variable for $k = k(t,x,y)$ and it will be object of a forthcoming paper.

The paper is organized as follows. In Section 2 we present some preliminary facts; in particular, Theorem 2.2 is a very general result on the solvability of Φ-Laplacian BVPs in compact intervals. Section 3 is devoted to our main result Theorem 3.9. Finally, in Section 4 we present a class of examples of functions Φ, k and f satisfying the assumptions used throughout the paper.

2 Preliminary results

In this section, we present an existence result for very general BVPs in compact real intervals, that is Theorem 2.2. The proof of Theorem 2.2 is based on the forthcoming lemma. Even if the proof of Lemma 2.1 somehow follows the proof of analogous results, cf. [2, Lemma 2.1] and [10, Lemma 2.6], for the sake of clarity we prefer to show it completely.

Lemma 2.1. *Let $T > 0$ be a fixed real number and denote by $I = [0, T] \subseteq \mathbb{R}$ and let $p > 1$ be fixed. Let $F : W^{2,p}(I) \to L^1(I)$, $v \mapsto F_v \in L^1(I)$, be a continuous operator for which there exists $\Theta \in L^1(I)$ such that*

$$|F_v(t)| \leq \Theta(t) \quad \text{for all } v \in W^{2,p}(I) \text{ and a.a. } t \in I. \tag{2.1}$$

Let $K : W^{2,p}(I) \subseteq C(I;\mathbb{R}) \to C(I;\mathbb{R})$, $v \mapsto K_v \in C(I;\mathbb{R})$, be continuous with respect to the uniform topology of $C(I;\mathbb{R})$ and suppose that there exist $k_1, k_2 \in C(I;\mathbb{R})$ satisfying

$$k_1, k_2 > 0 \quad \text{a.e. in } I \quad \text{and} \quad \frac{1}{k_1}, \frac{1}{k_2} \in L^p(I), \tag{2.2}$$

such that

$$k_1(t) \leq K_v(t) \leq k_2(t) \quad \text{for all } v \in W^{2,p}(I) \text{ and a.a. } t \in I. \tag{2.3}$$

Finally, let $\Psi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing homeomorphism. Then for all $v \in W^{2,p}(I)$ and for all $\delta_1, \delta_2 \in \mathbb{R}$ there exists a unique $\xi_v \in \mathbb{R}$ such that

$$\int_a^b \frac{1}{K_v(t)} \Psi^{-1}\left(\xi_v + \mathscr{F}_v(t)\right) dt = \delta_2 - \delta_1, \tag{2.4}$$

where

$$\mathscr{F} : W^{2,p}(I) \to C(I;\mathbb{R}), \quad v \mapsto \mathscr{F}_v(t) = \int_a^t F_v(s) ds, \quad t \in I.$$

Furthermore, there exists $c_0 > 0$, independent on v, such that:

$$|\xi_v| \leq c_0 \quad \text{for all } v \in W^{2,p}(I). \tag{2.5}$$

Proof. First, we observe that the operator $\mathscr{F}$ is well defined being $\mathscr{F}_v$ continuous in I for all $v \in W^{2,p}(I)$ by (2.1). Moreover, $\mathscr{F}$ is continuous from $W^{2,p}(I)$ in $C(I,\mathbb{R})$. Indeed, F is continuous from $W^{2,p}(I)$ in $L^1(I)$ by assumption and

$$\sup_{t \in I} |\mathscr{F}_u(t) - \mathscr{F}_v(t)| \leq \|F_u - F_v\|_{L^1(I)} \quad \text{for all } u, v \in W^{2,p}(I). \tag{2.6}$$

Furthermore

$$\sup_{t \in I} |\mathscr{F}_v(t)| \leq \|\Theta\|_{L^1(I)} \quad \text{for all } v \in W^{2,p}(I). \tag{2.7}$$

Now, we fix $v \in W^{2,p}(I)$ and define

$$\mathfrak{F}_v : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \mathfrak{F}_v(\xi) = \int_a^b \frac{1}{K_v(t)} \Psi^{-1}\left(\xi + \mathscr{F}_v(t)\right) dt.$$

Note that also $\mathfrak{F}_v$ is continuous in I. Indeed, $\mathscr{F}_v$ is continuous on I as noted above, the function Ψ^{-1} is continuous by assumption and so, by Lebesgue's Dominated Convergence Theorem, we can infer that $\mathfrak{F}_v \in C(\mathbb{R};\mathbb{R})$.

Moreover $\mathfrak{F}_v$ is strictly increasing in $\mathbb{R}$ since $K_v \geq 0$ and Ψ^{-1} is strictly increasing by assumption. Finally

$$\Psi^{-1}\left(\xi - \|\Theta\|_{L^1(I)}\right)\int_a^b \frac{1}{K_v(t)}dt \leq \mathfrak{F}_v(\xi) \leq \Psi^{-1}\left(\xi + \|\Theta\|_{L^1(I)}\right)\int_a^b \frac{1}{K_v(t)}dt$$

which implies that $\lim_{\xi\to\pm\infty}\mathfrak{F}_v(\xi) = \pm\infty$. Then, by Bolzano's Theorem, there exists a unique $\xi_v \in \mathbb{R}$ such that $\mathfrak{F}_v(\xi_v) = \delta_2 - \delta_1$ for any choice of $\delta_1, \delta_2 \in \mathbb{R}$.

In order to prove (2.5), note that, by the Mean Value Theorem, there exists $t_v \in I$ such that

$$\mathfrak{F}_v(\xi_v) = \int_a^b \frac{1}{K_v(t)}\Psi^{-1}\left(\xi_v + \mathscr{F}_v(t)\right) dt = \delta_2 - \delta_1 = \Psi^{-1}\left(\xi_v + \mathscr{F}_v(t_v)\right)\int_a^b \frac{1}{K_v(t)}dt$$

and so

$$\xi_v + \mathscr{F}_v(t_v) = \Psi\left((\delta_2 - \delta_1)\left(\int_a^b \frac{1}{K_v(t)}dt\right)^{-1}\right). \tag{2.8}$$

Now observe that, by (2.3)

$$(\delta_2 - \delta_1)\left(\int_a^b \frac{1}{K_v(t)}dt\right)^{-1} \leq |\delta_2 - \delta_1|\left(\int_a^b \frac{1}{k_2(t)}dt\right)^{-1} =: C. \tag{2.9}$$

Hence, denoted by $\hat{\Psi} = \max_{[-C,C]}|\Psi|$ and recalling that Ψ is strictly increasing, relations (2.8)–(2.9) give

$$\begin{aligned}|\xi_v| &\leq |\xi_v + \mathscr{F}_v(t_v)| + |\mathscr{F}_v(t_v)| = \left|\Psi\left((\delta_2 - \delta_1)\left(\int_a^b \frac{1}{K_v(t)}dt\right)^{-1}\right)\right| + |\mathscr{F}_v(t_v)| \\ &\leq |\Psi(C)| + |\mathscr{F}_v(t_v)| \leq \hat{\Psi} + \|\Theta\|_{L^1(I)}.\end{aligned}$$

Choosing $c_0 = \hat{\Psi} + \|\Theta\|_{L^1(I)}$ the proof is complete. □

Theorem 2.2. *Let $T > 0$, $p > 1$ and the operators F, K and Ψ be as in in Lemma 2.1.*

Then, for all $v_0, \omega_1, \omega_2 \in \mathbb{R}$ there exists a solution v of the problem

$$\begin{cases}\left(\Psi \circ K_v v''\right)'(t) = F_v(t), & \text{a.a. } t \in I, \\ v(0) = v_0,\ v'(0) = \omega_1,\ v'(T) = \omega_2, \end{cases} \tag{$\mathcal{PA}$}$$

that is a function $v \in W^{2,p}(I)$ such that

- $t \mapsto (\Psi \circ K_v v'')(t) \in W^{1,p}(I)$;
- $\left(\Psi \circ K_v v''\right)'(t) = F_v(t), \quad$ *a.a.* $t \in I$;
- $v(0) = v_0,\ v'(0) = \omega_1,\ v'(T) = \omega_2.$

Proof. The proof is an adaptation of [2, Theorem 2.2] and [11, Theorem 3.1]. First, we observe that, since the operators F, K and Ψ satisfy all the assumptions of Lemma 2.1, for all $v \in$

$W^{2,p}(I)$ and all $\delta_1, \delta_2 \in \mathbb{R}$, there exists a unique $\xi_v \in \mathbb{R}$ such that (2.4) and (2.5) still hold when $\delta_1 = w_1$ and $\delta_2 = w_2$. Set

$$\mathcal{W}_0 = \left\{ v \in W^{2,p}(I) : v(0) = v_0 \right\}$$

and define the operator $G : \mathcal{W}_0 \to \mathcal{W}_0$, with $v \mapsto G_v$, as follows

$$G_v(t) = v_0 + \omega_1 t + \int_0^t \int_0^s g_v(\tau) d\tau ds,$$

where

$$g_v(t) = \frac{1}{K_v(t)} \Psi^{-1} \left(\xi_v + \mathscr{F}_v(t) \right), \quad t \in I.$$

It is easy to see that G is well defined and that the solutions to ($\mathcal{PA}$) correspond to the fixed points of G. Finally, following [2] and [11], it is possible to show that G is bounded, continuous and compact, so that, by Schauder's Fixed Point Theorem, we get the existence of a function v which is a fixed point of G in I, that is a solution to the problem ($\mathcal{PA}$). □

3 Functional setting and main result

Throughout the paper we assume the following structural assumptions on Φ, k and f.

(A_1) $\Phi : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing homeomorphism such that* $\Phi(0) = 0$ *with*

$$\liminf_{s \to 0^+} \frac{\Phi(s)}{s^\rho} > 0 \quad \textit{for some } \rho > 0.$$

(A_2) $k : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that*

- $k(t,y) > 0$ *for a.a.* $(t,y) \in \mathbb{R}_0^+ \times \mathbb{R}$;
- $t \mapsto 1/k(t,y) \in L^p_{loc}(\mathbb{R}_0^+)$ *for all* $y \in \mathbb{R}$, *for some* $p > 1$.

(A_3) $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ is a *Carathéodory function, that is*

- $t \mapsto f(t,x,y,z)$ *is measurable for all* $(x,y,z) \in \mathbb{R}^3$;
- $(x,y,z) \mapsto f(t,x,y,z)$ *is continuous for a.a.* $t \in \mathbb{R}_0^+$,

which is also decreasing with respect to the second variable, that is

$$f(t,x_1,y,z) \geq f(t,x_2,y,z) \quad \textit{for a.a. } t \in \mathbb{R}_0^+,$$

for every $x_1, x_2, y, z \in \mathbb{R}$ *such that* $x_1 \leq x_2$.

Moreover, we shall refer to the equation in ($\mathcal{P}$) as (ODE), that is

$$(\Phi \circ \mathcal{K}_u)'(t) = f(t, u(t), u'(t), u''(t)), \quad \text{for a.a. } t \in \mathbb{R}_0^+, \tag{ODE}$$

where, for simplicity, we denote

$$\mathcal{K}_u(t) = k(t, u'(t)) u''(t) \quad \text{with } u \in W^{2,p}_{loc}(\mathbb{R}_0^+) \text{ and } t \in \mathbb{R}_0^+.$$

Definition 3.1. A function $u \in C^1(\mathbb{R}_0^+;\mathbb{R})$ is said to be a (weak) solution of $(\mathcal{P})$ if

- $u \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ and $t \mapsto (\Phi \circ \mathcal{K}_u)(t) \in W^{1,1}_{loc}(\mathbb{R}_0^+)$;
- $(\Phi \circ \mathcal{K}_u)'(t) = f(t,u(t),u'(t),u''(t))$ for a.a. $t \in \mathbb{R}_0^+$;
- $u(0) = u_0,\ u'(0) = \nu_1,\ u'(+\infty) = \nu_2$.

Remark 3.2. Since we allow the function k to vanish in a set having null measure, equation (ODE) can become singular. In this context, we search for solutions no more belonging to $C^2(\mathbb{R}_0^+)$, but to $W^{2,p}_{loc}(\mathbb{R}_0^+) \cap C^1(\mathbb{R}_0^+)$. The choice of this solution space is fairly natural if we consider that the map $t \mapsto 1/k(t,y)$ is assumed to be in $L^p_{loc}(\mathbb{R}_0^+)$ for all $y \in \mathbb{R}$.

Remark 3.3. Since $u \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ is a solution of $(\mathcal{P})$, and, in particular, the map $t \mapsto (\Phi \circ \mathcal{K}_u)(t)$ is in $W^{1,1}_{loc}(\mathbb{R}_0^+)$, and Φ is a homeomorphism, then $\mathcal{K}_u$ can be considered continuous in $\mathbb{R}_0^+$ (see [9, Remark 2.1]).

Definition 3.4. A function $\alpha \in C^1(\mathbb{R}_0^+;\mathbb{R})$ is said to be a (weak) lower solution of (ODE) if

- $\alpha \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ and $t \mapsto (\Phi \circ \mathcal{K}_\alpha)(t) \in W^{1,1}_{loc}(\mathbb{R}_0^+)$;
- $(\Phi \circ \mathcal{K}_\alpha)'(t) \geq f(t,\alpha(t),\alpha'(t),\alpha''(t))$ for a.a. $t \in \mathbb{R}_0^+$.

Definition 3.5. A function $\beta \in C^1(\mathbb{R}_0^+;\mathbb{R})$ is said to be a (weak) upper solution of (ODE) if

- $\beta \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ and $t \mapsto (\Phi \circ \mathcal{K}_\beta)(t) \in W^{1,1}_{loc}(\mathbb{R}_0^+)$;
- $(\Phi \circ \mathcal{K}_\beta)'(t) \leq f(t,\beta(t),\beta'(t),\beta''(t))$ for a.a. $t \in \mathbb{R}_0^+$.

Finally, we say that *a pair* (α,β) *of lower and upper solutions of* (ODE) *is ordered if*

$$\alpha'(t) \leq \beta'(t) \quad \text{for all } t \in \mathbb{R}_0^+.$$

Besides the structural assumptions (A_1)–(A_3) introduced before, we also consider some further natural requirements, including a suitable form of the so-called *Nagumo–Wintner* growth condition on f, see (3.2) below, which allows us to obtain a priori estimates on the derivatives of the solutions of $(\mathcal{P})$ on any compact interval of $\mathbb{R}_0^+$.

From now on, let $T_0 > 0$ be a fixed positive number and denote by $J = [0,T_0]$. Finally, assume the following conditions.

There exists an ordered pair (α,β) *of lower and upper solutions to* (ODE) *such that* $\alpha(0) = \beta(0) = u_0$, $\alpha'(0) = \nu_1$, β' *is increasing in* $(T_0,+\infty)$ *and* $\lim_{t\to+\infty}\beta'(t) = \nu_2$, *satisfying the following assumptions.*

(H_1) *Denoting by*

$$k_*(t) = \min\{k(t,y) : y \in [\alpha'(t),\beta'(t)]\}, \quad k^*(t) = \max\{k(t,y) : y \in [\alpha'(t),\beta'(t)]\},$$

assume that $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$.

(H_2) *There exist a constant* $H > 0$, *a non-negative function* $\ell \in L^1(J)$, *a non-negative function* $\mu \in L^q(J)$, *for some* $1 < q \leq +\infty$, *and a measurable function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying*

$$\frac{1}{\psi} \in L^1_{loc}(\mathbb{R}^+) \quad \text{and} \quad \int^\infty \frac{1}{\psi(t)}dt = +\infty, \tag{3.1}$$

such that

$$|f(t,x,y,z)| \leq \psi(|\Phi(k(t,y)z)|)\left(\ell(t)+\mu(t)|z|^{\frac{q-1}{q}}\right) \tag{3.2}$$

for a.a. $t \in J$ *and all* $x,y,z \in \mathbb{R}$ *such that* $x \in [\alpha(t), \beta(t)]$, $y \in [\alpha'(t), \beta'(t)]$ *and* $|z| \geq H$.

(H_3) *For any* $L > 0$ *there exist a non-negative function* $\eta_L \in L^1(\mathbb{R}_0^+)$ *and a continuous function* $K_L \in W^{1,1}_{loc}(\mathbb{R}_0^+)$*, with* K_L *null in* $[0, T_0]$ *and strictly increasing in* $[T_0, +\infty)$*, satisfying*

$$\int_{T_0}^{\infty} \frac{1}{k_*(t)} e^{-\frac{K_L(t)}{\rho}} dt < +\infty, \tag{3.3}$$

such that

- (i) $|f(t,x,y,z)| \geq K_L'(t)|\Phi(k(t,y)z)|$ *for a.a.* $t \in [T_0, +\infty)$*, all* $x \in [\alpha(t), \beta(t)]$*, all* $y \in [\alpha'(t), \beta'(t)]$ *and all* $z \in \mathbb{R}$ *with* $|z| \leq \mathcal{N}_L(t)/k(t,y)$*;*
- (ii) $|f(t,x,y,z)| \leq \eta_L(t)$ *for a.a.* $t \in \mathbb{R}_0^+$*, all* $x \in [\alpha(t), \beta(t)]$*, all* $y \in [\alpha'(t), \beta'(t)]$ *and all* $z \in \mathbb{R}$ *with* $|z| \leq \hat{\gamma}_L(t)$*;*
- (iii) $f(t,x,y,z) \leq 0$ *for a.a.* $t \in [T_0, +\infty)$*, all* $x \in [\alpha(t), \beta(t)]$*, all* $y \in [\alpha'(t), \beta'(t)]$ *and all* $z \in \mathbb{R}$ *with* $|z| \leq \hat{\gamma}_L(t)$*;*

where

$$\gamma_L(t) = \frac{\mathcal{N}_L(t)}{k_*(t)} \quad \text{and} \quad \hat{\gamma}_L(t) = \gamma_L(t) + |\alpha''(t)| + |\beta''(t)| \quad \text{a.a. } t \in \mathbb{R}_0^+,$$
$$\mathcal{N}_L(t) = \Phi^{-1}\left\{\Phi(L)e^{-K_L(t)}\right\} \qquad t \in \mathbb{R}_0^+.$$

Remark 3.6. Note that, since $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$ by (H_1), also $1/k^* \in L^p_{loc}(\mathbb{R}_0^+)$.

Moreover, by definition of k_*, we have

$$\frac{\mathcal{N}_L(t)}{k(t,y)} \leq \frac{\mathcal{N}_L(t)}{k_*(t)} = \gamma_L(t) \leq \hat{\gamma}_L(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+ \text{ and all } y \in [\alpha'(t), \beta'(t)],$$

so that, on account of (H_3)–(iii), we can rewrite (H_3)–(i) as follows

$$f(t,x,y,z) \leq -K_L'(t)|\Phi(k(t,y)z)|$$

for a.a. $t \in [T_0, +\infty)$, all $x \in [\alpha(t), \beta(t)]$, all $y \in [\alpha'(t), \beta'(t)]$ and all $z \in \mathbb{R}$ such that $|z| \leq \mathcal{N}_L(t)/k(t,y)$.

Remark 3.7. Despite their technicality, assumptions (H_1)–(H_3) are fulfilled in several remarkable cases, as it we be clear from the examples presented in Section 4.

In particular, the request (3.3) in (H_3) is compatible with the *critical* nature of the problem connected with the growth of f at infinity; see Section 4 and [11] for further details.

Remark 3.8. It is worth noting that $\mathcal{N}_L$ is continuous so that $\gamma_L = \mathcal{N}_L/k_* \in L^p_{loc}(\mathbb{R}_0^+)$, since $1/k_*$ in $L^p_{loc}(\mathbb{R}_0^+)$ by (A_2). Moreover, $\mathcal{N}_L$ is strictly positive by definition, being $\Phi(L) > 0$. Furthermore, recalling the monotonicity of K_L and the fact that Φ is a strictly increasing homeomorphism, we infer that $\mathcal{N}_L$ is strictly decreasing in $[T_0, +\infty)$. In particular, gathering the definition of $\mathcal{N}_L$ and the monotonicity of Φ, we deduce that $\mathcal{N}_L < L$ in $(T_0, +\infty)$ and $\mathcal{N}_L(t) = L$ for all $t \in J$.

Moreover, using the liminf condition in (A_1) we deduce that

$$\limsup_{\xi \to 0^+} \frac{\Phi^{-1}(\xi)}{\xi^{1/p}} < +\infty.$$

Consequently, combining the above considerations with (3.3) and the fact that $1/k_* \in L^p_{loc}(\mathbb{R}^+_0)$, we obtain that $\gamma_L = \mathcal{N}_L/k_* \in L^1(\mathbb{R}^+_0)$. Finally, since $\alpha, \beta \in W^{2,p}_{loc}(\mathbb{R}^+_0)$, we also have that $\hat{\gamma}_L = \gamma_L + |\alpha''| + |\beta''| \in L^1_{loc}(\mathbb{R}^+_0)$; see [9, Remark 3.7] and [11, Remark 1].

We are now ready to state our main result.

Theorem 3.9. *Assume (A_1)–(A_3) and (H_1)–(H_3). Then, problem $(\mathcal{P})$ admits at least a weak solution $u \in W^{2,p}_{loc}(\mathbb{R}^+_0)$ satisfying*

$$\alpha \le u \le \beta \quad \text{and} \quad \alpha' \le u' \le \beta' \ \text{ a.e. in } \mathbb{R}^+_0.$$

The proof of Theorem 3.9 is divided into two steps.

Step 1. Solvability on compact sets. Let $n \in \mathbb{N}$ be such that $n > T_0$ and consider problem $(\mathcal{P}_n)$. By solution to $(\mathcal{P}_n)$ we mean a function $u_n \in W^{2,p}(I_n)$ such that

- $t \mapsto \big(\Phi \circ \mathcal{K}_{u_n}\big)(t) \in W^{1,1}(I_n)$;
- $\big(\Phi \circ \mathcal{K}_{u_n}\big)'(t) = f(t, u_n(t), u_n'(t), u_n''(t))$ for a.a. $t \in I_n$;
- $u_n(0) = u_0,\ u_n'(0) = \alpha'(0),\ u_n'(n) = \beta(n)$.

Step 2. A limit argument. Once the existence on compact sets I_n is established, we construct a new sequence of functions $(x_n)_n$ by extending the functions u_n to $\mathbb{R}^+_0$ and we prove that the limit function of $(x_n)_n$ is a solution of $(\mathcal{P})$.

3.1 Solvability on compact sets

In order to prove the existence of solutions for $(\mathcal{P}_n)$ we first consider and intermediate *truncated problem*. Following [8], see also [2], for any pair of functions $\xi, \zeta \in L^1(I_n)$, satisfying the relation $\xi \le \zeta$ a.e. in I_n, we define the truncation operator

$$\mathcal{T}^{\xi,\zeta} : L^1(I_n) \to L^1(I_n), \quad \eta \mapsto \mathcal{T}^{\xi,\zeta}_\eta,$$
$$\mathcal{T}^{\xi,\zeta}_\eta(t) = \max\{\xi(t), \min\{\eta(t), \zeta(t)\}\}, \quad t \in I_n.$$

Observe that, by definition,

$$T^{\xi,\zeta}_\eta(t) \in [\xi(t), \zeta(t)] \quad \text{for all } \eta \in L^1(I_n) \text{ and for all } t \in I_n. \tag{3.4}$$

Moreover, by [8, Lemma A.1] we know that

$(\mathcal{T}_1)$ $|\mathcal{T}^{\xi,\zeta}_{\eta_1}(t) - \mathcal{T}^{\xi,\zeta}_{\eta_2}(t)| \le |\eta_1(t) - \eta_2(t)|$ *for all* $\eta_1, \eta_2 \in L^1(I_n)$ *and all* $t \in I_n$;

$(\mathcal{T}_2)$ *if* $\xi, \zeta \in W^{1,1}(I_n)$, *then* $\mathcal{T}^{\xi,\zeta}(W^{1,1}(I_n)) \subseteq W^{1,1}(I_n)$;

$(\mathcal{T}_3)$ *if* $\xi, \zeta \in W^{1,1}(I_n)$, *then* $\mathcal{T}^{\xi,\zeta}$ *is continuous from* $W^{1,1}(I_n)$ *into itself.*

Now, for all $u \in W^{2,p}(I_n)$ and for all $t \in I_n$ we denote

$$\mathcal{D}_{u'}(t) = \mathcal{T}^{-\hat{\gamma}_L,\hat{\gamma}_L}_{\left(\mathcal{T}^{\alpha',\beta'}_{u'}\right)'}(t),$$

where $\hat{\gamma}_L$ derives from assumption (H_3), and observe that this definition is well posed whenever $u \in W^{2,p}_{loc}(\mathbb{R}^+_0)$ by $(\mathcal{T}_2)$. Moreover, by property (3.4) it results that

$$|\mathcal{D}_{u'}(t)| \leq \hat{\gamma}_L(t) \quad \text{for all } t \in I_n. \tag{3.5}$$

Finally, we set

$$W_0 = \left\{u \in W^{2,p}(I_n) : \ u(0) = u_0\right\}$$

and consider the operator

$$\mathcal{F} : W_0 \to L^1(I_n), \quad u \mapsto \mathcal{F}_u,$$

defined as

$$\mathcal{F}_u(t) = f\left(t, \mathcal{T}^{\alpha,\beta}_u(t), \mathcal{T}^{\alpha',\beta'}_{u'}(t), \mathcal{D}_{u'}(t)\right) + \arctan\left(u'(t) - \mathcal{T}^{\alpha',\beta'}_{u'}(t)\right), \quad t \in I_n. \tag{3.6}$$

Then, we are in a position to introduce the truncated problem

$$\begin{cases} \left(\Phi \circ \mathcal{K}^{\mathcal{T}}_u u''\right)'(t) = \mathcal{F}_u(t), & \text{a.a. } t \in I_n, \\ u(0) = u_0, \ u'(0) = \alpha'(0), \ u'(n) = \beta'(n), \end{cases} \tag{$\mathcal{PT}_n$}$$

where

$$\mathcal{K}^{\mathcal{T}} : W^{2,p}(I_n) \to C(I_n; \mathbb{R}), \quad u \mapsto \mathcal{K}^{\mathcal{T}}_u(t) = k\left(t, \mathcal{T}^{\alpha',\beta'}_{u'}(t)\right). \tag{3.7}$$

We are going to prove that $(\mathcal{PT}_n)$ admits at least a weak solution, that is a function $u_n \in W_0$ such that

- $t \mapsto \left(\Phi \circ \mathcal{K}^{\mathcal{T}}_{u_n} u_n''\right)(t) \in W^{1,1}(I_n)$;
- $\left(\Phi \circ \mathcal{K}^{\mathcal{T}}_{u_n} u_n''\right)'(t) = \mathcal{F}_{u_n}(t)$ for a.a. $t \in I_n$;
- $u_n(0) = u_0, \ u_n'(0) = \alpha'(0), \ u_n'(n) = \beta(n)$.

To this aim, in the next result we show that $(\mathcal{PT}_n)$ can be framed into the functional setting of Theorem 2.2 and therefore it admits at least one solution. This is an intermediate step between the solvability of $(\mathcal{P}_n)$ and the solvability of $(\mathcal{P})$.

Theorem 3.10. ***Existence for*** $(\mathcal{PT}_n)$. *Assume (A_1)–(A_3) and (H_1)–(H_3). Then, problem $(\mathcal{PT}_n)$ admits at least a weak solution.*

Proof. We want to show that the operators $\mathcal{F}$ and $\mathcal{K}^{\mathcal{T}}$ defined in (3.6) and (3.7), respectively, satisfy the assumptions of Theorem 2.2.

First note that, since $\alpha'(t) \leq \beta'(t)$ for all $t \in \mathbb{R}^+_0$ and $\alpha(0) = \beta(0) = u_0$, we also have $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{R}^+_0$. Now, by (3.4), for all $t \in I_n$ and all $u \in W_0$ we have

$$\mathcal{T}^{\alpha,\beta}_u(t) \in [\alpha(t), \beta(t)] \quad \text{and} \quad \mathcal{T}^{\alpha',\beta'}_{u'}(t) \in [\alpha'(t), \beta'(t)],$$

so that, by assumption (H_3)–(ii) and (3.5), we get

$$\begin{aligned}|\mathcal{F}_u(t)| &= \left| f\left(t, \mathcal{T}_u^{\alpha,\beta}(t), \mathcal{T}_{u'}^{\alpha',\beta'}(t), \mathcal{D}_{u'}(t)\right) + \arctan\left(u'(t) - \mathcal{T}_{u'}^{\alpha',\beta'}(t)\right)\right| \\ &\leq \left| f\left(t, \mathcal{T}_u^{\alpha,\beta}(t), \mathcal{T}_{u'}^{\alpha',\beta'}(t), \mathcal{D}_{u'}(t)\right)\right| + \frac{\pi}{2} \\ &\leq \eta_L(t) + \frac{\pi}{2},\end{aligned}$$

for all $u \in W_0$ and all $t \in I_n$. Hence $\mathcal{F}$ satisfies assumption (2.1) of Lemma 2.1 with $\Theta = \eta_L + \pi/2$.

Now we shall prove that $\mathcal{F}_u$ is continuous from $W_0 \subseteq W^{2,p}(I_n)$ into $L^1(I_n)$. To this aim, let $(u_i)_i \subseteq W_0$ be a sequence in W_0 such that $u_i \to u$ in W_0. Fix a subsequence of $(\mathcal{F}u_i)_i$, still denoted by $(\mathcal{F}u_i)_i$ for simplicity. Since $u_i \to u$ in W_0 we also have

$$u_i \longrightarrow u \text{ in } W^{2,p}(I_n), \qquad u_i' \longrightarrow u' \text{ in } W^{1,p}(I_n), \qquad u_i'' \longrightarrow u'' \text{ in } L^p(I_n),$$

so that, by property $(\mathcal{T}_3)$, it follows that

$$\mathcal{T}_{u_i}^{\alpha,\beta} \longrightarrow \mathcal{T}_u^{\alpha,\beta} \quad \text{and} \quad \mathcal{T}_{u_i'}^{\alpha',\beta'} \longrightarrow \mathcal{T}_{u'}^{\alpha',\beta'} \quad \text{in } W^{1,1}(I_n),$$

and in turn

$$\left(\mathcal{T}_{u_i'}^{\alpha',\beta'}\right)' \longrightarrow \left(\mathcal{T}_{u'}^{\alpha',\beta'}\right)' \quad \text{in } L^1(I_n).$$

Hence, by Theorem 4.9 of [13], for a.a. $t \in I_n$ we have

$$\mathcal{T}_{u_i}^{\alpha,\beta}(t) \to \mathcal{T}_u^{\alpha,\beta}(t) \quad \text{and} \quad \left(\mathcal{T}_{u_i'}^{\alpha',\beta'}\right)'(t) \longrightarrow \left(\mathcal{T}_{u'}^{\alpha',\beta'}\right)'(t), \tag{3.8}$$

possibly up to a further subsequence. Moreover, using $(\mathcal{T}_1)$, for a.a. $t \in I_n$ we find

$$|\mathcal{D}_{u_i'}(t) - \mathcal{D}_{u'}(t)| = \left| \mathcal{T}^{-\hat{\gamma}_L,\hat{\gamma}_L}_{\left(\mathcal{T}_{u_i'}^{\alpha',\beta'}\right)'}(t) - \mathcal{T}^{-\hat{\gamma}_L,\hat{\gamma}_L}_{\left(\mathcal{T}_{u'}^{\alpha',\beta'}\right)'}(t)\right| \leq \left|\mathcal{T}_{u_i'}^{\alpha',\beta'}(t) - \mathcal{T}_{u'}^{\alpha',\beta'}(t)\right| \longrightarrow 0,$$

that is

$$\mathcal{D}_{u_i'}(t) \longrightarrow \mathcal{D}_{u'}(t) \quad \text{for a.a. } t \in I_n. \tag{3.9}$$

Combining (3.8)–(3.9), with the fact that f is a Carathéodory function, we get

$$\begin{aligned}\mathcal{F}_{u_i}(t) &= f\left(t, \mathcal{T}_{u_i}^{\alpha,\beta}(t), \mathcal{T}_{u_i'}^{\alpha',\beta'}(t), \mathcal{D}_{u_i'}(t)\right) + \arctan\left(u_i'(t) - \mathcal{T}_{u_i'}^{\alpha',\beta'}(t)\right) \\ &\longrightarrow f\left(t, \mathcal{T}_u^{\alpha,\beta}(t), \mathcal{T}_{u'}^{\alpha',\beta'}(t), \mathcal{D}_{u'}(t)\right) + \arctan\left(u'(t) - \mathcal{T}_{u'}^{\alpha',\beta'}(t)\right) \\ &= \mathcal{F}_u(t) \quad \text{for a.a. } t \in I_n.\end{aligned}$$

Therefore we get the continuity of $\mathcal{F}$ by the Lebesgue Dominated Convergence Theorem. Now, putting

$$\alpha_n = \min_{t\in I_n} \alpha(t), \quad \alpha_n' = \min_{t\in I_n} \alpha'(t), \quad \beta_n = \min_{t\in I_n} \beta(t), \quad \beta_n' = \min_{t\in I_n} \beta'(t),$$

it follows that for all $t \in I_n$

$$\alpha_n \leq \alpha(t) \leq \mathcal{T}_u^{\alpha,\beta}(t) \leq \beta(t) \leq \beta_n \quad \text{and} \quad \alpha_n' \leq \alpha'(t) \leq \mathcal{T}_{u'}^{\alpha',\beta'}(t) \leq \beta'(t) \leq \beta_n'.$$

Since $\mathcal{T}$ is a continuous operator in $C(I_n;\mathbb{R})$, the uniform continuity of k in $I_n \times [\alpha'_n, \beta'_n]$ implies that $\mathcal{K}^{\mathcal{T}}$ is continuous with respect to the uniform topology of $C(I_n;\mathbb{R})$. Moreover, if $u \in W^{2,p}(I_n)$, then for all $t \in I_n$

$$\mathcal{T}_u^{\alpha,\beta}(t) \in [\alpha(t), \beta(t)] \subseteq [\alpha_n, \beta_n] \quad \text{and} \quad \mathcal{T}_{u'}^{\alpha',\beta'}(t) \in [\alpha'(t), \beta'(t)] \subseteq [\alpha'_n, \beta'_n]$$

so that

$$0 < k_*(t) \le \mathcal{K}_u^{\mathcal{T}}(t) = k\left(t, \mathcal{T}_{u'}^{\alpha',\beta'}(t)\right) \le k^*(t),$$

where the functions k_* and k^* have been introduced in (H_1). Thus we have that $\mathcal{K}^{\mathcal{T}}$ satisfies the assumptions of Lemma 2.1 with $k_1 = k_*$ and $k_2 = k^*$.

Finally, taking $\omega_1 = \alpha'(0)$, $\omega_2 = \beta'(n)$, problem ($\mathcal{PT}_n$) becomes ($\mathcal{PA}$). Therefore ($\mathcal{PT}_n$) admits a solution by Theorem 2.2. □

Now, in order to prove the existence of a solution for $(\mathcal{P}_n)$, we consider some useful properties characterizing every solution of $(\mathcal{PT}_n)$ that will be proved in the next lemma. To this aim, we define

$$M = \max_{t\in J} \beta'(t) - \min_{t\in J} \alpha'(t). \tag{3.10}$$

Note that the constant M is well defined, being α, $\beta \in C^1(\mathbb{R}_0^+;\mathbb{R})$. Moreover, since Φ is a continuous and strictly increasing function with $\Phi(0) = 0$, there exists $N \in \mathbb{R}^+$ such that

$$\Phi(N) > 0, \quad \Phi(-N) < 0 \quad \text{and} \quad N > \max\left\{H, \frac{M}{2T_0}\right\} \|k^*\|_{L^\infty(J)}, \tag{3.11}$$

where H is the positive constant introduced in assumption (H_2).

Finally, take $L > N$ such that

$$\min\left\{\int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi(t)}dt, \int_{-\Phi(-N)}^{-\Phi-(L)} \frac{1}{\psi(t)}dt\right\} > \|\ell\|_{L^1(J)} + \|\mu\|_{L^q(J)} \cdot M^{\frac{q-1}{q}}, \tag{3.12}$$

which is possible by virtue of (3.1).

Lemma 3.11. *Assume (A_1)–(A_3) and (H_1)–(H_3). Let $u_n \in W^{2,p}(I_n)$ be a solution of $(\mathcal{PT}_n)$. Then the following properties hold:*

(i) $\alpha(t) \le u_n(t) \le \beta(t)$ *and* $\alpha'(t) \le u'_n(t) \le \beta'(t)$ *for all* $t \in I_n$;

(ii) $\min_{t\in J} |\mathcal{K}_{u_n}(t)| \le N$;

(iii) $|\mathcal{K}_{u_n}(t)| < L$ *for all* $t \in J$;

(iv) $\mathcal{K}_{u_n}$ *is decreasing in* $[T_0, n]$;

(v) $\mathcal{K}_{u_n} \ge 0$ *in* $[T_0, n]$;

(vi) *if there exists* $t_1 \in [T_0, n]$ *such that* $\mathcal{K}_{u_n}(t_1) = 0$, *then* $\mathcal{K}_{u_n}(t) = 0$ *for all* $t \in [t_1, n]$;

(vii) $|\mathcal{K}_{u_n}(t)| \le \mathcal{N}_L(t)$ *for all* $t \in I_n$;

(viii) $\mathcal{D}_{u'_n}(t) = u''_n(t) \le \gamma_L(t) \le \hat{\gamma}_L(t)$ *for all* $t \in I_n$.

Proof. The proof is similar to other results already known for second order BVPs, so that we present it here for the sake of clarity and completeness.

Let $u_n \in W^{2,p}(I_n)$ be a solution of ($\mathcal{PT}_n$).

Claim (i). First we show that

$$\alpha'(t) \le u_n'(t) \le \beta'(t) \quad \text{for all } t \in I_n. \tag{3.13}$$

Let us start with the first inequality in (3.13). Assume by contradiction that there exists $\bar{t} \in I_n$ such that $u_n'(\bar{t}) < \alpha'(\bar{t})$ and let us define

$$z(t) = u_n'(t) - \alpha'(t), \qquad t \in I_n.$$

Since u_n solves ($\mathcal{PT}_n$) we find that

$$z(0) = u_n'(0) - \alpha'(0) = 0, \qquad z(n) = u_n'(n) - \alpha'(n) = \beta'(n) - \alpha'(n) \ge 0 \quad \text{and} \quad z(\bar{t}) < 0.$$

Thus, by the continuity of z and the compactness of I_n, there exists $\hat{t} \in I_n$ such that

$$z(\hat{t}) = \min_{t \in I_n} z(t) < 0.$$

Therefore we can find $t_1 \in [0, \hat{t})$ and $t_2 \in (\hat{t}, n]$ such that

$$z(t_1) = z(t_2) = 0 \quad \text{and} \quad z(t) < 0 \quad \text{for all } t \in (t_1, t_2). \tag{3.14}$$

Thus, by the definition of the truncating operator $\mathcal{T}$ and the fact that $u_n'(t) < \alpha'(t)$ for all $t \in (t_1, t_2)$, it follows

$$\mathcal{T}_{u_n'}^{\alpha',\beta'}(t) = \alpha'(t) \quad \text{for all } t \in (t_1, t_2), \tag{3.15}$$

and consequently

$$\mathcal{D}_{u_n'}(t) = \mathcal{T}^{-\hat{\gamma}_L, \hat{\gamma}_L}_{\left(\mathcal{T}_{u_n'}^{\alpha',\beta'}\right)'}(t) = \mathcal{T}_{\alpha''}^{-\hat{\gamma}_L, \hat{\gamma}_L}(t) = \alpha''(t) \quad \text{for all } t \in (t_1, t_2), \tag{3.16}$$

the last equality in (3.16) being true since $|\alpha''(t)| \le \hat{\gamma}_L(t)$ for all $t \in (t_1, t_2)$ by the definition of $\hat{\gamma}_L$. Now, recalling that u_n is a weak solution of ($\mathcal{PT}_n$) and α is a lower solution of (ODE), by (3.6) and (3.14)–(3.16), we infer

$$\begin{aligned}
(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})'(t) = \mathcal{F}_{u_n}(t) &= f\left(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \mathcal{T}_{u_n'}^{\alpha',\beta'}(t), \mathcal{D}_{u_n'}(t)\right) + \arctan\left(u_n'(t) - \mathcal{T}_{u_n'}^{\alpha',\beta'}(t)\right) \\
&= f\left(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \alpha'(t), \alpha''(t)\right) + \arctan\left(u_n'(t) - \alpha'(t)\right) \\
&< f(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \alpha'(t), \alpha''(t)) \le f(t, \alpha(t), \alpha'(t), \alpha''(t)) \le (\Phi \circ \mathcal{K}_\alpha)'(t),
\end{aligned}$$

for all $t \in (t_1, t_2)$, that is

$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})'(t) < (\Phi \circ \mathcal{K}_\alpha)'(t) \quad \text{for all } t \in (t_1, t_2). \tag{3.17}$$

Now, we set

$$Z_1 = \{t \in (t_1, \hat{t}) : z'(t) < 0\} \quad \text{and} \quad Z_2 = \{t \in (t_1, \hat{t}) : z'(t) > 0\}.$$

Note that both Z_1 and Z_2 have positive Lebesgue measure so that, recalling that k is positive a.e. in $\mathbb{R}_0^+ \times \mathbb{R}$, we can find $t_1^* \in Z_1$ and $t_2^* \in Z_2$ such that

$$k\left(t_1^*, \mathcal{T}_{u_n'}^{\alpha',\beta'}(t_1^*)\right) > 0 \quad \text{and} \quad z'(t_1^*) < 0 \tag{3.18}$$

and
$$k\left(t_2^*, \mathcal{T}_{u_n'}^{\alpha',\beta'}(t_2^*)\right) > 0 \quad \text{and} \quad z'(t_2^*) > 0. \tag{3.19}$$
Integrating (3.17) in $[t_1^*, \hat{t}]$ we get
$$\int_{t_1^*}^{\hat{t}} \left(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}}\right)'(t)dt \leq \int_{t_1^*}^{\hat{t}} \left(\Phi \circ \mathcal{K}_\alpha\right)'(t)dt$$
which is equivalent to
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(\hat{t}) - (\Phi \circ \mathcal{K}_\alpha)(\hat{t}) \leq (\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_1^*) - (\Phi \circ \mathcal{K}_\alpha)(t_1^*). \tag{3.20}$$
Now observe that
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_1^*) - (\Phi \circ \mathcal{K}_\alpha)(t_1^*) < 0. \tag{3.21}$$
Indeed,
$$\begin{aligned}(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_1^*) - (\Phi \circ \mathcal{K}_\alpha)(t_1^*) &= \Phi\left(k\left(t_1^*, \mathcal{T}_{u_n'}^{\alpha',\beta'}(t_1^*)\right) u_n''(t_1^*)\right) - \Phi\left(k\left(t_1^*, \alpha'(t_1^*)\right) \alpha''(t_1^*)\right) \\ &= \Phi\left(k\left(t_1^*, \alpha'(t_1^*)\right) u_n''(t_1^*)\right) - \Phi\left(k\left(t_1^*, \alpha'(t_1^*)\right) \alpha''(t_1^*)\right) < 0,\end{aligned}$$
since $\mathcal{T}_{u_n}^{\alpha,\beta}(t_1^*) = \alpha(t_1^*)$, $u_n''(t_1^*) < \alpha''(t_1^*)$ by (3.18) and Φ is strictly increasing.

Thus, combining (3.20) and (3.21), we find
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(\hat{t}) - (\Phi \circ \mathcal{K}_\alpha)(\hat{t}) < 0. \tag{3.22}$$
Similarly, integrating (3.17) in $[\hat{t}, t_2^*]$, we get
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(\hat{t}) - (\Phi \circ \mathcal{K}_\alpha)(\hat{t}) \geq (\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_2^*) - (\Phi \circ \mathcal{K}_\alpha)(t_2^*).$$
Additionally,
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_2^*) - (\Phi \circ \mathcal{K}_\alpha)(t_2^*) > 0,$$
since $T_{u_n}^{\alpha,\beta}(t_2^*) = \alpha(t_2^*)$, $u_n''(t_2^*) > \alpha''(t_2^*)$ by (3.19) and Φ is strictly increasing. Therefore
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(\hat{t}) - (\Phi \circ \mathcal{K}_\alpha)(\hat{t}) > 0. \tag{3.23}$$
Now, (3.22) and (3.23) provide a contradiction. Therefore $\alpha'(t) \leq u_n'(t)$ for a.a. $t \in I_n$.

By adapting the previous argument, one is also able to prove $u_n'(t) \leq \beta'(t)$ for a.a. $t \in I_n$ finally obtaining $\alpha'(t) \leq u_n'(t) \leq \beta'(t)$ for a.a. $t \in I_n$.

Eventually, we get the thesis just by integrating (3.13) and recalling that $u_n(0) = \alpha(0) = \beta(0)$ by assumption.

Claim (ii). Suppose, by contradiction, that $\mathcal{K}_{u_n}(t) > N$ for a.a. $t \in J$. Note that
$$u_n''(t) = \frac{\mathcal{K}_{u_n}(t)}{k(t, u_n'(t))} > \frac{N}{k(t, u_n'(t))} > 0$$
for a.a. $t \in J$. Thus, applying (i) we find
$$\begin{aligned}NT_0 = \int_0^{T_0} N dt < \int_0^{T_0} \mathcal{K}_{u_n}(t)dt &= \int_0^{T_0} k\left(t, u_n'(t)\right) u_n''(t)dt \\ &\leq \|k^*\|_{L^\infty(J)} \int_0^{T_0} u_n''(t)dt = \|k^*\|_{L^\infty(J)} \left[u_n'(T_0) - u_n'(0)\right] \\ &\leq \|k^*\|_{L^\infty(J)} \left[\beta'(T_0) - \alpha'(0)\right] \leq M\|k^*\|_{L^\infty(J)} < NT_0,\end{aligned}$$

by the choice of N in (3.11), which is a contradiction.

Similarly, we would get a contradiction assuming that $\mathcal{K}_{u_n}(t) < -N$ for a.a. $t \in J$.

Claim (iii). Suppose, by contradiction, that there exists $\bar{t} \in J$ such that $|\mathcal{K}_{u_n}(\bar{t})| \geq L$. It follows that either $\mathcal{K}_{u_n}(\bar{t}) \geq L$ or $\mathcal{K}_{u_n}(\bar{t}) \leq -L$.

Let us assume that $\mathcal{K}_{u_n}(\bar{t}) \geq L$. By (ii) we know that

$$\min_{t \in J} \mathcal{K}_{u_n}(t) \leq \min_{t \in J} |\mathcal{K}_{u_n}(t)| \leq N < L$$

and so there exists $\hat{t} \in J$ such that $\mathcal{K}_{u_n}(\hat{t}) = \min_{t \in J} \mathcal{K}_{u_n}(t) \leq N$. By the continuity of $\mathcal{K}_{u_n}$ we can find $t_1, t_2 \in J$, with $t_1 \leq t_2$, such that $\mathcal{K}_{u_n}(t_1) = N$, $\mathcal{K}_{u_n}(t_2) = L$ and

$$N < \mathcal{K}_{u_n}(t) < L \quad \text{for all } t \in (t_1, t_2). \tag{3.24}$$

Consequently, recalling the definition of $\mathcal{K}_{u_n}$ and the fact that $k > 0$ a.e. in J, we find

$$\frac{N}{k(t, u_n'(t))} < u_n''(t) < \frac{L}{k(t, u_n'(t))} \quad \text{for all } t \in (t_1, t_2).$$

Hence, taking into account Remark 3.8, for a.a. $t \in (t_1, t_2)$

$$0 < H < \frac{N}{\|k^*\|_{L^\infty(J)}} \leq \frac{N}{k(t, u_n'(t))} < u_n''(t) < \frac{L}{k(t, u_n'(t))} \leq \frac{L}{k_*(t)} = \frac{\mathcal{N}_L(t)}{k_*(t)} = \gamma_L(t) \leq \hat{\gamma}_L(t),$$

which implies, in particular, that

$$\mathcal{D}_{u_n'(t)} = u_n''(t) \quad \text{for a.a. } t \in (t_1, t_2).$$

Therefore, recalling that u_n solves ($\mathcal{PT}_n$), using (H_2) and the fact that $\Phi \circ \mathcal{K}_{u_n}(t) > 0$ by (3.24), the monotonicity of Φ and the choice of N, for a.a. $t \in (t_1, t_2)$ it results

$$\begin{aligned}
\left| \left(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}} \right)'(t) \right| &= \left| \left(\Phi \left(k \left(t, \mathcal{T}_{u_n'}^{\alpha', \beta'}(t) \right) u_n''(t) \right) \right)'(t) \right| \\
&= \left| f \left(t, \mathcal{T}_{u_n}^{\alpha, \beta}(t), \mathcal{T}_{u_n'}^{\alpha', \beta'}(t), \mathcal{D}_{u_n'}(t) \right) + \arctan \left(u_n'(t) - \mathcal{T}_{u_n'}^{\alpha', \beta'}(t) \right) \right| \\
&= \left| f \left(t, u_n(t), u_n'(t), u_n''(t) \right) \right| \\
&\leq \psi(|\Phi \circ \mathcal{K}_{u_n}(t)|) \left(\ell(t) + \mu(t) |u_n''(t)|^{\frac{q-1}{q}} \right) \\
&= \psi(\Phi \circ \mathcal{K}_{u_n}(t)) \left(\ell(t) + \mu(t) |u_n''(t)|^{\frac{q-1}{q}} \right).
\end{aligned}$$

Hence, by Hölder's inequality, we get

$$\begin{aligned}
\int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi(\tau)} d\tau = \int_{\Phi(\mathcal{K}_{u_n}(t_1))}^{\Phi(\mathcal{K}_{u_n}(t_2))} \frac{1}{\psi(\tau)} d\tau &= \int_{t_1}^{t_2} \frac{(\Phi \circ \mathcal{K}_{u_n})'(t)}{\psi(\Phi \circ \mathcal{K}_{u_n}(t))} dt \\
&\leq \int_{t_1}^{t_2} \left[\ell(t) + \mu(t) (u_n''(t))^{\frac{q-1}{q}} \right] dt \\
&\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^q(J)} \left(\int_{t_1}^{t_2} u_n''(t) dt \right)^{\frac{q-1}{q}} \\
&\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^q(J)} \left[\beta'(t_2) - \alpha'(t_1) \right]^{\frac{q-1}{q}} \\
&\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^q(J)} M^{\frac{q-1}{q}}.
\end{aligned}$$

This relation contradicts the choice of L in (3.12). Similarly, if we suppose $\mathcal{K}_{u_n}(\bar{t}) \le -L$ we reach a contradiction again.

Claim (iv). Since u_n is a solution to $(\mathcal{PT}_n)$, using (i), we find that

$$(\Phi \circ \mathcal{K}_{u_n})'(t) = f\left(t, u_n(t), u_n'(t), \mathcal{D}_{u_n'}(t)\right) \quad \text{for a.a. } t \in I_n.$$

On the other hand, by (i) and (H_3)–(iii), we get

$$f\left(t, u_n(t), u_n'(t), \mathcal{D}_{u_n'}(t)\right) \le 0 \quad \text{for a.a. } t \in [T_0, n],$$

being $|D_{u_n'}(t)| \le \hat{\gamma}_L(t)$ by (3.5). Hence

$$(\Phi \circ \mathcal{K}_{u_n})'(t) \le 0 \quad \text{for a.a. } t \in [T_0, n]$$

and the claim follows since Φ is a strictly increasing homeomorphism.

Claim (v). By contradiction, suppose that there exists $t_1 \in [T_0, n]$ such that $\mathcal{K}_{u_n}(t_1) < 0$. Then, from (iv) we have that

$$\mathcal{K}_{u_n}(t) \le \mathcal{K}_{u_n}(t_1) < 0 \quad \text{for all } t \in [t_1, n].$$

Hence, considering the definition of $\mathcal{K}_{u_n}$ we deduce

$$u_n''(t) = \frac{\mathcal{K}_{u_n}(t)}{k(t, u_n'(t))} < 0 \quad \text{for a.a. } t \in [t_1, n].$$

Now, we recall that u_n solves $(\mathcal{PT}_n)$, and so, using (i) and assumption (H_1), we get

$$\beta'(n) = u_n'(n) = u_n'(t_1) + \int_{t_1}^{n} u_n''(\tau)d\tau < u_n'(t_1) \le \beta'(t_1) \le \beta'(n),$$

since β' is increasing in $(T_0, +\infty)$, which is a contradiction.

Claim (vi). The statement directly follows from (iv) and (v).

Claim (vii). Recalling that $\mathcal{N}_L = L$ in J, by virtue of (iii) and (v), it is sufficient to prove that

$$0 \le \mathcal{K}_{u_n}(t) \le \mathcal{N}_L(t) \qquad \forall t \in I_n \setminus J.$$

Put

$$t^* = \sup\left\{t \ge T_0 : \ \mathcal{K}_{u_n}(s) < \mathcal{N}_L(s) \qquad \forall\, s \in [T_0, t)\right\}.$$

Note that t^* is well defined, since $\mathcal{K}_{u_n}(T_0) < L = \mathcal{N}_L(T_0)$, and $t^* > T_0$.

We want to prove that $t^* > n$. Proceed by contradiction and suppose that $t^* \le n$. This implies that

$$\mathcal{K}_{u_n}(t) > 0 \quad \text{for all } t \in [T_0, t^*].$$

Indeed, if there exists $\bar{t} \in [T_0, t^*]$ such that $\mathcal{K}_{u_n}(\bar{t}) = 0$, we would get $\mathcal{K}_{u_n}(t) = 0 < \mathcal{N}_L(t)$ for all $t \in [\bar{t}, n]$ by (vi). On the other hand, by definition of t^*, it results $\mathcal{K}_{u_n}(t) < \mathcal{N}_L(t)$ for all $t \in [T_0, t^*)$. Consequently, since $\bar{t} \le t^*$, we would find $\mathcal{K}_{u_n}(t) < \mathcal{N}_L(t)$ for all $t \in [T_0, n]$ and this relation contradicts the maximality of t^*. Hence

$$0 < \mathcal{K}_{u_n}(t) = k(t, u_n'(t))u_n''(t) < \mathcal{N}_L(t) \quad \text{for a.a. } t \in [T_0, t^*),$$

and so

$$0 < u_n''(t) < \frac{\mathcal{N}_L(t)}{k(t, u_n'(t))} \le \frac{\mathcal{N}_L(t)}{k_*(t)} = \gamma_L(t) \le \hat{\gamma}_L(t) \quad \text{for a.a. } t \in [T_0, t^*).$$

Now, recalling that u_n'' is a solution to ($\mathcal{PT}_n$), assumptions (H_3)–(i) and (H_3)–(iii) give

$$(\Phi \circ \mathcal{K}_{u_n})'(t) = f\left(t, u_n(t), u_n'(t), u_n''(t)\right) \le -K_L'(t)\Phi \circ \mathcal{K}_{u_n}(t) \quad \text{for a.a. } t \in [T_0, t^*).$$

Recalling that $\mathcal{K}_{u_n} > 0$ a.e. in $[T_0, t^*)$ and that Φ is strictly increasing with $\Phi(0) = 0$, we infer

$$\frac{(\Phi \circ \mathcal{K}_{u_n})'(t)}{\Phi \circ \mathcal{K}_{u_n}(t)} \le -K_L'(t) \quad \text{for a.a. } t \in [T_0, t^*).$$

Integrating both sides of the previous estimate in $[T_0, t^*)$, we get

$$\log(\Phi \circ \mathcal{K}_{u_n})(t^*) - \log(\Phi \circ \mathcal{K}_{u_n})(T_0) \le -K_L(t^*),$$

since $K_L(T_0) = 0$. Now, by (iii) we know that $\mathcal{K}_{u_n}(T_0) < L$; therefore,

$$\log \frac{\Phi \circ \mathcal{K}_{u_n}(t^*)}{\Phi(L)} < -K_L(t^*),$$

being Φ strictly increasing, and in turn

$$\mathcal{K}_{u_n}(t^*) < \Phi^{-1}\left(\Phi(L)e^{-K_L(t^*)}\right) = \mathcal{N}_L(t^*).$$

This contradicts the maximality of t^*. Hence we conclude that $t^* > n$. This fact assures that $0 \le \mathcal{K}_{u_n}(t) < \mathcal{N}_L(t)$ for all $t \in [T_0, n]$.

Claim $(viii)$. Using (i) and (vii) we find

$$|u_n''(t)| = \frac{|k(t, u_n'(t))u_n''(t)|}{k(t, u_n'(t))} = \frac{|\mathcal{K}_{u_n}(t)|}{k(t, u_n'(t))} \le \frac{\mathcal{N}_L(t)}{k_*(t)} = \gamma_L(t) \le \hat{\gamma}_L(t) \quad \text{for a.a. } t \in I_n,$$

so that, by the definition of $\mathcal{D}$, we get the claim. □

Theorem 3.12. *Assume (A_1)–(A_3) and (H_1)–(H_3). Then, if $u_n \in W^{2,p}(I_n)$ is a solution of the truncated problem ($\mathcal{PT}_n$), then it is also a solution to ($\mathcal{P}_n$).*

Proof. Let $u_n \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ be a solution of ($\mathcal{PT}_n$). Then, by Lemma 3.11–(i) and $(viii)$, we find that for a.a. $t \in I_n$

$$\begin{aligned}
(\Phi \circ \mathcal{K}_{u_n})'(t) &= \left(\Phi\left(k(t, u_n'(t))u_n''(t)\right)\right)'(t) = \left(\Phi\left(k\left(t, \mathcal{T}_{u_n'}^{\alpha',\beta'}\right)u_n''(t)\right)\right)'(t) \\
&= \left(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}}\right)'(t) = \mathcal{F}_{u_n}(t) \\
&= f\left(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \mathcal{T}_{u_n'}^{\alpha',\beta'}(t), \mathcal{D}_{u_n'}(t)\right) + \arctan\left(u_n'(t) - \mathcal{T}_{u_n'}^{\alpha',\beta'}(t)\right) \\
&= f(t, u_n(t), u_n'(t), u_n''(t)),
\end{aligned}$$

that is u_n solves the equation in ($\mathcal{P}_n$). Moreover $u_n(0) = u_0$, $u_n'(0) = \alpha'(0)$ and $u_n'(n) = \beta'(n)$. Hence u_n is a weak solution to ($\mathcal{P}_n$). □

3.2 A limit argument

In order to complete the proof of Theorem 3.9 we consider a sequence $(u_n)_n$ of solutions to $(\mathcal{PT}_n)$. By Theorem 3.12, every u_n is also a solution to $(\mathcal{P}_n)$. We shall find a solution to $(\mathcal{P})$ via a limit argument.

To this aim, for any $n > T_0$ define

$$x_n : \mathbb{R}_0^+ \to \mathbb{R}, \qquad x_n(t) = \begin{cases} u_n(t), & t \in I_n, \\ u_n(n) + \beta'(n)(t-n), & t > n, \end{cases}$$

so that

$$x_n'(t) = \begin{cases} u_n'(t), & t \in I_n, \\ \beta'(n), & t > n. \end{cases}$$

Moreover, for all $t \in \mathbb{R}_0^+$ put

$$z_n(t) = x_n''(t) = \begin{cases} u_n''(t), & t \in I_n, \\ 0, & t > n; \end{cases} \qquad \Phi_n(t) = \begin{cases} (\Phi \circ \mathcal{K}_{u_n})'(t), & t \in I_n, \\ 0, & t > n. \end{cases}$$

By Lemma 3.11–$(viii)$ we have

$$|z_n(t)| = |x_n''(t)| = \begin{cases} |u_n''(t)|, & \text{a.a. } t \in I_n, \\ 0, & t > n, \end{cases} \le \gamma_L(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.25}$$

Now, since u_n is a solution to $(\mathcal{P}_n)$, by Lemma 3.11–(i) and $(viii)$, together with (H_3)–(ii), we also find

$$|\Phi_n(t)| = \begin{cases} |f(t, u_n(t), u_n'(t), u_n''(t))|, & \text{a.a. } t \in I_n, \\ 0, & t > n, \end{cases} \le \eta_L(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.26}$$

Moreover, since $\eta_L \in L^1(\mathbb{R}_0^+)$ by assumption and also $\gamma_L \in L^1(\mathbb{R}_0^+)$, see Remark 3.8, both $(z_n)_n$ and $(\Phi_n)_n$ are equi-integrable in $\mathbb{R}_0^+$ so that, by the Dunford–Pettis Theorem, there exist $z, \hat{\Phi} \in L^1(\mathbb{R}_0^+)$ such that

$$z_n \rightharpoonup z \quad \text{and} \quad \Phi_n \rightharpoonup \hat{\Phi} \quad \text{in } L^1(\mathbb{R}_0^+) \quad \text{as } n \to +\infty, \tag{3.27}$$

up to subsequences. Consequently, for all $s \in \mathbb{R}_0^+$

$$\int_0^s z_n(\tau)d\tau \longrightarrow \int_0^s z(\tau)d\tau \quad \text{and} \quad \int_0^s \Phi_n(\tau)d\tau \longrightarrow \int_0^s \hat{\Phi}(\tau)d\tau \quad \text{as } n \to +\infty. \tag{3.28}$$

On the other hand, by Lemma 3.11–(i) and (iii), the sequences $(u_n'(0))_n$ and $(\mathcal{K}_{u_n}(0))_n$ are bounded in $\mathbb{R}$ and so there exists $\mathcal{K}_0 \in \mathbb{R}$ such that

$$u_n'(0) = x_n'(0) \longrightarrow \alpha'(0) = \nu_1 \quad \text{and} \quad \mathcal{K}_{u_n}(0) \longrightarrow \mathcal{K}_0 \quad \text{as } n \to +\infty, \tag{3.29}$$

up to subsequences. Now, let us define

$$x(t) = u_0 + \nu_1 t + \int_0^t \int_0^s z(\tau)d\tau ds, \qquad t \in \mathbb{R}_0^+.$$

We want to show that x is a solution to ($\mathcal{P}$). Clearly, $x(0) = u_0$. Moreover, for all $t \in \mathbb{R}_0^+$

$$x'(t) = \nu_1 + \int_0^t z(s)ds, \quad \text{with} \quad x'(0) = \alpha'(0) = \nu_1, \quad \text{and} \quad x''(t) = z(t).$$

By (3.28)–(3.29) we have

$$x_n'(t) = x_n'(0) + \int_0^t z_n(s)ds \longrightarrow x'(t) \quad \text{for all } t \in \mathbb{R}_0^+. \tag{3.30}$$

Furthermore, for all $s \in \mathbb{R}_0^+$

$$\left| \int_0^s z_n(\tau)d\tau \right| \le \int_0^s |z_n(\tau)|d\tau \le \|\gamma_L\|_{L^1(\mathbb{R}_0^+)},$$

so that, by (3.28)

$$\int_0^t \int_0^s z_n(\tau)d\tau ds \longrightarrow \int_0^t \int_0^s z(\tau)d\tau ds \quad \text{for all } t \in \mathbb{R}_0^+,$$

and in turn, using also (3.29), it results

$$x_n(t) = u_0 + u_n'(0)t + \int_0^t \int_0^s z_n(\tau)d\tau ds \longrightarrow x(t) \quad \text{for all } t \in \mathbb{R}_0^+. \tag{3.31}$$

Moreover, recalling that $u_n'(t) = x_n'(t)$ and $u_n''(t) = x_n''(t)$ for a.a. $t \in I_n$, we find

$$\Phi(k(t, x_n'(t))x_n''(t)) = \Phi \circ \mathcal{K}_{u_n}(t) = \Phi \circ \mathcal{K}_{u_n}(0) + \int_0^t \Phi_n(s)ds,$$

that is

$$x_n''(t) = \frac{1}{k(t, x_n'(t))} \Phi^{-1} \left(\Phi \circ \mathcal{K}_{u_n}(0) + \int_0^t \Phi_n(s)ds \right) \quad \text{for a.a. } t \in I_n.$$

Hence, recalling that k and Φ^{-1} are continuous, and using (3.28)–(3.30), we get

$$z_n(t) = x_n''(t) \longrightarrow \frac{1}{k(t, x'(t))} \mathcal{U}(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+, \tag{3.32}$$

where

$$\mathcal{U}(t) = \begin{cases} \Phi^{-1} \left(\Phi(\mathcal{K}_0) + \int_0^t \hat{\Phi}(s)ds \right), & t \in I_n, \\ 0, & t > n. \end{cases}$$

Observe that $\mathcal{U} \in C(\mathbb{R}_0^+; \mathbb{R})$, $\Phi \circ \mathcal{U} \in AC(\mathbb{R}_0^+; \mathbb{R})$ and $(\Phi \circ \mathcal{U})' = \hat{\Phi} \in L^1(\mathbb{R}_0^+)$. Now, by (3.25) and (3.32) we obtain

$$z_n = x_n'' \longrightarrow \frac{\mathcal{U}(\cdot)}{k(\cdot, x')} \quad \text{in } L^1(\mathbb{R}),$$

so that, by (3.27), we get

$$z(t) = \frac{\mathcal{U}(t)}{k(t, x'(t))} \quad \text{for a.a. } t \in \mathbb{R}_0^+, \tag{3.33}$$

which implies

$$x_n''(t) = z_n(t) \longrightarrow z(t) = x''(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.34}$$

Combining (3.30), (3.31) and (3.34) with the fact that f is Carathéodory, we infer

$$f(t, x_n(t), x_n'(t), x_n''(t)) \longrightarrow f(t, x(t), x'(t), x''(t)) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.35}$$

Now, fix $t \in \mathbb{R}_0^+$; clearly there exists $\bar{n} > T_0$ such that $t \in I_n$ for all $n \geq \bar{n}$. Hence, recalling that for all $n > T_0$ the function u_n solves ($\mathcal{P}_n$), for any fixed $t \in \mathbb{R}_0^+$ we have

$$\Phi_n(t) = (\Phi \circ \mathcal{K}_{u_n})'(t) = f(t, u_n(t), u_n'(t), u_n''(t)) = f(t, x_n(t), x_n'(t), x_n''(t)) \quad \text{for all } n \geq \bar{n}.$$

Consequently, from (3.35) we obtain

$$\Phi_n(t) \longrightarrow f(t, x(t), x'(t), x''(t)) \quad \text{for a.a. } t \in \mathbb{R}_0^+,$$

and, by (3.26)

$$\Phi_n \longrightarrow f(\cdot, x, x', x'') \quad \text{in } L^1(\mathbb{R}_0^+).$$

Hence, by (3.33) we deduce that

$$(\Phi \circ \mathcal{K}_x)'(t) = (\Phi \circ \mathcal{U})'(t) = \hat{\Phi}(t) = f(t, x(t), x'(t), x''(t)) \quad \text{for a.a. } t \in \mathbb{R}_0^+,$$

that is x is a solution to (ODE).

Now, by Lemma 3.11–(i), we have

$$\alpha(t) \leq x_n(t) \leq \beta(t) \quad \text{and} \quad \alpha'(t) \leq x_n'(t) \leq \beta'(t) \quad \text{for a.a. } t \in I_n \text{ and all } n > T_0,$$

so that, by (3.30)–(3.31)

$$\alpha(t) \leq x(t) \leq \beta(t) \quad \text{and} \quad \alpha'(t) \leq x'(t) \leq \beta'(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.36}$$

Finally, by (3.25) and (3.34) we infer that $x_n'' \longrightarrow x''$ in $L^1(\mathbb{R}_0^+)$ and, recalling (3.29), we find

$$\sup_{t \in \mathbb{R}_0^+} |x_n'(t) - x'(t)| \leq |x_n'(0) - \nu_1| + \|x_n'' - x''\|_{L^1(\mathbb{R}_0^+)} \to 0,$$

so that $x_n' \to x'$ uniformly in $\mathbb{R}_0^+$. In particular,

$$\lim_{t \to +\infty} x'(t) = \lim_{n \to +\infty} \left(\lim_{t \to +\infty} x_n'(t) \right) = \lim_{n \to +\infty} \beta'(n) = \nu_2.$$

Concerning the regularity of x, we first observe that $x \in C^1(\mathbb{R}_0^+; \mathbb{R})$, being $x_n \in C^1(\mathbb{R}_0^+; \mathbb{R})$, and so $x \in L^p_{loc}(\mathbb{R}_0^+)$. Moreover, $\mathcal{U}$ is locally bounded in $\mathbb{R}_0^+$, since it is continuous, and $1/k(\cdot, x') \in L^p_{loc}(\mathbb{R}_0^+)$ by virtue of (3.36), being $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$ by assumption. Therefore $x'' = \mathcal{U}/k(\cdot, x') \in L^p_{loc}(\mathbb{R}_0^+)$. Furthermore also $x' \in L^p_{loc}(\mathbb{R}_0^+)$, being

$$\int_a^b |x'(t)|^p dt \leq 2^{p-1}|b-a| \left(|\nu_1|^p + \|x''\|_{L^1([a,b])} \right) < +\infty \quad \text{for all } a, b \in \mathbb{R}_0^+.$$

Hence $x \in W^{2,p}_{loc}(\mathbb{R}_0^+)$.

Finally, $\Phi \circ \mathcal{K}_x = \Phi \circ \mathcal{U} \in AC(\mathbb{R}_0^+; \mathbb{R})$ so that $\Phi \circ \mathcal{K}_x \in L^1_{loc}(\mathbb{R}_0^+)$ and $(\Phi \circ \mathcal{K}_x)' = \hat{\Phi} \in L^1(\mathbb{R}_0^+)$. Therefore $\Phi \circ \mathcal{K}_x \in W^{1,1}_{loc}(\mathbb{R}_0^+)$.

In conclusion, $x \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ is a solution to ($\mathcal{P}$) and the proof is complete.

4 Examples

In this section we present a class of examples of functions Φ, k and f satisfying conditions (A_1)–(A_3) and (H_1)–(H_3).

Let $u_0, \nu_1, \nu_2 \in \mathbb{R}$ be such that $\nu_1 < \nu_2$, and let's consider the following BVP

$$\begin{cases} \left(\Phi\left(k(t,u'(\cdot))u''(\cdot)\right)\right)'(t) = f_1\left(t,u(t),u'(t)\right) f_2(u''(t)), & \text{a.a. } t \in \mathbb{R}_0^+, \\ u(0) = u_0, \;\; u'(0) = \nu_1, \;\; u'(+\infty) = \nu_2, \end{cases} \tag{4.1}$$

where the functions Φ, k, f_1 and f_2 fulfill the assumptions listed below.

(I) $\Phi : \mathbb{R} \to \mathbb{R}$ is an *odd* strictly increasing homeomorphism, with $\Phi(0) = 0$, and there exists $\rho > 0$ such that
$$\liminf_{s \to 0^+} \frac{\Phi(s)}{s^\rho} > 0. \tag{4.2}$$

(II) $k : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is continuous, strictly positive a.e. in $\mathbb{R}_0^+ \times \mathbb{R}$ and bounded in $\mathbb{R}_0^+ \times [\nu_1, \nu_2]$.

Moreover, if we denote by
$$k_* = \min_{y \in [\nu_1, \nu_2]} k(t,y) \quad \text{and} \quad k^* = \max_{y \in [\nu_1, \nu_2]} k(t,y),$$
we suppose that there exist $p > 1$ and $\sigma > 0$ such that

(II)$_1$ $t \mapsto 1/k(t,y) \in L^p_{loc}(\mathbb{R}_0^+)$ for all $y \in \mathbb{R}$ and $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$;

(II)$_2$ $\displaystyle\int_1^\infty \frac{1}{t^\sigma k_*^p(t)} dt < +\infty.$

(III) $f_1 : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, decreasing with respect to the x variable, and there exists $T_0 > 0$ for which the following properties hold:

(III)$_1$ there exists $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}_0^+)$ such that
$$|f_1(t,x,y)| \le \tilde{f}_1(t)$$
for a.a. $t \in [0, T_0]$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$ and all $y \in [\nu_1, \nu_2]$;

(III)$_2$ there exist c_1, $c_2 > 0$ and $\delta \ge -1$ such that
$$c_1 t^{-1} \le |f_1(t,x,y)| \le c_2 t^\delta$$
for a.a. $t \ge T_0$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$ and all $y \in [\nu_1, \nu_2]$;

(III)$_3$ $f_1(t,x,y) \le 0$ for all $t \ge T_0$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$ and all $y \in [\nu_1, \nu_2]$.

(IV) $f_2 \in C(\mathbb{R}; \mathbb{R})$ and it verifies:

(IV)$_1$ $f_2(z) > 0$ for $z > 0$ and $f_2(0) = 0$;

(IV)$_2$ there exist $z^* > 0$, two real constants d_1, $d_2 > 0$ and a number $\gamma \le 1$ such that
$$d_1 |\Phi(z)| \le f_2(z) \le d_2 |\Phi(z)|^\gamma \quad \text{for all } z \in \mathbb{R} \text{ with } |z| < z^*;$$

(IV)$_3$ there exist $H > 0$ and $d_3 > 0$ such that if $z \in R$ and $|z| \geq H$ then

$$f_2(z) \leq d_3|z|^{\frac{q-1}{q}} \quad \text{for some } 1 < q \leq +\infty;$$

(IV)$_4$ f_2 is homogeneous of degree $d > 0$ in $\mathbb{R}$, with $d \leq p$, that is

$$f_2(tz) = t^d f_2(z) \quad \text{for all } t > 0 \text{ and } z \in \mathbb{R}.$$

Finally, put

$$K_M = \max_{t \in \mathbb{R}_0^+} k^*(t) = \sup\{k(t,y) : (t,y) \in \mathbb{R}_0^+ \times [\nu_1, \nu_2]\}$$

and suppose that

$$\frac{c_1 d_1}{K_M^d} \geq \sigma\rho \quad \text{and} \quad \gamma \frac{c_1 d_1}{K_M^d} \geq \sigma + \delta. \tag{4.3}$$

Remark 4.1. When $\delta = -1$ in (III)$_2$ we address the critical case.

Our aim is to prove that, in the present setting, all the hypotheses of Theorem 3.9 are satisfied. As a consequence, there exists a solution $u \in C^1(\mathbb{R}_0^+; \mathbb{R}) \cap W^{2,p}_{loc}(\mathbb{R}_0^+)$ of (4.1).

Remark 4.2. Before proceeding, we highlight, for future reference, a few consequences of the above assumptions (I)–(IV), that will also be employed in the sequel.
(A) For every $\nu \in (-\infty, p]$ it results

$$\int_1^\infty \frac{1}{t^\sigma k_*^\nu(t)} dt < +\infty.$$

Indeed, since $k(t,y) \geq k_*(t)$ for all for all $(t,y) \in \mathbb{R}_0^+ \times [\nu_1, \nu_2]$ and k is bounded in $\mathbb{R}_0^+ \times [\nu_1, \nu_2]$, it follows that k_* is bounded in $\mathbb{R}_0^+$. Therefore, recalling that the map $t \mapsto t^{-\sigma}/k_*^p(t)$ is integrable in $\{t \geq 1\}$ by (II)$_2$, for any $\nu \in (-\infty, p]$ we have

$$\int_1^\infty \frac{1}{t^\sigma k_*^\nu(t)} dt \leq \sup_{\mathbb{R}_0^+} k_*^{p-\nu} \int_1^\infty \frac{1}{t^\sigma k_*^p(t)} dt < +\infty.$$

(B) For all $\zeta > 0$ it is $\max_{|z| \leq \zeta} |\Phi(z)| = \Phi(\zeta)$. Indeed, if $z \in \mathbb{R}$ is such that $|z| \leq \zeta$, since Φ is odd and strictly increasing, we get $-\Phi(\zeta) = \Phi(-\zeta) \leq \Phi(z) \leq \Phi(\zeta)$.
(C) Combining (III)$_2$ with (III)$_3$ we have that

$$f_1(t,x,y) \leq -c_1 t^{-1} < 0$$

for a.a. $t \geq T_0$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$, all $y \in [\nu_1, \nu_2]$.

Obviously assumptions (A_1)–(A_3) are verified by virtue of (I)–(IV), with

$$f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}, \quad f(t,x,y,z) = f_1(t,x,y) f_2(z).$$

Now, we are going to prove that also assumptions (H_1)–(H_3) are verified. To this aim, let T_0 be the positive number introduced in (III), and define

$$\alpha(t) = u_0 + t\nu_1 \quad \text{and} \quad \beta(t) = u_0 + t\nu_2, \quad t \in \mathbb{R}_0^+.$$

Clearly $\alpha, \beta \in C^\infty(\mathbb{R}_0^+;\mathbb{R}) \subset W^{2,p}_{loc}(\mathbb{R}_0^+)$, with $p > 1$ introduced in (II), and they are, respectively, a lower and a upper solution to (ODE), since $f_2(0) = 0$ by (IV)$_1$ and $\Phi(0) = 0$ by (I). Finally, $\alpha(0) = \beta(0) = u_0$ and $\alpha'(t) = \nu_1 < \nu_2 = \beta'(t)$ for all $t \in \mathbb{R}_0^+$, so that the pair (α, β) is ordered in $\mathbb{R}_0^+$.

Hypothesis (H_1)**.** Obviously β' is increasing in $(T_0, +\infty)$, being $\beta''(t) = 0$ for all $t \in \mathbb{R}_0^+$, and $\lim_{t\to+\infty} \beta'(t) = \nu_2$.

Hypothesis (H_2)**.** Let $H > 0$, $d_3 > 0$ and $q \in (1, +\infty]$ be as in (IV)$_3$. Combining (IV)$_1$ with (III)$_1$ and (IV)$_3$, we obtain

$$|f(t,x,y,z)| = |f_1(t,x,y)| f_2(z) \le d_3 \tilde{f}_1(t) |z|^{\frac{q-1}{q}}$$

for a.a. $t \in [0, T_0]$, all $x \in [\alpha(t), \beta(t)]$, all $y \in [\nu_1, \nu_2]$ and all $z \in \mathbb{R}$ with $|z| \ge H$. Hence assumption (H_2) is verified with

$$\psi \equiv 1, \quad \ell \equiv 0, \quad \mu(t) = d_3 \tilde{f}_1(t).$$

Observe that $\mu = d_3 \tilde{f}_1 \in L^q([0, T_0])$, being $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}_0^+)$ by (III)$_1$.

Hypothesis (H_3)**.** Define the map $K_0 : \mathbb{R}_0^+ \to \mathbb{R}$ as follows

$$K_0(t) = \begin{cases} 0, & 0 \le t \le T_0, \\ \displaystyle\int_{T_0}^t f_0(s)ds, & t > T_0, \end{cases}$$

where

$$f_0(s) = \min \{ |f_1(s,x,y)| : (x,y) \in [\alpha(t), \beta(t)] \times [\nu_1, \nu_2] \}.$$

Observe that

- f_0 is well defined in $\mathbb{R}_0^+$ since f_1 is Carathéodory by assumption (III);
- $f_0 \in L^1_{loc}(\mathbb{R}_0^+)$, being $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}_0^+)$ by (III)$_1$ and

$$f_0(t) \le |f_1(t,x,y)| \le \tilde{f}_1(t) \quad \text{for all } t \in \mathbb{R}_0^+, \text{ all } x \in [\alpha(t), \beta(t)] \text{ and all } y \in [\nu_1, \nu_2].$$

Hence $K_0 \in AC(\mathbb{R}_0^+;\mathbb{R}) \cap W^{1,1}_{loc}(\mathbb{R}_0^+)$. Furthermore, K_0 is strictly increasing in $[T_0, +\infty)$, and

$$K_0(t) \ge c_1 \int_{T_0}^t \frac{ds}{s} = c_1 \log \frac{t}{T_0} \quad \text{for all } t \ge T_0, \tag{4.4}$$

by (III)$_2$ and (III)$_3$; see also Remark 4.2–(C).

Now, let $L > 0$ be fixed arbitrarily and put

$$m(L) = \min_{z^* \le |z| \le L} f_2(z), \quad M(L) = \max_{z^* \le |z| \le L} |\Phi(z)|, \quad c(L) = \min \left\{ \frac{d_1}{K_M^d}, \frac{m(L)}{M(L) K_M^d} \right\} > 0.$$

Define the function $K_L : \mathbb{R}_0^+ \to \mathbb{R}$ as follows

$$H_L(t) = \Phi^{-1} \left(\Phi(L) e^{-c(L) K_0(t)} \right).$$

By (4.4) it follows that $K_0(t) \to +\infty$ as $t \to +\infty$, so that $H_L(t) \to 0$ as $t \to +\infty$. Therefore it is possible to find $t_L > T_0$ such that

$$H_L(t) \le z^* \quad \text{for all } t \ge t_L. \tag{4.5}$$

We then claim that the function $K_L(t) : \mathbb{R}_0^+ \to \mathbb{R}$ defined by

$$K_L(t) = \begin{cases} c(L)K_0(t), & 0 \le t \le t_L, \\ c(L)K_0(t_L) + \dfrac{d_1}{K_M^d} \displaystyle\int_{t_L}^t f_0(s)ds, & t > t_L, \end{cases}$$

satisfies all the properties in assumption (H_3). Observe that

- K_L is continuous in $\mathbb{R}_0^+$, being K_0 continuous in $\mathbb{R}_0^+$;
- $K_L \equiv K_0 \equiv 0$ in $[0, T_0]$ and K_L is strictly increasing in $[T_0, +\infty)$ since the same holds for K_0 and $t_L > T_0$; see Remark 4.2–(C).

Moreover, $K_L \in W^{1,1}_{loc}(\mathbb{R}_0^+)$, since the same is true for K_0, and for a.a. $t \in R_0^+$, all $x \in [\alpha(t), \beta(t)]$ and all $y \in [\nu_1, \nu_2]$, it is

$$K_L'(t) = \begin{cases} 0, & 0 \le t \le T_0 \\ c(L)f_0(t), & T_0 < t \le t_L \\ \dfrac{d_1}{K_M^d} f_0(t), & t > t_L \end{cases} \le \begin{cases} 0, & 0 \le t \le T_0, \\ c(L)|f_1(t,x,y)|, & T_0 < t \le t_L, \\ \dfrac{d_1}{K_M^d}|f_1(t,x,y)|, & t > t_L. \end{cases} \tag{4.6}$$

Now, we observe that $K_L(t) \ge c(L)K_0(t)$ for all $t \in \mathbb{R}_0^+$ so that

$$\mathcal{N}_L(t) = \Phi^{-1}\left(\Phi(L)e^{-K_L(t)}\right) \le H_L(t) \quad \text{for all } t \in \mathbb{R}_0^+,$$

and in turn, by (4.5), it follows that

$$\mathcal{N}_L(t) \le z^* \quad \text{for all } t \ge t_L. \tag{4.7}$$

Consequently, using (IV)$_4$ and (IV)$_2$, by (4.6), we obtain

$$\begin{aligned} |f(t,x,y,z)| &= \frac{|f_1(t,x,y)|}{k^d(t,y)} f_2(k(t,y)z) \ge \frac{d_1}{k^d(t,y)}|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)| \\ &\ge \frac{d_1}{K_M^d}|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)| \ge K_L'(t)|\Phi(k(t,y)z)|, \end{aligned} \tag{4.8}$$

for a.a. $t > t_L$, $x \in [\alpha(t), \beta(t)]$, $y \in [\nu_1, \nu_2]$ and $z \in \mathbb{R}$ such that $|k(t,y)z| \le \mathcal{N}_L(t)$.

On the other hand, by definition of $\mathcal{N}_L$ it is $\mathcal{N}_L(t) \le L$ for all $t \in \mathbb{R}_0^+$. Therefore, using again (IV)$_4$ and (IV)$_2$ and (4.6), it results

$$\begin{aligned} |f(t,x,y,z)| &= \frac{|f_1(t,x,y)|}{k^d(t,y)} f_2(k(t,y)z) \ge \frac{1}{K_M^d}|f_1(t,x,y)| f_2(k(t,y)z) \\ &\ge \begin{cases} \dfrac{d_1}{K_M^d}|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)|, & |k(t,y)z| \le z^*, \\ \dfrac{m_L}{M_L K_M^d}|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)|, & z^* \le |k(t,y)z| \le \mathcal{N}_L(t), \end{cases} \\ &\ge c(L)|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)| \\ &\ge K_L'(t)|\Phi(k(t,y)z)|, \end{aligned} \tag{4.9}$$

a.a. $t \in [T_0, t_L]$, $x \in [\alpha(t), \beta(t)]$, $y \in [\nu_1, \nu_2]$ and $z \in \mathbb{R}$ such that $|k(t,y)z| \le \mathcal{N}_L(t)$.

Combining (4.8) with (4.9), it follows that

$$|f(t,x,y,z)| \geq K_L'(t)|\Phi(k(t,y)z)|$$

for a.a. $t \geq T_0$, all $x \in [\alpha(t), \beta(t)]$, all $y \in [\nu_1, \nu_2]$ and all $z \in \mathbb{R}$ such that $|z| \leq \mathcal{N}_L(t)/k(t,y)$, that is condition (H_3)–(i) is satisfied.

Now we are going to prove the validity of (3.3). To this aim, observe that, since K_L is continuous in $\mathbb{R}_0^+$ and $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$ by (II), it results

$$\int_{T_0}^{t_L} \frac{1}{k_*(t)} e^{-\frac{K_L(t)}{\rho}} dt < +\infty. \tag{4.10}$$

On the other hand, using the definition of K_L and (III)$_2$, we get

$$\begin{aligned} \int_{t_L}^{\infty} \frac{1}{k_*(t)} e^{-\frac{K_L(t)}{\rho}} dt &= e^{-\frac{c(L)K_0(t_L)}{\rho}} \int_{t_L}^{\infty} \frac{1}{k_*(t)} e^{-\frac{d_1}{\rho K_M^d}\int_{t_L}^t f_0(s)ds} dt \\ &\leq e^{-\frac{c(L)K_0(t_L)}{\rho}} \int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{t_L}{t}\right)^{\frac{c_1 d_1}{\rho K_M^d}} dt \\ &= t_L^{\frac{c_1 d_1}{\rho K_M^d}} e^{-\frac{c(L)K_0(t_L)}{\rho}} \int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{1}{t}\right)^{\frac{c_1 d_1}{\rho K_M^d}} dt. \end{aligned} \tag{4.11}$$

Now, recalling that the map $t \mapsto t^{-\sigma}/k_*(t)$ is integrable in $\{t \geq 1\}$, see Remark 4.2–(A), it follows that also the map $t \mapsto t^{-\frac{c_1 d_1}{\rho K_M^d}}/k_*(t)$ is integrable in $\{t \geq 1\}$, since

$$\int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{1}{t}\right)^{\frac{c_1 d_1}{\rho K_M^d}} dt \leq \int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{1}{t}\right)^{\sigma} dt < +\infty,$$

being $\sigma \leq c_1 d_1/\rho K_M^d$ by (4.3), which implies

$$\int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{1}{t}\right)^{\frac{c_1 d_1}{\rho K_M^d}} dt < +\infty. \tag{4.12}$$

Combining (4.11) with (4.12) we get (3.3).

Now, we want to prove the existence of a non-negative function $\eta_L \in L^1(\mathbb{R}_0^+)$ satisfying (H_3)–(ii) (note that $\gamma_L = \hat{\gamma}_L$ in $\mathbb{R}_0^+$, being $\alpha'' \equiv \beta'' \equiv 0$ in $\mathbb{R}_0^+$). By (4.7), using also (IV)$_4$, (III)$_2$ and (IV)$_2$, see also Remark 4.2–(C), for a.a. $t \in \mathbb{R}_0^+$, all $x \in [\alpha(t), \beta(t)]$, all $y \in [\nu_1, \nu_2]$ and all $z \in \mathbb{R}$ such that $|k_*(t)z| \leq \mathcal{N}_L(t)$ we find

$$\begin{aligned} |f(t,x,y,z)| &= \frac{|f_1(t,x,y)|}{k_*^d(t)} f_2(k_*(t)z) \\ &\leq \begin{cases} c_2 d_2 \dfrac{t^\delta}{k_*^d(t)} |\Phi(k_*(t)z)|^\gamma \leq c_2 d_2 \dfrac{t^\delta}{k_*^d(t)} \left[\Phi(\mathcal{N}_L(t))\right]^\gamma, & t > t_L, \\ \max\limits_{[0,L]} f_2 \cdot \dfrac{\tilde{f}_1(t)}{k_*^d(t)}, & 0 \leq t \leq t_L, \end{cases} \\ &=: \eta_L(t). \end{aligned}$$

Finally, we claim that $\eta_L \in L^1(\mathbb{R}_0^+)$. First observe that, since $d \le p$, $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}_0^+)$ and $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$, we get

$$\begin{aligned}\int_0^{t_l} \eta_L(t)dt &\le \max_{[0,L]} f_2 \cdot \|\tilde{f}_1\|_{L^\infty([0,t_L])} \int_0^{t_L} \frac{1}{k_*^d(t)}dt \\ &\le \max_{[0,L]} f_2 \cdot \|\tilde{f}_1\|_{L^\infty([0,t_L])} \sup_{\mathbb{R}_0^+} k_*^{p-d} \int_0^{t_L} \frac{1}{k_*^p(t)}dt \\ &< +\infty.\end{aligned} \tag{4.13}$$

On the other hand, by (III)$_2$, from the definition of η_L, it is

$$\begin{aligned}\int_{t_L}^\infty \eta_L(t)dt &= c_2 d_2 \int_{t_L}^\infty \frac{t^\delta}{k_*^d(t)} [\Phi(\mathcal{N}_L(t))]^\gamma \, dt \\ &= c_2 d_2 \Phi(L)^\gamma \int_{t_L}^\infty \frac{t^\delta}{k_*^d(t)} e^{-\gamma K_L(t)} dt \\ &= c_2 d_2 \Phi(L)^\gamma e^{-\gamma c(L) K_0(t_L)} \int_{t_L}^\infty \frac{t^\delta}{k_*^d(t)} e^{-\gamma \frac{d_1}{K_M^d} \int_{t_L}^t f_0(s)ds} dt \\ &\le c_3 \int_{t_L}^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt,\end{aligned} \tag{4.14}$$

with $c_3 > 0$ appropriate constant.

Now, observe that, if $t_L \ge 1$, recalling that $\gamma \frac{c_1 d_1}{K_M^d} - \delta \ge \sigma$ by (4.3), we get

$$\int_{t_L}^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt \le \int_1^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt \le \int_1^\infty \frac{1}{k_*^d(t) t^\sigma} dt < +\infty,$$

since $d \le p$ by assumption, see Remark 4.2–(A); on the other hand, if $t_L < 1$, then

$$\begin{aligned}\int_{t_L}^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt &= \int_{t_L}^1 \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt + \int_1^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt \\ &\le \int_{t_L}^1 \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt + \int_1^\infty \frac{1}{k_*^d(t) t^\sigma} dt \\ &< +\infty,\end{aligned}$$

since the map $t \mapsto k_*^{-d}(t) t^{\delta - \gamma \frac{c_1 d_1}{K_M^d}}$ is bounded in $[t_L, 1]$ and, as before, $\int_1^\infty \frac{1}{k_*^d(t) t^\sigma} dt < +\infty$, being $d \le p$. Therefore, from (4.14), we infer that

$$\int_{t_L}^\infty \eta_L(t)dt < +\infty. \tag{4.15}$$

Combining (4.13) with (4.15) we get the claim.

Gathering together all these facts, we are entitled to apply Theorem 3.9 to the BVP (4.1), getting at least a solution $u \in C^1(\mathbb{R}_0^+) \cap W^{2,p}_{loc}(\mathbb{R}_0^+)$ such that

$$u_0 + t\nu_1 \le u(t) \le u_0 + t\nu_2 \quad \text{and} \quad \nu_1 \le u'(t) \le \nu_2 \quad \text{for a.a. } t \in \mathbb{R}_0^+.$$

Remark 4.3. It is worth noting that, in the particular case when the homeomorphism Φ in (I) is also homogeneous of degree $r \in (0, p]$, the growth assumption (IV)$_3$ can be replaced with the following one:

$(\mathrm{IV})'_3$ there exists $H > 0$ and $d_3 > 0$ such that, if $z \in \mathbb{R}$ and $|z| \geq H$, then

$$f_2(z) \leq d_3 |\Phi(z)|^{\overline{\alpha}} \quad \text{for some } \overline{\alpha} \leq 1. \tag{4.16}$$

Indeed, if (4.16) holds, it follows

$$\begin{aligned} |f_1(t,x,y)| f_2(z) &\leq d_3 |f_1(t,x,y)| \cdot |\Phi(z)|^{\overline{\alpha}} = d_3 \frac{|f_1(t,x,y)|}{k(t,y)^{r\overline{\alpha}}} |\Phi(k(t,y)z)|^{\overline{\alpha}} \\ &\leq d_3 \frac{\tilde{f}_1(t)}{k_*^{r\overline{\alpha}}(t)} |\Phi(k(t,y)z)|^{\overline{\alpha}}, \end{aligned}$$

for a.a. $t \in [0, T_0]$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$, all $y \in [\nu_1, \nu_2]$ and all $z \in \mathbb{R}$ such that $|z| \geq H$.
As a consequence, hypothesis (H_2) in Theorem 3.9 is fulfilled with

$$\psi(s) = s^{\overline{\alpha}}, \quad \ell(t) = d_3 \frac{\tilde{f}_1(t)}{k_*^{\overline{\alpha}r}(t)}, \quad \mu(t) \equiv 0.$$

Note that, since $\overline{\alpha} \leq 1$, the function ψ satisfies (H_2)–(i); furthermore, since $\tilde{f}_1 \in L^{\infty}_{loc}(\mathbb{R}^+_0)$, the function $1/k_* \in L^p_{loc}(\mathbb{R}^+_0)$ and $\overline{\alpha}r \leq r \leq p$, we infer that $\ell \in L^1([0, T_0])$.

We conclude this section by presenting some concrete examples of BVPs of the form (4.1), satisfying assumptions (I)–(IV) introduced above.

Example 4.4. Let us consider the following boundary value problem

$$\begin{cases} \left(\Phi\left(e^{-u'(t)^2} \min\left\{\sqrt{t}, \dfrac{1}{t^2}\right\} u'(t)\right)\right)' = f(t, u, u'(t), u''(t)), \quad \text{a.a. } t \in \mathbb{R}^+_0, \\ u(0) = 0, \quad u'(0) = 0, \quad u'(+\infty) = 1, \end{cases} \tag{4.17}$$

with $m > 1$ to be fixed later, $\theta \in (0, 1)$ and

$$\Phi(z) = z + \sin z, \quad f(t, x, y, z) = -m[\arctan(x^3 + y^2) + \pi] \cdot |z|^{\theta} \frac{t}{1 + t^2}.$$

Obviously, problem (4.17) takes the form (4.1) with

- $\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(z) = z + \sin z;$
- $k : \mathbb{R}^+_0 \times \mathbb{R} \to \mathbb{R}, \quad k(t, y) = e^{-y^2} \min\{\sqrt{t}, 1/t^2\};$
- $f_1 : \mathbb{R}^+_0 \times \mathbb{R}^2 \to \mathbb{R}, \quad f_1(t, x, y) = -[\arctan(x^3 + y^2) + \pi] \dfrac{mt}{1 + t^2};$
- $f_2 : \mathbb{R} \to \mathbb{R}, \quad f_2(z) = |z|^{\theta}.$

We claim that the functions Φ, k, f_1 and f_2 satisfy all the assumptions (I)–(IV) introduced in this section, with suitable constants fulfilling (4.3).

Assumption (I) It is straightforward to recognize that Φ is an odd strictly increasing homeomorphism. Moreover, since

$$\lim_{z \to 0^+} \frac{\Phi(z)}{z} = \lim_{z \to 0^+} \frac{z + \sin z}{z} = 2,$$

condition (4.2) is satisfied with $\rho = 1$.

Assumption (II) Clearly k is continuous, strictly positive a.e. in $\mathbb{R}_0^+ \times \mathbb{R}$ and bounded in $\mathbb{R}_0^+ \times [0,1]$. Moreover, it is very easy to see that the map $t \mapsto 1/k(t,y) \in L^p_{loc}(\mathbb{R}_0^+)$ for all $y \in \mathbb{R}$, for every fixed $p \in (1,2)$, and the same is true for $1/k_*(t) = e/\min\{\sqrt{t}, t^2\}$. Moreover,

$$\int_1^\infty \frac{1}{t^\sigma k_*^p(t)} dt = e^p \int_1^\infty \frac{1}{t^{\sigma-2p}} dt < +\infty, \quad \text{for every } \sigma > 2p+1.$$

Assumption (III) f_1 is a Carathéodory function in $\mathbb{R}_0^+ \times \mathbb{R}^2$, being continuous in the same set, and it is also decreasing with respect to x in $\mathbb{R}_0^+ \times \mathbb{R}^2$.

Moreover, for all $t \in \mathbb{R}_0^+$, all $x \in [0,t]$ and all $y \in [0,1]$ it is

$$\begin{aligned} |f_1(t,x,y)| &= \left[\arctan(x^3+y^2)+\pi\right] \frac{mt}{1+t^2} \\ &\le \left[\arctan(1+t^3)+\pi\right] \frac{mt}{1+t^2} =: \tilde{f}_1(t) \in C(\mathbb{R}_0^+;\mathbb{R}) \subseteq L^\infty_{loc}(\mathbb{R}_0^+) \end{aligned}$$

and also

$$\frac{\pi}{2} \cdot \frac{mt^2}{1+t^2} = \left(\pi - \frac{\pi}{2}\right) \frac{mt^2}{1+t^2} \le t|f_1(t,x,y)| \le t\tilde{f}_1(t) = \left[\arctan(1+t^3)+\pi\right] \frac{mt^2}{1+t^2}. \tag{4.18}$$

Now, since

$$\lim_{t\to+\infty} \frac{\pi}{2} \cdot \frac{mt^2}{1+t^2} = m\frac{\pi}{2} \quad \text{and} \quad \lim_{t\to+\infty} \left[\arctan(1+t^3)+\pi\right] \frac{mt^2}{1+t^2} = \frac{3}{2}m\pi,$$

from (4.18) we deduce that for any $\varepsilon \in (0, m\pi/2)$ there exists $T_0 = T_0(\varepsilon) > 0$ such that

$$\left(m\frac{\pi}{2} - \varepsilon\right) \frac{1}{t} \le |f_1(t,x,y)| \le \left(\frac{3}{2}m\pi + \varepsilon\right) \frac{1}{t}$$

for all $t \ge T_0$, all $x \in [0,t]$ and all $y \in [0,1]$, so that (III)$_2$ holds with

$$c_1 = m\frac{\pi}{2} - \varepsilon, \quad c_2 = \frac{3}{2}m\pi + \varepsilon, \quad \delta = -1. \tag{4.19}$$

Note that $c_1 = m\frac{\pi}{2} - \varepsilon > 0$. Finally, condition (III)$_3$ is trivially satisfied.

Assumption (IV) Clearly f_2 is continuous in $\mathbb{R}$, being $\theta > 0$, and (IV)$_1$ trivially holds. Moreover,

$$\lim_{z\to0} \frac{f_2(z)}{|\Phi(z)|^\theta} = \lim_{z\to0} \frac{|z|^\theta}{|z+\sin z|^\theta} = \frac{1}{2^\theta},$$

so that for any $\varepsilon \in (0, 1/2^\theta)$ there exists $z^* = z^*(\varepsilon) > 0$ such that

$$\left(\frac{1}{2^\theta} - \varepsilon\right) |\Phi(z)| \le f_2(z) \le \left(\frac{1}{2^\theta} + \varepsilon\right) |\Phi(z)|^\theta \quad \text{for all } z \in \mathbb{R}: \ |z| < z^*.$$

Hence condition (IV)$_2$ is satisfied with

$$d_1 = \frac{1}{2^\theta} - \varepsilon, \quad d_2 = \frac{1}{2^\theta} + \varepsilon, \quad \gamma = \theta. \tag{4.20}$$

Furthermore, condition (IV)$_3$ holds with any constant $H > 0$, any number $d_3 \ge 1$ and $q = 1/(1-\theta) > 1$. Finally f_2 is homogeneous of degree $d = \theta < 1 < p$.

Now, we claim that, for any fixed $p \in (1,2)$, it is possible to choose

$$m > 1 \quad \text{and} \quad 0 < \varepsilon < \frac{1}{2^\theta} < m\frac{\pi}{2}$$

in such a way that (4.3) holds. To prove the claim, first note that

$$K_M = \sup_{\mathbb{R}_0^+ \times [0,1]} e^{-y^2} \min\{\sqrt{t}, 1/t^2\} = 1.$$

Now, since

$$\lim_{\varepsilon \to 0^+} (m-\varepsilon) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) = \frac{m}{2^\theta},$$

we can choose $m > 1$ and $\varepsilon \in (0, 1/2^\theta)$ satisfying

$$(m-\varepsilon) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) > \max\left\{1 + 2p, \frac{2p}{\theta}\right\}. \tag{4.21}$$

Consequently, by (4.19)–(4.20) and recalling that $\rho = 1$, one gets

$$\frac{c_1 d_1}{K_M^d} = (m-\varepsilon) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) > 1 + 2p = (1+2p)\rho.$$

On the other hand, using again (4.19)–(4.20) and recalling that $\delta = -1$, we see that

$$\gamma \frac{c_1 d_1}{K_M^d} = \theta(m-\varepsilon) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) > 2p = (1+2p) + \delta.$$

From this, since σ can be chosen arbitrarily close to $1 + 2p$, we conclude that (4.3) is satisfied.

We are then entitled to apply Theorem 3.9 which ensures the existence of a solution $u \in C^1(\mathbb{R}_0^+; \mathbb{R}) \cap W_{loc}^{2,p}(\mathbb{R}_0^+)$ of (4.17).

Remark 4.5. A further example which covers the critical case can be constructed following Example 4.4, but choosing $\nu_1 = 1 < \nu_2$ and substituting the function f_1 with the following one

$$f_1(t,x,y) = m\left[e^{-(x+y)} - 1\right] t \sin\left(\frac{1}{t^2+1}\right).$$

Indeed, $f_1 \in C(\mathbb{R}_0^+ \times \mathbb{R}^2)$, it is non positive whenever $x + y \geq 0$ and decreasing with respect to x in the whole of $\mathbb{R}_0^+ \times \mathbb{R}^2$.

Furthermore, for all $t \geq 1$, all $x \in [t, t\nu_2]$ and all $y \in [1, \nu_2]$ we have

$$\frac{m}{2} t \sin\left(\frac{1}{t^2+1}\right) \leq m\left[1 - e^{-(x+y)}\right] t \sin\left(\frac{1}{t^2+1}\right) = |f_1(t,x,y)| \leq mt \sin\left(\frac{1}{t^2+1}\right) =: \tilde{f}_1(t),$$

with $\tilde{f}_1 \in C(\mathbb{R}_0^+)$, which implies

$$\frac{m}{2} t^2 \sin\left(\frac{1}{t^2+1}\right) \leq t|f_1(t,x,y)| \leq mt^2 \sin\left(\frac{1}{t^2+1}\right).$$

Therefore, for any $\varepsilon \in (0, m/2)$ there exists $T_0 = T_0(\varepsilon)$ that can be chosen greater or equal than 1, such that, for all $t \geq T_0$, all $x \in [t, t\nu_2]$ and all $y \in [1, \nu_2]$, it is

$$\left(\frac{m}{2} - \varepsilon\right) \frac{1}{t} \leq |f_1(t,x,y)| \leq (m-\varepsilon)\frac{1}{t},$$

so that condition (III) holds with

$$T_0 = 1, \quad c_1 = \frac{m}{2} - \varepsilon, \quad c_2 = m - \varepsilon, \quad \delta = -1.$$

Finally, we conclude by choosing $m > 1$ and $\varepsilon \in (0, 1/2^\theta)$ satisfying

$$\left(\frac{m}{2} - \varepsilon\right) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) > \max\left\{1 + 2p, \frac{2p}{\theta}\right\}$$

in place of (4.21).

Example 4.6. Let $\overline{p} \in (1, +\infty)$ be fixed and let $r, \theta \in \mathbb{R}$ be such that

$$1 < r < \overline{p} + 1 \quad \text{and} \quad 0 < \theta < r - 1.$$

Let us consider the following boundary value problem

$$\begin{cases} \left(\Phi\left(\dfrac{|\sin t|^{1/\overline{p}} + |\sin^2 u'(t)|}{2} u''(t)\right)\right)' = f(t, u(t), u'(t), u''(t)), & \text{a.a. } t \in \mathbb{R}_0^+, \\ u(0) = 0, \;\; u'(0) = 1, \;\; u'(+\infty) = \nu_2, \end{cases} \tag{4.22}$$

with $\nu_2 > 1$, $m > 1$ to be fixed later and

$$\Phi(z) = \Phi_r(z) = |z|^{r-2} z \quad \text{and} \quad f(t, x, y, z) = -\frac{m + t|\cos t|}{1 + t} (x + |\cos y|)^3 |z|^\theta.$$

Problem (4.22) takes the form (4.1) with

- $\Phi : \mathbb{R} \to \mathbb{R}$ is the *r-Laplacian*;
- $k : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}, \quad k(t, y) = \frac{|\sin t|^{1/\overline{p}} + |\sin^2 y|}{2}$;
- $f_1 : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}, \quad f_1(t, x, y) = -\frac{m + t|\cos t|}{1+t}(x + |\cos y|)^3$;
- $f_2 : \mathbb{R} \to \mathbb{R}, \quad f_2(z) = |z|^\theta$.

We shall see that the functions Φ, k, f_1 and f_2 satisfy all the assumptions (I)–(IV) introduced in this section, with suitable constants fulfilling (4.3).

Assumption (I) Clearly the *r-Laplacian* is an odd strictly increasing homeomorphism, for which condition (4.2) is satisfied with $\rho = r - 1$, being $\lim_{z \to 0^+} \frac{\Phi(z)}{z^{r-1}} = 1$.

Assumption (II) The function k is continuous and bounded in $\mathbb{R}_0^+ \times \mathbb{R}$, and it is strictly positive a.e. in $\mathbb{R}_0^+ \times \mathbb{R}$.

Moreover, it is very easy to check that the map $t \mapsto 1/k(t, y) \in L^p_{loc}(\mathbb{R}_0^+)$ for all $y \in \mathbb{R}$, for every fixed $p \in (1, \overline{p})$, and the same is true for $1/k_*(t) = 2/|\sin t|^{1/\overline{p}}$, so that $(\text{II})_1$ holds.

Now, choose $p > 1$ in such a way that

$$p \in (\max\{1, r - 1\}, \overline{p}).$$

For any $\sigma > 1$ it results

$$\begin{aligned} \int_{2\pi}^{\infty} \frac{1}{t^\sigma k_*^p(t)} dt &= 2^p \int_{2\pi}^{\infty} \frac{1}{t^\sigma |\sin t|^{p/\overline{p}}} dt = 2^p \sum_{n=1}^{\infty} \int_{2n\pi}^{2(n+1)\pi} \frac{1}{t^\sigma |\sin t|^{p/\overline{p}}} dt \\ &= 2^p \sum_{n=1}^{+\infty} \int_0^{2\pi} \frac{1}{(t + 2n\pi)^\sigma |\sin t|^{p/\overline{p}}} dt \le 2^p \sum_{n=1}^{+\infty} \int_0^{2\pi} \frac{1}{(2n\pi)^\sigma |\sin t|^{p/\overline{p}}} dt \\ &= 2^p \sum_{n=1}^{+\infty} \frac{1}{(2n\pi)^\sigma} \int_0^{2\pi} \frac{1}{|\sin t|^{p/\overline{p}}} dt < +\infty, \end{aligned}$$

and, consequently, assumption $(\mathrm{II})_2$ is fulfilled.
Assumption (III) f_1 is a Carathéodory function in $\mathbb{R}_0^+ \times \mathbb{R}^2$, being continuous in the same set, and it is also decreasing with respect to x in $\mathbb{R}_0^+ \times \mathbb{R}^2$.

Now, take $T_0 = 1$. For all $t \geq T_0$, all $x \in [t, t\nu_2]$ and all $y \in \mathbb{R}$ it results $f_1(t,x,y) \leq 0$ and

$$|f_1(t,x,y)| = \frac{m + t|\cos t|}{1+t}(x + |\cos y|)^3 \leq 4\frac{(m + t|\cos t|)}{1+t}(x^3 + |\cos y|^3)$$
$$\leq \frac{4(m+t)(t^3\nu_2^3 + 1)}{1+t} =: \tilde{f}_1(t),$$

with $\tilde{f}_1 \in C(\mathbb{R}_0^+; \mathbb{R}) \subseteq L^\infty_{loc}(\mathbb{R}_0^+)$. Furthermore, for all $t \geq 1$, all $x \in [t, t\nu_2]$ and all $y \in \mathbb{R}$

$$|f_1(t,x,y)| \leq \tilde{f}_1(t) \leq 4(m+1)(\nu_2^3 + 1)t^3. \tag{4.23}$$

On the other hand, for all $t \geq 1$, all $x \in [t, t\nu_2]$ and all $y \in \mathbb{R}$, it is also true that

$$|f_1(t,x,y)| \geq \frac{m + t|\cos t|}{1+t}x^3 \geq \frac{m}{1+t} \geq \frac{m}{2} \cdot \frac{1}{t}. \tag{4.24}$$

Hence, combining (4.23) with (4.24) we see that $(\mathrm{III})_2$ holds with

$$T_0 = 1, \quad c_1 = \frac{m}{2}, \quad c_2 = 2(m+1)(1+\nu_2^2), \quad \delta = 3. \tag{4.25}$$

Assumption (IV) Clearly $f_2 > 0$ in $\mathbb{R}^+$ and $f_2(0) = 0$. Moreover, if $z \in \mathbb{R}$ and $|z| < 1$, then

$$|\Phi(z)| = |z|^{r-1} < |z|^\theta = f_2(z) = |z|^{\gamma(r-1)} = |\Phi(z)|^\gamma,$$

provided that $\gamma = \theta/(r-1) < 1$. Therefore, condition $(\mathrm{IV})_2$ is satisfied with $z^* = 1$ and

$$d_1 = d_2 = 1. \tag{4.26}$$

Now, observe that, since Φ is homogeneous of degree $r - 1 < p$ and

$$f_2(z) = |z|^\theta \leq |z|^{r-1} = |\Phi(z)| \quad \text{for all } z \in \mathbb{R} \text{ with } |z| \geq 1,$$

we find that assumption $(\mathrm{IV})_3'$ in Remark 4.3 holds with

$$H = 1, \quad d_3 = 1 \quad \overline{\alpha} = 1.$$

Finally, f_2 is homogeneous of degree $d = \theta$.

Now, in order to prove the validity of (4.3), take $m > 1$ satisfying

$$m > 2(r-1)\max\left\{1, \frac{4}{\theta}\right\}.$$

By definition

$$K_M = \sup_{\mathbb{R}_0^+ \times [1,\nu_2]} \frac{|\sin t|^{1/\overline{p}} + \sin^2 y}{2} \leq 1,$$

so that, by (4.25)–(4.26), and recalling that $\rho = r - 1$, it is

$$\frac{c_1 d_1}{K_M^d} \geq c_1 d_1 = \frac{m}{2} > r - 1 = \rho.$$

On the other hand, using again (4.25)–(4.26), we find

$$\gamma\frac{c_1 d_1}{K_M^d} \geq \gamma c_1 d_1 = \frac{\theta}{r-1} \cdot \frac{m}{2} > 4 = 1 + \delta.$$

Choosing σ arbitrarily close to 1 we obtain (4.3).

In conclusion, by Theorem 3.9 there exists a solution $u \in C^1(\mathbb{R}_0^+; \mathbb{R}) \cap W^{2,p}_{loc}(\mathbb{R}_0^+)$ of (4.22).

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