Energy conservation and conditional regularity for the incompressible Navier–Stokes–Maxwell system

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Abstract. In this paper, we study a hydrodynamic system modeling the evolution of a plasma subject to a self-induced electromagnetic Lorentz force in incompressible viscous fluids. The system consists of the Navier–Stokes equations coupled with a Maxwell equation. In the three dimensional case, we show that every weak solution verifies the energy equality for the incompressible Navier–Stokes–Maxwell equations with damping. We also establish some non-explosion criteria in terms of the velocity and magnetic of local strong solution for standard Navier–Stokes–Maxwell system.

Keywords: Navier–Stokes–Maxwell system, damping term, energy conservation, non-explosion criteria.

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1 Introduction

In this paper, we are interested in studying the unconditional energy conservation for the weak solutions of Navier–Stokes–Maxwell (NSM for short) equations with damping (or the tamed NSM equations)

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nu |u|^{\alpha-1} u = -\nabla P + j \times B, & \operatorname{div} u = 0, \\ \frac{1}{c} \partial_t E - \nabla \times B = -j, & j = \sigma(cE + u \times B), \\ \frac{1}{c} \partial_t B + \nabla \times E + \nu |B|^{\beta-1} B = 0, & \operatorname{div} B = 0, \end{cases} \tag{1.1}$$

with initial data

$$u(0,x) = u_0, \quad E(0,x) = E_0, \quad B(0,x) = B_0 \tag{1.2}$$

for $(t,x) \in \mathbb{R}^+ \times \Omega$, where Ω is a periodic domain $\mathbb{T}^3$ or whole space $\mathbb{R}^3$. Here $c > 0$ denotes the speed of light, μ and ν denote respectively the positive viscosity and damping coefficients of the fluid, and $\sigma > 0$ is the electrical conductivity. In the above system (1.1),

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$u = (u_1, u_2, u_3) = u(t, x)$ stands for the velocity field of the (incompressible) fluid, while $E = (E_1, E_2, E_3) = E(t, x)$ and $B = (B_1, B_2, B_3) = B(t, x)$ are the electric and magnetic fields, respectively. The scalar function $P = P(t, x)$ is the pressure and is also an unknown. Observe, though, that the electric current $j = j(t, x)$ is not an unknown, for it is fully determined by (u, E, B) through Ohm's law. The exponent α, β can be greater than or equal to 1.

The standard Navier–Stokes–Maxwell system (i.e., $\alpha = \beta = 1$) describes the evolution of a plasma (i.e., a charged fluid) subject to a self-induced electromagnetic Lorentz force $j \times B$.

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + j \times B, & \operatorname{div} u = 0, \\ \frac{1}{c}\partial_t E - \nabla \times B = -j, & j = \sigma(cE + u \times B), \\ \frac{1}{c}\partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0. \end{cases} \tag{1.3}$$

Mathematically, NSM equations is a coupled system, constituted by the parabolic nature of the Navier–Stokes equations from fluid dynamics and the hyperbolic of the Maxwell equation from electromagnetism. Moreover, it can be derived from the Vlasov–Maxwell–Boltzmann system [3]. In the 2D case, Masmoudi [14] prove global existence of regular solutions to the Maxwell–Navier–Stokes system (1.3), and also provide an exponential growth estimate for the H^s norm of the solution when the time goes to infinity. In the 3D case, Ibrahim and Keraani [7] showed the existence of global small mild solutions of NSM system (1.3). Recently, Arsenio and Gallagher [1] have made important and new progress in this direction, they established that global existence of solutions to the 3D system (1.3) holds whenever the initial data tum (u_0, E_0, B_0) is chosen in the natural energy space L^2, while the electromagnetic field (E_0, B_0) alone lies in $\dot{H}^s$, for some given $s \in \left[\frac{1}{2}, \frac{3}{2}\right)$, and is sufficiently small when compared to some non-linear function of the initial energy

$$\mathcal{E}_0 := \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2\right).$$

Kang and Lee [8] showed that the maximal existence time of the local strong solution T^* is finite if and only if

$$\int_0^{T^*} \|u\|_{L^\infty}^2 + \|B\|_{L^\infty}^{\frac{8}{3}} \, dt = \infty.$$

This was improved by Fan-Zhou [4] to be

$$\int_0^{T^*} \|u\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \, dt = \infty.$$

Thereafter, Ma, Jiang and Zhu [12] proved the following three regularity criteria:

- $u \in L^2\left(0, T; L^\infty\left(\mathbb{R}^3\right)\right)$ and $\nabla u \in L^2\left(0, T; L^3\left(\mathbb{R}^3\right)\right)$;

- $u \in L^p\left(0, T; L^q\left(\mathbb{R}^3\right)\right), \frac{2}{p} + \frac{3}{q} = 1, 3 < q \leq \infty$ and $\nabla B \in L^2\left(0, T; L^3\left(\mathbb{R}^3\right)\right)$;

- $\nabla u \in L^p\left(0, T; L^q\left(\mathbb{R}^3\right)\right), \frac{2}{p} + \frac{3}{q} = 2, \frac{3}{2} < q \leqslant \infty$ and $\nabla B \in L^2\left(0, T; L^3\left(\mathbb{R}^3\right)\right)$.

More recently, this result has been improved by Zhang, Pan and iu [20], who proved that if

$$u \in L^{\frac{2}{1-r}}\left(0, T; \dot{B}_{\infty,\infty}^{-r}\left(\mathbb{R}^3\right)\right), \qquad -1 < r < 1$$

and

$$\nabla B \in L^p\left(0, T; L^q\left(\mathbb{R}^3\right)\right), \qquad \frac{2}{p} + \frac{3}{q} = 2 \quad \text{with } 2 \leq q \leq 3,$$

then the strong solution to the NSM system (1.3) can be smoothly extended beyond T. In addition, the energy balance of distributional solutions/Leray–Hopf weak solutions for the NSM system was obtained in [13, 19]. In particular, for a Leray–Hopf weak solution (u, E, B) satisfies

$$u \in L^q\left(0, T; L^p(\Omega)\right), \qquad \frac{1}{q} + \frac{1}{p} \leq \frac{1}{2}, \qquad p \geq 4, \tag{1.4}$$

and

$$B \in L^r\left(0, T; L^s(\Omega)\right), \qquad \frac{1}{r} + \frac{1}{s} \leq \frac{1}{2}, \qquad s \geq 4. \tag{1.5}$$

Then it keeps energy balance.

The damping comes from the resistance to the motion of the flow. It describes various physical phenomena such as porous media flow, drag or friction effects, and some dissipative mechanisms [2]. Recently, Liu, Sun and Xin [11] considered the following 3D NSM system with damping:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + |u|^{\alpha-1} u = -\nabla P + j \times B, & \operatorname{div} u = 0, \\ \partial_t E - \nabla \times B + |E|E = -j, & j = \sigma(E + u \times B), \\ \partial_t B + \nabla \times E + |B|^4 B = 0, & \operatorname{div} B = 0, \end{cases} \tag{1.6}$$

and proved the existence and uniqueness of strong solutions for system (1.6) provided that $\alpha \geq 3$.

When $E = B \equiv 0$, system (1.1) reduces to the Navier–Stokes system with damping(or the tamed Navier–Stokes equations). When viscosity and damping coefficients μ and ν equal to one, Cai and Jiu [2] first established the global existence of strong solutions provided that $\alpha \geq \frac{7}{2}$. Later, it was improved to the $\alpha \geq 3$ by Zhou [21]. Very recently, Hajduk and Robinson in [6] give a simple proof of the existence of global-in-time smooth solutions for tamed Navier–Stokes equations on a 3D periodic domain, for values of the absorption exponent α larger than 3. Furthermore, they prove that global, regular solutions exist also for the critical value of exponent $\alpha = 3$, provided that the coefficients satisfy the relation $4\mu\nu \geq 1$. Additionally, they showed that in the critical case every Leray-Hopf weak solution verifies the energy identity:

$$\|u(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(s)\|_{L^2}^2 \, \mathrm{d}s + 2\nu \int_0^t \|u(s)\|_{L^4}^4 \, \mathrm{d}s = \|u(0)\|_{L^2}^2, \qquad 0 < t < T.$$

To the best of our knowledge, the validity of the energy equality is not to date verified for the NSM equations with damping (1.1) for the range of exponent values $\alpha, \beta \in [1, 3]$. For larger values of the exponent α, β with lower order damping term $|E|E$, it was already shown that the tamed NSM equations (1.6) enjoy existence of global-in-time strong solutions (see proof for whole spaces $\mathbb{R}^3$ in [11]) and hence the energy equality is satisfied. A natural question immediately arises: **does any Leray–Hopf weak solution of the tamed NSM equations (1.1) automatically satisfy the energy balance?** In this work, we will try to answer this question, and show the energy balance to the critical case $\alpha = \beta = 3$. In addition, for the standard NSM system (1.3) (i.e., $\alpha = \beta = 1$ in (1.1)), we will show some blow-up criteria under scaling invariant conditions on gradient of the velocity and the magnetic in some different Banach spaces, including the homogeneous Sobolev space, the weighted L^∞-space and the Morrey space, etc.

In a same fashion with [6], we first state the definition of the Leray–Hopf weak solutions to system (1.1).

Definition 1.1 (Leray–Hopf weak solution). Let $(u_0, E_0, B_0) \in L^2(\Omega)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$, $T > 0$. The function (u, E, B) is said to be a Leray–Hopf weak solution to tamed NSM system(1.1) if

1.
$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \dot{H}^1((\Omega))) \cap L^{\alpha+1}(0, T; L^{\alpha+1}(\Omega))$$
$$B \in L^\infty(0, T; L^2(\Omega)) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))$$
$$E \in L^\infty(0, T; L^2(\Omega)), \quad j \in L^2(0, T; L^2(\Omega));$$

2. for any smooth test function $\varphi \in \mathcal{C}_c^\infty(\Omega \times [0, T))$ and $\nabla \cdot \varphi = 0$, holds that

$$-\int_0^T \int_\Omega u \cdot \partial_t \varphi \, dx dt + \mu \int_0^T \int_\Omega \nabla u : \nabla \varphi \, dx dt + \int_0^T \int_\Omega [u \cdot \nabla u - j \times B + v|u|^{\alpha-1}u] \cdot \varphi \, dx dt$$
$$= \int_\Omega u_0 \cdot \varphi(x, 0) dx,$$

$$-\frac{1}{c} \int_0^T \int_\Omega E \cdot \partial_t \varphi \, dx dt + \int_0^T \int_\Omega B \nabla \times \varphi \, dx dt + \int_0^T \int_\Omega j \cdot \varphi \, dx dt = \int_\Omega E_0 \cdot \varphi(x, 0) dx,$$

and

$$-\frac{1}{c} \int_0^T \int_\Omega B \cdot \partial_t \varphi \, dx dt - \int_0^T \int_\Omega E \nabla \times \varphi \, dx dt + \int_0^T \int_\Omega v|B|^{\beta-1}B \cdot \varphi \, dx dt = \int_\Omega B_0 \cdot \varphi(x, 0) dx,$$

3. for any $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, it holds that

$$\int_\Omega u \cdot \nabla \Phi dx = \int_\Omega B \cdot \nabla \Phi dx = 0$$

a.e. $t \in (0, T)$;

4. (u, E, B) satisfies the energy inequality
$$\frac{1}{2}(\|u(\cdot, t)\|_{L^2}^2 + \|E(\cdot, t)\|_{L^2}^2 + \|B(\cdot, t)\|_{L^2}^2) + \int_0^T (\mu \|\nabla u\|_{L^2}^2 + \frac{1}{\sigma}\|j\|_{L^2}^2) dt$$
$$+ v \int_0^T \|u\|_{L^{\alpha+1}}^{\alpha+1} + c\|B\|_{L^{\beta+1}}^{\beta+1} dt \leq \frac{1}{2} \int_\Omega (|u_0|^2 + |E_0|^2 + |B_0|^2) dx,$$

for all $t \in [0, T]$.

We now make the observation that Leray–Hopf weak solutions of the tamed NSM system (1.1) with $\alpha = \beta = 3$ by Definition 1.1 satisfy the condition (1.4)–(1.5). This suggests that the energy balance holds for all weak solutions of this problem, and we will prove this in the following unconditional energy balance theorem.

Theorem 1.2. *Let $\alpha = \beta = 3$ in tamed NSM system (1.1), then every Leray–Hopf weak solution (u, E, B) with $(u_0, E_0, B_0) \in L^2(\Omega)$ satisfies the energy balance:*

$$\frac{1}{2}(\|u(\cdot, t)\|_{L^2}^2 + \|E(\cdot, t)\|_{L^2}^2 + \|B(\cdot, t)\|_{L^2}^2) + \int_0^T (\mu \|\nabla u\|_{L^2}^2 + \frac{1}{\sigma}\|j\|_{L^2}^2) dt$$
$$+ v \int_0^T \|u\|_{L^4}^4 + c\|B\|_{L^4}^4 dt = \frac{1}{2}(\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2), \tag{1.7}$$

for all $t \in [0, T]$.

Remark 1.3. In [5], the authors approximate functions defined on smooth bounded domains by elements of the eigenspaces of the Laplacian or the Stokes operator in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces. As a direct application, they prove that all weak solutions of the tamed NS equations posed on a bounded domain in $\mathbb{R}^3$ satisfy the energy equality. One may establish similar result to tamed NSM equations (1.1) in smooth bounded domain by the the method developed in [4].

As mentioned in the introduction, we will give several new non-explosion criteria in terms of the velocity and magnetic for standard NSM system on the framework of different Banach spaces. (Definitions of various Banach spaces can be found in Section 3.)

Theorem 1.4. *Let* $(u_0, E_0, B_0) \in H^2(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$*. Assume that* (u, E, B) *be the local strong solution of the standard NSM system* (1.3)*. If*

$$S, \nabla B \in L^2\left(0, T; \mathcal{X}\left(\mathbb{R}^3\right)\right), \tag{1.8}$$

where $\mathcal{X}$ *is one of the Banach spaces:*

- $\mathcal{X} = \dot{H}^{1/2}\left(\mathbb{R}^3\right)$ *(the homogeneous Sobolev space),*

- $\mathcal{X} = L^3\left(\mathbb{R}^3\right)$ *(the Lebesgue space),*

- $\mathcal{X} = \left\{f \in L^\infty_{loc}\left(\mathbb{R}^3\right) : \|f\| = \sup_{x \in \mathcal{X}} |x| \|f(x)| < \infty\right\}$ *(the weighted L^∞-space),*

- $\mathcal{X} = \mathcal{PM}^2\left(\mathbb{R}^3\right)$ *(the Le Jan–Sznitman space),*

- $\mathcal{X} = L^{3,\infty}\left(\mathbb{R}^3\right)$ *(the Marcinkiewicz space),*

- $\mathcal{X} = \dot{M}^3_p\left(\mathbb{R}^3\right)$ *for each* $2 < p \leqslant 3$ *(the Morrey space).*

Then the local strong solution can be smoothly extended beyond T. Here, $S = \nabla_{sym}(u)_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$*.*

Remark 1.5. The result in Theorem 1.4 is more weaker condition to that in [12] or [20]. On the one hand, the deformation tensor S in theorem 1.4 can be replaced ∇u or $\nabla \times u$, on the other hand, we see that the framework of several Banach spaces in theorem 1.4 is more flexible and larger than that of Lebesgue spaces.

2 Proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2. The main idea is to use a Leray–Hopf weak solution as a test function. We cannot do this directly since it is not sufficiently regular in space or time. Therefore, for the sake of simplicity, we will proceed as if the solution is differentiable in time. The extra arguments needed to mollify in time are straightforward. To this end we recall here some standard facts of the theory of mollification and introduce a crucial lemma. The key lemma is as follows which was proved by Lions in [10].

Let us to define $\eta : \mathbb{R}^d \to \mathbb{R}$ be a standard mollifier, i.e. $\eta(x) = Ce^{\frac{1}{|x|^2-1}}$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geqslant 1$, where constant $C > 0$ selected such that $\int_{\mathbb{R}^d} \eta(x)dx = 1$. For any $\varepsilon > 0$, we define the rescaled mollifier $\eta_\varepsilon(x) = \varepsilon^{-d}\eta\left(\frac{x}{\varepsilon}\right)$. For any function $f \in L^1_{loc}(\Omega)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_\Omega \eta_\varepsilon(x-y)f(y)dy.$$

If $f \in W^{1,p}(\Omega)$, the following local approximation is well known

$$f^\varepsilon(x) \to f \quad \text{in } W^{1,p}_{loc}(\Omega) \quad \forall p \in [1, \infty).$$

Lemma 2.1. *Let ∂ be a partial derivative in one direction. Let $f, \partial f \in L^p(\mathbb{R}^+ \times \Omega)$, $g \in L^q(\mathbb{R}^+ \times \Omega)$ with $1 \le p, q \le \infty$, and $\frac{1}{p} + \frac{1}{q} \le 1$. Then, we have*

$$\|\partial(fg) * \eta_\varepsilon - \partial(f(g*\eta_\varepsilon))\|_{L^r(\mathbb{R}^+\times\Omega)} \le C\|\partial f\|_{L^p(\mathbb{R}^+\times\Omega)}\|g\|_{L^q(\mathbb{R}^+\times\Omega)}$$

for some constant $C > 0$ independent of ε, f and g, and with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$\partial(fg) * \eta_\varepsilon - \partial(f(g*\eta_\varepsilon)) \to 0 \quad \text{in } L^r(\mathbb{R}^+ \times \Omega)$$

as $\varepsilon \to 0$, if $r < \infty$.

Proof of Theorem 1.2. First, testing system $(1.1)_{1,2,3}$ by $(u^\varepsilon)^\varepsilon$, $(E^\varepsilon)^\varepsilon$ and $(B^\varepsilon)^\varepsilon$, respectively, we infer that

$$\begin{cases} \int_\Omega u^\varepsilon \left(\partial_t u + u \cdot \nabla u - \mu\Delta u + \nu|u|^2 u + \nabla P - j \times B\right)^\varepsilon dx = 0, \\ \int_\Omega E^\varepsilon \left(\partial_t E - c\nabla \times B + cj\right)^\varepsilon dx = 0, \\ \int_\Omega B^\varepsilon \left(\partial_t B + c\nabla \times E + c\nu|B|^2 B\right)^\varepsilon dx = 0. \end{cases} \tag{2.1}$$

Moreover, it yields that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|u^\varepsilon|^2 + |E^\varepsilon|^2 + |B^\varepsilon|^2\right) dx + \mu\int_\Omega |\nabla u^\varepsilon|^2 dx \\ &+ \nu\int_\Omega \left((|u|^2 u)^\varepsilon \cdot u^\varepsilon + c(|B|^2 B)^\varepsilon \cdot B^\varepsilon\right) dx \\ &= -\int_\Omega \text{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx + \int_\Omega (j \times B)^\varepsilon \cdot u^\varepsilon dx + c\int_\Omega (\nabla \times B)^\varepsilon \cdot E^\varepsilon dx \\ &- c\int_\Omega j^\varepsilon \cdot E^\varepsilon dx - c\int_\Omega (\nabla \times E)^\varepsilon \cdot B^\varepsilon dx. \end{aligned} \tag{2.2}$$

Clearly,

$$\begin{aligned} &\int_\Omega \left(|u^\varepsilon|^2 + |E^\varepsilon|^2 + |B^\varepsilon|^2\right) dx - \int_\Omega \left(|u_0^\varepsilon|^2 + |E_0^\varepsilon|^2 + |B_0^\varepsilon|^2\right) dxdt + 2\mu\int_0^T \int_\Omega |\nabla u^\varepsilon|^2 dxdt \\ &+ 2\nu\int_0^T \int_\Omega \left((|u|^2 u)^\varepsilon \cdot u^\varepsilon + c(|B|^2 B)^\varepsilon \cdot B^\varepsilon\right) dxdt \\ &= -2\int_0^T \int_\Omega \text{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dxdt + 2\int_0^T \int_\Omega (j \times B)^\varepsilon \cdot u^\varepsilon dxdt \\ &+ 2c\int_0^T \int_\Omega (\nabla \times B)^\varepsilon \cdot E^\varepsilon dxdt - 2c\int_0^T \int_\Omega j^\varepsilon \cdot E^\varepsilon dxdt \\ &- 2c\int_0^T \int_\Omega (\nabla \times E)^\varepsilon \cdot B^\varepsilon dxdt \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{2.3}$$

We want to pass to the limit in (2.3) as $\varepsilon \to 0$. To this end, using Hölder's inequality, we

observe the following estimates for the nonlinear terms:

$$
\left| \int_0^T \int_\Omega (|u|^2 u)^\varepsilon \cdot u^\varepsilon \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega |u|^2 u \cdot u \mathrm{d}x\mathrm{d}t \right|
$$
$$
\leq \left| \int_0^T \int_\Omega (|u|^2 u)^\varepsilon - |u|^2 u) \cdot u^\varepsilon + |u|^2 u (u^\varepsilon - u) \mathrm{d}x\mathrm{d}t \right|
$$
$$
\leq \|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))} \left\| (|u|^2 u)^\varepsilon - |u|^2 u) \right\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)}
$$
$$
+ \|u^\varepsilon - u\|_{L^4(0,T;L^4(\Omega))} \left\| |u|^2 u \right\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.4}
$$

Similarly,

$$
\left| \int_0^T \int_\Omega (|B|^2 B)^\varepsilon \cdot B^\varepsilon \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega |B|^2 B \cdot u \mathrm{d}x\mathrm{d}t \right| \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.5}
$$

In addition, since

$$
\mathrm{div}\,(u \otimes u)^\varepsilon = [\mathrm{div}(u \otimes u)^\varepsilon - \mathrm{div}(u \otimes u^\varepsilon)] + [\mathrm{div}(u \otimes u^\varepsilon) - \mathrm{div}(u^\varepsilon \otimes u^\varepsilon)] + \mathrm{div}(u^\varepsilon \otimes u^\varepsilon)
$$
$$
= I_{11} + I_{12} + I_{13},
$$

and so

$$
I_1 = -2 \int_0^T \int_\Omega (I_{11} + I_{12} + I_{13}) \cdot u^\varepsilon \mathrm{d}x\mathrm{d}t. \tag{2.6}
$$

From Lemma 2.1, one obtains

$$
\|I_{11}\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \leq C \|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))} \|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\Omega))}
$$

and it converges to zero in $L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)$ as ε tends to zero. Thus, as ε goes to zero, it follows that

$$
\left| -2 \int_0^T \int_\Omega I_{11} u^\varepsilon \mathrm{d}x\mathrm{d}t \right| = \left| \int_0^T \int_\Omega [\mathrm{div}(u \otimes u)^\varepsilon - \mathrm{div}(u \otimes u^\varepsilon)] \cdot u^\varepsilon \mathrm{d}x\mathrm{d}t \right|
$$
$$
\leq \|I_{11}\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))} \tag{2.7}
$$
$$
\to 0.
$$

Moreover, we have

$$
\left| -2 \int_0^T \int_\Omega I_{12} u^\varepsilon \mathrm{d}x\mathrm{d}t \right| = \left| \int_0^T \int_\Omega [\mathrm{div}(u \otimes u^\varepsilon) - \mathrm{div}(u^\varepsilon \otimes u^\varepsilon)] \cdot u^\varepsilon \mathrm{d}x\mathrm{d}t \right|
$$
$$
= \left| \int_0^T \int_\Omega (u \otimes u^\varepsilon - u^\varepsilon \otimes u^\varepsilon) \cdot \nabla u^\varepsilon \mathrm{d}x\mathrm{d}t \right| \tag{2.8}
$$
$$
\leq \|u - u^\varepsilon\|_{L^4(0,T;L^4(\Omega))} \|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))} \|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\Omega))}
$$
$$
\to 0 \quad \text{as } \varepsilon \to 0.
$$

Since $\mathrm{div}\, u^\varepsilon = 0$, one has

$$
-2 \int_0^T \int_\Omega I_{13} \cdot u^\varepsilon \mathrm{d}x = 0. \tag{2.9}
$$

Combining (2.6)–(2.9), we know that

$$
I_1 \to 0, \quad \text{as } \varepsilon \to 0. \tag{2.10}
$$

For the term I_2, we claim that

$$I_2 = 2\int_0^T \int_\Omega (j \times B)^\varepsilon \cdot u^\varepsilon \mathrm{d}x\mathrm{d}t \to 2\int_0^T \int_\Omega (j \times B) \cdot u \mathrm{d}x\mathrm{d}t, \quad \text{as } \varepsilon \to 0. \tag{2.11}$$

Indeed,

$$
\begin{aligned}
2\int_0^T \int_\Omega &(j \times B)^\varepsilon \cdot u^\varepsilon - (j \times B) \cdot u \mathrm{d}x\mathrm{d}t \\
&= 2\int_0^T \int_\Omega (j \times B)^\varepsilon \cdot u^\varepsilon - (j \times B) \cdot u^\varepsilon + (j \times B) \cdot u^\varepsilon - (j \times B) \cdot u \mathrm{d}x\mathrm{d}t \\
&= 2\int_0^T \int_\Omega [(j \times B)^\varepsilon - (j \times B)] \cdot u^\varepsilon + (j \times B) \cdot (u^\varepsilon - u)\, \mathrm{d}x\mathrm{d}t \\
&\leq 2\, \|(j \times B)^\varepsilon - (j \times B)\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))} \\
&\quad + 2\, \|j \times B\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \|u^\varepsilon - u\|_{L^4(0,T;L^4(\Omega))} \\
&\to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned}
\tag{2.12}
$$

where we used fact that

$$\|j \times B\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \leq \|j\|_{L^2(0,T;L^2(\Omega))} \|B\|_{L^4(0,T;L^4(\Omega))} \leq C.$$

By using the same trick, we find that

$$I_4 = -2c\int_0^T \int_\Omega j^\varepsilon \cdot E^\varepsilon \mathrm{d}x\mathrm{d}t \to -2c\int_0^T \int_\Omega j \cdot E \mathrm{d}x\mathrm{d}t, \quad \text{as } \varepsilon \to 0. \tag{2.13}$$

After integration by part, the term I_3 can be dominated as

$$
\begin{aligned}
I_3 &= 2c\int_0^T \int_\Omega (\nabla \times B)^\varepsilon \cdot E^\varepsilon \mathrm{d}x\mathrm{d}t \\
&= 2c\int_0^T \int_\Omega \left(\epsilon_{ijk}\partial_j B_k\right)^\varepsilon \cdot E_i^\varepsilon \mathrm{d}x\mathrm{d}t \\
&= -2c\int_0^T \int_\Omega \epsilon_{ijk} B_k^\varepsilon \cdot \partial_j E_i^\varepsilon \mathrm{d}x\mathrm{d}t \\
&= 2c\int_0^T \int_\Omega \epsilon_{kji} B_k^\varepsilon \cdot \partial_j E_i^\varepsilon \mathrm{d}x\mathrm{d}t \\
&= 2c\int_0^T \int_\Omega B^\varepsilon \cdot (\nabla \times E)^\varepsilon \mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{2.14}
$$

So, it follows that

$$I_3 + I_5 = 0. \tag{2.15}$$

Letting ε goes to zero in (2.3), and using the facts (2.4)–(2.5) and (2.10)–(2.15), we infer that

$$
\begin{aligned}
(\|u(\cdot,t)\|_{L^2}^2 + \|E(\cdot,t)\|_{L^2}^2 &+ \|B(\cdot,t)\|_{L^2}^2) + 2\int_0^T \left(\mu\|\nabla u\|_{L^2}^2 + \frac{1}{\sigma}\|j\|_{L^2}^2\right) dt \\
&+ 2\nu \int_0^T \|u\|_{L^4}^4 + c\|B\|_{L^4}^4 dt = (\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2),
\end{aligned}
\tag{2.16}
$$

where we have used the facts that

$$2 \int_0^T \int_\Omega (j \times B) \cdot u \, dx dt - 2c \int_0^T \int_\Omega j \cdot E \, dx dt = -2 \int_\Omega (u \times B) \cdot j \, dx - 2c \int_\Omega E \cdot j \, dx$$

and

$$j = \sigma(cE + u \times B).$$

The energy equality (1.7) follows easily from (2.16). □

3 Proof of Theorem 1.4

In this section, we first introduce several definitions used throughout proof of theorem 1.4 and also recall the well-known results for our analysis (see e.g. [9, 16, 17] and [18]). And then show the derivation details of theorem 1.4.

Definition 3.1. The weighted L^∞-space is defined by

$$\text{Weighted } L^\infty = \left\{ f \in L^\infty_{loc}\left(\mathbb{R}^3\right) : \|f\|_{\mathcal{X}} = \sup_{x \in \mathcal{X}} |x| |f(x)| < \infty \right\}.$$

Definition 3.2. The Le Jan–Sznitman space is defined by

$$\mathcal{PM}^2 = \left\{ v \in \mathcal{S}'\left(\mathbb{R}^3\right) : \widehat{v} \in L^1_{loc}\left(\mathbb{R}^3\right), \quad \|v\|_{\mathcal{PM}^2} = \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^3} |\xi|^2 |\widehat{v}(\xi)| < \infty \right\}.$$

Definition 3.3. Let $1 < p \leq q < \infty$, the homogeneous Morrey spaces are defined as

$$\dot{M}_p^q\left(\mathbb{R}^3\right) = \left\{ f \in L^q_{loc}\left(\mathbb{R}^3\right) : \|f\|_{\dot{M}_p^q} = \sup_{R > 0} \sup_{x \in \mathbb{R}^3} R^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \left(\int_{B_R(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty \right\}.$$

Lemma 3.4 ([15]). *For all $-\frac{3}{2} < \alpha < \frac{3}{2}$ and for all u divergence free in the sense that $\xi \cdot \hat{u}(\xi) = 0$ almost everywhere,*

$$\|S\|_{\dot{H}^\alpha}^2 = \|A\|_{\dot{H}^\alpha}^2 = \frac{1}{2} \|\omega\|_{\dot{H}^\alpha}^2 = \frac{1}{2} \|\nabla \otimes u\|_{\dot{H}^\alpha}^2, \tag{3.1}$$

where symmetric part $S = S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$, which we refer to as the strain tensor, anti-symmetric part $A = A_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$, $\omega = \nabla \times u$.

Lemma 3.5 (Hardy-type inequalities [9]). *There exists a constant $K > 0$ such that the following inequality holds true*

$$\left| \int_{\mathbb{R}^3} W \cdot (g \cdot \nabla) h \, dx \right| \leqslant K \|W\|_{\mathcal{X}_\sigma} \|\nabla g\|_{L^2} \|\nabla h\|_{L^2},$$

for all vector fields $g, h \in \dot{H}^1\left(\mathbb{R}^3\right)$ and all $W \in \mathcal{X}_\sigma$, where $\mathcal{X}_\sigma$ (the subspace of $\mathcal{X}$ of divergence-free vector functions) is one of the Banach spaces

- $\mathcal{X}_\sigma = \dot{H}_\sigma^{1/2}\left(\mathbb{R}^3\right)$ *(the homogeneous Sobolev space),*
- $\mathcal{X}_\sigma = L_\sigma^3\left(\mathbb{R}^3\right)$ *(the Lebesgue space),*

- $\mathcal{X}_\sigma = \left\{ f \in L^\infty_{loc} \left(\mathbb{R}^3 \right) : \|f\| = \sup_{x \in \mathcal{X}_\sigma} |x| |f(x)| < \infty \right\}$ *(the weighted L^∞-space),*

- $\mathcal{X}_\sigma = \mathcal{PM}^2 \left(\mathbb{R}^3 \right)$ *(the Le Jan–Sznitman space),*

- $\mathcal{X}_\sigma = L^{3,\infty}_\sigma \left(\mathbb{R}^3 \right)$ *(the Marcinkiewicz space),*

- $\mathcal{X}_\sigma = \dot{M}^3_p \left(\mathbb{R}^3 \right)$ *for each $2 < p \leqslant 3$ (the Morrey space).*

The following proposition is an ad hoc variant of the preceding lemma 3.5. It will be more applicable to proving theorem 1.4. This proposition is inspired by the details of the proof from lemma 3.5.

Proposition 3.6. *There exists a constant $C > 0$ such that the following inequality holds true*

$$\left| \int_{\mathbb{R}^3} Wgh \, \mathrm{d}x \right| \leqslant C \|W\|_{\mathcal{X}} \|\nabla g\|_{L^2} \|h\|_{L^2},$$

for all vector fields $g \in \dot{H}^1 \left(\mathbb{R}^3 \right)$, $h \in L^2 \left(\mathbb{R}^3 \right)$ and all $W \in \mathcal{X}$, where $\mathcal{X}$ is one of the Banach spaces

- $\mathcal{X} = \dot{H}^{1/2} \left(\mathbb{R}^3 \right)$ *(the homogeneous Sobolev space),*

- $\mathcal{X} = L^3 \left(\mathbb{R}^3 \right)$ *(the Lebesgue space),*

- $\mathcal{X} = \left\{ f \in L^\infty_{loc} \left(\mathbb{R}^3 \right) : \|f\| = \sup_{x \in \mathcal{X}} |x| |f(x)| < \infty \right\}$ *(the weighted L^∞-space),*

- $\mathcal{X} = \mathcal{PM}^2 \left(\mathbb{R}^3 \right)$ *(the Le Jan–Sznitman space),*

- $\mathcal{X} = L^{3,\infty} \left(\mathbb{R}^3 \right)$ *(the Marcinkiewicz space),*

- $\mathcal{X} = \dot{M}^3_p \left(\mathbb{R}^3 \right)$ *for each $2 < p \leqslant 3$ (the Morrey space).*

Proof of Theorem 1.4. We now shall prove Theorem 1.4, we only need to establish a priori estimates. Without loss of generality, we may assume $c = \sigma = 1$ in system (1.3).

Step 1: Energy estimates. Testing $(1.3)_{1,2,3}$ by u, E, B respectively, and adding up the results, we have the well-known energy equality

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|(u,E,B)\|^2_{L^2} &+ \|\nabla u\|^2_{L^2} \\
&= \int_{\mathbb{R}^3} (j \times B) \cdot u \mathrm{d}x + \int_{\mathbb{R}^3} \mathrm{curl}\, B \cdot E \mathrm{d}x - \int_{\mathbb{R}^3} j \cdot E \mathrm{d}x - \int_{\mathbb{R}^3} \mathrm{curl}\, E \cdot B \mathrm{d}x \\
&= - \int_{\mathbb{R}^3} (u \times B) \cdot j \mathrm{d}x - \int_{\mathbb{R}^3} E \cdot j \mathrm{d}x \\
&= - \|j\|^2_{L^2},
\end{aligned}$$

where we have used Ohm's law: $j = (E + u \times B)$. Which gives

$$\|(u,E,B)(t)\|^2_{L^2} + 2 \int_0^T \|(\nabla u, j)(t)\|^2_{L^2} \mathrm{d}t = \|(u_0, E_0, B_0)\|^2_{L^2}, \quad \forall t \geq 0.$$

Step 2: $\dot{H}^1$ estimates. First, taking $\nabla \times$ on the first equation of (1.3), we get

$$\partial_t \omega + (u \cdot \nabla)\omega - \Delta \omega - S\omega = \nabla \times (j \times B), \tag{3.2}$$

the vortex stretching term $S\omega$ is often written $(\omega \cdot \nabla)u$, that is to say $S\omega = (\omega \cdot \nabla)u$. Indeed, the symmetric part S is given by

$$S_{ij} = \nabla_{sym}(u)_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

and the anti-symmetric part A is given by

$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).$$

Naturally, $\nabla u = S + A$. In addition, we know that in three spatial dimensions the anti-symmetric matrix A can be represented as a vector. Here, we write as:

$$A = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

Then we have $A\omega = 0$.

Next, multiplying (3.2) by ω and integrating by parts over $\mathbb{R}^3$, one has

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 = \int_{\mathbb{R}^3} S\omega \cdot \omega dx + \int_{\mathbb{R}^3} \operatorname{curl}(j \times B) \cdot \omega dx$$
$$= \int_{\mathbb{R}^3} S\omega \cdot \omega dx - \int_{\mathbb{R}^3} (j \times B) \cdot \Delta u dx. \tag{3.3}$$

Testing (1.3)$_{2,3}$ by $-\Delta E, -\Delta B$ in $L^2(\mathbb{R}^3)$ respectively, and putting together, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\nabla E, \nabla B)\|_{L^2}^2 = -\int_{\mathbb{R}^3} \operatorname{curl} B \cdot \Delta E dx + \int_{\mathbb{R}^3} j \cdot \Delta E dx + \int_{\mathbb{R}^3} \operatorname{curl} E \cdot \Delta B dx$$
$$= -\int_{\mathbb{R}^3} \partial_i j \cdot \partial_i E dx. \tag{3.4}$$

Since

$$-\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i j \cdot \partial_i E dx = -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i j \cdot \partial_i (j - u \times B) dx$$
$$= -\|\nabla j\|_{L^2}^2 + \int_{\mathbb{R}^3} \partial_i (u \times B) \cdot \partial_i j dx. \tag{3.5}$$

From the equalities in (3.3)–(3.5), it follows that

$$\frac{1}{2}\frac{d}{dt}\|(\omega, \nabla E, \nabla B)\|_{L^2}^2 + \|(\nabla\omega, \nabla j)\|_{L^2}^2$$
$$= \int_{\mathbb{R}^3} S\omega \cdot \omega dx + \int_{\mathbb{R}^3} \partial_i (j \times B) \cdot \partial_i u dx + \int_{\mathbb{R}^3} \partial_i (u \times B) \cdot \partial_i j dx$$
$$\leq \int_{\mathbb{R}^3} S\omega \cdot \omega dx + \int_{\mathbb{R}^3} |j||\nabla u||\nabla B| dx + \int_{\mathbb{R}^3} |u||\nabla B||\nabla j| dx$$
$$= I + J + K, \tag{3.6}$$

where we have used the following cancellation property

$$\int_{\mathbb{R}^3} \partial_i j \times B \cdot \partial_i u dx + \int_{\mathbb{R}^3} \partial_i j \cdot \partial_i u \times B dx = 0.$$

Let (1.8) hold true. We use proposition 3.6 and lemma 3.4 to bound I as follows

$$
\begin{aligned}
I &\le C\|S\|_{\mathcal{X}}\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}\\
&\le C\,\|S\|_{\mathcal{X}}^2\,\|\omega\|_{L^2}^2 + \epsilon\|\nabla\omega\|_{L^2}^2,
\end{aligned}
\tag{3.7}
$$

Similarly, we bound $J + K$ as follows

$$
J + K \le 2\epsilon\,\|\nabla j\|_{L^2}^2 + C\,\|\nabla B\|_{\mathcal{X}}^2\,\|\nabla u\|_{L^2}^2.
\tag{3.8}
$$

Where $\mathcal{X}$ is one of the Banach spaces:

- $\mathcal{X} = \dot{H}^{1/2}\left(\mathbb{R}^3\right)$ (the homogeneous Sobolev space),

- $\mathcal{X} = L^3\left(\mathbb{R}^3\right)$ (the Lebesgue space),

- $\mathcal{X} = \{f \in L^\infty_{loc}\left(\mathbb{R}^3\right) : \|f\| = \sup_{x \in \mathcal{X}}|x|\|f(x)| < \infty\}$ (the weighted L^∞-space),

- $\mathcal{X} = \mathcal{PM}^2\left(\mathbb{R}^3\right)$ (the Le Jan–Sznitman space),

- $\mathcal{X} = L^{3,\infty}\left(\mathbb{R}^3\right)$ (the Marcinkiewicz space),

- $\mathcal{X} = \dot{M}_p^3\left(\mathbb{R}^3\right)$ for each $2 < p \le 3$ (the Morrey space).

Applying the entropy identity (i.e., Lemma 3.4) and collecting (3.7) and (3.8) into (3.6), we find

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|(\omega, \nabla E, \nabla B)\|_{L^2}^2 + \|(\nabla\omega, \nabla j)\|_{L^2}^2 \le C\left(\|S\|_{\mathcal{X}}^2 + \|\nabla B\|_{\mathcal{X}}^2\right)\|\omega\|_{L^2}^2.
\tag{3.9}
$$

Using the Gronwall inequality, we conclude that

$$
\sup_{0 \le t \le T}\int |\nabla u|^2 + |\nabla E|^2 + |\nabla B|^2 \mathrm{d}x \le C
$$

and

$$
\int_0^T \|\Delta u\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 \mathrm{d}t \le C.
$$

Step 3: $\dot{H}^2$ estimates. Applying Δ to $(1.3)_{1,2,3}$ and multiplying by Δu, ΔE and ΔB, respectively, and add the result equations, one has

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}&\|\left(\Delta u, \Delta B, \Delta E\right)\|_{L^2}^2 + \|\left(\nabla\Delta u, \Delta j\right)\|_{L^2}^2\\
&= \int_{\mathbb{R}^3} -\Delta(u \cdot \nabla u)\cdot \Delta u + \Delta(j \times B)\cdot \Delta u + \Delta j \cdot \Delta(u \times B)\mathrm{d}x\\
&= C\left(\|\nabla u\|_{L^3}^2 + 1\right)\|\Delta u\|_{L^2}^2 + \epsilon\|\nabla\Delta u\|_{L^2}^2\\
&\quad + \int_{\mathbb{R}^3}\partial_{kk}j \times B \cdot \Delta u + \partial_k j \times \partial_k B \cdot \Delta u + j \times \partial_{kk}B \cdot \Delta u\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^3}\Delta j \cdot \partial_{kk}u \times B + \Delta j \cdot \partial_k u \times \partial_k B + \Delta j \cdot u \times \partial_{kk}B\,\mathrm{d}x\\
&= C\left(\|\nabla u\|_{L^3}^2 + 1\right)\|\Delta u\|_{L^2}^2 + \epsilon\|\nabla\Delta u\|_{L^2}^2 + \underbrace{\int_{\mathbb{R}^3}\partial_k j \times \partial_k B \cdot \Delta u + j \times \partial_{kk}B \cdot \Delta u\,\mathrm{d}x}_{I}\\
&\quad + \underbrace{\int_{\mathbb{R}^3}\Delta j \cdot \partial_k u \times \partial_k B + \Delta j \cdot u \times \partial_{kk}B\,\mathrm{d}x}_{J},
\end{aligned}
\tag{3.10}
$$

where used the following cancellation property:

$$\int_{\mathbb{R}^3} \partial_{kk} j \times B \cdot \Delta u \mathrm{d}x + \int_{\mathbb{R}^3} \Delta j \cdot \partial_{kk} u \times B \mathrm{d}x = 0.$$

I can be estimated as

$$
\begin{aligned}
I &\le C\|\Delta u\|_{L^2}\|\nabla B\|_{L^3}\|\nabla j\|_{L^6} + C\|j\|_{L^3}\|\Delta B\|_{L^2}\|\Delta u\|_{L^6} \\
&\le C\|\Delta u\|_{L^2}^2 \left(\|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right) + C\left(\|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right)\|\Delta B\|_{L^2}^2 + \epsilon\|\Delta j\|_{L^2} + \epsilon\|\nabla \Delta u\|_{L^2}^2
\end{aligned}
\tag{3.11}
$$

Similarly,

$$J \le C\left(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right)\|\Delta B\|_{L^2}^2 + C\left(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right)\|\Delta B\|_{L^2}^2 + 2\epsilon\|\Delta j\|_{L^2}^2 \tag{3.12}$$

Putting (3.11) and (3.12) into (3.10), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\| \left(\Delta u, \Delta B, \Delta E\right) \|_{L^2}^2 + \| \left(\nabla \Delta u, \Delta j\right) \|_{L^2}^2 \\
\le C\left(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right)\| \left(\Delta u, \Delta B, \Delta E\right) \|_{L^2}^2,
\end{aligned}
\tag{3.13}
$$

and by Gronwall's lemma, we get that

$$
\begin{aligned}
u, B, E &\in L^\infty\left(0, T, \dot{H}^2\right); \\
\nabla u, j &\in L^2\left(0, T, \dot{H}^2\right).
\end{aligned}
$$

This completes the proof of Theorem 1.4. □

Acknowledgements

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