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Analysis of stochastic SEIR(S) models with random total populations and variable diffusion rates

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Abstract. A stochastic SEIR(S) model with random total population, overall saturation constant $K > 0$ and general, local Lipschitz continuous diffusion rates is presented. We prove the existence of unique, Markovian, continuous time solutions w.r.t. filtered, complete probability spaces on certain, bounded 4D prisms. The total population $N(t)$ is governed by kind of stochastic logistic equations, which allows to have an asymptotically stable maximum population constant $K > 0$. Under natural conditions on our SEIR(S) model, we establish asymptotic stochastic and moment stability of the disease-free and endemic equilibria. Those conditions naturally depend on the basic reproduction number $\mathcal{R}_0$, the growth parameter $\mu > 0$ and environmental noise intensity σ_5^2 coupled with the maximum threshold K^2 of total population $N(t)$. For the mathematical proofs, the technique of appropriate Lyapunov functionals $V(S(t), E(t), I(t), R(t))$ is exploited. Some numerical simulations of the expected Lyapunov functionals $\mathbb{E}[V(S, E, I, R)]$ depending on several parameters and time t support our findings.

Keywords: stochastic SEIR(S) model, stochastic differential equations, random transition functions, variable diffusion rates, Lyapunov functionals, asymptotic stability.

2020 Mathematics Subject Classification: 34F05, 60H10, 92C42, 92D25, 92D30.

1 Introduction to stochastic SEIR(S) model based on SDEs

Research on epidemic modeling has gone quite far since the seminal contributions of Kermack and McKendrick [15]. The random, erratic nature of evolution of populations forces us to incorporate stochastic terms in modeling and analysis. For modeling of diseases with

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sensitive, randomly fluctuating transmissions such as COVID, here we suggest to make use of and analyze the stochastic SEIR(S) models based on Itô-interpreted SDEs

$$\begin{aligned}
dS &= \Big(-\beta SI + \mu(K-S) + \alpha I + \zeta R\Big)dt - \sigma_1 SI \cdot F_1(S,E,I,R)dW_1 \\
&\quad + \sigma_4 R \cdot F_4(S,E,I,R)dW_4 + \sigma_5 S(K-N)dW_5 \\
dE &= \Big(\beta SI - (\mu+\eta)E\Big)dt + \sigma_1 SI \cdot F_1(S,E,I,R)dW_1 - \sigma_2 E \cdot F_2(S,E,I,R)dW_2 \\
&\quad + \sigma_5 E(K-N)dW_5 \\
dI &= \Big(\eta E - (\alpha+\gamma+\mu)I\Big)dt + \sigma_2 E \cdot F_2(S,E,I,R)dW_2 - \sigma_3 I \cdot F_3(S,E,I,R)dW_3 \\
&\quad + \sigma_5 I(K-N)dW_5 \\
dR &= \Big(\gamma I - (\mu+\zeta)R\Big)dt + \sigma_3 I \cdot F_3(S,E,I,R)dW_3 - \sigma_4 R \cdot F_4(S,E,I,R)dW_4 \\
&\quad + \sigma_5 R(K-N)dW_5,
\end{aligned} \tag{1.1}$$

driven by independent, standard Wiener processes $W_k = (W_k(t))_{t\geq 0}$ on a complete, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the total initial population (note that they can be supposed to be nonrandom since start values are known from real-time data)

$$0 < N(0) := S(0) + E(0) + I(0) + R(0) < K$$

with nonrandom constant $K > 0$ of maximum possible threshold for total population. These models (1.1) are stochastic generalizations of deterministic counterparts in mathematical epidemiology (cf. [21,22]). For an introduction to mathematical models in population biology and epidemiology, see the textbooks [2,5,6]. In biological modeling Itô calculus has to be used since the dynamics of offsprings can only depend on its past, parental generations. For an overview on the theory of Itô-interpreted stochastic differential equations (SDEs), see [1,3,10,11,14,23,24,33] for stochastic calculus with Wiener processes. Deterministic model variants of SEIR(S), SI, SIR, SIS, etc. are well-understood nowadays. The construction and analysis of dynamics along Lyapunov functions plays a key role in understanding those models, cf. [7,9,13,17–19]. This is also the case with stochastic settings, cf. [12,30–32,34,36]. Extinction, ergodicity, stability and recurrence of some random SEIR(S) models with constant or absent F_k are studied in [35,37–39], restricted to unbounded cones $\mathbb{R}^d_+$. Our models (1.1) allow all solutions to live exclusively a.s. on bounded prisms of $\mathbb{R}^4$ or $\mathbb{R}^5$, resp., which represents a real requirement for biologically relevant application (due to finite resources in real life of organisms).

To the best of our knowledge, the class of SEIR(S) models (1.1) is fairly new to the literature. Our model focuses on the possible sensitivity of diseases to random transitions between compartments S of susceptible, E of exposed, I of infected, and R of recovering sub-populations, which are controlled by noise intensity functions $\sigma_k F_k$ in a fairly general manner. Those random transitions can be interpreted as random perturbations of the incidence terms βSI (i.e. **direct contact terms**), motivated by the CLT (=Central Limit Theorem, cf. Shiryaev [33]). Moreover, we allow a possible return of a share of the recovered sub-populations R to the susceptible ones as an expression for the possible loss of immunity w.r.t. the modelled disease-type, represented by the parameter ζ, and a possible switch of the infected sub-populations I to the susceptible ones, represented by the parameter α. The parameter γ stands for the rate of transitions from the infected to the recovering sub-populations. Our main focus in this paper is to verify several qualitative properties such as the boundedness and stability of all dynamics of (1.1) on certain positive prisms - properties which relate to biologically relevant models in order to to be able to replicate real scenarios.

We shall show that the new SEIR(S) model (1.1) is well-defined ($\mathbb{P}$-a.s.) on 4D prism

$$\mathbb{D} = \Big\{(S, E, I, R) \in \mathbb{R}_+^4 \,:\, 0 < S, E, I, R < K,\ S + E + I + R < K\Big\}.$$

For this purpose, first we shall analyze the total population $N(t)$ of the SEIR(S) model (1.1) at time t, which is defined by

$$N(t) = S(t) + E(t) + I(t) + R(t).$$

The understanding of their dynamics plays a crucial role in establishing qualitative properties of the solutions of SDEs (1.1). By summing up all equations of SEIR(S) model (1.1), the SDE for the total population $N = S + E + I + R$ is found to be of the form

$$dN(t) = \mu(K - N)\,dt + \sigma_5 N(K - N)\,dW_5 \tag{1.2}$$

on natural domain $\mathbb{D}_0 = \big(0, K\big)$, which can be treated in a separated fashion from the original system (1.1).

The paper is organized as follows. Section 2 studies the boundedness of the total population N and proves the existence of strong solutions of SDE (1.2) on open domain $\mathbb{D}_0 = (0, K)$ for all times $t \geq 0$. Section 3 is devoted to establish the existence of unique, Markovian, continuous time, strong solutions of the SEIR(S) model (1.1) on certain, positive 4D prisms. There we present the two types of equilibrium solutions, namely the disease-free and the endemic ones. Section 4 investigates stochastic stability of the disease-free equilibrium and the endemic equilibrium of SEIR(S) model (1.1). As usual, the associated basic reproduction number decides in which stable state the system is in (in the long-term sense). Moreover, we also discuss the moment and stochastic stability of its saturation equilibrium $n^* = K$ for the SDE (1.2) of the total populations. Finally, Section 5 is reporting on some graphical illustrations of simulation results related to the associated mean Lyapunov functionals depending on diverse parameters. Section 6 concludes the paper with a brief summary and outlook. An appendix recalls a general standard result on the existence of bounded, unique solutions of systems of Itô SDEs and the structure of associated infinitesimal generator, which plays a key role in our studies.

2 Existence of bounded, unique solution of (1.2) on $\mathbb{D}_0 = (0, K)$

The proof of existence of global, unique solutions of nonlinear SDE (1.2) is far from trivial, due to the quadratic nonlinearity in its diffusion term. For the sake of abbreviation, take $\sigma = \sigma_5$. Let $N(t_0) = N_0 \in \mathbb{D}_0$ with $\mathbb{D}_0 = (0, K)$ and $\mathbb{D}_r = \left(\frac{1}{r}, K - \frac{1}{r}\right), r > 1/K$. Now, consider the events $[N(t) = n]$. Define

$$n \in \mathbb{D}_0 \quad \mapsto \quad V(n) := c - \ln\Big(n(K - n)\Big) = c - \ln(n) - \ln(K - n).$$

Choose c sufficiently large such that $V \geq 0$ on $\mathbb{D}_0$. e.g. $c = \ln\left(\frac{K^2}{4}\right)$. The infinitesimal generator $\mathcal{L}$ of SDE (1.2) applied to the function V (see the general formula (A.2) in appendix) takes the form

$$\forall n \in \mathbb{D}_0: \quad \mathcal{L}V(n) = \mu(K-n)\left(\frac{-1}{n} + \frac{1}{K-n}\right) + \frac{1}{2}\sigma^2 n^2 (K-n)^2 \left[\frac{1}{n^2} + \frac{1}{(K-n)^2}\right]$$
$$= -\mu\left(\frac{K-n}{n}\right) + \mu + \frac{1}{2}\sigma^2 (K-n)^2 + \frac{1}{2}\sigma^2 n^2.$$
$$\Longrightarrow \quad \mathcal{L}V(n) \overset{\mu>0}{\leq} \mu + \frac{1}{2}\sigma^2(K-n)^2 + \frac{1}{2}\sigma^2 n^2 = \mu + \frac{1}{2}\sigma^2\Big[(K-n)^2 + n^2\Big]$$
$$< \mu + \frac{1}{2}\sigma^2 K^2 =: c_0 \quad \text{since} \quad g(n) := (K-n)^2 + n^2 < K^2 \quad \text{on } \mathbb{D}_0.$$

Hence, by Dynkin's formula (1965) (cf. Dynkin [8]), we arrive at

$$\forall t \geq 0: \quad \mathbb{E}\Big[V\Big(N(t)\Big)\Big] = \mathbb{E}\Big[V\Big(N(0)\Big)\Big] + \mathbb{E}\Big[\int_0^t \mathcal{L}V(s)\,ds\Big] \leq \mathbb{E}\Big[V\Big(N(0)\Big)\Big] + c_0 \cdot t \; < \; +\infty$$

with constant $c_0 = \mu + \sigma^2 K^2/2$. Obviously, we have

$$\lim_{r\to+\infty} \inf_{t\geq 0, n\in\partial\mathbb{D}_r} V(n) = \lim_{r\to+\infty} \min\left(V\Big(\frac{1}{r}\Big), V\Big(K - \frac{1}{r}\Big)\right) = c - \lim_{r\to+\infty} \ln\left(\frac{1}{r}\Big(K - \frac{1}{r}\Big)\right) = +\infty.$$

By the remark below Theorem A.1, there exists exactly one strong, global, continuous time, unique Markovian solution $N = (N(t))_{t\geq 0}$ of SDE (1.2) with $N(t) \in \mathbb{D}_0 = (0, K)$ (a.s.) for all $t \geq 0$. This gives the positivity of N (a.s.) and boundedness $N(t) < K$ (a.s.). Of course, the equilibrium $n^* = K$ represents a solution itself (i.e. the trivial solution). Consequently, we verified the following theorem.

Theorem 2.1 (Solvability and boundedness of total population SDE (1.2))**.** *Assume that either $N(0) = K$ or $N(0) \in (0, K)$ (a.s.) is independent of sigma-algebra $\sigma(W) = \sigma(W(t) : t \geq 0)$ with*

$$\mathbb{E}\Big[\ln\Big(N(0)(K - N(0))\Big)\Big] \; < \; +\infty.$$

Then, *there is a unique, strong solution process $N = (N(t))_{t\geq 0}$ satisfying SDE (1.2) and $\forall$ nonrandom $0 < T < +\infty$ $\forall 0 < N(0) < K$*

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[\ln\Big(N(t)(K - N(t))\Big)\Big] \leq \mathbb{E}\Big[\ln\Big(N(0)(K - N(0))\Big)\Big] + \Big(\mu + \frac{\sigma^2}{2}K^2\Big) \cdot T < +\infty.$$

3 Existence of bounded, unique solution of (1.1) on 4D prism $\mathbb{D}$

The following theorem establishes the existence of strong, unique solutions of SEIR(S) models (1.1) bounded to stay on certain positive prisms (a.s.).

Theorem 3.1 (Existence theorem of unique solutions of SEIR(S) model on prisms)**.** *Let $\big(S(t_0), E(t_0), I(t_0), R(t_0)\big) = (S_0, E_0, I_0, R_0) \in \mathbb{D}$ with*

$$\mathbb{D} = \Big\{(S, E, I, R) \in \mathbb{R}^4_+ : 0 < S, E, I, R < K, \; S + E + I + R < K\Big\}.$$

Consider the stochastic SEIR(S) model with random, nonconstant total populations

$$\begin{aligned}
dS &= \big(-\beta SI + \mu(K-S) + \alpha I + \zeta R\big)dt - \sigma_1 SI \cdot F_1(S,E,I,R)dW_1 \\
&\quad + \sigma_4 R \cdot F_4(S,E,I,R)dW_4 + \sigma_5 S(K-N)dW_5 \\
dE &= \big(\beta SI - (\mu+\eta)E\big)dt + \sigma_1 SI \cdot F_1(S,E,I,R)dW_1 - \sigma_2 E \cdot F_2(S,E,I,R)dW_2 \\
&\quad + \sigma_5 E(K-N)dW_5 \\
dI &= \big(\eta E - (\alpha+\gamma+\mu)I\big)dt + \sigma_2 E \cdot F_2(S,E,I,R)dW_2 - \sigma_3 I \cdot F_3(S,E,I,R)dW_3 \\
&\quad + \sigma_5 I(K-N)dW_5 \\
dR &= \big(\gamma I - (\mu+\zeta)R\big)dt + \sigma_3 I \cdot F_3(S,E,I,R)dW_3 - \sigma_4 R \cdot F_4(S,E,I,R)dW_4 \\
&\quad + \sigma_5 R(K-N)dW_5 \\
dN &= \mu(K-N)\,dt + \sigma_5 N(K-N)\,dW_5.
\end{aligned} \tag{3.1}$$

Assume that all constants $\alpha,\beta,\eta,\gamma,\zeta,\mu \geq 0$ *and*

(i) $(S_0,\ E_0,\ I_0,\ R_0) \in \mathbb{D}$ *is independent of* $\sigma\big(W_k : 1 \leq k \leq 5\big)$*,*

(ii) $\forall k = 1,\ 2,\ 3,\ 4,\ 5$: $F_k \in C^0_{locLip}(\mathbb{D})$ *(i.e. local Lipschitz continuous on interior* $\mathbb{D}$*)* $\bigcap C^0(\overline{\mathbb{D}})$*,*

(iii) $\mathbb{E}[V(S_0,E_0,I_0,R_0)] < +\infty$ *with*

$$V(S,E,I,R) = \begin{cases} R - \ln(R) + I - \ln(I) + E - \ln(E) + S - \ln(S), \\ +K - S - E - I - R - \ln(K - S - E - I - R), \end{cases}$$

(iv) $\sup_{(S,E,I,R)\in\mathbb{D}} \frac{S^2 I^2 F_1^2(S,E,I,R)}{E^2} + \sup_{(S,E,I,R)\in\mathbb{D}} \frac{E^2 F_2^2(S,E,I,R)}{I^2} + \sup_{(S,E,I,R)\in\mathbb{D}} \frac{I^2 F_3^2(S,E,I,R)}{R^2} + + \sup_{(S,E,I,R)\in\mathbb{D}} \frac{R^2 F_4^2(S,E,I,R)}{S^2} < +\infty.$

Then*, the stochastic SEIR model* (1.1) *with random total population size* $N(t)$ *admits*

(1) a unique, continuous time, Markovian, global strong solution $\big(S(t),\ E(t),\ I(t),\ R(t)\big)$ *on* $t \geq t_0$*,*

(2) an a.s. $\mathbb{D}$*-invariant solution (i.e. a.s. uniform boundedness of solutions on positive cone of* $\mathbb{R}^4$*),*

(3) a uniform estimate of moments ($\forall T < +\infty$ *nonrandom)*

$$\sup_{0\leq t\leq T} \mathbb{E}[V(S(t),E(t),S(t),R(t))] \leq \mathbb{E}[V(S_0,E_0,S_0,R_0)] + [\beta K + 4\mu + \eta + \alpha + \gamma + \zeta + c_1]\cdot T,$$

where c_1 *is an appropriate constant (one may extract that from proof below).*

Proof. Define

$$\mathbb{D}_n := \Big\{(S,E,I,R,N)\ \in\ \mathbb{R}^5_+ : e^{-n} < S,E,I,R < K - e^{-n},\ N = S+E+I+R < K(1-e^{-n})\Big\}$$

for $n \in \mathbb{N}$. Then, due to its local Lipschitz continuous drift and diffusion coefficients, system (3.1) has a unique solution up to stopping time $\tau(\mathbb{D}_n)$ hitting the boundary of open sets $\mathbb{D}_n$ (see [3,11,16]). Furthermore, define

$$\begin{aligned}
V(S,E,I,R,N) &= \begin{cases} R - \ln(R) + I - \ln(I) + E - \ln(E) + S - \ln(S) \\ +K - N - \ln(K-N) \end{cases} \\
&= V_1(S,E,I,R) + V_2(N)
\end{aligned}$$

where $\quad V_1(S,E,I,R) = R - \ln(R) + I - \ln(I) + E - \ln(E) + S - \ln(S),$

$$V_2(N) = \widetilde{V}_2(S,E,I,R) = K - S - E - I - R - \ln(K-S-E-I-R) \ = \ K - N - \ln(K-N)$$

$$\text{on} \quad \widetilde{\mathbb{D}} = \big\{(S,E,I,R,N) \in \mathbb{R}^5_+ : 0 < S,E,I,R < K, N = S+E+I+R < K\big\}.$$

Suppose that $\mathbb{E}\big[V(S_0, E_0, I_0, R_0, N_0)\big] < +\infty$. Note that $V(S,E,I,R,N) \geq 5$ for $(S,E,I,R,N) \in \widetilde{\mathbb{D}}$ (this fact will be used below to estimate $\mathcal{L}V$). Now, calculate the infinitesimal generator $\mathcal{L}V$ applied to our SEIR(S) model 3.1 using the general formula (A.2) as stated in appendix). One encounters with the 2nd differential operator

$$\begin{aligned}
\mathcal{L}V(S,E,I,R,N) &= (-\beta SI + \mu(K-S) + \alpha I + \zeta R)\,\frac{\partial V}{\partial S} + (\beta SI - (\mu+\eta)E)\,\frac{\partial V}{\partial E} \\
&+ (\eta E - (\alpha+\gamma+\mu)I)\,\frac{\partial V}{\partial I} + (\gamma I - (\mu+\zeta)R)\,\frac{\partial V}{\partial R} + \mu(K-N)\frac{\partial V}{\partial N} \\
&+ \frac{\sigma_1^2}{2} S^2 I^2 [F_1(S,E,I,R)]^2 \left(\frac{\partial^2 V}{\partial S^2} - 2\frac{\partial^2 V}{\partial S \partial E} + \frac{\partial^2 V}{\partial E^2}\right) \\
&+ \frac{\sigma_2^2}{2} E^2 [F_2(S,E,I,R)]^2 \left(\frac{\partial^2 V}{\partial E^2} - 2\frac{\partial^2 V}{\partial E \partial I} + \frac{\partial^2 V}{\partial I^2}\right) \\
&+ \frac{\sigma_3^2}{2} I^2 [F_3(S,E,I,R)]^2 \left(\frac{\partial^2 V}{\partial I^2} - 2\frac{\partial^2 V}{\partial I \partial R} + \frac{\partial^2 V}{\partial R^2}\right) \\
&+ \frac{\sigma_4^2}{2} R^2 [F_4(S,E,I,R)]^2 \left(\frac{\partial^2 V}{\partial R^2} - 2\frac{\partial^2 V}{\partial R \partial S} + \frac{\partial^2 V}{\partial S^2}\right) \\
&+ \frac{\sigma_5^2}{2} [K-N]^2 \left(S^2\frac{\partial^2 V}{\partial S^2} + 2ES\frac{\partial^2 V}{\partial S \partial E} + E^2\frac{\partial^2 V}{\partial E^2}\right) \\
&+ \sigma_5^2 [K-N]^2 \left(SI\frac{\partial^2 V}{\partial S \partial I} + SR\frac{\partial^2 V}{\partial S \partial R} + SN\frac{\partial^2 V}{\partial S \partial N}\right) \\
&+ \sigma_5^2 [K-N]^2 \left(EI\frac{\partial^2 V}{\partial E \partial I} + ER\frac{\partial^2 V}{\partial E \partial R} + EN\frac{\partial^2 V}{\partial E \partial N} + IR\frac{\partial^2 V}{\partial I \partial R} + IN\frac{\partial^2 V}{\partial I \partial N} + RN\frac{\partial^2 V}{\partial R \partial N}\right) \\
&+ \frac{\sigma_5^2}{2} [K-N]^2 \left(I^2\frac{\partial^2 V}{\partial I^2} + R^2\frac{\partial^2 V}{\partial R^2} + N^2\frac{\partial^2 V}{\partial N^2}\right)
\end{aligned} \tag{3.2}$$

for any twice continuously differentiable function $V \in C^2(\widetilde{\mathbb{D}})$. Next, an application $\mathcal{L}V$ to our specific functional V yields that

$$\begin{aligned}
\mathcal{L}V(S,E,I,R,N) &= \mathcal{L}V_1(S,E,I,R) + \mathcal{L}V_2(N), \\
\mathcal{L}V_1(S,E,I,R) &= \mu(K-S-E-I-R) + \beta I - \frac{1}{S}\Big(\mu(K-S) + \alpha I + \zeta R\Big) - \frac{1}{E}\,\beta SI \\
&\quad + \mu + \eta - \frac{1}{I}\,\eta E + \alpha + \gamma + \mu - \frac{1}{R}\,\gamma I + \mu + \zeta + \frac{\sigma_1^2}{2}\, I^2\, F_1^2(S,E,I,R) \\
&\quad + \frac{\sigma_4^2}{2}\,\frac{R^2\,F_4^2(S,E,I,R)}{S^2} + \frac{\sigma_1^2}{2}\,\frac{S^2\,I^2\,F_1^2(S,E,I,R)}{E^2} + \frac{\sigma_2^2}{2}\,F_2^2(S,E,I,R) \\
&\quad + \frac{\sigma_2^2}{2}\,\frac{E^2\,F_2^2(S,E,I,R)}{I^2} + \frac{\sigma_3^2}{2}\,F_3^2(S,E,I,R) \\
&\quad + \frac{\sigma_3^2}{2}\,\frac{I^2\,F_3^2(S,E,I,R)}{R^2} + \frac{\sigma_4^2}{2}\,F_4^2(S,E,I,R) + 2\sigma_5^2(K-N)^2 \\
&\leq \mu(K-S-E-I-R) + \beta I + 3\mu + \alpha + \gamma + \eta + \zeta + 2\sigma_5^2(K-N)^2 \\
&\quad + \frac{\sigma_1^2}{2}\sup_{(S,E,I,R)\in\mathbb{D}} \frac{S^2 I^2 F_1^2(S,E,I,R)}{E^2} + \frac{\sigma_1^2}{2}\max_{(S,E,I,R)\in\mathbb{D}} I^2 F_1^2(S,E,I,R) \\
&\quad + \frac{\sigma_4^2}{2}\sup_{(S,E,I,R)\in\mathbb{D}} \frac{R^2 F_4^2(S,E,I,R)}{S^2} + \frac{\sigma_4^2}{2}\max_{(S,E,I,R)\in\mathbb{D}} F_4^2(S,E,I,R)
\end{aligned}$$

$$+\frac{\sigma_3^2}{2}\sup_{(S,E,I,R)\in\mathbb{D}}\frac{I^2F_3^2(S,E,I,R)}{R^2}+\frac{\sigma_3^2}{2}\max_{(S,E,I,R)\in\mathbb{D}}F_3^2(S,E,I,R)$$
$$+\frac{\sigma_2^2}{2}\sup_{(S,E,I,R)\in\mathbb{D}}\frac{E^2F_2^2(S,E,I,R)}{I^2}+\frac{\sigma_2^2}{2}\max_{(S,E,I,R)\in\mathbb{D}}F_2^2(S,E,I,R)<+\infty$$

since Theorem 3.1 (iv). Similarly, we find

$$\mathcal{L}V_2(N)=\mathcal{L}\widetilde{V}_2(S,\,E,\,I,\,R)=-\mu(K-N)+\mu+\frac{\sigma_5^2}{2}N^2.$$

Note that we may estimate $\sigma_5^2[4(K-N)^2+N^2]\le 4\sigma_5^2K^2/5$ on $0<N<K$. Thus, we have

$$\mathcal{L}V(S,E,I,R,N)\ \le\ \beta I+4\mu+\eta+\alpha+\gamma+\zeta+c_1\ \le\ \beta K+4\mu+\eta+\alpha+\gamma+\zeta+c_1$$

on $\widetilde{\mathbb{D}}$, where

$$c_1=\begin{cases}\dfrac{2\sigma_5^2}{5}K^2+\dfrac{\sigma_1^2}{2}\left[\sup\limits_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}\dfrac{S^2I^2F_1^2(S,E,I,R)}{E^2}+\max\limits_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}I^2F_1^2(S,E,I,R)\right]\\ +\dfrac{\sigma_4^2}{2}\left[\sup\limits_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}\dfrac{R^2F_4^2(S,E,I,R)}{S^2}+\max\limits_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}F_4^2(S,E,I,R)\right]\\ +\dfrac{\sigma_3^2}{2}\left[\sup\limits_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}\dfrac{I^2F_3^2(S,E,I,R)}{R^2}+\max\limits_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}F_3^2(S,E,I,R)\right]\\ +\dfrac{\sigma_2^2}{2}\left[\sup\limits_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}\dfrac{E^2F_2^2(S,E,I,R)}{I^2}+\max\limits_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}F_2^2(S,E,I,R)\right].\end{cases} \tag{3.3}$$

Note that c_1 is finite due to hypotheses (ii) and (iv). Next, let $\tau_n(t):=\min(\tau(\mathbb{D}_n),t)$ where $\tau(\mathbb{D}_n)$ is the stopping time of the first exit from the domain $\mathbb{D}_n$. An application of Dynkin's formula [8] (1965) provides us the estimate

$$\begin{aligned}&\mathbb{E}[V(S(t),E(t),I(t),R(t),N(t))]\\ &\quad=\mathbb{E}[V(S_0,E_0,I_0,R_0,N_0)]+\mathbb{E}\left[\int_0^{\tau_n(t)}\mathcal{L}V(S(s),E(s),I(s),R(s),N(s))\,ds\right]\\ &\quad\le\mathbb{E}[V(S_0,E_0,I_0,R_0,N_0)]+[\beta K+4\mu+\eta+\alpha+\gamma+\zeta+c_1]\cdot\mathbb{E}[\tau_n(t)]\\ &\quad\le\mathbb{E}[V(S_0,E_0,I_0,R_0,N_0)]+[\beta K+4\mu+\eta+\alpha+\gamma+\zeta+c_1]\cdot t\quad\text{since }\tau_n(t)\le t,\end{aligned}$$

for all nonrandom times $t>t_0$, as long as the solution $\big(S(s),E(s),I(s),R(s),N(s)\big)$ on $\widetilde{\mathbb{D}}$. Note that $\forall n\in\mathbb{N}:n>0$ and $n>\ln(K)/5$

$$\inf_{(S,E,I,R,N)\,\in\,\partial\,\mathbb{D}_n}V(S,E,I,R,N)>5n-\ln(K). \tag{3.4}$$

Recall that we have defined the stopping time $\tau_n(t):=\min\{t,\tau(\mathbb{D}_n)\}$ based on the stopping time $\tau(\mathbb{D}_n)$ arriving the first time at the boundary of $\mathbb{D}_n$. Now, apply the above estimate to get to

$$
\begin{aligned}
0 \;&\leq\; \mathbb{P}\Big(\Big[\tau(\widetilde{\mathbb{D}}) < t\Big]\Big) \overset{\mathbb{D}_n \subseteq \widetilde{\mathbb{D}}}{\leq} \mathbb{P}\Big(\Big[\tau(\mathbb{D}_n) < t\Big]\Big) = \mathbb{P}\Big(\Big[\tau_n(t) < t\Big]\Big) \\
&= \mathbb{E}\big[\mathbf{1}_{\tau_n(t)<t}\big] \quad \text{where } \mathbf{1} \text{ is the indicator function} \\
&\leq \mathbb{E}\left[\frac{V\Big(S\big(\tau(\mathbb{D}_n)\big), E\big(\tau(\mathbb{D}_n)\big), I\big(\tau(\mathbb{D}_n)\big), R\big(\tau(\mathbb{D}_n)\big), N\big(\tau(\mathbb{D}_n)\big)\Big)}{\inf_{(S,E,I,R,N)\in\partial\mathbb{D}_n} V(S,E,I,R,N)} \cdot \mathbf{1}_{\tau_n<t}\right] \\
&\overset{(3)}{\leq} \frac{\mathbb{E}[V(S_0,E_0,I_0,R_0.N_0)] + \mathbb{E}[\tau_n(t)]\cdot[\beta K + 4\mu + \eta + \alpha + \gamma + \zeta + c_1]}{\inf_{(x,y,z,v,w)\in\partial\mathbb{D}_n} V(x,y,z,v,w)} \\
&\overset{(3.4)}{\leq} \frac{\mathbb{E}[V(S_0,E_0,I_0,R_0,N_0)] + t[\beta K + 4\mu + \eta + \alpha + \gamma + \zeta + c_1]}{5n - \ln(K)} \longrightarrow 0 \quad \text{as } n \to \infty
\end{aligned}
$$

for all $(S_0, E_0, I_0, R_0, N_0) \in \mathbb{D}_n$, and for all fixed, nonrandom $t \in [s, \infty)$. Thus

$$
\Longrightarrow \quad \mathbb{P}\Big(\Big[\tau(\widetilde{\mathbb{D}}) < t\Big]\Big) = \lim_{n\to+\infty} \mathbb{P}\Big(\Big[\tau(\mathbb{D}_n) < t\Big]\Big) = 0
$$

for all adapted $(S_0, E_0, I_0, R_0, N_0) \in \widetilde{\mathbb{D}}$ and all $t \geq t_0$. That means that

$$
\mathbb{P}\Big(\Big[\tau(\widetilde{\mathbb{D}}) = +\infty\Big]\Big) = 1.
$$

This proves the invariance property and global existence of solutions $\big(S(t), I(t), R(t), N(t)\big)$ on $\widetilde{\mathbb{D}}$ for any finite time t. Thus, the proof of Theorem (3.1) is complete. □

4 Asymptotic moment and stochastic stability, stability exponents

Let $p > 0$ be a real constant. Consider the d-dimensional, autonomous, Itô-interpreted SDEs

$$
dX(t) = a\big(X(t)\big)dt + b\big(X(t)\big)\,dW(t). \tag{4.1}
$$

Definition 4.1. SDE (4.1) has a **globally asymptotically p-th moment stable** equilibrium (solution) $X = x^*$ if and only if $a(x^*) = b(x^*) = 0$ and $\forall\, X(s) \in L^p(\Omega, \mathcal{F}_s, \mathbb{P})$, $s \geq 0$, $X(s) \neq x^*$ we have

$$
\lim_{t\to+\infty} \mathbb{E}\Big[\|X_{s,X(s)}(t) - x^*\|_d^p\Big] = 0
$$

(where d is the state-space dimension of the stochastic process X).

Definition 4.2. The equilibrium solution x^* of SDE (4.1) is **stochastically stable** (stable in probability) iff, for every $\varepsilon > 0$ and $s \geq t_0$, we have

$$
\lim_{x_0\to x^*} \mathbb{P}\left(\left[\sup_{t_0\leq s<\infty} \|X_{s,x_0}(t) - x^*\| \geq \varepsilon\right]\right) = 0 \tag{4.2}
$$

where $X_{s,x_0}(t)$ denotes the solution of SDE (4.1) satisfying $X(s) = x_0$ at time $t \geq s$.

Definition 4.3. The equilibrium solution x^* of SDE (4.1) is said to be **(locally) asymptotically stochastically stable** iff it is stochastically stable and

$$
\forall x_0 \in N(x^*) : \mathbb{P}\left(\left[\lim_{t\to\infty} X_{s,x_0}(t) = x^*\right]\right) = 1. \tag{4.3}
$$

Definition 4.4. The equilibrium solution x^* of SDE (4.1) is said to be **globally asymptotically stochastically stable** iff it is stochastically stable and, for every x_0 and every s, we have

$$\mathbb{P}\left(\left[\lim_{t\to\infty} X_{s,x_0}(t) = x^*\right]\right) \;=\; 1. \tag{4.4}$$

Theorem 4.5 (Stability theorem of Arnold [3])**.** *Assume that the SDE (4.1) has a unique solution started at every nonrandom x_0 in the nonrandom, a.s. invariant, open neighborhood $N(x^*) \subseteq \mathbb{R}^d$.* **Then**, *the equilibrium solution $x^* \in \overline{N(x^*)} \subseteq \mathbb{R}^d$ for SDE (4.1) is stochastically stable if $\exists$ positive definite*

$$V = V(t,x) \in C^{1,2}\Big([t_0,\infty) \times N(x^*), \mathbb{R}^1_+\Big)$$

on $N(x^*)$ such that $\forall\,(t,x) \in [t_0,\infty) \times N(x^*)$:

$$\mathcal{L}V(t,x) \le 0.$$

If additionally V is decrescent on $N(x^*)$ and

$$\forall\,(t,x) \in [t_0,\infty) \times N(x^*) \setminus \{x^*\} : \quad \mathcal{L}V(t,x) < 0,$$

then *x^* is (locally) asymptotically stochastically stable for SDE (4.1).*

We also call the equilibrium x^* of SDE (4.1) to be **globally asymptotically stochastically stable** iff it is asymptotically stochastically stable and $\mathcal{L}V < 0$ on the entire domain $\mathbb{D}$ where the dynamics of X live on (a.s.) (i.e., in this case, we may extend $N(x^*) = \mathbb{D}$ as the relevant neighborhood of x^* in above definition of stochastic stability). Note that the equilibria x^* do not have to be in the neighborhood $N(x^*)$, but $x^* \in \overline{N(x^*)}$. In fact, the Theorem 4.5 remains valid for the cases like neighborhoods of the form $N(K) = (0,K)$ or $N(K) = [\varepsilon,K)$ with equilibrium $x^* = K$ (or multidimensional variants of those examples) in order to cover the important cases of semi-stability too. For the SDE of the total population N of our SEIR(S) model, we can establish both stochastic and moment stability of the saturation constant $x^* = K$.

Theorem 4.6 (Stability of equilibrium $n^* = K$ for total populations)**.** *Consider the SDE for random total population*

$$dN = \mu(K-N)\,dt + \sigma_5 N(K-N)\,dW_5. \tag{4.5}$$

Then, *the equilibrium point $n^* = K$ of SDE (4.5) is*

(1) global asymptotically stochastically stable if $2\mu > K^2\sigma_5^2$,

(2) p-th moment exponentially stable if $2\mu > (2p-1)K^2\sigma_5^2$ and $p \ge \frac{1}{2}$, and

(3) almost surely asymptotically stable if $2\mu > (2p-1)K^2\sigma_5^2$ and $p \ge \frac{1}{2}$.

Proof. The proof for the equilibrium $n^* = K$ is naturally divided into the items (1)–(3).

(1) Let $n \in (0,K)$ for $n^* = K$. Define the Lyapunov function V by

$$n \in (0,K) \mapsto V(n) = (K-n)^2.$$

Then, we find that

$$\begin{aligned}\mathcal{L}V(n) &= \mu(K-n)\cdot 2(K-n)\cdot(-1)+\frac{1}{2}\sigma_5^2 n^2(K-n)^2\cdot 2\\ &= (-2\mu+\sigma_5^2 n^2)\,V(n)\\ &\overset{\text{on }\mathbb{D}_0}{\leq} (-2\mu+\sigma_5^2 K^2)\,V(n).\end{aligned}$$

Now, note that $\mathcal{L}V(n) < 0$ for all $n \in (0,K)$ if $2\mu > K^2\sigma_5^2$. Therefore, an application of Arnold's Stability Theorem 4.5 confirms the claim of stochastic stability.

(2) Next, consider the Lyapunov function

$$n \in (0,K) \mapsto V(n) = (K-n)^{2p} = \left[(K-n)^2\right]^p.$$

Then, for $p \geq \frac{1}{2}$, we have

$$\begin{aligned}\mathcal{L}V(n) &= \mu(K-n)\cdot 2p(K-n)^{2p-1}\cdot(-1)+\frac{1}{2}\sigma_5^2 n^2(K-n)^2\cdot 2p(2p-1)(K-n)^{2p-2}\\ &= 2p\left[-\mu+\frac{1}{2}(2p-1)\sigma_5^2 n^2\right]\cdot V(n)\\ &\leq 2p\left[-\mu+\frac{1}{2}(2p-1)\sigma_5^2 K^2\right]\cdot V(n).\end{aligned}$$

An application of Dynkin's formula (cf. [8]) will give the conclusion that

$$\begin{aligned}\mathbb{E}\left[V\Big(N(t)\Big)\right] &= \mathbb{E}\left[|K-N(t)|^{2p}\right]\\ &\leq \mathbb{E}\left[V\Big(N(s)\Big)\right]\cdot e^{2p\left[-\mu+\frac{1}{2}(2p-1)\sigma_5^2K^2\right](t-s)} \qquad (4.6)\\ &\overset{t\to+\infty}{\longrightarrow} 0\end{aligned}$$

if $2\mu > (2p-1)K^2\sigma_5^2$ and $p \geq \frac{1}{2}$.

(3) The property of a.s. asymptotical stability of $n^* = K$ follows directly from the item (2) due to the fact that all exponentially moment stable equilibria also possess a.s. asymptotically stable pathwise solutions (for a proof of this fact, see [26]). This completes the proof of Theorem 4.6. □

As a by-product of the previous proof, we gain the following result on the asymptotic behavior of Lyapunov functionals $V(n) = |K-n|^{2p} = \|K-n\|_{\mathbb{R}^1}^{2p}$.

Theorem 4.7 (Uniform estimation of moment V-exponents)**.** *Consider Itô SDEs* (4.5) *for random total population* $N = (N(t))_{t\geq 0}$. **Then**

$$\begin{aligned}\forall p>0\ \forall N(0)=n_0\in(0,K):\ \lambda_{2p}(n_0) &:= \lim_{t\to+\infty}\frac{\ln\left(\mathbb{E}\left[|K-N(t)|^{2p}\right]\right)^{1/2p}}{t}\\ &\leq -\mu+\frac{1}{2}\max(2p-1,0)\sigma_5^2K^2,\end{aligned}$$

which represents a uniform estimation of moment V-exponents $\lambda_{2p}(n_0)$ on $(0,K)$.

Proof. Return to the identity (4.6) with $V(n) = |K - n|^{2p}$. Taking $2p$-th root and the natural logarithm yield that

$$\ln\left(\mathbb{E}\left[|K - N(t)|^{2p}\right]\right)^{1/2p} \leq \ln\left(\mathbb{E}\left[|K - N(0)|^{2p}\right]\right)^{1/2p} + \left(-\mu + \frac{1}{2}\max(2p-1,0)\sigma_5^2 K^2\right) \cdot t.$$

Thus, dividing by $t > 0$ and taking the limit as $t \to +\infty$ confirms the conclusion on the asymptotic behavior of $\ln\left(\mathbb{E}[V(N(t))]\right)/t$ in the moment sense. □

Remark 4.8 (Moment V-exponents)**.** The moment V-exponents λ_{2p} measure the speed of exponential convergence of total populations $N(t)$ to the saturation constant K as time $t \to +\infty$ in the $2p$-th moment sense. The definition of moment V-exponents is made in a consistent manner (to incorporate the deterministic case). In passing, note that nonlinear V-exponents may depend on initial quantity $N(0) = n_0$, whereas V-exponents for linear systems do not depend on $N(0) = n_0$. Remarkably, for sufficiently small powers p or noise intensities σ_5 or very small constants $K > 0$, we find exponentially stable $2p$-moments (i.e. exponential moment convergence of $N(t)$ to equilibrium $n^* = K$) due to the birth parameter $\mu > 0$ in our model.

The following lemma states the form of all existing equilibria (trivial solutions) of SEIR(S) models (1.1). Its proof is an elementary exercise of algebra, hence it is omitted here.

Lemma 4.9 (Disease-free and endemic equilibria)**.** *For the drift coefficients of our SEIR(S) model* (1.1)*, we have two equilibrium points. One is disease-free and the other is the endemic equilibrium. The disease-free equilibrium of* (1.1) *is given by*

$$(S_1, E_1, I_1, R_1) = (K, 0, 0, 0) \in \overline{\mathbb{D}}$$

with its total sum $N_1 := S_1 + E_1 + I_1 + R_1 = K$ and the endemic equilibrium by

$$(S_2, E_2, I_2, R_2) \in \overline{\mathbb{D}},$$

$$\begin{aligned} \text{where} \quad S_2 &= \frac{(\mu+\eta)(\alpha+\gamma+\mu)}{\beta\eta} \\ E_2 &= \frac{(\mu+\zeta)(\alpha+\gamma+\mu)}{\beta\eta}\left[\frac{\beta\eta K - (\mu+\eta)(\alpha+\gamma+\mu)}{(\mu+\zeta)(\alpha+\gamma+\mu) + \eta(\gamma+\mu+\zeta)}\right] \\ I_2 &= \left(\frac{\mu+\zeta}{\beta}\right)\left[\frac{\beta\eta K - (\mu+\eta)(\alpha+\gamma+\mu)}{(\mu+\zeta)(\alpha+\gamma+\mu) + \eta(\gamma+\mu+\zeta)}\right] \\ R_2 &= \frac{\gamma}{\beta}\left[\frac{\beta\eta K - (\mu+\eta)(\alpha+\gamma+\mu)}{(\mu+\zeta)(\alpha+\gamma+\mu) + \eta(\gamma+\mu+\zeta)}\right]. \end{aligned} \tag{4.7}$$

The disease-free equilibrium is also an equilibrium of the diffusion coefficients. For biologically meaningful occurrence of endemic equilibrium (i.e. $(S_2, E_2, I_2, R_2) \in \overline{\mathbb{D}}$), we need to require that

$$\beta\eta K > (\mu+\eta)(\alpha+\gamma+\mu) \tag{$*$}$$

– a condition, which is equivalent to $\mathcal{R}_0 > 1$. For the classic concept of endemic equilibrium of both drift and diffusion terms at the same location, vanishing $F_k(S_2, E_2, I_2, R_2) = 0$ are imposed. Moreover, at the endemic equilibrium, we have total sum

$$N_2 := S_2 + E_2 + I_2 + R_2 = K.$$

Proof. Elementary calculus exercise. □

Biologists and Ecologists usually express the qualitative state of systems in terms of **basic reproduction numbers** $\mathcal{R}_0$. For our SEIR(S) model (1.1), this quantity takes the form

$$\mathcal{R}_0 = \frac{\beta\eta K}{(\mu+\eta)(\alpha+\gamma+\mu)}.$$

Indeed, as we shall see below, this quantity decides about the long-term **stability mode** in which the stochastic SEIR(S) models is and the value $\mathcal{R}_0 = 1$ serves as a bifurcation parameter (cf. stability analysis in what follows). Moreover, the endemic equilibrium (S_2, E_2, I_2, R_2) can be expressed in terms of $\mathcal{R}_0$ by

$$(S_2, E_2, I_2, R_2) = \left(\frac{K}{\mathcal{R}_0}, (\mu+\zeta)\frac{K}{\mathcal{R}_0}\rho, \frac{(\mu+\zeta)}{\beta}\rho, \frac{\gamma}{\beta}\rho\right)$$

with

$$\rho := \frac{\mathcal{R}_0 - 1}{\frac{(\mu+\zeta)}{(\mu+\eta)} + \frac{(\gamma+\mu+\zeta)\mathcal{R}_0}{\beta K}}.$$

Clearly, the 2nd, 3rd and 4th components of (S_2, E_2, I_2, R_2) are positive iff $\rho > 0$ iff $\mathcal{R}_0 > 1$.

First, for stability investigation of SEIR(S) models (1.1), note that all equilibria $(S_k^*, E_k^*, I_k^*, R_k^*)$ of SDEs (1.1) possess the total sum

$$N_k^* = S_k^* + E_k^* + I_k^* + R_k^* = K$$

and are at the boundary of domain $\mathbb{D}$, i.e.

$$N_k^* \in \overline{\mathbb{D}}.$$

This is also the unique equilibrium of dynamics $N = (N(t))_{t\geq 0}$ of total populations governed by SDE (1.2). Consequently, it remains to prove that the asymptotic stability of the disease-free equilibrium when reproduction number $\mathcal{R}_0 < 1$ and the endemic equilibrium when reproduction number $\mathcal{R}_0 > 1$.

Asymptotic stability of general epidemic or environmental systems has already been investigated in [4, 13, 20, 31, 32, 34], but not our SEIR(S) model (1.1) to the best of our knowledge. These investigations are associated to appropriate Lyapunov functions or functionals (cf. [4,7,9,10,17–19] among others).

Theorem 4.10 (Asymptotic stochastic stability of disease-free equilibrium)**.** *The disease-free equilibrium solution* $(S_1, E_1, I_1, R_1) = (K, 0, 0, 0)$ *of* (1.1) *is (globally) asymptotically stochastically stable if*

$$\sigma^2 K^2 < 2\mu, \qquad \zeta \geq 0, \qquad \beta K \leq \alpha. \tag{4.8}$$

Proof. We shall apply Theorem 4.5. For this purpose, define the Lyapunov function

$$\begin{aligned} V_4(S,E,I,R) &= \frac{1}{2}(S - K + E + I + R)^2 + KE + KI + KR \\ &= \frac{1}{2}(K-N)^2 + KE + KI + KR = \hat{V}_4(E,I,R,N) \end{aligned}$$

on $\mathbb{D}$. The infinitesimal generator $\mathcal{L}$ (cf. (3.2) and generally presented one in Appendix A) acting on the Lyapunov function V_4 can be written as:

$$\begin{aligned}
\mathcal{L}V_4(S,E,I,R) &= \big(-\beta SI + \mu(K-S) + \alpha I + \zeta R\big)(S-K+E+I+R) \\
&\quad + \big(\beta SI - (\mu+\eta)E\big)(S-K+E+I+R+K) \\
&\quad + \big(\eta E - (\alpha+\gamma+\mu)I\big)(S-K+E+I+R+K) \\
&\quad + \big(\gamma I - (\mu+\zeta)R\big)(S+E+I+R+K-K) + \frac{\sigma_5^2}{2}N^2(K-N)^2 \\
&= -\mu(S+E+I+R-K)^2 + K\big[\beta SI - \mu(E+I+R) - \alpha I - \zeta R\big] \\
&\quad + \frac{\sigma_5^2}{2}N^2(K-N)^2 \\
&= -\mu(N-K)^2 - \mu(KE+KI+KR) + K\big[(\beta S-\alpha)I - \zeta R\big] + \frac{\sigma_5^2}{2}N^2(K-N)^2 \\
&\overset{0<\delta<1}{=} -\mu(1-\delta)(N-K)^2 - \mu(KE+KI+KR) + K\big[(\beta K-\alpha)I - \zeta R\big] \\
&\quad - \Big[\delta\mu - \frac{\sigma_5^2}{2}N^2\Big](K-N)^2 \\
&\le -\mu(1-\delta)V_4 + K\big[(\beta K-\alpha)I - \zeta R\big] - \big[\delta\mu - \frac{\sigma_5^2}{2}K^2\big](K-N)^2.
\end{aligned}$$

Now, let $\delta \to 1-$. Then, from some $\delta > 0$ onwards as $\delta \uparrow 1$, we find a $\delta_0 < 1$ such that, for all $\delta \in (\delta_0, 1]$, we have $-\delta\mu + \sigma_5^2 K^2/2 \le 0$ by hypothesis $2\mu > \sigma_5^2 K^2$.

Thus, $\mathcal{L}V_4(S,E,I,R) \le 0$ is indeed negative-definite on $\mathbb{D}$ under the presumptions that $\beta K - \alpha \le 0$, $\zeta \ge 0$ and $2\mu > \sigma_5^2 K^2$. It remains to apply stochastic stability Theorem 4.5 to confirm Theorem 4.10. □

Remark 4.11 (Role of basic reproduction number)**.** One of the most important quantities in epidemiology is the basic reproduction number $\mathcal{R}_0$, expected number of secondary infections produced when one infected individual entered a fully susceptible population [15]. It usually determines whether there is an epidemic or not. If $\mathcal{R}_0 < 1$ then the outbreak will disappear. On the contrary, if $\mathcal{R}_0 > 1$ then the epidemic will spread a population. Recall that the basic reproduction number of our SEIR(S) model is $\mathcal{R}_0 = \frac{\eta\beta K}{(\mu+\eta)(\alpha+\gamma+\mu)}$. Later we will see that this number $\mathcal{R}_0$, the magnitude of $\mu > 0$ and the parameter $\sigma_5^2 K^2$ involving environmental noise intensity σ_5 decide about whether the disease-free or the endemic equilibrium is (asymptotically) stochastically stable (cf. Theorems 4.10 and 4.15).

Remark 4.12 (Possible extinction of disease)**.** Theorem 4.10 concludes that, if $\alpha - \beta K \ge 0$ and the environmental noise level σ_5^2 is so small such that $2\mu \ge \sigma_5^2 K^2$, then the disease will die out. This statement does not contradict to the fact $\mathcal{R}_0 < 1$. Because the stability condition $\alpha - \beta K \ge 0$ can be written in terms of the basic reproduction number as follows

$$\beta K \le \alpha < (\alpha+\gamma+\mu)\frac{(\mu+\eta)}{\eta} \quad \Rightarrow \quad \frac{\eta\beta K}{(\mu+\eta)(\alpha+\gamma+\mu)} = \mathcal{R}_0 < 1.$$

Corollary 4.13 (Exponential moment stability of disease-free equilibrium)**.** *Since*

$$\mathcal{L}V_4(S,E,I,R) \le -\mu V_4(S,E,I,R)$$

under condition that $\beta K \leq \alpha$, $\zeta \geq 0$ *and* $\mu \geq \sigma_5^2 K^2$, *by Dynkin's formula, the disease-free equilibrium* $(K, 0, 0, 0)$ *is exponentially moment* V*-stable [26,27] with rate* $-\mu$, *i.e.* $\forall t \geq 0$:

$$\mathbb{E}\Big[V_4(S(t), E(t), I(t), R(t))\Big] \leq \mathbb{E}\Big[V_4(S(0), E(0), I(0), R(0))\Big] \cdot \exp\Big(-\mu \cdot t\Big),$$

$$\text{hence} \quad \lim_{t\to\infty} \frac{\ln\big[\mathbb{E}[V_4(S(t), E(t), I(t), R(t))]\big]}{t} = -\mu < 0. \tag{4.9}$$

Proof. Recall the structure of associated infinitesimal generator $\mathcal{L}$ and the computations of $\mathcal{L}V_4$ in the proof of Theorem 4.10. There we have found that

$$\mathcal{L}V_4(S, E, I, R) = -\mu(N-K)^2 - \mu(KE + KI + KR) + K\big[(\beta S - \alpha)I - \zeta R\big] + \frac{\sigma_5^2}{2} N^2 (K-N)^2.$$

Under $\mu \geq \sigma_5^2 K^2$, we further estimate

$$\begin{aligned}\mathcal{L}V_4(S, E, I, R) &= -\frac{\mu}{2}(N-K)^2 - \mu(KE + KI + KR) + K\big[(\beta S - \alpha)I - \zeta R\big] - \frac{\mu - \sigma_5^2 N^2}{2}(K-N)^2 \\ &\leq -\mu V_4 + \underbrace{K\big[(\beta S - \alpha)I - \zeta R\big] - \frac{\mu - \sigma_5^2 K^2}{2}(K-N)^2}_{\leq\ 0 \text{ on } \mathbb{D} \text{ since } \beta K \leq \alpha,\ \zeta \geq 0,\ \mu \geq \sigma_5^2 K^2} \\ &\leq -\mu V_4.\end{aligned}$$

Then, Dynkin's formula [8] gives

$$\begin{aligned}\mathbb{E}\big[V_4\big(S(t), E(t), I(t), R(t)\big)\big] &= \mathbb{E}\big[V_4\big(S_0, E_0, I_0, R_0\big)\big] + \mathbb{E}\Big[\int_0^t \mathcal{L}V_4\big(S(s), E(s), I(s), R(s)\big)ds\Big] \\ &\leq \mathbb{E}\big[V_4\big(S_0, E_0, I_0, R_0\big)\big] - \mu\mathbb{E}\Big[\int_0^t V_4\big(S(s), E(s), I(s), R(s)\big)ds\Big].\end{aligned}$$

Now, apply the well-known Bellman–Gronwall lemma to the dynamics of

$$v(t) := \mathbb{E}\big[V_4\big(S(t), E(t), I(t), R(t)\big)\big]$$

to conclude exponentially moment V-stability with $V = V_4$ (for the general concept of moment V-stability, see [26,27]). This finishes the proof of Corollary 4.13. □

Remark 4.14 (Extension of exponential stability at reduced rates)**.** There is a verification of a small extension of the range of exponential stability of disease-free equilibrium possible for the case $\sigma_5^2 K^2/2 < \mu < \sigma_5^2 K^2$. However, this is verified only at reduced rate $-\mu + \sigma_5^2 K^2/2$ of exponential convergence, compared to rate $-\mu < 0$ of Corollary 4.13. For this, one may establish the estimates $\mathcal{L}V_4 \leq [-\mu + \sigma_5^2 K^2/2] V_4$ from the above proof.

Now, let us turn to the study of asymptotic stability of the endemic equilibrium.

Theorem 4.15 (Asymptotic stochastic stability of endemic equilibrium)**.** *Assume that*

$$\beta\eta K > (\mu + \eta)(\alpha + \gamma + \mu)$$

(i.e. $\mathcal{R}_0 > 1$*) and* $2\mu \geq \sigma_5^2 K^2$*.* **Then***, the endemic equilibrium solution* (S_2, E_2, I_2, R_2) *of the system (1.1) is (globally) stochastically stable on*

$$\mathbb{D} = \big\{(S, E, I, R) : S > 0, E > 0, I > 0, R > 0, S + E + I + R < K\big\}.$$

If even $2\mu > \sigma_5^2 K^2$, **then** *the endemic equilibrium* (S_2, E_2, I_2, R_2) *of (1.1) is (globally) asymptotically stochastically stable on* $\mathbb{D}$.

Proof. Introduce the function

$$V_5(S,E,I,R) = \begin{cases} S - S_2 + E - E_2 + I - I_2 + R - R_2 \\ -(S_2+E_2+I_2+R_2)\ln\left(\dfrac{S+E+I+R}{S_2+E_2+I_2+R_2}\right) \end{cases} \tag{4.10}$$

on $\mathbb{D}$. Note that $V_5 \geq 0$ on $\mathbb{D}$ and $V_5 = 0 \iff S+E+I+R = K$ on $\mathbb{D}$ (by elementary calculus applied to $V_5(S,E,I,R) = \widetilde{V}_5(N)$ with $N = S+E+I+R \in (0,K]$. Actually $\widetilde{V}_5$ is strictly decreasing in $N \in [0,K))$. Thus, it is fairly easy to recognize that V_5 possesses all properties of a Lyapunov function on $\mathbb{D}$. Then, we have $\forall (S,E,I,R) \in \mathbb{D}$

$$\begin{aligned}
\mathcal{L}V_5(S,E,I,R) &= \big(-\beta SI + \mu(K-S) + \alpha I + \zeta R\big)\left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right) \\
&\quad + \big(\beta SI - (\mu+\eta)E\big)\left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right) \\
&\quad + \big(\eta E - (\alpha+\gamma+\mu)I\big)\left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right) \\
&\quad + \big(\gamma I - (\mu+\zeta)R\big)\left(1 - \frac{S_2+E_2+I_2+R_2}{(S+E+I+R)}\right) + \frac{\sigma_5^2}{2}N^2(K-N)^2 \cdot \frac{K}{N^2} \\
&= \left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right)\mu(K-S-E-I-R) + \frac{\sigma_5^2}{2}K(K-N)^2.
\end{aligned} \tag{4.11}$$

Note that, with $N = S+E+I+R$, we find that

$$\mu(K-N) = \mu(K-S-E-I-R) \overset{\text{LEM}\,4.9}{=} -\mu(S+E+I+R-S_2-E_2-I_2-R_2).$$

Hence, we arrive at

$$\begin{aligned}
\mathcal{L}V_5(S,E,I,R) \;&=\; -\mu(S+E+I+R-S_2-E_2-I_2-R_2)\left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right) \\
&\qquad + \frac{\sigma_5^2}{2}K(K-N)^2 \\
&=\; -\mu\frac{(S-S_2+E-E_2+I-I_2+R-R_2)^2}{S+E+I+R} + \frac{\sigma_5^2}{2}K(K-N)^2 \\
&=\; -\mu\frac{(K-N)^2}{N} + \frac{\sigma_5^2}{2}KN\frac{(K-N)^2}{N} \\
&\overset{\sigma_5^2 \geq 0}{\leq}\; -\left(\mu - \frac{\sigma_5^2}{2}K^2\right)\frac{(K-N)^2}{N} \leq 0
\end{aligned} \tag{4.12}$$

since $0 < N < K$ on $\mathbb{D}$ and by hypothesis $2\mu \geq \sigma_5^2 K^2$.

Therefore, by Theorem 4.5, the endemic equilibrium (S_2, E_2, I_2, R_2) is stochastically stable (globally on $\mathbb{D}$) if $\mathcal{R}_0 > 1$ and $2\mu \geq \sigma_5^2 K^2$. Moreover, when additionally $2\mu > \sigma_5^2 K^2$, a careful look again at estimation (4.12) yields that

$$\mathcal{L}V_5(S,E,I,R) \leq -\left(\mu - \frac{\sigma_5^2}{2}K^2\right)\frac{(K-N)^2}{N} \;<\; 0$$

on $\mathbb{D}$. Consequently, by Theorem 4.5, the endemic equilibrium (S_2, E_2, I_2, R_2) of SDEs (1.1) indeed is asymptotically stochastically stable (globally on $\mathbb{D}$) if $\mathcal{R}_0 > 1$ and $2\mu > \sigma_5^2 K^2$. This conclusion completes the proof of Theorem 4.15. $\square$

Remark 4.16 (Nonlinear distance measure to endemic equilibrium)**.** The function V_5 of the form (4.10) measures the distance of solutions (S, E, I, R) to the endemic equilibrium (S_2, E_2, I_2, R_2) in a nonlinear fashion.

Corollary 4.17 (Exponential moment stability of endemic equilibrium)**.** *Assume that*

$$\beta\eta K > (\mu+\eta)(\alpha+\gamma+\mu)$$

(i.e. $\mathcal{R}_0 > 1$*),* $2\mu > \sigma_5^2 K^2$ *and the initial total population* $0 < N(0) := S(0)+E(0)+I(0)+R(0) < K$ *is nonrandom.*
Then, *the endemic equilibrium solution* (S_2, E_2, I_2, R_2) *of system* (1.1) *is exponentially moment* V_5-*stable with rate* $-(\mu - \sigma_5^2 K^2/2)N(0)/K$*, i.e.* $\forall t \geq 0$:

$$\mathbb{E}[V_5(S(t),E(t),I(t),R(t))] \leq \mathbb{E}[V_5(S(0),E(0),I(0),R(0))] \cdot \exp\left(-\left(\mu - \frac{\sigma_5^2 K^2}{2}\right)\frac{N(0)}{K}\cdot t\right),$$

hence

$$\lim_{t\to+\infty} \frac{\ln\Big[\mathbb{E}[V_5(S(t),E(t),I(t),R(t))]\Big]}{t} \leq -\left(\mu - \frac{\sigma_5^2K^2}{2}\right)\frac{N(0)}{K} < 0.$$

Proof. Define the total population $N(t) = S(t)+E(t)+I(t)+R(t)$ for $t \geq 0$. Suppose that $N(0)$ is nonrandom. Recall that $S_2+E_2+I_2+R_2 = K$ by Lemma 4.9. Now, return to the proof of Theorem 4.15 where we have computed

$$\begin{aligned}
\mathcal{L}V_5(S,E,I,R) &= \left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right)\mu(K-S-E-I-R) \\
&= \left(1-\frac{K}{N}\right)\left(\mu - \frac{\sigma_5^2}{2}K^2\right)(K-N) = -\left(\mu - \frac{\sigma_5^2}{2}K^2\right)\cdot\frac{(N-K)^2}{N} \\
&\leq -\left(\mu - \frac{\sigma_5^2}{2}K^2\right)\frac{N(0)}{K}\cdot V_5(S,E,I,R) \quad \text{for } N \geq N(0)
\end{aligned}$$

since the total population $N(t)$ is monotonically increasing for our SEIR model and Lyapunov functional $V_5(S,E,I,R) = N - K - K\cdot\ln\left[\frac{N}{K}\right] =: \widetilde{V}_5(N)$ on $\mathbb{D}$ with monotonically decreasing $\widetilde{V}_5(N)$ in N (calculate $\widetilde{V}_5'(N) = (N-K)/N < 0$ on $N \in (0,K)$ and the simple calculus fact that

$$-\frac{(N-K)^2}{N} < -\frac{N}{K}\widetilde{V}_5(N) < -\frac{N(0)}{K}\widetilde{V}_5(N)$$

for all $N \geq N(0)$. Finally, with nonrandom initial $N(0) = S(0)+E(0)+I(0)+R(0) < K$, apply Dynkin's formula to arrive at

$$\begin{aligned}
\mathbb{E}\left[\widetilde{V}_5(N(t))\right] &= \mathbb{E}\left[\widetilde{V}_5(N(0))\right] + \mathbb{E}\left[\int_0^t \mathcal{L}\widetilde{V}_5(N(s))\,ds\right] \\
&\leq \mathbb{E}\left[\widetilde{V}_5(N(0))\right] - \left(\mu - \frac{\sigma_5^2}{2}K^2\right)\frac{N(0)}{K}\int_0^t \mathbb{E}\left[\widetilde{V}_5(N(s))\right]ds.
\end{aligned}$$

It remains to use the well-known Bellman–Gronwall lemma to conclude that

$$v(t) := \mathbb{E}\left[\widetilde{V}_5(N(t))\right]$$

(recall that $V_5(S(t),E(t),I(t),R(t)) = \widetilde{V}_5(N(t))$) in order to verify exponential moment stability along functional V_5 with a "least" rate estimated by $-(\mu - \frac{\sigma_5^2}{2}K^2)\frac{N(0)}{K}$. □

Remark 4.18 (A.s. stability and rates of exponential stability)**.** Since exponential moment stability also implies a.s. asymptotic stability, from Corollary 4.17, we also gain the conclusion on a.s. asymptotic V_5-stability of the endemic equilibrium (S_2, E_2, I_2, R_2) on $\overline{\mathbb{D}}$ under the hypothesis that $\mathcal{R}_0 > 1$ and $2\mu > \sigma_5^2 K^2$ (by dissipative techniques from [27]). Besides, the continuous time and discrete time moment attractivity exponents of other appropriate functionals $V \geq 0$ can also be estimated by some results from [26]. But, this would sprinkle the frame of this paper. Note, it is common that the rates of stability or attractivity of nonlinear dynamical systems depend on the initial values like $N(0)$ above (in contrast to linear systems).

5 Illustrations of moment functionals and reproduction number

Here we illustrate the behavior of moment Lyapunov functionals along the solutions of SEIR(S) model (1.1) and the structure of reproduction number. First, we plot the 2D surface of reproduction number $\mathcal{R}_0$ depending on growth parameter μ and transition parameters $\alpha + \gamma$. The conceivable hyperplane $\mathcal{R}_0 = 1$ decides whether the system (1.1) has an asymptotically stable disease-free or endemic equilibrium. For examples, above the hyperplane $\mathcal{R}_0 = 1$ we locate the region where the endemic equilibrium is asymptotically stochastically stable (similar below that plane for stability of disease-free equilibrium). Figure 5.1 shows that increasing μ stabilizes the dynamics of SEIR model (1.1) toward the disease-free equilibrium. This also happens with increasing the transition parameter sum $\alpha + \gamma$, but at a much slower scale. For sufficiently small μ and small $\alpha + \gamma$, the endemic equilibrium is asymptotically stochastically stable since the reproduction number is well above the hyperplane $\mathcal{R}_0 = 1$, as clearly seen in left corner of Figure 5.1.

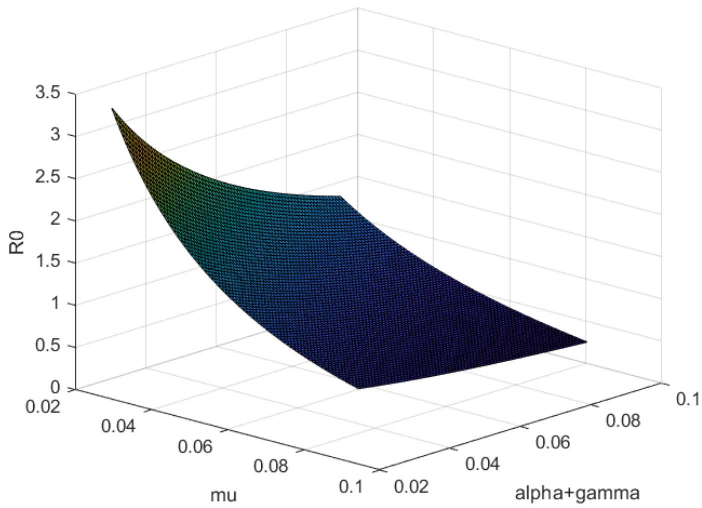

Figure 5.1: Reproduction number $\mathcal{R}_0(\mu, \alpha+\gamma)$ with $\beta = 25 \cdot 10^{-5}$, $\eta = 0.005$, $\zeta = 0.002$, $K = 1000$ depending on $\mu = mu$ and $\alpha + \gamma = r$.

Next, we illustrate the dynamics of total population process $N = (N(t))_{t \geq 0}$ in pathwise (a.s.) and mean sense. Figure 5.2 shows several paths of total population $N(t)$ generated by

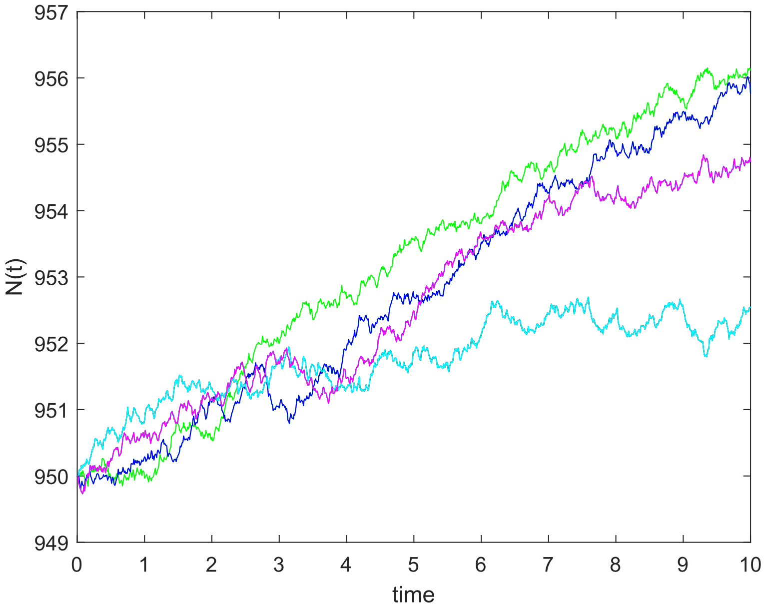

Figure 5.2: Several trajectories of total population with $\mu = 10^{-2}$, $K = 1000$, $\sigma = 10^{-5}$, $T = 10$, step size $h = 10^{-2}$, started at $N(0) = 950$.

the Euler–Maruyama method in MATLAB. This demonstrates the variety and erratic effect of noise on the solution-paths.

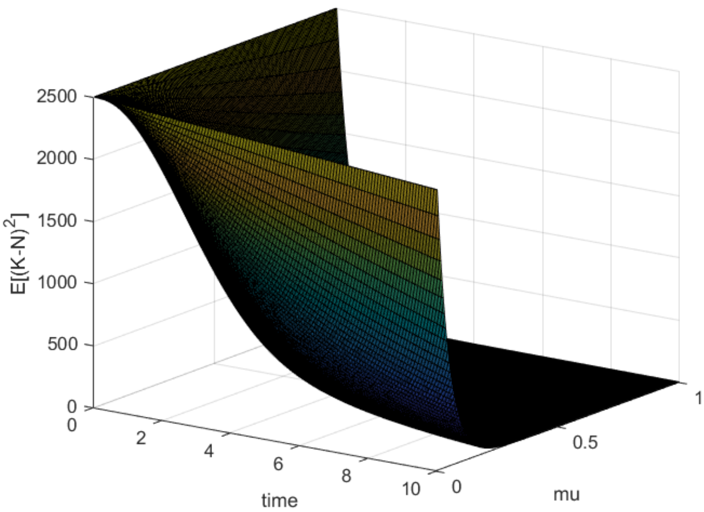

Figure 5.3: Expected Lyapunov functional $\mathbb{E}[K - N(t)]^2$ versus t and μ with $K = 1000$, $\sigma = 10^{-5}$, $T = 10$, $M = 10^6$, step size $h = 10^{-2}$, started at $N(0) = 950$.

Figure 5.3 displays the expected Lyapunov functional $\mathbb{E}[K - N(t)]^2$ versus time t and parameter μ, generated by $M = 10^6$ samples started with same total population size $N(0) = 950 < K = 10^3$ and discretized by standard Euler–Maruyama method with uniform step size $h = 10^{-2}$ in MATLAB. As seen there, the dynamics stabilize with increasing parameter μ and with advancing time t. The decline of 2D surface of $\mathbb{E}[K - N(t)]^2$ in time t and μ also confirms Theorem 4.6 that, for sufficiently large μ, we find asymptotically stable equilibrium $n^* = K$ for total population.

In Figure 5.4 the expected Lyapunov functional $\mathbb{E}[K - N(10)]^2$ versus parameters μ and σ is depicted, generated by $M = 10^6$ samples started with same total population size $N(0) = 950 < K = 10^3$ and discretized by standard Euler–Maruyama method with uniform step size $h = 10^{-2}$ in MATLAB. Clearly, we reckon that the dynamics of that functional is "destabilized" with increasing noise intensity σ and "stabilized" with increasing parameter μ. This gives us some statistical evidence for our Theorems 4.7 (i.e. decline of moments with growing $\mu > 0$) and 4.6 (i.e. the destabilizing effect of growing σ^2 on moments and stability). Of course, care is needed since growing variance with increasing σ^2 reduces our confidence in the estimation process and perhaps larger sample sizes are needed to confirm simulation results. All in all, larger noise intensities reveal a fairly nontrivial, nonlinear dependence of functionals $\mathbb{E}[K - N(10)]^2$ on model parameters (μ, σ).

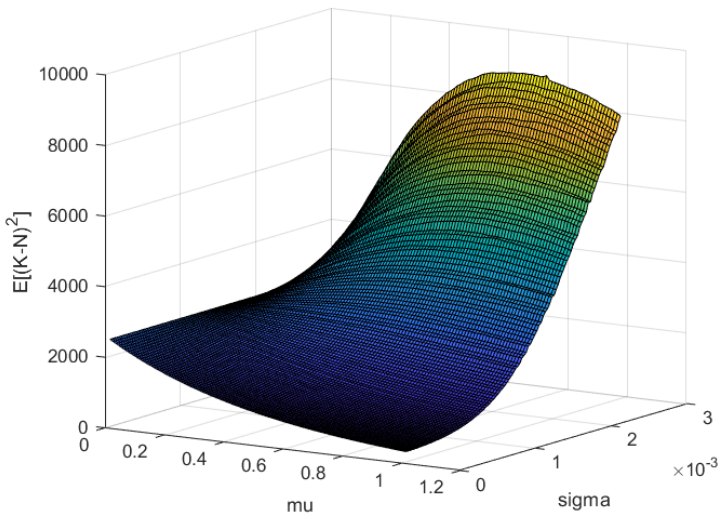

Figure 5.4: Expected Lyapunov functional $\mathbb{E}[K - N(10)]^2$ versus μ and σ with $K = 1000$, $T = 10$, $M = 10^6$, step size $h = 10^{-2}$, started at $N(0) = 950$.

Figure 5.5 plots the 2D surface of expected Lyapunov functional $\mathbb{E}[K-N(t)]^2$ versus time t and parameter σ with $\mu = 0.09$, generated by $M = 10^6$ samples started with same total population size $N(0) = 950 < K = 10^3$ and discretized by standard Euler–Maruyama method with step size $h = 10^{-2}$ in MATLAB. For small $\sigma > 0$, the surface of this functional declines at lower right corner of Figure 5.5. That is an empirical indicator that the SEIR(S) model is in the stable regime. However, for larger, increasing values of $\sigma > 0$, the 2D surface gets "destablized" as time t advances, as we especially reckon at upper right corner of Figure 5.5.

We could continue with showing more and more simulation results. Clearly, we have demonstrated the applicability of our analysis and have suggested to plot 2D surface of multi-dimensional expected Lyapunov functionals in order to get empirical evidence about which stable or unstable mode the SEIR(S) model is in. Eventually, by Figure 5.6, we display 2D surfaces of expected Lyapunov functional $m(t, p) = (\mathbb{E}[|K - N(t)|^{2p}])^{1/2p}$ depending on powers $p \geq 0.5$ and time t, while $\mu = 1.0$, $K = 1000$ and $\sigma = 10^{-5}$ are fixed. This shows the depen-

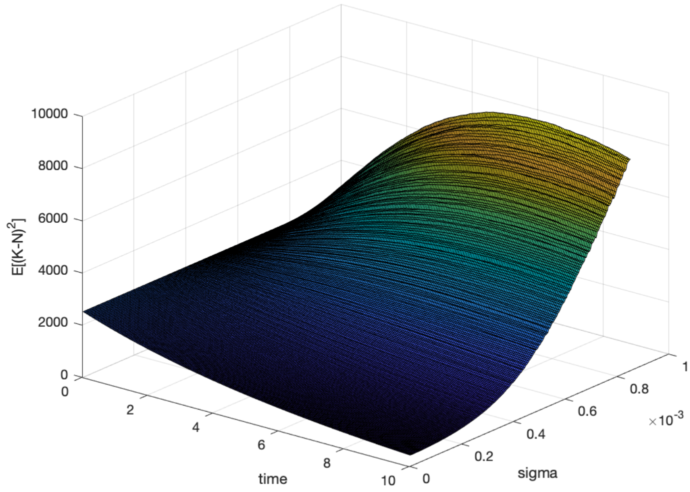

Figure 5.5: Expected Lyapunov functional $\mathbb{E}[K - N(t)]^2$ versus time t and σ with $\mu = 0.09$, $K = 1000$, $T = 10$, $M = 10^6$, step size $h = 10^{-2}$, started at $N(0) = 950$.

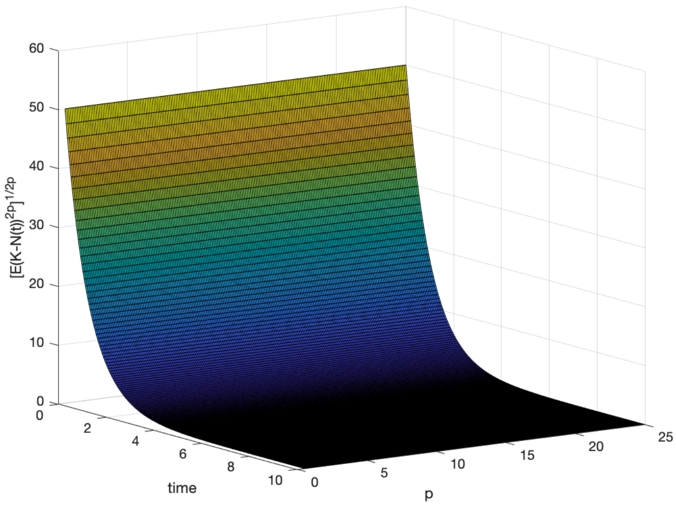

Figure 5.6: Expected Lyapunov functional $\left(\mathbb{E}[K - N(t)]^{2p}\right)^{1/2p}$ versus time t and power p with $\mu = 1.0$, $K = 1000$, $T = 10$, $M = 10^6$, step size $h = 0.05$, started at $N(0) = 950$.

dence of expected Lyapunov functionals $m(t,p)$ on powers $p \geq 0.5$ and time $t \geq 0$. As time t advances,the depicted hyperplane declines toward zero, giving some evidence of the stable mode of our SEIR(S) model since $2\mu > \sigma^2 K^2$ in our simulation. There are only small changes in p in that range. The decline of 2D-surface $m(t,p)$ in Figure 5.6 with increasing time t confirms the findings of Theorems 4.6 and 4.7 on asymptotic stability of equilibrium $n^* = N$ using Lyapunov functionals. The simulation has been conducted with the Euler–Maruyama method using step size $h = 0.05$ in MATLAB. The crude step size $h = 0.05$ is applied since the limited computational capacity of our computers, and we put our main emphasis on large sample sizes M and fine discretization of parameter space $p \in (0, 25]$ in order to get more statistical evidence (instead of higher numerical accuracy). To get some assurance of our graphical plots and stable computations, we repeated the experiments to get some confirmation by much smaller step sizes h (but at the expense of reduced sample sizes).

Similar experiments can be conducted for other functionals of biological interest like the convergence to all their equilibria. Such an endeavor is left to interested reader.

6 Summary, conclusions and outlook

This paper introduced a stochastic SEIR(S) model (1.1) based on Itô stochastic differential equations (SDEs) with a deterministic maximum saturation constant $K > 0$. The main emphasis is on the incorporation of possible random transitions from one compartment to another (sub-populations). As one of the major differences to previously introduced SEIR(S) models, our model (1.1) possesses a random total population $N = (N(t))_{t\geq 0}$, which itself is governed by a logistic Itô SDE with the equilibrium $n^* = K$. It was shown that the total population $N(t)$ is a.s. positive and bounded by the saturation constant $K > 0$ - a requirement for the practical relevance of any SEIR(S) models. Moreover, conditions have been worked out for the asymptotic stochastic and moment stability of the equilibrium K of the total population process $N = (N(t))_{t\geq 0}$. The analysis of dynamics of the total population N is essential for the understanding and qualitative control of the solutions of SEIR models (1.1).

The paper proves the existence of unique, strong solutions (S, E, I, R) of original SEIR(S) models (1.1) on bounded, positive prisms $\mathbb{D} \subset \mathbb{R}^4_+$ for all adapted, initial data residing inside $\mathbb{D}$ (with finite initial "energy"). We have also verified reasonable criteria for the asymptotic stochastic and moment stability of the disease-free and the endemic equilibria of (1.1). As commonly expected, the basic reproduction number $\mathcal{R}_0$ decides about the stable character of the equilibria ($\mathcal{R}_0 < 1$ for stability of the disease-free equilibrium and $\mathcal{R}_0 > 1$ for stability of the endemic equilibrium). Finally, we illustrated our major findings w.r.t. declining moment Lyapunov functionals, depending on several parameters. Very recently during submission of this paper, it came to our attention that there is already a generalization of SEIR(S) models with stochastic transmission by [36]. However, his model only allows back-and-forth transitions from S to E to S and there is no back coupling from R or I back to S and E, and he does not incorporate general functions F_k controlling the rates of nonlinearities (i.e. just the case of constant rates in the incidence terms). Moreover, a verification of a.s. exponential stability of equilibria is only conducted there. Our model also admits random transitions from the remaining population $K - N$ to the sub-populations S, E, I, R with $N = S + E + I + R$.

There are plenty of possible generalizations. One could try out Levy-type- or jump-processes for the random noise sources or Markovian switching or non-Markovian regimes. However, all generalizations should be done through semi-martingale theory due to the continuity requirement of the underlying integration operator in biologically relevant applications.

Itô calculus interpretations are the commonly adopted models for the sake of the fact that the offspring populations should only depend on the past, i.e. the closest parental generations. At the end, statistical matching to real data would decide on the relevance of each SEIR(S) model. We leave the practical execution of all of those ideas to the interested reader. We are convinced that our model class already offers enough flexibility and interesting phenomena, however restricted to Markovian modeling by this contribution.

A Appendix: A general existence result of solutions of SDEs

Consider d-dimensional, Itô-interpreted stochastic differential equations (SDEs) of the form

$$dX(t) = f\big(X(t),t\big)dt + g\big(X(t),t\big)dW(t) \tag{A.1}$$

with initial value $X(t_0) = X_0, t_0 \leq t \leq T < +\infty$, where $f : \mathbb{R}^d \times [t_0,T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0,T] \to \mathbb{R}^{d\times m}$ are Borel measurable functions, $W = \{W(t)\}_{t\geq t_0}$ is a $\mathbb{R}^m-$valued Wiener process and X_0 is a $\mathbb{R}^d-$valued random variable. Recall that its infinitesimal generator $\mathcal{L}$ associated with the above SDE (A.1) is given by

$$\mathcal{L} \;=\; \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{d}\sum_{k=1}^{d} g_i^j(x,t)g_k^j(x,t)\frac{\partial^2}{\partial x_i \partial x_k}. \tag{A.2}$$

Theorem A.1 (Improved version of a theorem from Khas'minskii (1980))**.** *Assume that*

(i) $f,g \in C^0_{locLip(L)}\big(\mathbb{D}\times[0,T]\big)$,

(ii) $(\mathbb{D}_r)_{r>0}$ *nondecreasing, bounded, connected, all* $\mathbb{D}_r \subseteq \mathbb{R}^d$ *and* $\mathbb{D} = \cup_{r>0}\mathbb{D}_r$,

(iii) $\sigma\big(X(0)\big)$ *is independent of* $\sigma\big(W(s) : s \leq T\big)$ *and* $X(0) \in \mathbb{D}$,

(iv) $\exists\, V \in C^{2,1}\big(\mathbb{D}\times[0,T]\big)$ *with* $V : \mathbb{D}\times[0,T] \to \mathbb{R}^1_+$, $\exists\, a \in L^1\big([0,T]\big)$

$$\forall\, x \in \mathbb{D}\ \forall\, t \in [0,T] : \quad \mathcal{L}V(x,t) \leq a \cdot V(x,t),$$

(v) $\mathbb{E}\big[V\big(X(0),0\big)\big] < +\infty$,

(vi) $\inf_{t>0,\ x\in\partial\mathbb{D}_r} V(x,t) \overset{r\to+\infty}{\longrightarrow} +\infty$.

Then, $\exists$ *strong, unique, continuous time, Markovian solution* X *of SDE* (A.1) *with* $X(0) = X_0$ *and* $X(t) \in \mathbb{D}$ *for all* $t > 0$.

Remark A.2 (Linear versus exponential moment bounds)**.** The conclusion of Theorem A.1 remains valid if one replaces the assumption (iv) by the hypothesis

(iv)' $\exists\, V \in C^{2,1}\big(\mathbb{D}\times[0,T]\big)$ with $V : \mathbb{D}\times[0,T] \to \mathbb{R}^1_+$, $\exists\, a \in L^1\big([0,T]\big)$

$$\forall\, x \in \mathbb{D}\ \forall t \in [0,T] : \quad \mathcal{L}V(x,t) \leq c_0,$$

where c_0 is an appropriate constant. In this case, one is able to prove the uniform boundedness

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[V(X(t),t)\Big] \leq \mathbb{E}\Big[V(X(0),0)\Big] + [c_0]_+ \cdot T$$

of the moments along the functionals V of solutions X. In contrast to that fact, the original assumption (iv) of Theorem A.1 with any constant $a(t) = c_1$ guarantees the uniform exponential bounds

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[V(X(t),t)\Big] \leq \mathbb{E}\Big[V(X(0),0)\Big] \cdot \exp\big([c_1]_+ \cdot T\big)$$

of the moments $\mathbb{E}[V(X(t),t)]$ as worst-case estimate. Here, $[\cdot]_+$ denotes the nonnegative part of the inscribed mathematical expression.

Remark A.3 (Comment on uniqueness of solutions)**.** Uniqueness of strong solutions X of SDEs (A.1) with local Lipschitz continuous coefficients f, g on open, connected sets $\mathbb{D} \subseteq \mathbb{R}^d$ can only be lost when the solutions explode on the boundary of $\mathbb{D}$. Common (nonrandom) equilibria x^* of both f and g are considered unique solutions $X = x^*$ of SDEs (A.1) itself, sometimes called **trivial solutions** or **equilibrium solutions** (i.e., in this case, applied to SDEs with extended drift and diffusion coefficients vanishing on entire $\mathbb{D}$). In our paper the existence of local solutions is established for SDEs with Lipschitz coefficients inside the open prism $\mathbb{D}$. The uniqueness of such local solutions inside $\mathbb{D}$ is clear from standard texts on SDEs (such as [3], [14] and [23]) since the closed prism $\overline{\mathbb{D}}$ is a compact set and we do not hit the boundary of $\mathbb{D}$ at any finite time, provided that we start inside the prism (that latter is what we presumed anyway). Recall that the equilibria of our SEIR(S) model are located on the boundary of the open prism $\mathbb{D}$. Hence, they can not be reached in any finite time from the interior of $\mathbb{D}$. Moreover, we have proved the boundedness of moments along certain Lyapunov functionals V, which implies that the solutions can not hit the boundary of the prism $\mathbb{D}$. This is obvious from the application of Khasminskij's Theorem A.1 in this appendix. We just had to construct and verify a related Lyapunov functional V and the appropriate set $\mathbb{D}$ for our SEIR(S) model.

Contribution statement

The first author suggested the work on this SEIR(S) model (1.1) and its extension (3.1). The 2nd and 3rd authors simulated all pictures (Figures 5.1–5.6) of this paper by MATLAB and checked the calculations for the related Lyapunov functionals.

Interest statement

There is not any conflict of interest and any other competitive interest of whatever nature.

Data statement

We have not used any real-world data. All our plotted data have arisen from the simulations.

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