Normalized solutions for Schrödinger equations with potential and general nonlinearities involving critical case on large convex domains

ⓘ Jun Wang✉[1] and **ⓘ Zhaoyang Yin**[1, 2]

[1]Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, China
[2]School of Science, Shenzhen Campus of Sun Yat-sen University, Shenzhen, 518107, China

Abstract. In this paper, we study the following Schrödinger equations with potentials and general nonlinearities

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{q-2}u + \beta f(u), \\ \int |u|^2 dx = \Theta, \end{cases}$$

both on $\mathbb{R}^N$ as well as on domains Ω_r where $\Omega_r \subset \mathbb{R}^N$ is an open bounded convex domain and $r > 0$ is large. The exponent satisfies $2 + \frac{4}{N} \leq q \leq 2^* = \frac{2N}{N-2}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies L^2-subcritical or L^2-critical growth. This paper generalizes the conclusion of Bartsch et al. in [4]. Moreover, we consider the Sobolev critical case and L^2-critical case of the above problem.

Keywords: Schrödinger equations, normalized solutions, variational methods, mixed nonlinearity.

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1 Introduction and main results

This paper studies the existence of normalized solutions for the following Schrödinger equations with potentials and general nonlinearities

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{q-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r, \end{cases} \tag{1.1}$$

where $\Omega_r \subset \mathbb{R}^N$ is either all of $\mathbb{R}^N$ or a bounded smooth convex domain, $N \geq 3$, $2 + \frac{4}{N} \leq q \leq 2^* = \frac{2N}{N-2}$, the mass $\Theta > 0$ and the parameter $\beta \in \mathbb{R}$ are prescribed. The frequency λ is unknown and to be determined.

✉Corresponding author. Email: wangj937@mail2.sysu.edu.cn

Such problems are motivated in particular by searching for solitary waves (stationary states) in nonlinear equations of the Schrödinger type. Specifically, consider the following nonlinear Schrödinger equation

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Psi = \Delta\Psi - V(x)\Psi + f\left(|\Psi|^2\right)\Psi = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ \Psi = \Psi(x,t), & (x,t) \in \mathbb{C}, \end{cases}$$

where $N \geq 1$. Researchers are interested in finding the existence of standing wave solutions to the above equations, that is, $\Psi(x,t) = e^{i\lambda t}u(x), \lambda \in \mathbb{R}$, and $u : \mathbb{R}^N \to \mathbb{R}$, so we get the equation

$$-\Delta u + (V(x) + \lambda)u = Q(u), \qquad x \in \mathbb{R}^N,$$

where $Q(u) = f(|u|^2)u$. For physical reasons, we focus on the existence of normalized solutions for the following problem

$$\begin{cases} -\Delta u + (V(x) + \lambda)u = Q(u), & x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = \Theta, & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

For more physical background about the above equation, please refer to [9, 16].

If potential $V(x)$ in (1.2) is constant, we call (1.2) is autonomous. In this case, recalling paper [21], Jeanjean developed an approach based on the Pohozaev identity which has been used successfully in recent years. The key to this method is to find a bounded Palais–Smale sequences by using the transformation $s * u(x) = e^{\frac{sN}{2}}u(e^s x)$. After that, by weakening the conditions in [21], Jeanjean [22] and Bieganowski [8] improved these results. Of course, these articles only consider the problem of a single nonlinear term. Recently, there have been many studies on mixed nonlinear terms. For example, Soave [26, 27] studied normalized solution of (1.2) with mixed nonlinearity $f(|u|)u = \mu|u|^{q-2}u + |u|^{p-2}u$, $2 < p < 2 + \frac{4}{N} < q \leq 2^* = \frac{2N}{N-2}$. Specifically, Soave in [26] obtained many results of existence and non-existence. More precisely, if $2 < q < p = 2 + \frac{4}{N}$, that is, the leading nonlinearity is L^2-critical and a L^2-subcritical lower order term. (1.2) had a real-valued positive and radially symmetric solution for some $\lambda < 0$ in $\mathbb{R}^N$ provided $\mu > 0$ and $\Theta > 0$ small enough. Moreover, if $\mu < 0$, (1.2) had no solution. If $2 + \frac{4}{N} = q < p < 2^*$, that is, the leading term is L^2-critical and L^2-supercritical, (1.2) had a real-valued positive, radially symmetric solution for some $\lambda < 0$ in $\mathbb{R}^N$ provided $\mu > 0$ and μ, Θ satisfy the appropriate conditions. If $2 < q < 2 + \frac{4}{N} < p < 2^*$, that is, the leading term L^2-subcritical and L^2-supercritical, (1.2) also had a real-valued positive and radially symmetric solution for some $\lambda < 0$ in $\mathbb{R}^N$ provided $\Theta > 0, \mu < 0$ and μ, Θ satisfy the appropriate conditions. Soave in [27] considered the Sobolev critical case and obtained some similar results. In particular, the Sobolev critical case also has been considered in [1, 2, 24, 25](see also the references therein). It is worth mentioning that many researchers are also interested in the existence of normalized multiple solutions. In [23], Jeanjean et al. obtained the existence of normalized multiple solutions for Sobolev critical case in (1.2). For more results on this aspect, please refer to [5–7, 10, 29] and its references.

If (1.2) is non-autonomous, Ikoma and Miyamoto in [19] considered question (1.2) with $V(x) \in C(\mathbb{R}^N), 0 \not\equiv V(x) \leq 0, V(x) \to 0(|x| \to \infty)$, they obtained some existence and non-existence results. After that, Ding and Zhong in [14] proved the existence of normalized solutions to the following Schrödinger equation

$$\begin{cases} -\Delta u(x) + V(x)u(x) + \lambda u(x) = g(u(x)), & x \in \mathbb{R}^N, \\ 0 \leq u(x) \in H^1(\mathbb{R}^N), \ N \geq 3, \end{cases}$$

where g satisfies:

(G1) $g : \mathbb{R} \to \mathbb{R}$ is C^1 and odd.

(G2)   There exists some $(\alpha, \beta) \in \mathbb{R}_+^2$ satisfying $2 + \frac{4}{N} < \alpha \leq \beta < \frac{2N}{N-2}$ such that

$$\alpha G(s) \leq g(s)s \leq \beta G(s) \text{ with } G(s) = \int_0^s g(t)dt.$$

(G3)   The functional defined by $\widetilde{G}(s) := \frac{1}{2}g(s)s - G(s)$ is of class C^1 and

$$\widetilde{G}'(s)s \geq \alpha \widetilde{G}(s), \forall s \in \mathbb{R},$$

where α is given by $(G2)$.

Note that, $(G3)$ plays a crucial role in the uniqueness of t_u(see [14] or [21, Lemma 2.9]). However, we do not need this condition, since we directly perform scaling and complex calculations on energy functionals. Recently, Bartsch et al. in [4] considered following Schrödinger equations with potentials and inhomogeneous nonlinearities on large convex domains

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{q-2}u + \beta|u|^{p-2}u, \\ \int |u|^2 dx = \Theta, \end{cases}$$

they developed a robust method to study the existence of normalized solutions of nonlinear Schrödinger equations with potential. Under the stimulation of [4], our goal is to generalize its conclusion to general nonlinear terms and the Sobolev critical case.

In order to state our main results, we introduce some notations. Set $s_+ = \max\{s, 0\}$, $s_- = \min\{s, 0\}$ for $s \in \mathbb{R}$. The Aubin–Talenti constant [3] is denoted by S, that is, S is the best constant in the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{\mathcal{D}^{1,2}} := \|\nabla u\|_2$. It is well known [28] that the optimal constant is achieved by (any multiple of)

$$U_{\varepsilon,y}(x) = [N(N-2)]^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{\frac{N-2}{2}}, \qquad \varepsilon > 0, \ y \in \mathbb{R}^N, \tag{1.3}$$

which are the only positive classical solutions to the critical Lane–Emden equation

$$-\Delta w = w^{2^*-1}, \qquad w > 0 \text{ in } \mathbb{R}^N.$$

Let $C_{N,s}$ be the best constant in the Gagliardo–Nirenberg inequality

$$\|u\|_s^s \leq C_{N,s} \|u\|_2^{\frac{2s-N(s-2)}{2}} \|\nabla u\|_2^{\frac{N(s-2)}{2}}, \qquad 2 < s < 2^*.$$

For some results, we expect that V is C^1 and consider the function

$$\widetilde{V} : \mathbb{R}^N \to \mathbb{R}, \qquad \widetilde{V}(x) = \nabla V(x) \cdot x.$$

For $\Omega \subset \mathbb{R}^N$ and $r > 0$, let

$$\Omega_r = \left\{ rx \in \mathbb{R}^N : x \in \Omega \right\}$$

and

$$S_{r,\Theta} := S_\Theta \cap H_0^1(\Omega_r) = \left\{ u \in H_0^1(\Omega_r) : \|u\|_{L^2(\Omega_r)}^2 = \Theta \right\}.$$

From now on we assume that $\Omega \subset \mathbb{R}^N$ is a bounded smooth convex domain with $0 \in \Omega$.

Our assumptions on V are:

(V_0) $V \in C^1(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ is bounded and $\|V_-\|_{\frac{N}{2}} < S$.

$(\widetilde{V_0})$ $V \in C^1(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ is bounded and $\|V_-\|_{\frac{N}{2}} < \frac{N(q-p_2)-2[N(p_2-2)-4]}{N(q-p_2)}S$.

$(\widehat{V_0})$ $V \in C^1(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ is bounded and $\|V_-\|_{\frac{N}{2}} < \left(1 - \frac{NC_N\Theta^{\frac{2}{N}}}{N+2}\right)S$.

(V_1) V is of class $C^1, \lim_{|x|\to\infty} V(x) = 0$, and there exists $\rho \in (0,1)$ such that

$$\liminf_{|x|\to\infty} \inf_{y\in B(x,\rho|x|)} (x \cdot \nabla V(y))e^{\tau|x|} > 0 \quad \text{for any } \tau > 0.$$

Remark 1.1. In order to obtain the existence of normalized solutions in $\mathbb{R}^N$ by taking $\Omega = B_1$, the unit ball centered at the origin in $\mathbb{R}^N$, and analyzing the compactness of the solutions $u_{r,\Theta}$ established in Theorems 1.3, 1.4 and 1.5 as r tends to infinity, we require the condition (V_1).

Now, we make the following assumptions on the nonlinearity f:

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and f is odd.

(f_2) There exists some $(p_1, p_2) \in \mathbb{R}^2_+$ satisfying $2 < p_2 \le p_1 < 2 + \frac{4}{N}$ such that

$$p_2F(\tau) \le f(\tau)\tau \le p_1F(\tau) \text{ with } F(\tau) = \int_0^\tau f(t)dt.$$

$(\widetilde{f_2})$ There exists some $(p_1, p_2) \in \mathbb{R}^2_+$ satisfying $2 < p_2 < p_1 = 2 + \frac{4}{N}$ such that

$$p_2F(\tau) \le f(\tau)\tau \le p_1F(\tau).$$

Remark 1.2. If $f(u) = \sum_{i=1}^m a_i|u|^{\sigma_i-2}u$, where $a_i > 0$ and $2 < \sigma_i < 2 + \frac{4}{N}$, then the assumption (f_1) can be weakened to $f \in C(\mathbb{R}, \mathbb{R})$ and f is odd. In order to ensure the boundedness of Palais–Smale sequence under constraint conditions in Lemma 3.3, we need to slightly strengthen the conditions for the nonlinear term f, that is, $f \in C^1(\mathbb{R}, \mathbb{R})$.

The main results of this paper are as follows. Firstly, we consider the Sobolev subcritical case, that is, $2 + \frac{4}{N} < q < 2^*$.

Theorem 1.3 (case $\beta \le 0$). *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1)–(f_2). There hold:*

(i) *For every $\Theta > 0$, there exists $r_\Theta > 0$ such that (1.1) on Ω_r with $r > r_\Theta$ has a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and positive energy $I_r(u_{r,\Theta}) > 0$. Moreover, there exists $C_\Theta > 0$ such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

(ii) *If in addition $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, then there exists $\widetilde{\Theta} > 0$ such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \text{for any } 0 < \Theta < \widetilde{\Theta}.$$

Theorem 1.4 (case $\beta > 0$). *Assume V satisfies (V_0), f satisfies (f_1)–(f_2) and set*

$$\Theta_V = \left[\frac{1 - \|V_-\|_{\frac{N}{2}} S^{-1}}{2N(q - p_1)} \right]^{\frac{N}{2}} \left[\frac{q(4 - N(p_1 - 2))}{C_{N,q}} \right]^{\frac{4 - N(p_1 - 2)}{2(q - p_1)}} \left[\frac{N(q - 2) - 4}{\alpha\beta C_{N,p_1}} \right]^{\frac{N(q-2)-4}{2(q-p_1)}}.$$

Then the following hold for $0 < \Theta < \Theta_V$:

(i) *There exists $r_\Theta > 0$ such that (1.1) on Ω_r with $r > r_\Theta$ has a local minimum type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and negative energy $I_r(u_{r,\Theta}) < 0$.*

(ii) *There exists $C_\Theta > 0$ such that*

$$\limsup_{r \to \infty} \max_{x \in \Omega_r} u_{r,\Theta}(x) < C_\Theta, \qquad \liminf_{r \to \infty} \lambda_{r,\Theta} > 0.$$

Theorem 1.5 (case $\beta > 0$). *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1)–(f_2). Set*

$$\widetilde{\Theta}_V = \frac{1}{2} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} \left(\frac{C_{N,q}}{q} A_{p_1,q} + \frac{C_{N,q}}{q} \right)^{-\frac{N}{2}} \left(\frac{\alpha\beta q C_{N,p_1}}{C_{N,q} A_{p_1,q}} \right)^{\frac{N(q-2)-4}{2N(q-p_1)}},$$

where

$$A_{p_1,q} = \frac{(q-2)(N(q-2)-4)}{(p_1-2)(4-N(p_1-2))}.$$

Then the following hold for $0 < \Theta < \widetilde{\Theta}_V$:

(i) *There exists $\widetilde{r}_\Theta > 0$ such that (1.1) in Ω_r admits for $r > r_\Theta$ a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and positive energy $I_r(u_{r,\Theta}) > 0$. Moreover, there exists $C_\Theta > 0$ such that*

$$\limsup_{r \to \infty} \max_{x \in \Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

(ii) *There exists $0 < \bar{\Theta} \le \widetilde{\Theta}_V$ such that*

$$\liminf_{r \to \infty} \lambda_{r,\Theta} > 0 \quad \text{for any } 0 < \Theta \le \bar{\Theta}.$$

If $\Omega = \mathbb{R}^N$, (V_1) is significant for obtaining the following results.

Theorem 1.6 (case $\beta > 0$). *Assume V satisfies (V_0)–(V_1). Then problem (1.1) with $\Omega = \mathbb{R}^N$ admits for any $0 < \Theta < \Theta_V$, where Θ_V is as in Theorem 1.4, a solution $(\lambda_\Theta, u_\Theta)$ with $u_\Theta > 0, \lambda_\Theta > 0$, and $I(u_\Theta) < 0$.*

Theorem 1.7 (case $\beta > 0$). *Assume V satisfies (V_0)–(V_1). Then (1.1) with $\Omega = \mathbb{R}^N$ admits for $0 < \Theta < \bar{\Theta}, \bar{\Theta} > 0$ as in Theorem 1.5 (ii), a solution $(\lambda_\Theta, u_\Theta)$ with $u_\Theta > 0, \lambda_\Theta > 0$, and $I(u_\Theta) > 0$. Moreover, $\lim_{\Theta \to 0} I(u_\Theta) = \infty$.*

Theorem 1.8 (case $\beta \le 0$). *Assume V satisfies (V_0)–(V_1), and $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$. Then problem (1.1) with $\Omega = \mathbb{R}^N$ admits for $0 < \Theta < \widetilde{\Theta}, \widetilde{\Theta} > 0$ as in Theorem 1.3, a solution $(\lambda_\Theta, u_\Theta)$ with $u_\Theta > 0, \lambda_\Theta > 0$, and $I(u_\Theta) > 0$. Moreover, $\lim_{\Theta \to 0} I(u_\Theta) = \infty$.*

For the Sobolev critical case, that is $q = 2^*$, we have the following results.

Theorem 1.9 (case $\beta > 0$). *Assume V satisfies (V_0), f satisfies (f_1)–(f_2). Set*

$$\Theta_V = \left(\frac{1}{N\alpha\beta C_{N,p_1}}\right)^{\frac{4}{2p_1-N(p_1-2)}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}\cdot\frac{4-N(p_1-2)}{2p_1-N(p_1-2)}}.$$

Then the following hold for $0 < \Theta < \Theta_V$:

  (i) *There exists $r_\Theta > 0$ such that (6.2) on Ω_r with $r > r_\Theta$ has a local minimum type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and negative energy $\mathcal{I}_r(u_{r,\Theta}) < 0$.*

  (ii) *There exists $C_\Theta > 0$ such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta, \qquad \liminf_{r\to\infty} \lambda_{r,\Theta} > 0.$$

Theorem 1.10 (case $\beta \leq 0$). *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1)–(f_2). There hold:*

  (i) *There exists $r_\Theta > 0$ such that (7.1) on Ω_r with $r > r_\Theta$ has a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} \geq 0$ in Ω_r and positive energy $\mathcal{I}_r(u_{r,\Theta}) > 0$.*

  (ii) *There exists $C_\Theta > 0$ such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

Theorem 1.11 (case $\beta > 0$). *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1)–(f_2). Set*

$$\widetilde{\Theta}_V = \left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}}}{A_{p_1}}\right)^{-\frac{4}{2p_1-N(p_1-2)}} \left[\frac{S^{\frac{2^*}{2}}}{2\cdot 2^*}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)(2^* A_{p_1}+1)\right]^{\frac{2[2\cdot2^*-N(p_1-2)]}{(2^*-2)[2p_1-N(p_1-2)]}}$$

where
$$A_{p_1} = \frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))}.$$

Then the following hold for $0 < \Theta < \widetilde{\Theta}_V$:

  (i) *There exists $\widetilde{r}_\Theta > 0$ such that (8.1) in Ω_r admits for $r > r_\Theta$ a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} \geq 0$ in Ω_r and positive energy $\mathcal{I}_r(u_{r,\Theta}) > 0$.*

  (ii) *There exists $C_\Theta > 0$ such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

For the L^2-critical case, that is $p_1 = 2 + \frac{4}{N}$ or $q = 2 + \frac{4}{N}$, we have the following results.

Theorem 1.12 (case $\beta > 0$ and $p_1 = 2 + \frac{4}{N}$). *Assume V satisfies $(\widetilde{V_0})$, f satisfies (f_1) and $(\widetilde{f_2})$. Set*

$$\widetilde{\Theta}_V = \left[\frac{N(q-p_2)-4}{N\alpha\beta(q-p_2)C_N}\right]^{\frac{N}{2}}.$$

Then the following hold for $0 < \Theta < \widetilde{\Theta}_V$:

(i) *There exists $\widetilde{r}_\Theta > 0$ such that (1.1) in Ω_r admits for $r > r_\Theta$ a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and positive energy $I_r(u_{r,\Theta}) > 0$. Moreover, there exists $C_\Theta > 0$ such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

(ii) *There exists $0 < \bar\Theta \le \widetilde{\Theta}_V$ such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \text{for any } 0 < \Theta \le \bar\Theta.$$

Theorem 1.13 (case $\beta \le 0$ and $p_1 = 2 + \frac{4}{N}$). *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1) and $(\widetilde{f}_2)$. Set*

$$\widehat{\Theta}_V = \left[\frac{(N-2)q - 2N}{2N\alpha\beta(q - p_2)C_N} \right]^{\frac{N}{2}}.$$

Then the following hold for $0 < \Theta < \widehat{\Theta}_V$:

(i) *There exists $r_\Theta > 0$ such that (1.1) on Ω_r with $r > r_\Theta$ has a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and positive energy $I_r(u_{r,\Theta}) > 0$. Moreover, there exists $C_\Theta > 0$ such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

(ii) *If in addition $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, then there exists $\widetilde{\Theta} > 0$ such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \text{for any } 0 < \Theta < \widetilde{\Theta}.$$

Theorem 1.14. *(case $\beta > 0$ and $q = 2 + \frac{4}{N}$) Assume V satisfies $(\widehat{V}_0)$, f satisfies (f_1)–(f_2) and set*

$$\Theta_V = \left(\frac{N+2}{NC_N} \right)^{\frac{N}{2}}.$$

Then the following hold for $0 < \Theta < \Theta_V$:

(i) *There exists $r_\Theta > 0$ such that (1.1) on Ω_r with $r > r_\Theta$ has a global minimum type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and negative energy $I_r(u_{r,\Theta}) < 0$.*

(ii) *There exists $C_\Theta > 0$ such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta, \qquad \liminf_{r\to\infty} \lambda_{r,\Theta} > 0.$$

Remark 1.15.

(i) Theorems 1.3–1.11 are valid if $2 = p_2 < p_1 < 2 + \frac{4}{N}$ in (f_2). Moreover, the proof of Theorems 1.6–1.8 is very similar to [4], so we omit it in this paper.

(ii) Our conclusion also applies to $p_1 = p_2 = 2 + \frac{4}{N}$ if $2 + \frac{4}{N} < q < 2^*$, such as $f(u) = |u|^{\frac{4}{N}}u$. Therefore, our results cover certain conclusions in [26].

Remark 1.16. Theorems 1.4 and 1.9 (resp. Theorems 1.5 and 1.11) both require some limitations on Θ_V (resp. $\widetilde{\Theta}_V$), although their values are different, they all stem from changes in the geometric structure of the energy functional. In addition, there are still some unknown results for the Sobolev critical case, that is, $\liminf_{r\to\infty} \lambda_{r,\Theta} > 0$ may not necessarily hold when $\beta > 0$ or $\beta \leq 0$. In fact, the methods and techniques in Theorem 1.4 (or Theorem 1.5) cannot be applied to the Sobolev critical case since $\frac{(N-2)q-2N}{2Nq} = 0$, thus $\lambda_\Theta > 0$ cannot be obtained $0 < \Theta \leq \bar{\Theta}$.

Remark 1.17. In this paper, whether in subcritical or critical situations, the monotonicity trick in [20] is one of the keys to get the conclusion. Proposition 3.2 does not ensure the existence of a mountain pass solution for the original problem obtained when $s = 1$. However, it gives the existence of a sequence $s_n \to 1^-$, with a corresponding sequence of mountain pass critical points u_{r,s_n} of I_{r,s_n}, constrained on $S_{r,\Theta}$. We aim to show that u_{r,s_n} strongly converges to a constrained critical point of I_r. For this purpose, it is sufficient to prove that u_{r,s_n} is bounded in $H_0^1(\Omega_r)$, thanks to Proposition 3.1 in [15].

The structure of this paper is arranged as follows. In section 2, we provide some ideas in the proof of main theorems. In section 3, we obtain the mountain pass type positive solution in the case $\beta \leq 0$ and have completed the proof of Theorem 1.3. If $\beta > 0$, there are two situations, that is, Theorems 1.4 and 1.5. We get the two results in sections 3 and 4 by using different geometric analysis. After that, we consider the Sobolev critical case. Finally, we consider the L^2-critical case and give some comments.

2   Preliminary

Consider the problem

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{q-2}u + \beta f(u), & x \in \Omega_r, \\ \displaystyle\int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r, \end{cases} \tag{2.1}$$

where $N \geq 3, 2 + \frac{4}{N} \leq q \leq 2^* = \frac{2N}{N-2}$, the mass $\Theta > 0$ and the parameter $\beta \in \mathbb{R}$ are prescribed. The frequency λ is unknown and to be determined. The energy functional $I_r : H_0^1(\Omega_r) \to \mathbb{R}$ is defined by

$$I_r(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} V(x)u^2 dx - \frac{1}{q}\int_{\Omega_r} |u|^q dx - \beta\int_{\Omega_r} F(u)dx \tag{2.2}$$

and the mass constraint manifold is defined by

$$S_{r,\Theta} = \left\{ u \in H_0^1(\Omega_r) : \|u\|_2^2 = \Theta \right\}. \tag{2.3}$$

If $\Omega = \mathbb{R}^N$, the energy functional $I : H_0^1(\Omega_r) \to \mathbb{R}$ is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \beta\int_{\mathbb{R}^N} F(u)dx \tag{2.4}$$

and the mass constraint manifold is defined by

$$S_\Theta = \left\{ u \in H_0^1(\mathbb{R}^N) : \|u\|_2^2 = \Theta \right\}. \tag{2.5}$$

The proof idea of Theorem 1.3 is as follows. In order to find a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$, we first need to analyze the geometric structure of the energy functional corresponding to the equation (3.1). In Lemma 3.1, we perform precise geometric analysis on the energy functional corresponding to (3.1) and know that the energy functional $I_{r,s}$ has a global maximum. Next, we obtain the bounded Palais–Smale sequence by using [11, Theorem 1] and get a solution $(\lambda_{r,s}, u_{r,s})$ for (3.1). Finally, we consider Lagrange multiplier and establish an a priori estimate for the solutions of (1.1). Theorem 1.4 is relatively simple because the energy functional has a local minimum, which can be proved using the method of constrained minimization. The proof of Theorem 1.5 is similar to Theorem 1.3, but the geometric structures of the two cases are significantly different and require refined estimate of energy.

Note that, there are some differences between the proof of Lemma 5.3 and Lemma 3.4, and we cannot directly use the method of Lemma 3.4, even if q can be reduced to p_2 according to condition (f_2) and $p_2 < 2 + \frac{4}{N} < q < 2^*$. More precisely, it then follows from $\beta > 0$ and (f_2) that

$$
\begin{aligned}
& \frac{1}{N} \int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2N} \int_{\partial \Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma - \frac{1}{2N} \int_{\Omega_r} (\nabla V \cdot x) u^2 dx \\
&= \frac{(q-2)s}{2q} \int_{\Omega_r} |u|^q dx + s \int_{\Omega_r} (\frac{\beta}{2} f(u)u - \beta F(u)) dx \\
&\geq \frac{p_2 - 2}{2} \left(\frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V u^2 dx - m_{r,s}(\Theta) \right).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\frac{p_2 - 2}{2} m_{r,s}(\Theta) \geq{} & \frac{p_2 - 2}{2} \left(\frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V u^2 dx \right) - \frac{1}{N} \int_{\Omega_r} |\nabla u|^2 dx \\
& + \frac{1}{2N} \int_{\partial \Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma + \frac{1}{2N} \int_{\Omega_r} (\nabla V \cdot x) u^2 dx \\
\geq{} & \frac{N(p_2 - 2) - 4}{4N} \int_{\Omega_r} |\nabla u|^2 dx - \Theta \left(\frac{1}{2N} \|\nabla V \cdot x\|_\infty + \frac{p_2 - 2}{4} \|V\|_\infty \right).
\end{aligned}
$$

However, this method is useless because $\frac{N(p_2-2)-4}{4N} < 0$, we cannot obtain that $\int_{\Omega_r} |\nabla u|^2 dx$ is uniformly bounded in s and r.

3   Proof of Theorem 1.3

In this section, we assume $\beta \leq 0$ and the assumptions of Theorem 1.3 hold. In order to obtain a bounded Palais–Smale sequence, we will use the monotonicity trick inspired by [20]. For $\frac{1}{2} \leq s \leq 1$, we define the functional $I_{r,s} : S_{r,\Theta} \to \mathbb{R}$ by

$$
I_{r,s}(u) = \frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V u^2 dx - \frac{s}{q} \int_{\Omega_r} |u|^q dx - \beta \int_{\Omega_r} F(u) dx. \tag{3.1}
$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $I_{r,s}$, then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the equation

$$
\begin{cases}
-\Delta u + V u + \lambda u = s|u|^{q-2}u + \beta f(u), & x \in \Omega_r, \\
\int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r.
\end{cases} \tag{3.2}
$$

Lemma 3.1. *For any $\Theta > 0$, there exist $r_\Theta > 0$ and $u^0, u^1 \in S_{r_\Theta,\Theta}$ such that*

(i) $I_{r,s}(u^1) \leq 0$ for any $r > r_\Theta$ and $s \in \left[\frac{1}{2}, 1\right]$,

$$\left\|\nabla u^0\right\|_2^2 < \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}} < \left\|\nabla u^1\right\|_2^2$$

and

$$I_{r,s}\left(u^0\right) < \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)}\left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

(ii) If $u \in S_{r,\Theta}$ satisfies

$$\|\nabla u\|_2^2 = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}},$$

then there holds

$$I_{r,s}(u) \geq \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)}\left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

(iii) Set

$$m_{r,s}(\Theta) = \inf_{\gamma \in \Gamma_{r,\Theta}} \sup_{t \in [0,1]} I_{r,s}(\gamma(t))$$

with

$$\Gamma_{r,\Theta} = \left\{\gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1\right\}.$$

Then

$$\frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)}\left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}} \leq m_{r,s}(\Theta) \leq h(T_\Theta),$$

where $h(T_\Theta) = \max_{t \in \mathbb{R}^+} h(t)$, the function $h : \mathbb{R}^+ \to \mathbb{R}$ being defined by

$$h(t) = \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)t^2\theta\Theta - \alpha\beta C_{N,p_1}\Theta^{\frac{p_1}{2}}\theta^{\frac{N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{2}} - \frac{1}{2q}\Theta^{\frac{q}{2}}|\Omega|^{\frac{2-q}{2}}t^{\frac{N(q-2)}{2}}.$$

Here θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω, and $|\Omega|$ is the volume of Ω.

Proof. (i) Clearly, the set $S_{r,\Theta}$ is path connected. Since $v_1 \in S_{1,\Theta}$ be the positive eigenfunction associated to θ and note that θ is the principal eigenvalue of $-\Delta$, then

$$\int_\Omega |\nabla v_1|^2 \, dx = \theta\Theta. \tag{3.3}$$

By the Hölder inequality, we know that

$$\Theta = \int_\Omega |v_1(x)|^2 dx \leq \left(\int_\Omega |v_1(x)|^q dx\right)^{\frac{2}{q}} \cdot |\Omega|^{\frac{q-2}{q}},$$

which implies

$$\int_{\Omega} |v_1(x)|^q dx \geq \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}}. \tag{3.4}$$

According to (f_2), there exists a constant $\alpha > 0$ such that

$$F(\tau) \leq \alpha \tau^{p_1}. \tag{3.5}$$

For $x \in \Omega_{\frac{1}{t}}$ and $t > 0$, define $v_t(x) := t^{\frac{N}{2}} v_1(tx)$. Using (3.3), (3.4), (3.5) and $\frac{1}{2} \leq s \leq 1$, it holds

$$
\begin{aligned}
I_{\frac{1}{t},s}(v_t) &\leq \frac{1}{2} \int_{\Omega_r} |\nabla v_t|^2 dx + \frac{1}{2} \int_{\Omega_r} V v_t^2 dx - \frac{1}{2q} \int_{\Omega_r} |v_t|^q dx - \alpha\beta \int_{\Omega_r} |v_t|^{p_1} dx \\
&\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) \int_{\Omega_r} |\nabla v_t|^2 dx - \alpha\beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \left(\int_{\Omega_r} |\nabla v_t|^2 dx\right)^{\frac{N(p_1-2)}{4}} \\
&\quad - \frac{1}{2q} \int_{\Omega_r} |v_t|^q dx \\
&\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \int_{\Omega} |\nabla v_1|^2 dx - \alpha\beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \left(t^2 \int_{\Omega} |\nabla v_1|^2 dx\right)^{\frac{N(p_1-2)}{4}} \\
&\quad - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \int_{\Omega} |v_1|^q dx \\
&\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta\Theta - \alpha\beta C_{N,p_1} \Theta^{\frac{p_1}{2}} \theta^{\frac{N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{2}} - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}} \\
&=: h(t). \tag{3.6}
\end{aligned}
$$

Note that since $2 < p_1 < 2 + \frac{4}{N} < q < 2^*$ and $\beta \leq 0$ there exist $0 < T_\Theta < t_0$ such that $h(t_0) = 0, h(t) < 0$ for any $t > t_0, h(t) > 0$ for any $0 < t < t_0$ and $h(T_\Theta) = \max_{t \in \mathbb{R}^+} h(t)$. As a consequence, there holds

$$I_{r,s}(v_{t_0}) = I_{\frac{1}{t_0},s}(v_{t_0}) \leq h(t_0) = 0 \tag{3.7}$$

for any $r \geq \frac{1}{t_0}$ and $s \in \left[\frac{1}{2}, 1\right]$. Moreover, there exists $0 < t_1 < T_\Theta$ such that

$$h(t) < \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)} \left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}} \tag{3.8}$$

for $t \in [0, t_1]$. On the other hand, it follows from the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$
\begin{aligned}
I_{r,s}(u) &\geq \frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V u^2 dx - \frac{1}{q} \int_{\Omega_r} |u|^q dx \\
&\geq \frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2} \int_{\Omega_r} |\nabla u|^2 dx - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(q-2)}{4}}. \tag{3.9}
\end{aligned}
$$

Define

$$g(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}}$$

and

$$\widetilde{t} = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) \Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}},$$

it is easy to see that g is increasing on $(0, \widetilde{t})$ and decreasing on $(\widetilde{t}, \infty)$, and

$$g(\widetilde{t}) = \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)} \left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

For $r \geq \widetilde{r}_\Theta := \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{\widetilde{t}}}\right\}$, we have $v_{\frac{1}{\widetilde{r}_\Theta}} \in S_{r,\Theta}$ and

$$\|\nabla v_{\frac{1}{\widetilde{r}_\Theta}}\|_2^2 = \left(\frac{1}{\widetilde{r}_\Theta}\right)^2 \|\nabla v_1\|_2^2$$

$$< \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}}. \tag{3.10}$$

Moreover, there holds

$$I_{\widetilde{r}_\Theta,s}\left(v_{\frac{1}{\widetilde{r}_\Theta}}\right) \leq h\left(\frac{1}{\widetilde{r}_\Theta}\right) \leq h(t_1). \tag{3.11}$$

Setting $u^0 = v_{\frac{1}{\widetilde{r}_\Theta}}, u^1 = v_{t_0}$ and

$$r_\Theta = \max\left\{\frac{1}{t_0}, \widetilde{r}_\Theta\right\}. \tag{3.12}$$

Combining (3.7), (3.8), (3.10) and (3.11), (i) holds.

(ii) By (3.9) and a direct calculation, (ii) holds.

(iii) Since $I_{r,s}\left(u^1\right) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$, we have

$$\|\nabla\gamma(0)\|_2^2 < \widetilde{t} < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (3.9) that

$$\max_{t\in[0,1]} I_{r,s}(\gamma(t)) \geq g(\widetilde{t})$$

$$= \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)} \left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}$$

for any $\gamma \in \Gamma_{r,\Theta}$, hence the first inequality in (iii) holds. Now we define a path $\gamma \in \Gamma_{r,\Theta}$ by

$$\gamma(\tau)(x) = \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)x\right)$$

for $\tau \in [0,1]$ and $x \in \Omega_r$. Then by (3.6) we have $m_{r,s}(\Theta) \leq h(T_\Theta)$, where $h(T_\Theta) = \max_{t\in\mathbb{R}^+} h(t)$. Note that T_Θ is independent of r and s. $\square$

By using Lemma 3.1, the energy functional $I_{r,s}$ possesses the mountain pass geometry. To obtain bounded Palais–Smale sequence, we recall a proposition from [11,13].

Proposition 3.2 (see [11, Theorem 1]). *Let $(E, \langle\cdot,\cdot\rangle)$ and $(H, (\cdot,\cdot))$ be two infinite-dimensional Hilbert spaces and assume there are continuous injections*

$$E \hookrightarrow H \hookrightarrow E'.$$

Let

$$\|u\|^2 = \langle u, u \rangle, \quad |u|^2 = (u, u) \quad \text{for } u \in E,$$

and

$$S_\mu = \{ u \in E : |u|^2 = \mu \}, \qquad T_u S_\mu = \{ v \in E : (u, v) = 0 \} \quad \text{for } \mu \in (0, +\infty).$$

Let $I \subset (0, +\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \to \mathbb{R}$ of the form

$$\Phi_\rho(u) = A(u) - \rho B(u), \quad \text{for } \rho \in I,$$

with $B(u) \geq 0$ for every $u \in E$, and

$$A(u) \to +\infty \quad \text{or} \quad B(u) \to +\infty \quad \text{as } u \in E \text{ and } \|u\| \to +\infty. \tag{3.13}$$

Suppose moreover that Φ_ρ' and Φ_ρ'' are τ-Hölder continuous, $\tau \in (0, 1]$, on bounded sets in the following sense: for every $R > 0$ there exists $M = M(R) > 0$ such that

$$\left\| \Phi_\rho'(u) - \Phi_\rho'(v) \right\| \leq M \|u - v\|^\tau \quad \text{and} \quad \left\| \Phi_\rho''(u) - \Phi_\rho''(v) \right\| \leq M \|u - v\|^\tau \tag{3.14}$$

for every $u, v \in B(0, R)$. Finally, suppose that there exist $w_1, w_2 \in S_\mu$ independent of ρ such that

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\rho(\gamma(t)) > \max \left\{ \Phi_\rho(w_1), \Phi_\rho(w_2) \right\} \quad \text{for all } \rho \in I,$$

where

$$\Gamma = \left\{ \gamma \in C\left([0,1], S_\mu\right) : \gamma(0) = w_1, \gamma(1) = w_2 \right\}.$$

Then for almost every $\rho \in I$, there exists a sequence $\{u_n\} \subset S_\mu$ such that

(i) $\Phi_\rho(u_n) \to c_\rho$,

(ii) $\Phi_\rho'\big|_{S_\mu}(u_n) \to 0$,

(iii) $\{u_n\}$ is bounded in E.

Lemma 3.3. *For any $\Theta > 0$, let $r > r_\Theta$, where r_Θ is defined in Lemma 3.1. Then problem (3.1) has a solution $(\lambda_{r,s}, u_{r,s})$ for almost every $s \in \left[\frac{1}{2}, 1\right]$. Moreover, $u_{r,s} \geq 0$ and $I_{r,s}(u_{r,s}) = m_{r,s}(\Theta)$.*

Proof. By Proposition 3.2, it follows that

$$A(u) = \frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V(x) u^2 dx - \beta \int_{\Omega_r} F(u) dx \quad \text{and} \quad B(u) = \frac{1}{q} \int_{\Omega_r} |u|^q dx.$$

Note that the assumptions in Proposition 3.2 hold due to $\beta \leq 0$ and Lemma 3.1. Hence, for almost every $s \in \left[\frac{1}{2}, 1\right]$, there exists a bounded Palais–Smale sequence $\{u_n\}$ satisfying

$$I_{r,s}(u_n) \to m_{r,s}(\Theta) \quad \text{and} \quad I_{r,s}'(u_n)\big|_{T_{u_n} S_{r,\Theta}} \to 0,$$

where $T_{u_n} S_{r,\Theta}$ denotes the tangent space of $S_{r,\Theta}$ at u_n. Then

$$\lambda_n = -\frac{1}{\Theta} \left(\int_{\Omega_r} |\nabla u_n|^2 \, dx + \int_{\Omega_r} V(x) u_n^2 dx - \beta \int_{\Omega_r} f(u_n) u_n dx - s \int_{\Omega_r} |u_n|^q \, dx \right)$$

is bounded and

$$I_{r,s}'(u_n) + \lambda_n u_n \to 0 \quad \text{in } H^{-1}(\Omega_r). \tag{3.15}$$

Moreover, since $\{u_n\}$ is a bounded Palais–Smale sequence, there exist $u_0 \in H_0^1(\Omega_r)$ and $\lambda \in \mathbb{R}$ such that, up to a subsequence,

$$
\begin{aligned}
\lambda_n &\to \lambda && \text{in } \mathbb{R}, \\
u_n &\rightharpoonup u_0 && \text{in } H_0^1(\Omega_r), \\
u_n &\to u_0 && \text{in } L^t(\Omega_r) \text{ for all } 2 \le t < 2^*,
\end{aligned}
$$

where u_0 satisfies

$$
\begin{cases}
-\Delta u_0 + V u_0 + \lambda u_0 = s\,|u_0|^{q-2}\,u_0 + \beta f(u_0) & \text{in } \Omega_r \\
u_0 \in H_0^1(\Omega_r), \quad \int_{\Omega_r} |u_0|^2\, dx = \Theta.
\end{cases}
$$

Using (3.15), we have

$$
I'_{r,s}(u_n)\, u_0 + \lambda_n \int_{\Omega_r} u_n u_0 dx \to 0 \quad \text{as } n \to \infty
$$

and

$$
I'_{r,s}(u_n)\, u_n + \lambda_n \Theta \to 0 \quad \text{as } n \to \infty.
$$

Note that

$$
\lim_{n\to\infty} \int_{\Omega_r} V(x) u_n^2 dx = \int_{\Omega_r} V(x) u_0^2 dx,
$$

$$
\lim_{n\to\infty} \int_{\Omega_r} f(u_n) u_n dx = \int_{\Omega_r} f(u_0) u_0 dx,
$$

$$
\lim_{n\to\infty} \int_{\Omega_r} f(u_n) u_0 dx = \int_{\Omega_r} f(u_0) u_0 dx,
$$

so we get $u_n \to u_0$ in $H_0^1(\Omega_r)$, hence $I_{r,s}(u_0) = m_{r,s}(\Theta)$.

Now, we show that $u_{r,s} \ge 0$. In order to obtain it, we only need to modify the proof of Proposition 3.2. In fact, for almost every $s \in \left[\frac{1}{2}, 1\right]$, the derivative $m'_{r,s}$ with respect to s is well defined since the function $s \mapsto m_{r,s}$ is nonincreasing, where $m_{r,s}$ denotes $m_{r,s}(\Theta)$ for fixed Θ. Let s be such that $m'_{r,s}$ exists and $\{s_n\} \subset \left[\frac{1}{2}, 1\right]$ be a monotone increasing sequence converging to s. Similar to the proof of Proposition 3.2, there exist $\{\gamma_n\} \subset \Gamma_{r,\Theta}$ and $K = K\left(m'_{r,s}\right)$ such that:

(i) if $I_{r,s}(\gamma_n(t)) \ge m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$, then $\int_{\Omega_r} |\nabla \gamma_n(t)|^2\, dx \le K$.

(ii) $\max_{t\in[0,1]} I_{r,s}(\gamma_n(t)) \le m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$.

Letting $\widetilde{\gamma}_n(t) = |\gamma_n(t)|$ for any $t \in [0,1]$, it follows that $\{\widetilde{\gamma}_n\} \subset \Gamma_{r,\Theta}$. Observe that $\left\|\nabla|u|\right\|_2^2 \le \|\nabla u\|_2^2$ for any $u \in H^1(\mathbb{R}^N)$. Now we have:

(I) if $I_{r,s}(\widetilde{\gamma}_n(t)) \ge m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$, then $I_{r,s}(\gamma_n(t)) \ge m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$. By (i), there holds $\int_{\Omega_r} |\nabla \gamma_n(t)|^2\, dx \le K$, and hence $\int_{\Omega_r} |\nabla \widetilde{\gamma}_n(t)|^2\, dx \le K$. Thus (i) also holds for $\widetilde{\gamma}_n$.

(II) $\max_{t\in[0,1]} I_{r,s}(\widetilde{\gamma}_n(t)) \le \max_{t\in[0,1]} I_{r,s}(\gamma_n(t)) \le m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$.

By replacing γ_n with $\widetilde{\gamma}_n$ in the proof of Proposition 3.2, we obtain a nonnegative bounded Palais–Smale sequence $\{u_n\}$. Consequently, there exists a nonnegative normalized solution to (3.1) for almost every $s \in \left[\frac{1}{2}, 1\right]$ as above. □

In order to obtain a solution of (1.1), we need to prove a uniform estimate for the solutions of (3.1) established in Lemma 3.3.

Lemma 3.4. *If $(\lambda_{r,s}, u_{r,s}) \in \mathbb{R} \times S_{r,\Theta}$ is a solution of (3.1) established in Lemma 3.3 for some r and s, then*

$$\int_{\Omega_r} |\nabla u|^2 dx \leq \frac{4N}{N(q-2)-4} \left[\frac{q-2}{2} h(T_\Theta) + \Theta \left(\frac{1}{2N} \|\widetilde{V}\|_\infty + \frac{q-2}{4} \|V\|_\infty \right) \right],$$

where the constant $h(T_\Theta)$ is defined in (iii) of Lemma 3.1 and is independent of r and s.

Proof. For simplicity, we denote $(\lambda_{r,s}, u_{r,s})$ as (λ, u) in this lemma. Since u is a solution of (3.1), we have

$$\int_{\Omega_r} |\nabla u|^2 dx + \int_{\Omega_r} V(x)u^2 dx = s \int_{\Omega_r} |u|^q dx + \beta \int_{\Omega_r} f(u)u dx - \lambda \int_{\Omega_r} |u|^2 dx. \tag{3.16}$$

The Pohozaev identity implies

$$\frac{N-2}{2N} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma + \frac{1}{2N} \int_{\Omega_r} \widetilde{V}(x)u^2 dx + \frac{1}{2} \int_{\Omega_r} Vu^2 dx$$
$$= -\frac{\lambda}{2} \int_{\Omega_r} |u|^2 dx + \frac{s}{q} \int_{\Omega_r} |u|^q dx + \beta \int_{\Omega_r} F(u) dx,$$

where $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Omega_r$. It then follows from $\beta \leq 0$ and (f_2) that

$$\frac{1}{N} \int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma - \frac{1}{2N} \int_{\Omega_r} (\nabla V \cdot x)u^2 dx$$
$$= \frac{(q-2)s}{2q} \int_{\Omega_r} |u|^q dx + \int_{\Omega_r} (\frac{\beta}{2} f(u)u - \beta F(u)) dx$$
$$\geq \frac{(q-2)s}{2q} \int_{\Omega_r} |u|^q dx + \frac{\beta(q-2)}{2} \int_{\Omega_r} F(u) dx$$
$$= \frac{q-2}{2} \left(\frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} Vu^2 dx - m_{r,s}(\Theta) \right).$$

Consequently, we have

$$\frac{q-2}{2} m_{r,s}(\Theta) \geq \frac{q-2}{2} \left(\frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} Vu^2 dx \right) - \frac{1}{N} \int_{\Omega_r} |\nabla u|^2 dx$$
$$+ \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma + \frac{1}{2N} \int_{\Omega_r} (\nabla V \cdot x)u^2 dx$$
$$\geq \frac{N(q-2)-4}{4N} \int_{\Omega_r} |\nabla u|^2 dx - \Theta \left(\frac{1}{2N} \|\nabla V \cdot x\|_\infty + \frac{q-2}{4} \|V\|_\infty \right),$$

where the last inequality holds since $x \cdot \mathbf{n}(x) \geq 0$ for any $x \in \partial\Omega_r$ due to the convexity of Ω_r. Using Lemma 3.1, we have

$$\frac{N(q-2)-4}{4N} \int_{\Omega_r} |\nabla u|^2 dx - \Theta \left(\frac{1}{2N} \|\nabla V \cdot x\|_\infty + \frac{q-2}{4} \|V\|_\infty \right) \leq \frac{q-2}{2} h(T_\Theta),$$

which implies

$$\int_{\Omega_r} |\nabla u|^2 dx \leq \frac{4N}{N(q-2)-4} \left[\frac{q-2}{2} h(T_\Theta) + \Theta \left(\frac{1}{2N} \|\widetilde{V}\|_\infty + \frac{q-2}{4} \|V\|_\infty \right) \right].$$

This completes the proof of lemma. $\qquad\square$

Now, we obtain a solution of (1.1) by letting $s \to 1$.

Lemma 3.5. *For every $\Theta > 0$, problem (1.1) has a solution (λ_r, u_r) provided $r > r_\Theta$ where r_Θ is as in Lemma 3.1. Moreover, $u_r \geq 0$ in Ω_r.*

Proof. By using Lemma 3.3, there is a nonnegative solution $(\lambda_{r,s}, u_{r,s})$ to (3.1) for almost every $s \in \left[\frac{1}{2}, 1\right]$. In view of Lemma 3.4, $\{u_{r,s}\}$ is bounded. By an argument similar to that in Lemma 3.3, there exist $u_r \in S_{r,\Theta}$ and λ_r such that, going if necessary to a subsequence,

$$\lambda_{r,s} \to \lambda_r \quad \text{and} \quad u_{r,s} \to u_r \quad \text{in } H_0^1(\Omega_r) \quad \text{as } s \to 1.$$

Hence u_r is a nonnegative solution of problem (1.1). □

Next, we will consider the Lagrange multiplier. we first establish an a priori estimate for the solutions of (1.1).

Lemma 3.6. *If $\{(\lambda_r, u_r)\}$ is a family of nonnegative solutions of (1.1) such that $\|u_r\|_{H^1} \leq C$ with $C > 0$ independent of r, then $\limsup_{r \to \infty} \|u_r\|_\infty < \infty$.*

Proof. Using the regularity theory of elliptic partial differential equations, we know that $u_r \in C(\Omega_r)$. Assume to the contrary that there exist a sequence, for simplicity denoted by $\{u_r\}$, and $x_r \in \Omega_r$ such that

$$M_r := \max_{x \in \Omega_r} u_r(x) = u_r(x_r) \to \infty \quad \text{as } r \to \infty.$$

Suppose without loss of generality that, up to a subsequence, $\lim_{r \to \infty} \frac{x_r}{|x_r|} = (1, 0, \ldots, 0)$. Set

$$v_r(x) = \frac{u_r(x_r + \tau_r x)}{M_r} \quad \text{for } x \in \Sigma^r := \left\{ x \in \mathbb{R}^N : x_r + \tau_r x \in \Omega_r \right\},$$

where $\tau_r = M_r^{\frac{2-q}{2}}$. Then $\tau_r \to 0$ as $r \to \infty$, $\|v_r\|_{L^\infty(\Sigma^r)} \leq 1$, and v_r satisfies

$$-\Delta v_r + \tau_r^2 V(x_r + \tau_r x) v_r + \tau_r^2 \lambda_r v_r = |v_r|^{q-2} v_r + \beta M_r^{1-q} f(M_r v_r) \quad \text{in } \Sigma^r. \tag{3.17}$$

In fact, since u_r is a nonnegative solution of (1.1), we obtain

$$-\Delta u_r(x_r + \tau_r x) + V(x_r + \tau_r x) u_r(x_r + \tau_r x) + \lambda_r u_r(x_r + \tau_r x)$$
$$= |u_r(x_r + \tau_r x)|^{q-2} u_r(x_r + \tau_r x) + \beta f(u_r(x_r + \tau_r x)) \quad \text{in } \Omega_r,$$

then by a direct calculation and the definition of $v_r(x)$, τ_r, we know that (3.17) holds. In view of (1.1), the Gagliardo–Nirenberg inequality and $\|u_r\|_{H^1} \leq C$ with C independent of r, we infer that the sequence $\{\lambda_r\}$ is bounded. It then follows from the regularity theory of elliptic partial differential equations and the Arzelà–Ascoli theorem that there exists v such that, up to a subsequence

$$v_r \to v \quad \text{in } H_0^1(\Sigma) \quad \text{and} \quad v_r \to v \quad \text{in } C_{loc}^\beta(\Sigma) \text{ for some } \beta \in (0, 1),$$

where $\Sigma := \lim_{r \to \infty} \Sigma^r$.
Similar to the proof of [4, Lemma 2.7], we have

$$\liminf_{r \to \infty} \frac{\text{dist}(x_r, \partial \Omega_r)}{\tau_r} = \liminf_{r \to \infty} \frac{|y_r - x_r|}{\tau_r} \geq d > 0,$$

where $y_r \in \partial\Omega_r$ is such that $\text{dist}(x_r, \partial\Omega_r) = |y_r - x_r|$ for any large r. As a result, by letting $r \to \infty$ in (3.17), we obtain that $v \in H_0^1(\Sigma)$ is a nonnegative solution of

$$-\Delta v = |v|^{q-2}v \quad \text{in } \Sigma,$$

where

$$\Sigma = \begin{cases} \mathbb{R}^N & \text{if } \liminf\limits_{r\to\infty} \frac{\text{dist}(x_r, \partial\Omega_r)}{\tau_r} = \infty, \\ \{x \in \mathbb{R}^N : x_1 > -d\} & \text{if } \liminf\limits_{r\to\infty} \frac{\text{dist}(x_r, \partial\Omega_r)}{\tau_r} > 0. \end{cases}$$

It then follows from the Liouville theorems (see [17]) that $v = 0$ in $H_0^1(\Sigma)$, which contradicts $v(0) = \lim_{r\to\infty} v_r(0) = 1$. $\qquad\square$

Clearly, the proof of Lemma 3.6 does not depend on β.

Lemma 3.7. *Let $(\lambda_{r,\Theta}, u_{r,\Theta})$ be the solution of (1.1) from Lemma 3.5. If $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, then there exists $\bar{\Theta} > 0$ such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \text{for } 0 < \Theta < \bar{\Theta}.$$

Proof. Let $(\lambda_{r,\Theta}, u_{r,\Theta})$ be the solution of (1.1) established in Theorem 3.5. By the regularity theory of elliptic partial differential equations, we have $u_{r,\Theta} \in C(\Omega_r)$. Using Lemma 3.6, it holds

$$\limsup_{r\to\infty} \max_{\Omega_r} u_{r,\Theta} < \infty.$$

Setting

$$Q(\Theta) = \liminf_{r\to\infty} \max_{\Omega_r} u_{r,\Theta},$$

we claim that there is $\Theta_1 > 0$ such that $Q(\Theta) > 0$ for any $0 < \Theta < \Theta_1$. Assume to the contrary that there exists a sequence $\{\Theta_k\}$ tending to 0 as $k \to \infty$ such that $Q(\Theta_k) = 0$ for any k, that is,

$$\liminf_{r\to\infty} \max_{\Omega_r} u_{r,\Theta_k} = 0 \quad \text{for any } k. \tag{3.18}$$

As a consequence of (iii) in Lemma 3.1, for any $r > r_{\Theta_k}$, we have

$$I_r(u_{r,\Theta_k}) = m_{r,1}(\Theta_k) \to \infty \quad \text{as } k \to \infty. \tag{3.19}$$

For any given k, it follows from (3.18) and $u_{r,\Theta_k} \in S_{r,\Theta_k}$ that, up to a subsequence,

$$\int_{\Omega_r} |u_{r,\Theta_k}|^s \, dx = \int_{\Omega_r} |u_{r,\Theta_k}|^{s-2}|u_{r,\Theta_k}|^2 \, dx \le \left|\max_{\Omega_r} u_{r,\Theta_k}\right|^{s-2} \Theta_k \to 0 \quad \text{as } r \to \infty \tag{3.20}$$

for any $s > 2$. Hence, for any given large k, there exists $\bar{r}_k > r_{\Theta_k}$ such that

$$\left|\frac{1}{q}\int_{\Omega_r} |u_{r,\Theta_k}|^q dx + \beta \int_{\Omega_r} f(u_{r,\Theta_k}) dx\right| < \frac{m_{r,1}(\Theta_k)}{2} \quad \text{for any } r \ge \bar{r}_k.$$

In view of (3.19) and $I_r(u_{r,\Theta_k}) = m_{r,1}(\Theta_k)$, we further have

$$\int_{\Omega_r} |\nabla u_{r,\Theta_k}|^2 \, dx + \int_{\Omega_r} V(x) u_{r,\Theta_k}^2 \, dx \ge \frac{m_{r,1}(\Theta_k)}{2} \quad \text{for any large } k \text{ and } r \ge \bar{r}_k. \tag{3.21}$$

It follows from (3.18), (3.20) and (3.21) that there exists $r_k \ge \bar{r}_k$ with $r_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \max_{\Omega_{r_k}} u_{r_k,\Theta_k} = 0, \tag{3.22}$$

$$\int_{\Omega_{r_k}} |u_{r_k,\Theta_k}|^s \, dx \le \left| \max_{\Omega_{r_k}} u_{r_k,\Theta_k} \right|^{s-2} \Theta_k \to 0 \quad \text{as } k \to \infty \text{ for any } s > 2 \tag{3.23}$$

and

$$\int_{\Omega_{r_k}} |\nabla u_{r_k,\Theta_k}|^2 \, dx + \int_{\Omega_r} V u_{r_k,\Theta_k}^2 \, dx \to \infty \quad \text{as } k \to \infty. \tag{3.24}$$

By (1.1), (3.23) and (3.24), we have

$$\lambda_{r_k,\Theta_k} \to -\infty \quad \text{as } k \to \infty. \tag{3.25}$$

Now (1.1) implies

$$-\Delta u_{r_k,\Theta_k} + V(x) u_{r_k,\Theta_k} + \lambda_{r_k,\Theta_k} u_{r_k,\Theta_k} = |u_{r_k,\Theta_k}|^{q-2} u_{r_k,\Theta_k} + \beta f(u_{r_k,\Theta_k}),$$

so

$$-\Delta u_{r_k,\Theta_k} + \left(\|V\|_\infty + \frac{\lambda_{r_k,\Theta_k}}{2} \right) u_{r_k,\Theta_k} \ge -\frac{\lambda_{r_k,\Theta_k}}{2} u_{r_k,\Theta_k} + |u_{r_k,\Theta_k}|^{q-2} u_{r_k,\Theta_k} + \beta f(u_{r_k,\Theta_k}).$$

Using (3.25) and (3.22), it follows that

$$-\Delta u_{r_k,\Theta_k} + \left(\|V\|_\infty + \frac{\lambda_{r_k,\Theta_k}}{2} \right) u_{r_k,\Theta_k} \ge 0$$

for large k. Let θ_{r_k} be the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω_{r_k}, and $v_{r_k} > 0$ be the corresponding normalized eigenfunction. It follows that

$$\left(\theta_{r_k} + \|V\|_\infty + \frac{\lambda_{r_k,\Theta_k}}{2} \right) \int_{\Omega_{r_k}} u_{r_k,\Theta_k} v_{r_k} \, dx \ge 0.$$

Since $\int_{\Omega_{r_k}} u_{r_k,\Theta_k} v_{r_k} \, dx > 0$, we have

$$\theta_{r_k} + \|V\|_\infty + \frac{\lambda_{r_k,\Theta_k}}{2} \ge 0,$$

which contradicts (3.25) for large k. Hence the claim holds, that is, there exists $\Theta_1 > 0$ such that

$$Q(\Theta) = \liminf_{r \to \infty} \max_{\Omega_r} u_{r,\Theta} > 0 \tag{3.26}$$

for any $0 < \Theta < \Theta_1$.

We consider $H^1(\Omega_r)$ as a subspace of $H^1(\mathbb{R}^N)$ for any $r > 0$. It follows from Lemma 3.4 that the set of solutions $\{u_{r,\Theta} : r > r_\Theta\}$ established in Lemma 3.5 is bounded in $H^1(\mathbb{R}^N)$, so there exist $u_\Theta \in H^1(\mathbb{R}^N)$ and $\lambda_\Theta \in \mathbb{R}$ such that up to a subsequence:

$$\lambda_{r,\Theta} \to \lambda_\Theta,$$
$$u_{r,\Theta} \rightharpoonup u_\Theta \quad \text{in } H^1(\mathbb{R}^N),$$
$$u_{r,\Theta} \to u_\Theta \quad \text{in } L_{loc}^k(\mathbb{R}^N) \text{ for all } 2 \le k < 2^*,$$
$$u_{r,\Theta} \to u_\Theta \quad \text{a.e. in } \mathbb{R}^N$$

and u_Θ is a solution of the equation

$$-\Delta u + V(x)u + \lambda_\Theta u = |u|^{q-2} u + \beta f(u) \quad \text{in } \mathbb{R}^N.$$

Hence,

$$\int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx + \int_{\mathbb{R}^N} V(x) u_{\Theta}^2 dx + \lambda_{\Theta} \int_{\mathbb{R}^N} u_{\Theta}^2 dx = \int_{\mathbb{R}^N} |u_{\Theta}|^q \, dx + \beta \int_{\mathbb{R}^N} f(u_{\Theta}) u_{\Theta} dx \quad (3.27)$$

and the Pohozaev identity gives

$$\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx + \frac{1}{2N} \int_{\mathbb{R}^N} \widetilde{V} u_{\Theta}^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u_{\Theta}^2 dx + \frac{\lambda_{\Theta}}{2} \int_{\mathbb{R}^N} u_{\Theta}^2 dx$$

$$= \frac{1}{q} \int_{\mathbb{R}^N} |u_{\Theta}|^q \, dx + \beta \int_{\mathbb{R}^N} F(u_{\Theta}) dx. \quad (3.28)$$

It follows from (3.27), (3.28), (f_2), the Gagliardo–Nirenberg inequality and the fact $\beta \leq 0$ that

$$\frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx + \frac{1}{2N} \int_{\mathbb{R}^N} \widetilde{V}(x) u_{\Theta}^2 dx$$

$$= \left(\frac{1}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u_{\Theta}|^q \, dx + \frac{\beta}{2} \int_{\mathbb{R}^N} (f(u_{\Theta}) u_{\Theta} - 2F(u_{\Theta})) dx$$

$$\leq \frac{C_{N,q}(q-2)}{2q} \left(\int_{\mathbb{R}^N} u_{\Theta}^2 dx \right)^{\frac{2q-N(q-2)}{4}} \left(\int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx \right)^{\frac{N(q-2)}{4}}.$$

By using the Hölder inequality, we have

$$\left(\frac{1}{N} - \frac{\|\widetilde{V}_+\|_{\frac{N}{2}} S^{-1}}{2N} \right) \int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx \leq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx + \frac{1}{2N} \int_{\mathbb{R}^N} \widetilde{V}(x) u_{\Theta}^2 dx.$$

Therefore,

$$\left(\frac{1}{N} - \frac{\|\widetilde{V}_+\|_{\frac{N}{2}} S^{-1}}{2N} \right) \int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx$$

$$\leq \frac{C_{N,q}(q-2)}{2q} \left(\int_{\mathbb{R}^N} u_{\Theta}^2 dx \right)^{\frac{2q-N(q-2)}{4}} \left(\int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx \right)^{\frac{N(q-2)}{4}}.$$

If $u_{\Theta} \neq 0$, Using $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, we obtain that

$$\int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx \geq \left[\frac{q \left(2 - \|\widetilde{V}_+\|_{\frac{N}{2}} S^{-1} \right)}{N C_{N,q}(q-2)} \right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{q(N-2)-2N}{N(q-2)-4}}. \quad (3.29)$$

Next, it follows from (3.5), (3.27), (3.28), (3.29), (f_2) and $2 + \frac{4}{N} < q < 2^*$ that

$$\left(\frac{1}{q} - \frac{1}{2} \right) \lambda_{\Theta} \int_{\mathbb{R}^N} u_{\Theta}^2 dx = \left(\frac{N-2}{2N} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx + \frac{1}{2N} \int_{\mathbb{R}^N} \widetilde{V}(x) u_{\Theta}^2 dx$$

$$+ \left(\frac{1}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} V(x) u_{\Theta}^2 dx - \frac{\beta}{q} \int_{\mathbb{R}^N} (qF(u_{\Theta}) - f(u_{\Theta})u_{\Theta})) \, dx$$

$$\leq \frac{(N-2)q - 2N}{2Nq} \int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx + \frac{\|\widetilde{V}\|_{\infty}}{2N} \Theta + \frac{(q-2)\|V\|_{\infty}}{2q} \Theta$$

$$- \frac{\beta(q - p_2)\alpha}{q} C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \left(\int_{\mathbb{R}^N} |\nabla u_{\Theta}|^2 \, dx \right)^{\frac{N(p_1-2)}{4}}$$

$$\to -\infty \quad \text{as } \Theta \to 0,$$

since $\frac{(N-2)q-2N}{2Nq} < 0$. Therefore, if $u_\Theta \neq 0$ for $\Theta > 0$ small there exists $\Theta_0 > 0$ such that $\lambda_\Theta > 0$ for $0 < \Theta < \Theta_0$.

In order to complete the proof, we consider the case that there is a sequence $\Theta_k \to 0$ such that $u_{\Theta_k} = 0$ for any k. Assume without loss of generality that $u_\Theta = 0$ for any $\Theta \in (0, \Theta_1)$. Let $x_{r,\Theta} \in \Omega_r$ be such that $u_{r,\Theta}(x_{r,\Theta}) = \max_{\Omega_r} u_{r,\Theta}$. In view of (3.26), there holds $|x_{r,\Theta}| \to \infty$ as $r \to \infty$. Otherwise, there exists $x_0 \in \mathbb{R}$ such that, up to a subsequence, $x_{r,\Theta} \to x_0$, and hence $u_\Theta(x_0) \geq d_\Theta > 0$. This contradicts $u_\Theta = 0$. We claim that $\operatorname{dist}(x_{r,\Theta}, \partial\Omega_r) \to \infty$ as $r \to \infty$. Arguing by contradiction we assume that $\liminf_{r\to\infty} \operatorname{dist}(x_{r,\Theta}, \partial\Omega_r) = l < \infty$. It follows from (3.26) that $l > 0$. Let $w_r(x) = u_{r,\Theta}(x + x_{r,\Theta})$ for any $x \in \Sigma^r := \{x \in \mathbb{R}^N : x + x_{r,\Theta} \in \Omega_r\}$. Then w_r is bounded in $H^1(\mathbb{R}^N)$, and there is $w \in H^1(\mathbb{R}^N)$ such that $w_r \rightharpoonup w$ as $r \to \infty$. By the regularity theory of elliptic partial equations and $\liminf_{r\to\infty} u_{r,\Theta}(x_{r,\Theta}) > d_\Theta > 0$, we infer that $w(0) \geq d_\Theta > 0$. Assume without loss of the generality that, up to a subsequence,

$$\lim_{r\to\infty} \frac{x_{r,\Theta}}{|x_{r,\Theta}|} = e_1.$$

Setting

$$\Sigma = \left\{ x \in \mathbb{R}^N : x \cdot e_1 < l \right\} = \left\{ x \in \mathbb{R}^N : x_1 < l \right\},$$

we have $\phi(\cdot - x_{r,\Theta}) \in C_c^\infty(\Omega_r)$ for any $\phi \in C_c^\infty(\Sigma)$ and r large enough. It then follows that

$$\int_{\Omega_r} \nabla u_{r,\Theta} \nabla \phi\,(\cdot - x_{r,\Theta})\,dx + \int_{\Omega_r} V u_{r,\Theta} \phi\,(\cdot - x_{r,\Theta})\,dx + \lambda_{r,\Theta} \int_{\Omega_r} u_{r,\Theta} \phi\,(\cdot - x_{r,\Theta})\,dx$$

$$= \int_{\Omega_r} |u_{r,\Theta}|^{q-2} u_{r,\Theta} \phi\,(\cdot - x_{r,\Theta})\,dx + \beta \int_{\Omega_r} f(u_{r,\Theta}) \phi\,(\cdot - x_{r,\Theta})\,dx. \tag{3.30}$$

Since $|x_{r,\Theta}| \to \infty$ as $r \to \infty$, it holds

$$\left| \int_{\Omega_r} V u_{r,\Theta} \phi\,(\cdot - x_{r,\Theta})\,dx \right| \leq \int_{\operatorname{Supp}\phi} |V\,(\cdot + x_{r,\Theta})\,w_r \phi|\,dx$$

$$\leq \|w_r\|_{2^*} \|\phi\|_{2^*} \left(\int_{\operatorname{Supp}\phi} |V\,(\cdot + x_{r,\Theta})|^{\frac{N}{2}}\,dx \right)^{\frac{2}{N}}$$

$$\leq \|w_r\|_{2^*} \|\phi\|_{2^*} \left(\int_{\mathbb{R}^N \setminus B_{\frac{|x_{r,\Theta}|}{2}}} |V|^{\frac{N}{2}}\,dx \right)^{\frac{2}{N}}$$

$$\to 0 \quad \text{as } r \to \infty. \tag{3.31}$$

Letting $r \to \infty$ in (3.30), we obtain for $\phi \in C_c^\infty(\Sigma)$:

$$\int_\Sigma \nabla w \cdot \nabla \phi\,dx + \lambda_\Theta \int_\Sigma w\phi\,dx = \int_\Sigma |w|^{q-2} w\phi\,dx + \beta \int_\Sigma f(w)\phi\,dx.$$

Thus $w \in H_0^1(\Sigma)$ is a weak solution of the equation

$$-\Delta w + \lambda_\Theta w = |w|^{q-2} w + \beta f(w) \quad \text{in } \Sigma. \tag{3.32}$$

Hence we obtain a nontrivial nonnegative solution of (3.32) on a half space which is impossible by the Liouville theorem (see [17]). This proves that $\operatorname{dist}(x_{r,\Theta}, \partial\Omega_r) \to \infty$ as $r \to \infty$. A similar argument as above shows that (3.32) holds for $\Sigma = \mathbb{R}^N$. Now we argue as in the case $u_\Theta \neq 0$ above that there exists Θ_2 such that $\lambda_\Theta > 0$ for any $0 < \Theta < \Theta_2$.

Setting $\bar\Theta = \min\{\Theta_0, \Theta_1, \Theta_2\}$, the proof is complete. $\qquad\qquad\square$

Proof of Theorem 1.3. The proof is an immediate consequence of Lemmas 3.5, 3.6 and 3.7. $\quad\square$

4 Proof of Theorem 1.4

In this section, we assume that the assumptions of Theorem 1.4 hold. Since $\beta > 0$,

$$I_r(u) \geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) \int_{\Omega_r} |\nabla u|^2 dx - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q}\left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(q-2)}{4}}$$

$$- \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}$$

$$= h_1(t),$$

where

$$h_1(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t^2 - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{2}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{2}}$$

$$= t^{\frac{N(p_1-2)}{2}}\left[\frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t^{\frac{4-N(p_1-2)}{2}} - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-p_1)}{2}}\right]$$

$$- \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{2}}.$$

Consider

$$\psi(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t^{\frac{4-N(p_1-2)}{2}} - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-p_1)}{2}}.$$

Note that ψ admits a unique maximum at

$$\bar{t} = \left[\frac{q(4-N(p_1-2))\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1}\right)}{2N(q-p_1)C_{N,q}}\right]^{\frac{2}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{2(N(q-2)-4)}}.$$

By a direct calculation, we obtain

$$\psi(\bar{t}) = \left[\frac{1 - \|V_-\|_{\frac{N}{2}} S^{-1}}{2N(q-p_1)}\right]^{\frac{N(q-p_1)}{N(q-2)-4}}\left[\frac{q(4-N(p_1-2))}{C_{N,q}}\right]^{\frac{4-N(p_1-2)}{N(q-2)-4}}[N(q-2)-4].$$

Hence,

$$\psi(\bar{t}) > \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}$$

as long as

$$\Theta_V = \left[\frac{1 - \|V_-\|_{\frac{N}{2}} S^{-1}}{2N(q-p_1)}\right]^{\frac{N}{2}}\left[\frac{q(4-N(p_1-2))}{C_{N,q}}\right]^{\frac{4-N(p_1-2)}{2(q-p_1)}}\left[\frac{N(q-2)-4}{\alpha\beta C_{N,p_1}}\right]^{\frac{N(q-2)-4}{2(q-p_1)}}.$$

Now, let $0 < \Theta < \Theta_V$ be fixed, we obtain

$$\psi(\bar{t}) > \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} \tag{4.1}$$

and $h_1(\bar{t}) > 0$. In view of $2 < p_1 < 2 + \frac{4}{N} < q < 2^*$ and (4.1), there exist $0 < R_1 < T_\Theta < R_2$ such that $h_1(t) < 0$ for $0 < t < R_1$ and for $t > R_2$, $h_1(t) > 0$ for $R_1 < t < R_2$, and $h_1(T_\Theta) = \max_{t \in \mathbb{R}^+} h_1(t) > 0$.

Define

$$\mathcal{V}_{r,\Theta} = \left\{ u \in S_{r,\Theta} : \|\nabla u\|_2^2 \leq T_\Theta^2 \right\}.$$

Let θ be the principal eigenvalue of operator $-\Delta$ with Dirichlet boundary condition in Ω, and let $|\Omega|$ be the volume of Ω.

Lemma 4.1.

    (i) If $r < \frac{\sqrt{C\Theta}}{T_\Theta}$, then $\mathcal{V}_{r,\Theta} = \emptyset$.

    (ii) If

$$r > \max \left\{ \frac{\sqrt{C\Theta}}{T_\Theta}, \left(\frac{\theta \left(1 + \|V\|_{\frac{N}{2}} S^{-1} \right)}{2\alpha_1 \beta} \Theta^{\frac{2-p_2}{2}} |\Omega|^{\frac{p_2-2}{2}} \right)^{\frac{2}{N(p_2-2)+4}} \right\},$$

    then $\mathcal{V}_{r,\Theta} \neq \emptyset$ and

$$e_{r,\Theta} := \inf_{u \in \mathcal{V}_{r,\Theta}} I_r(u) < 0$$

    is attained at some interior point $u_r > 0$ of $\mathcal{V}_{r,\Theta}$. As a consequence, there exists a Lagrange multiplier $\lambda_r \in \mathbb{R}$ such that (λ_r, u_r) is a solution of (2.1). Moreover $\liminf_{r\to\infty} \lambda_r > 0$ holds true.

Proof. (i) The Poincaré inequality implies there exists a positive constant C (only depending on Ω) such that

$$\int_{\Omega_r} |\nabla u|^2 dx = \frac{1}{r^2} \int_\Omega |\nabla u|^2 dx \geq \frac{C}{r^2} \int_\Omega |u|^2 dx = \frac{C\Theta}{r^2}$$

for any $u \in S_{r,\Theta}$. Since T_Θ is independent of r, there holds $\mathcal{V}_{r,\Theta} = \emptyset$ if and only if $r < \frac{\sqrt{C\Theta}}{T_\Theta}$.

    (ii) Let $v_1 \in S_{1,\Theta}$ be the positive normalized eigenfunction corresponding to θ. Setting

$$r_\Theta = \max \left\{ \frac{\sqrt{C\Theta}}{T_\Theta}, \left[\frac{\theta \left(1 + \|V\|_{\frac{N}{2}} S^{-1} \right)}{2\alpha_1 \beta} \Theta^{\frac{2-p_2}{2}} |\Omega|^{\frac{p_2-2}{2}} \right]^{\frac{2}{N(p_2-2)+4}} \right\}. \tag{4.2}$$

Now, we construct for $r > r_\Theta$ a function $u_r \in S_{r,\Theta}$ such that $u_r \in \mathcal{V}_{r,\Theta}$ and $I_r(u_r) < 0$. Clearly,

$$\int_\Omega |\nabla v_1|^2 \, dx = \theta\Theta, \quad \Theta = \int_\Omega |v_1|^2 \, dx \leq \left(\int_\Omega |v_1|^{p_2} \, dx \right)^{\frac{2}{p_2}} |\Omega|^{\frac{p_2-2}{p_2}}.$$

Define $u_r \in S_{r,\Theta}$ by $u_r(x) = r^{-\frac{N}{2}} v_1 \left(r^{-1} x \right)$ for $x \in \Omega_r$. Then

$$\int_{\Omega_r} |\nabla u_r|^2 \, dx = r^{-2}\theta\Theta \quad \text{and} \quad \int_{\Omega_r} |u_r|^{p_2} \, dx \geq r^{\frac{N(2-p_2)}{2}} \Theta^{\frac{p_2}{2}} |\Omega|^{\frac{2-p_2}{2}}. \tag{4.3}$$

According to (f_2), there exists a constant $\alpha_1 > 0$ such that

$$F(\tau) \geq \alpha_1 \tau^{p_2}. \tag{4.4}$$

By (4.2), (4.3), (4.4), $2 < p_2 < 2 + \frac{4}{N}$ and a direct calculation we have $u_r \in \mathcal{V}_{r,\Theta}$ and

$$I_r(u_r) \leq \frac{1}{2} \left(1 + \|V\|_{\frac{N}{2}} S^{-1} \right) r^{-2}\theta\Theta - \alpha_1 \beta r^{\frac{N(2-p_2)}{2}} \Theta^{\frac{p_2}{2}} |\Omega|^{\frac{2-p_2}{2}}$$
$$< 0.$$

It then follows from the Gagliardo–Nirenberg inequality that

$$I_r(u_r) \geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1}\right)\int_{\Omega_r}|\nabla u|^2 dx - C_{N,p_1}\beta\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}$$

$$- \frac{C_{N,q}}{q}\Theta^{\frac{2q-N(q-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2 dx\right)^{\frac{N(q-2)}{4}}. \tag{4.5}$$

As a consequence I_r is bounded from below in $\mathcal{V}_{r,\Theta}$. By the Ekeland principle there exists a sequence $\{u_{n,r}\} \subset \mathcal{V}_{r,\Theta}$ such that

$$I_r(u_{n,r}) \to \inf_{u \in \mathcal{V}_{r,\Theta}} I_r(u), \ I_r'(u_{n,r})|_{T_{u_{n,r}}S_{r,\Theta}} \to 0 \quad \text{as } n \to \infty.$$

Consequently there exists $u_r \in H_0^1(\Omega_r)$ such that $u_{n,r} \rightharpoonup u_r$ in $H_0^1(\Omega_r)$ and

$$u_{n,r} \to u_r \quad \text{in } L^k(\Omega_r) \text{ for all } 2 \leq k < 2^*.$$

Moreover, $\|\nabla u_r\|_2^2 \leq \liminf_{n\to\infty}\|\nabla u_{n,r}\|_2^2 \leq T_\Theta^2$, that is, $u_r \in \mathcal{V}_{r,\Theta}$. Note that

$$\int_{\Omega_r} Vu_{n,r}^2 dx \to \int_{\Omega_r} Vu_r^2 dx \text{ as } n \to \infty,$$

hence

$$e_{r,\Theta} \leq I_r(u_r) \leq \liminf_{n\to\infty} I_r(u_{n,r}) = e_{r,\Theta}.$$

It follows that $u_{n,r} \to u_r$ in $H_0^1(\Omega_r)$, so $I_r(u_r) < 0$. Therefore u is an interior point of $\mathcal{V}_{r,\Theta}$ because $I_r(u) \geq h_1(T_\Theta) > 0$ for any $u \in \partial\mathcal{V}_{r,\Theta}$ by (4.5). The Lagrange multiplier theorem implies that there exists $\lambda_r \in \mathbb{R}$ such that (λ_r, u_r) is a solution of (2.1). Moreover,

$$\lambda_r\Theta = \int_{\Omega_r}|u_r|^q\, dx + \beta\int_{\Omega_r} f(u_r)u_r dx - \int_{\Omega_r}|\nabla u_r|^2\, dx - \int_{\Omega_r} Vu_r^2 dx$$

$$= \int_{\Omega_r}|u_r|^q\, dx + \beta\int_{\Omega_r} f(u_r)u_r dx - \frac{2}{q}\int_{\Omega_r}|u_r|^q dx - 2\beta\int_{\Omega_r} F(u_r)dx - 2I_r(u_r)$$

$$> -2I_r(u_r) = -2e_{r,\Theta}. \tag{4.6}$$

It follows from the definition of $e_{r,\Theta}$ that $e_{r,\Theta}$ is nonincreasing with respect to r. Hence, $e_{r,\Theta} \leq e_{r_\Theta,\Theta} < 0$ for any $r > r_\Theta$ and $0 < \Theta < \Theta_V$. In view of (4.6), we have $\liminf_{r\to\infty}\lambda_r > 0$. Finally, the strong maximum principle implies $u_r > 0$. $\qquad\square$

Proof of Theorem 1.4. The proof is a direct consequence of Lemma 4.1 and Lemma 3.6. $\qquad\square$

5   Proof of Theorem 1.5

In this subsection we assume that the assumptions of Theorem 1.5 hold. For $s \in \left[\frac{1}{2}, 1\right]$, $\beta > 0$, we define the functional $J_{r,s} : S_{r,\Theta} \to \mathbb{R}$ by

$$J_{r,s}(u) = \frac{1}{2}\int_{\Omega_r}|\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} Vu^2 dx - s\left(\frac{1}{q}\int_{\Omega_r}|u|^q dx + \beta\int_{\Omega_r} F(u)dx\right).$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $J_{r,s}$ then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the problem

$$\begin{cases} -\Delta u + Vu + \lambda u = s|u|^{q-2}u + s\beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r}|u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r, \end{cases} \tag{5.1}$$

Lemma 5.1. *For $0 < \Theta < \widetilde{\Theta}_V$ where $\widetilde{\Theta}_V$ is defined in Theorem 1.5, there exist $\widetilde{r}_\Theta > 0$ and $u^0, u^1 \in S_{r_\Theta, \Theta}$ such that*

(i) *For $r > \widetilde{r}_\Theta$ and $s \in \left[\frac{1}{2}, 1\right]$ we have $J_{r,s}\left(u^1\right) \leq 0$ and*

$$J_{r,s}\left(u^0\right) < \frac{N(q-2)-4}{4} \left(\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S\right)}{N(q-2)}\right)^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}},$$

where

$$A = \left(\frac{C_{N,q}(q-2)(N(q-2)-4)}{q(p_1-2)(4-N(p_1-2))} + \frac{C_{N,q}}{q}\right).$$

Moreover,

$$\left\|\nabla u^0\right\|_2^2 < \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)A}\right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}$$

and

$$\left\|\nabla u^1\right\|_2^2 > \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)A}\right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

(ii) *If $u \in S_{r,\Theta}$ satisfies*

$$\|\nabla u\|_2^2 = \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)A}\right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}},$$

then there holds

$$J_{r,s}(u) \geq \frac{N(q-2)-4}{4} \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S\right)}{N(q-2)}\right]^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

(iii) *Let*

$$m_{r,s}(\Theta) = \inf_{\gamma \in \Gamma_{r,\Theta}} \sup_{t \in [0,1]} J_{r,s}(\gamma(t)),$$

where

$$\Gamma_{r,\Theta} = \left\{\gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1\right\}.$$

Then

$$m_{r,s}(\Theta) \geq \frac{N(q-2)-4}{4} \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S\right)}{N(q-2)}\right]^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}$$

and

$$m_{r,s}(\Theta) \leq \frac{N(q-2)-4}{2} \left(\frac{\theta\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)}\right)^{\frac{N(q-2)}{N(q-2)-4}} (4q)^{\frac{4}{N(q-2)-4}} |\Omega|^{\frac{2(q-2)}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

where θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω.

Proof. Let $v_1 \in S_{1,\Theta}$ be the positive normalized eigenfunction of $-\Delta$ with Dirichlet boundary condition in Ω associated to θ, then we have

$$\int_\Omega |\nabla v_1|^2 \, dx = \theta\Theta. \tag{5.2}$$

By the Hölder inequality, we know

$$\int_\Omega |v_1(x)|^{p_2} dx \geq \Theta^{\frac{p_2}{2}} \cdot |\Omega|^{\frac{2-p_2}{2}}. \tag{5.3}$$

Setting $v_t(x) = t^{\frac{N}{2}} v_1(tx)$ for $x \in B_{\frac{1}{t}}$ and $t > 0$. Using (4.4), (5.2), (5.3) and $\frac{1}{2} \leq s \leq 1$, we get

$$J_{\frac{1}{t},s}(v_t) \leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta\Theta - \frac{\beta}{2}\alpha_1 t^{\frac{N(p_2-2)}{2}} \int_\Omega |v_1|^{p_2} dx - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}}$$
$$\leq h_2(t), \tag{5.4}$$

where

$$h_2(t) = \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta\Theta - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}}.$$

A simple computation shows that $h_2(t_0) = 0$ for

$$t_0 := \left[\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) q\theta\Theta^{\frac{2-q}{2}} |\Omega|^{\frac{q-2}{2}}\right]^{\frac{2}{N(q-2)-4}}$$

and $h_2(t) < 0$ for any $t > t_0$, $h_2(t) > 0$ for any $0 < t < t_0$. Moreover, $h_2(t)$ achieves its maximum at

$$t_\Theta = \left[\frac{4q\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)\theta}{N(q-2)}\Theta^{\frac{2-q}{2}}|\Omega|^{\frac{q-2}{2}}\right]^{\frac{2}{N(q-2)-4}}.$$

This implies

$$J_{r,s}(v_{t_0}) = J_{\frac{1}{t_0},s}(v_{t_0}) \leq h_2(t_0) = 0 \tag{5.5}$$

for any $r \geq \frac{1}{t_0}$ and $s \in \left[\frac{1}{2}, 1\right]$. There exists $0 < t_1 < t_\Theta$ such that for any $t \in [0, t_1]$,

$$h_2(t) < \frac{N(q-2)-4}{4}\left(\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S\right)}{N(q-2)}\right)^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}. \tag{5.6}$$

On the other hand, it follows from (3.5), the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$J_{r,s}(u) \geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\int_{\Omega_r} |\nabla u|^2 dx - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q}\left(\int_{\Omega_r}|\nabla u|^2 dx\right)^{\frac{N(q-2)}{4}}$$
$$- \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}. \tag{5.7}$$

Define

$$
\begin{aligned}
g_1(t) &:= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}} \\
&= t^{\frac{N(p_1-2)}{4}}\left[\frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t^{\frac{4-N(p_1-2)}{4}} - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-p_1)}{4}}\right] \\
&\quad - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}}.
\end{aligned}
$$

In view of $2 < p < 2 + \frac{2}{N} < q < 2^*$ and the definition of $\widetilde{\Theta}_V$, there exist $0 < l_1 < l_M < l_2$ such that $g_1(t) < 0$ for any $0 < t < l_1$ and $t > l_2$, $g_1(t) > 0$ for $l_1 < t < l_2$ and $g_1(l_M) = \max\limits_{t\in\mathbb{R}^+} g_1(t) > 0$. Let

$$
t_2 = \left(\frac{\alpha\beta q C_{N,p_1}(p_1-2)(4-N(p_1-2))}{C_{N,q}(q-2)(N(q-2)-4)}\right)^{\frac{4}{N(q-p_1)}} \Theta^{\frac{N-2}{N}}.
$$

Then by a direct calculation, we have $g_1''(t) \leq 0$ if and only if $t \geq t_2$. Hence

$$
\max_{t\in\mathbb{R}^+} g_1(t) = \max_{t\in[t_2,\infty)} g_1(t).
$$

Note that for any $t \geq t_2$,

$$
\begin{aligned}
g_1(t) &= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}} \\
&= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \alpha\beta C_{N,p_1}\Theta^{\frac{(q-p_1)(N-2)}{4}} \cdot \Theta^{\frac{2q-N(q-2)}{4}} t^{\frac{N(p_1-2)}{4}} \\
&\quad - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}} \\
&\geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \left(\frac{C_{N,q}(q-2)(N(q-2)-4)}{q(p_1-2)(4-N(p_1-2))} + \frac{C_{N,q}}{q}\right)\Theta^{\frac{2q-N(q-2)}{4}} \cdot t^{\frac{N(q-2)}{4}} \\
&=: g_2(t). \tag{5.8}
\end{aligned}
$$

Now, we will determine the value of $\widetilde{\Theta}_V$. In fact, $g_1(l_M) = \max_{t\in\mathbb{R}^+} g_1(t) > 0$ as long as $g_2(t_2) > 0$, that is,

$$
\begin{aligned}
g_2(t_2) &= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\left(\frac{\alpha\beta q C_{N,p_1}}{C_{N,q}A_{p_1,q}}\right)^{\frac{4}{N(q-p_1)}} \Theta^{\frac{N-2}{N}} - \left(\frac{C_{N,q}}{q}A_{p_1,q} + \frac{C_{N,q}}{q}\right)\cdot\Theta \\
&\quad \cdot \left(\frac{\alpha\beta q C_{N,p_1}}{C_{N,q}A_{p_1,q}}\right)^{\frac{q-2}{q-p_1}} \\
&> 0,
\end{aligned}
$$

where

$$
A_{p_1,q} = \frac{(q-2)(N(q-2)-4)}{(p_1-2)(4-N(p_1-2))}.
$$

Hence, we take

$$
\widetilde{\Theta}_V = \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}}\left(\frac{C_{N,q}}{q}A_{p_1,q} + \frac{C_{N,q}}{q}\right)^{-\frac{N}{2}}\left(\frac{\alpha\beta q C_{N,p_1}}{C_{N,q}A_{p_1,q}}\right)^{\frac{N(q-2)-4}{2N(q-p_1)}}.
$$

Let

$$A = \left(\frac{C_{N,q}(q-2)(N(q-2)-4)}{q(p_1-2)(4-N(p_1-2))} + \frac{C_{N,q}}{q} \right)$$

and

$$t_g = \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)A} \right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}},$$

so that $t_g > t_2$ by the definition of $\widetilde{\Theta}_V$, $\max_{t \in [t_2,\infty)} g_2(t) = g_2(t_g)$ and

$$\max_{t \in \mathbb{R}^+} g_1(t) \geq \max_{t \in [t_2,\infty)} g_2(t)$$

$$= \frac{(N(q-2)-4)}{4} \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S\right)}{N(q-2)} \right]^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

Set $\bar{r}_\Theta = \max\left\{ \frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{t_g}} \right\}$, then $v_{\frac{1}{\bar{r}_\Theta}} \in S_{r,\Theta}$ for any $r > \bar{r}_\Theta$, and

$$\left\| \nabla v_{\frac{1}{\bar{r}_\Theta}} \right\|_2^2 = \left(\frac{1}{\bar{r}_\Theta} \right)^2 \|\nabla v_1\|_2^2 < t_g = \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)A} \right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}. \tag{5.9}$$

Moreover,

$$J_{\bar{r}_\Theta,s}\left(v_{\frac{1}{\bar{r}_\Theta}} \right) \leq h_2\left(\frac{1}{\bar{r}_\Theta} \right) \leq h_2(t_1). \tag{5.10}$$

Let $u^0 = v_{\frac{1}{\bar{r}_\Theta}}, u^1 = v_{t_0}$ and

$$\widetilde{r}_\Theta = \max\left\{ \frac{1}{t_0}, \bar{r}_\Theta \right\}.$$

Then the statement (i) holds by (5.5), (5.6), (5.9), (5.10).

    (ii) holds by (5.8) and a direct calculation.

    (iii) In view of $J_{r,s}(u^1) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$ and the definition of t_0, we have

$$\|\nabla\gamma(0)\|_2^2 < t_g < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (5.8) that

$$\max_{t \in [0,1]} J_{r,s}(\gamma(t)) \geq g_2(t_g)$$

$$= \frac{N(q-2)-4}{4} \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)} \right]^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}$$

for any $\gamma \in \Gamma_{r,\Theta}$, hence the first inequality in (iii) holds. We define a path $\gamma : [0,1] \to S_{r,\Theta}$ by

$$\gamma(t) : \Omega_r \to \mathbb{R}, \qquad x \mapsto \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta} \right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta} \right) x \right).$$

Then $\gamma \in \Gamma_{r,\Theta}$, and the second inequality in (iii) follows from (5.4). $\square$

Lemma 5.2. *Assume $0 < \Theta < \widetilde{\Theta}_V$ where $\widetilde{\Theta}_V$ is given in Theorem 1.5. Let $r > \widetilde{r}_\Theta$, where $\widetilde{r}_\Theta$ is defined in Lemma 5.1. Then problem (5.1) admits a solution $(\lambda_{r,s}, u_{r,s})$ for almost every $s \in \left[\frac{1}{2}, 1\right]$. Moreover, there hold $u_{r,s} > 0$ and $J_{r,s}(u_{r,s}) = m_{r,s}(\Theta)$.*

Proof. The proof is similar to the Lemma 3.3. We omit it here. □

Lemma 5.3. *For fixed $\Theta > 0$ the set of solutions $u \in S_{r,\Theta}$ of (5.1) is bounded uniformly in s and r.*

Proof. Since u is a solution of (5.1), we have

$$\int_{\Omega_r} |\nabla u|^2 dx + \int_{\Omega_r} V u^2 dx = s \int_{\Omega_r} |u|^q dx + s\beta \int_{\Omega_r} f(u) u \, dx - \lambda \int_{\Omega_r} |u|^2 dx.$$

The Pohozaev identity implies

$$\frac{N-2}{2N} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma + \frac{1}{2N} \int_{\Omega_r} \widetilde{V}(x) u^2 + \frac{1}{2} \int_{\Omega_r} V u^2 dx$$
$$= -\frac{\lambda}{2} \int_{\Omega_r} |u|^2 dx + \frac{s}{q} \int_{\Omega_r} |u|^q dx + s\beta \int_{\Omega_r} F(u) dx.$$

It then follows from $\beta > 0$ and (f_2) that

$$\frac{1}{N} \int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma - \frac{1}{2N} \int_{\Omega_r} (\nabla V \cdot x) u^2 dx$$
$$= \frac{(q-2)s}{2q} \int_{\Omega_r} |u|^q dx + s \int_{\Omega_r} \left(\frac{\beta}{2} f(u) u - \beta F(u)\right) dx$$
$$\geq \frac{q-2}{2} \left(\frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V u^2 dx - m_{r,s}(\Theta)\right) + s\frac{\beta(p_2 - q)}{2} \int_{\Omega_r} F(u) dx.$$

Using Gagliardo–Nirenberg inequality, (3.5) and (iii) in Lemma 5.1, we have

$$\frac{q-2}{2} m_{r,s}(\Theta) \geq \frac{N(p_2-2)-4}{4N} \int_{\Omega_r} |\nabla u|^2 dx - \Theta\left(\frac{1}{2N} \|\nabla V \cdot x\|_\infty + \frac{p_2-2}{4} \|V\|_\infty\right)$$
$$+ \frac{s\alpha\beta(p_2-q)}{2} C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}.$$

Since $2 < p_1 < 2 + \frac{4}{N}$, we can bound $\int_{\Omega_r} |\nabla u|^2 dx$ uniformly in s and r. □

Lemma 5.4. *Assume $0 < \Theta < \widetilde{\Theta}_V$, where $\widetilde{\Theta}_V$ is given in Theorem 1.5, and let $r > \widetilde{r}_\Theta$, where $\widetilde{r}_\Theta$ is defined in Lemma 5.1. Then the following hold:*

(i) *Equation (2.1) admits a solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ for every $r > \widetilde{r}_\Theta$ such that $u_{r,\Theta} > 0$ in Ω_r.*

(ii) *There is $0 < \bar{\Theta} \leq \widetilde{\Theta}_V$ such that*

$$\liminf_{r \to \infty} \lambda_{r,\Theta} > 0 \quad \text{for any } 0 < \Theta < \bar{\Theta}.$$

Proof. The proof of (i) is similar to that of Lemma 3.5, we omit it. As be consider $H^1_0(\Omega_r)$ as a subspace of $H^1(\mathbb{R}^N)$ for every $r > 0$. In view of Lemma 5.3, there are λ_Θ and $u_\Theta \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_{r,\Theta} \rightharpoonup u_\Theta \quad \text{in } H^1(\mathbb{R}^N) \quad \text{and} \quad \lim_{r \to \infty} \lambda_{r,\Theta} \to \lambda_\Theta.$$

Arguing by contradiction, we assume that $\lambda_{\Theta_n} \leq 0$ for some sequence $\Theta_n \to 0$. Let θ_r be the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω_r and let $v_r > 0$ be the corresponding normalized eigenfunction. Testing (2.1) with v_r, it holds

$$(\theta_r + \lambda_{r,\Theta_n}) \int_{\Omega_r} u_{r,\Theta_n} v_r dx + \int_{\Omega_r} V u_{r,\Theta_n} v_r dx \geq 0.$$

In view of $\int_{\Omega_r} u_{r,\Theta_n} v_r dx > 0$ and $\theta_r = r^{-2}\theta_1$, there holds

$$\max_{x \in \mathbb{R}^N} V + \lambda_{r,\Theta_n} + r^{-2}\theta_1 \geq 0.$$

Hence there exists $C > 0$ independent of n such that $|\lambda_{\Theta_n}| \leq C$ for any n.

Case 1. There is subsequence denoted still by $\{\Theta_n\}$ such that $u_{\Theta_n} = 0$. We first claim that there exists $d_n > 0$ for any n such that

$$\liminf_{r \to \infty} \sup_{z \in \mathbb{R}^N} \int_{B(z,1)} u_{r,\Theta_n}^2 dx \geq d_n. \tag{5.11}$$

Otherwise, the concentration compactness principle implies for every n that

$$u_{r,\Theta_n} \to 0 \text{ in } L^t(\mathbb{R}^N) \quad \text{as } r \to \infty, \quad \text{for all } 2 < t < 2^*.$$

By the diagonal principle, (2.1) and $|\lambda_{r,\Theta_n}| \leq 2C$ for large r, there exists $r_n \to \infty$ such that

$$\int_{\Omega_r} |\nabla u_{r_n,\Theta_n}|^2 dx \leq C$$

for some C independent of n, contradicting (iii) in Lemma 5.1 for large n. As a consequence (5.11) holds, and there is $z_{r,\Theta_n} \in \Omega_r$ with $|z_{r,\Theta_n}| \to \infty$ such that

$$\int_{B(z_{r,\Theta_n},1)} u_{r,\Theta_n}^2 dx \geq \frac{d_n}{2}.$$

Moreover, $\text{dist}(z_{r,\Theta_n}, \partial \Omega_r) \to \infty$ as $r \to \infty$ by an argument similar to that in Lemma 3.7. Now, for n fixed let $v_r(x) = u_{r,\Theta_n}(x + z_{r,\Theta_n})$ for $x \in \Sigma^r := \{x \in \mathbb{R}^N : x + z_{r,\Theta_n} \in \Omega_r\}$. It follows from Lemma 5.3 that there is $v \in H^1(\mathbb{R}^N)$ with $v \neq 0$ such that $v_r \rightharpoonup v$. Observe that for every $\phi \in C_c^\infty(\mathbb{R}^N)$ there is r large such that $\phi(\cdot - z_{r,\Theta_n}) \in C_c^\infty(\Omega_r)$ due to $\text{dist}(z_{r,\Theta_n}, \partial \Omega_r) \to \infty$ as $r \to \infty$. It follows that

$$\int_{\Omega_r} \nabla u_{r,\Theta_n} \nabla \phi(\cdot - z_{r,\Theta_n}) dx + \int_{\Omega_r} V u_{r,\Theta_n} \phi(\cdot - z_{r,\Theta_n}) dx + \lambda_{r,\Theta_n} \int_{\Omega_r} u_{r,\Theta_n} \phi(\cdot - z_{r,\Theta_n}) dx$$

$$= \int_{\Omega_r} |u_{r,\Theta_n}|^{q-2} u_{r,\Theta_n} \phi(\cdot - z_{r,\Theta_n}) dx + \beta \int_{\Omega_r} f(u_{r,\Theta_n}) \phi(\cdot - z_{r,\Theta_n}) dx. \tag{5.12}$$

Using $|z_{r,\Theta_n}| \to \infty$ as $r \to \infty$, it follows that

$$\left| \int_{\Omega_r} V u_{r,\Theta_n} \phi(\cdot - z_{r,\Theta_n}) dx \right| \leq \int_{\text{Supp}\,\phi} |V(\cdot + z_{r,\Theta_n}) v_r \phi| dx$$

$$\leq \|v_r\|_{2^*} \|\phi\|_{2^*} \left(\int_{\mathbb{R}^N \setminus B_{\frac{|z_{r,\Theta_n}|}{2}}} |V|^{\frac{N}{2}} dx \right)^{\frac{2}{N}}$$

$$\to 0 \quad \text{as } r \to \infty.$$

Letting $r \to \infty$ in (5.12), we get for every $\phi \in C_c^\infty(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} \nabla v \cdot \nabla \phi \, dx + \lambda_{\Theta_n} \int_{\mathbb{R}^N} v \phi \, dx = \int_{\mathbb{R}^N} |v|^{q-2} v \phi \, dx + \beta \int_{\mathbb{R}^N} f(v) \phi \, dx.$$

Therefore $v \in H^1(\mathbb{R}^N)$ is a weak solution of the equation

$$-\Delta v + \lambda_{\Theta_n} v = \beta f(v) + |v|^{q-2} v \quad \text{in } \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx + \lambda_{\Theta_n} \int_{\mathbb{R}^N} |v|^2 dx = \beta \int_{\mathbb{R}^N} f(v) v \, dx + \int_{\mathbb{R}^N} |v|^q dx.$$

The Pohozaev identity implies

$$\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{\lambda_{\Theta_n}}{2} \int_{\mathbb{R}^N} |v|^2 dx = \beta \int_{\mathbb{R}^N} F(v) dx + \frac{1}{q} \int_{\mathbb{R}^N} |v|^q dx,$$

hence

$$\frac{\lambda_{\Theta_n}}{N} \int_{\mathbb{R}^N} |v|^2 dx$$
$$= \frac{\beta(N-2)}{2N} \int_{\mathbb{R}^N} \left[\frac{2N}{N-2} F(v) - f(v) v \right] dx + \frac{2N - q(N-2)}{2Nq} \int_{\mathbb{R}^N} |v|^q dx$$
$$\geq \frac{\beta(N-2)}{2N} \left(\frac{2N}{N-2} - p_1 \right) \int_{\mathbb{R}^N} F(v) dx + \frac{2N - q(N-2)}{2Nq} \int_{\mathbb{R}^N} |v|^q dx. \tag{5.13}$$

We have $\lambda_{\Theta_n} > 0$ because of $2 < p_1 < 2 + \frac{4}{N} < q < 2^*$, which is a contradiction.

Case 2. $u_{\Theta_n} \neq 0$ for n large. Note that u_{Θ_n} satisfies

$$-\Delta u_{\Theta_n} + V u_{\Theta_n} + \lambda_{\Theta_n} u_{\Theta_n} = \beta f(u_{\Theta_n}) + |u_{\Theta_n}|^{q-2} u_{\Theta_n}. \tag{5.14}$$

If $v_{r,\Theta_n} := u_{r,\Theta_n} - u_{\Theta_n}$ satisfies

$$\limsup_{r \to \infty} \max_{z \in \mathbb{R}^N} \int_{B(z,1)} v_{r,\Theta_n}^2 dx = 0, \tag{5.15}$$

then the concentration compactness principle implies $u_{r,\Theta_n} \to u_{\Theta_n}$ in $L^t(\mathbb{R}^N)$ for any $2 < t < 2^*$. It then follows from (2.1) and (5.14) that

$$\int_{\Omega_r} |\nabla u_{r,\Theta_n}|^2 dx + \Theta_n \lambda_{r,\Theta_n} = \beta \int_{\Omega_r} f(u_{r,\Theta_n}) u_{r,\Theta_n} dx + \int_{\Omega_r} |u_{r,\Theta_n}|^q dx - \int_{\Omega_r} V u_{r,\Theta_n}^2 dx$$
$$\to \beta \int_{\mathbb{R}^N} f(u_{\Theta_n}) u_{r,\Theta_n} dx + \int_{\mathbb{R}^N} |u_{\Theta_n}|^q dx - \int_{\mathbb{R}^N} V u_{\Theta_n}^2 dx$$
$$= \int_{\mathbb{R}^N} |\nabla u_{\Theta_n}|^2 dx + \lambda_{\Theta_n} \int_{\mathbb{R}^N} u_{\Theta_n}^2 dx.$$

Using $\lambda_{r,\Theta_n} \to \lambda_{\Theta_n}$ as $r \to \infty$, we further have

$$\int_{\Omega_r} |\nabla u_{r,\Theta_n}|^2 dx + \Theta_n \lambda_{\Theta_n} \to \int_{\mathbb{R}^N} |\nabla u_{\Theta_n}|^2 dx + \lambda_{\Theta_n} \int_{\mathbb{R}^N} u_{\Theta_n}^2 dx \quad \text{as } r \to \infty. \tag{5.16}$$

Using (5.16), (iii) in Lemma 5.1 and $|\lambda_{\Theta_n}| \leq C$ for large n, there holds

$$\int_{\mathbb{R}^N} |\nabla u_{\Theta_n}|^2 dx \to \infty \quad \text{as } n \to \infty.$$

By (5.14) and the Pohozaev identity

$$\frac{N-2}{2N}\int_{\mathbb{R}^N}|\nabla u_{\Theta_n}|^2\,dx + \frac{1}{2N}\int_{\mathbb{R}^N}\widetilde{V}u_{\Theta_n}^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u_{\Theta_n}^2\,dx + \frac{\lambda_\Theta}{2}\int_{\mathbb{R}^N}u_{\Theta_n}^2\,dx$$
$$= \frac{1}{q}\int_{\mathbb{R}^N}|u_{\Theta_n}|^q\,dx + \beta\int_{\mathbb{R}^N}F(u_{\Theta_n})\,dx.$$

It holds that

$$0 \le \frac{(2-q)\lambda_{\Theta_n}}{2q}\int_{\mathbb{R}^N}u_{\Theta_n}^2\,dx$$
$$\le \frac{(N-2)q-2N}{2Nq}\int_{\mathbb{R}^N}|\nabla u_{\Theta_n}|^2\,dx + \frac{\|\widetilde{V}\|_\infty}{2N}\Theta_n + \frac{(q-2)\|V\|_\infty}{2q}\Theta_n$$
$$\to -\infty \quad \text{as } n \to \infty.$$

Therefore (5.15) cannot occur. Consequently there exist $d_n > 0$ and $z_{r,\Theta_n} \in \Omega_r$ with $|z_{r,\Theta_n}| \to \infty$ as $r \to \infty$ such that

$$\int_{B(z_{r,\Theta_n},1)} v_{r,\Theta_n}^2\,dx > d_n.$$

Then $\widetilde{v}_{r,\Theta_n} := v_{r,\Theta_n}(\cdot + z_{r,\Theta_n}) \rightharpoonup \widetilde{v}_{\Theta_n} \ne 0$, and $\widetilde{v}_{\Theta_n}$ is a nonnegative solution of

$$-\Delta v + \lambda_{\Theta_n}v = \beta f(v)v + |v|^{q-2}v \quad \text{in } \mathbb{R}^N.$$

In fact, we have $\liminf_{r\to\infty}\text{dist}(z_{r,\Theta_n}, \partial\Omega_r) = \infty$ by the Liouville theorem on the half space. It follows from an argument similar to that of (5.13) that $\lambda_{\Theta_n} > 0$ for large n, which is a contradiction. $\qquad\square$

Proof of Theorem 1.5. The proof is a direct consequence of Lemma 5.4 and Lemma 3.6. $\qquad\square$

6   Proof of Theorem 1.9

In this section we assume that the assumptions of Theorem 1.9 hold. Define the functional $\mathcal{I}_r : S_{r,\Theta} \to \mathbb{R}$ by

$$\mathcal{I}_r(u) = \frac{1}{2}\int_{\Omega_r}|\nabla u|^2\,dx + \frac{1}{2}\int_{\Omega_r}Vu^2\,dx - \frac{1}{2^*}\int_{\Omega_r}|u|^{2^*}\,dx - \beta\int_{\Omega_r}F(u)\,dx. \tag{6.1}$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $\mathcal{I}_{r,s}$, then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the equation

$$\begin{cases} -\Delta u + Vu + \lambda u = |u|^{2^*-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r}|u|^2\,dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{6.2}$$

Since $\beta > 0$,

$$\mathcal{I}_r(u) \ge \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1}\right)\int_{\Omega_r}|\nabla u|^2\,dx - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}\left(\int_{\Omega_r}|\nabla u|^2\,dx\right)^{\frac{2^*}{2}}$$
$$- \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2\,dx\right)^{\frac{N(p_1-2)}{4}}$$
$$= \widetilde{h}_1(t),$$

where

$$\widetilde{h}_1(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1}\right)t^2 - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}t^{2^*} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1 - N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{2}}.$$

Consider

$$\widehat{\psi}(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1}\right)t^2 - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}t^{2^*}.$$

Note that $\widehat{\psi}$ admits a unique maximum at

$$\widehat{t} = \left[\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1}\right)S^{\frac{2^*}{2}}\right]^{\frac{1}{2^*-2}}.$$

By a direct calculation, we obtain

$$\widehat{\psi}(\widehat{t}) = \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1}\right)^{\frac{N}{2}}S^{\frac{N}{2}}.$$

Hence,

$$\widehat{\psi}(\widehat{t}) > \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\widehat{t}^{\frac{N(p_1-2)}{2}}$$

as long as

$$\Theta_V = \left(\frac{1}{N\alpha\beta C_{N,p_1}}\right)^{\frac{4}{2p_1-N(p_1-2)}}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1}\right)^{\frac{N}{2}}S^{\frac{N}{2}\cdot\frac{4-N(p_1-2)}{2p_1-N(p_1-2)}}.$$

Now, let $0 < \Theta < \Theta_V$ be fixed, we obtain

$$\widehat{\psi}(\widehat{t}) > \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\widehat{t}^{\frac{N(p_1-2)}{2}}$$

and $\widetilde{h}_1(\widehat{t}) > 0$. In view of $2 < p_1 < 2 + \frac{4}{N} < 2^*$, there exist $0 < \widetilde{R}_1 < \widetilde{T}_\Theta < \widetilde{R}_2$ such that $\widetilde{h}_1(t) < 0$ for $0 < t < \widetilde{R}_1$ and for $t > \widetilde{R}_2$, $\widetilde{h}_1(t) > 0$ for $\widetilde{R}_1 < t < \widetilde{R}_2$, and $\widetilde{h}_1(\widetilde{T}_\Theta) = \max_{t\in\mathbb{R}^+}\widetilde{h}_1(t) > 0$.

Define

$$\widetilde{\mathcal{V}}_{r,\Theta} = \left\{u \in S_{r,\Theta} : \|\nabla u\|_2^2 \le \widetilde{T}_\Theta^2\right\}.$$

Let θ be the principal eigenvalue of operator $-\Delta$ with Dirichlet boundary condition in Ω, and let $|\Omega|$ be the volume of Ω.

Lemma 6.1.

(i) *If* $r < \frac{\sqrt{C\Theta}}{\widetilde{T}_\Theta}$, *then* $\widetilde{\mathcal{V}}_{r,\Theta} = \emptyset$.

(ii) *If*

$$r > \max\left\{\frac{\sqrt{C\Theta}}{\widetilde{T}_\Theta}, \left(\frac{\theta\left(1 + \|V\|_{\frac{N}{2}}S^{-1}\right)}{2\alpha_1\beta}\Theta^{\frac{2-p_2}{2}}|\Omega|^{\frac{p_2-2}{2}}\right)^{\frac{2}{N(p_2-2)+4}}\right\}$$

then $\widetilde{\mathcal{V}}_{r,\Theta} \ne \emptyset$ *and*

$$\widetilde{e}_{r,\Theta} := \inf_{u\in\widetilde{\mathcal{V}}_{r,\Theta}}\mathcal{I}_r(u) < 0$$

is attained at some interior point $u_r > 0$ *of* $\widetilde{\mathcal{V}}_{r,\Theta}$. *As a consequence, there exists a Lagrange multiplier* $\lambda_r \in \mathbb{R}$ *such that* (λ_r, u_r) *is a solution of* (6.1). *Moreover* $\liminf_{r\to\infty}\lambda_r > 0$ *holds true.*

Proof. (i) The proof is similar to the Lemma 4.1. (ii) Let $v_1 \in S_{1,\Theta}$ be the positive normalized eigenfunction corresponding to θ. Setting

$$r_\Theta = \max\left\{ \frac{\sqrt{C\Theta}}{\widetilde{T_\Theta}}, \left(\frac{\theta\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)}{2\alpha_1 \beta} \Theta^{\frac{2-p_2}{2}} |\Omega|^{\frac{p_2-2}{2}} \right)^{\frac{2}{N(p_2-2)+4}} \right\}. \tag{6.3}$$

Now, we construct for $r > r_\Theta$ a function $u_r \in S_{r,\Theta}$ such that $u_r \in \widetilde{\mathcal{V}}_{r,\Theta}$ and $\mathcal{I}_r(u_r) < 0$. By (6.3), (4.3), (4.4), $2 < p_2 < 2 + \frac{4}{N}$ and a direct calculation, we have $u_r \in \widetilde{\mathcal{V}}_{r,\Theta}$ and

$$\mathcal{I}_r(u_r) \leq \frac{1}{2}\int_{\Omega_r} |\nabla u_r|^2\, dx + \frac{1}{2}\int_{\Omega_r} V u_r^2 dx - \frac{1}{2^*}\int_{\Omega_r} |u_r|^{2^*}\, dx - \alpha_1 \beta \int_{\Omega_r} |u_r|^{p_2}\, dx$$

$$\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) r^{-2}\theta\Theta - \alpha_1\beta r^{\frac{N(2-p_2)}{2}} \Theta^{\frac{p_2}{2}} |\Omega|^{\frac{2-p_2}{2}}$$

$$< 0.$$

It then follows from the Gagliardo–Nirenberg inequality that

$$\mathcal{I}_r(u_r) \geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\int_{\Omega_r} |\nabla u|^2 dx - C_{N,p_1}\beta\Theta^{\frac{2p_1 - N(p_1-2)}{4}} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}$$

$$- \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{2^*}{2}}. \tag{6.4}$$

As a consequence $\mathcal{I}_r$ is bounded from below in $\widetilde{\mathcal{V}}_{r,\Theta}$. By the Ekeland principle there exists a sequence $\{u_{n,r}\} \subset \widetilde{\mathcal{V}}_{r,\Theta}$ such that

$$\mathcal{I}_r(u_{n,r}) \to \inf_{u \in \widetilde{\mathcal{V}}_{r,\Theta}} \mathcal{I}_r(u), \ \mathcal{I}'_r(u_{n,r})|_{T_{u_{n,r}} S_{r,\Theta}} \to 0 \quad \text{as } n \to \infty \tag{6.5}$$

Consequently there exists $u_r \in H_0^1(\Omega_r)$ such that

$$u_{n,r} \rightharpoonup u_r \quad \text{in } H_0^1(\Omega_r)$$

and

$$u_{n,r} \to u_r \quad \text{in } L^k(\Omega_r) \text{ for all } 2 \leq k < 2^*. \tag{6.6}$$

We claim now that the weak limit u_r does not vanish identically. Suppose by contradiction that $u_r \equiv 0$. Since $\{u_{n,r}\}$ is bounded in $H^1(\Omega_r)$, up to a subsequence we have that $\|\nabla u_{n,r}\|_2^2 \to \ell \in \mathbb{R}$. Using (f_2), (6.5), (6.6), we have

$$\langle \mathcal{I}'_r(u_{n,r}), u_{n,r}\rangle = \int_{\Omega_r} |\nabla u_{n,r}|^2\, dx + \int_{\Omega_r} V u_{n,r}^2 dx - \int_{\Omega_r} |u_{n,r}|^{2^*}\, dx - \beta\int_{\Omega_r} f(u_{n,r})u_{n,r} dx$$

$$\to 0,$$

hence

$$\|u_{n,r}\|_{2^*}^{2^*} = \|\nabla u_{n,r}\|_2^2 \to \ell$$

as well. Therefore, by the Sobolev inequality $\ell \geq S\ell^{\frac{2}{2^*}}$, and we deduce that either $\ell = 0$, or $\ell \geq S^{\frac{N}{2}}$. Let us suppose at first that $\ell \geq S^{N/2}$. Since $\mathcal{I}_r(u_{n,r}) \to \widetilde{e}_{r,\Theta} < 0$, we have that

$$0 > \widetilde{e}_{r,\Theta} + o(1) = I_r(u_{n,r})$$

$$= \frac{1}{2}\int_{\Omega_r} |\nabla u_{n,r}|^2\, dx + \frac{1}{2}\int_{\Omega_r} V u_{n,r}^2 dx - \frac{1}{2^*}\int_{\Omega_r} |u_{n,r}|^{2^*}\, dx - \beta\int_{\Omega_r} F(u_{n,r}) dx$$

$$= \frac{1}{N}\|\nabla u_{n,r}\|_2^2 + o(1) = \frac{\ell}{N} + o(1),$$

which is not possible. If instead $\ell = 0$, we have $\|u_{n,r}\|_{2^*} \to 0$, $\|\nabla u_{n,r}\|_2 \to 0$ and $F(u_{n,r}) \to 0$. But then $\mathcal{I}_r(u_{n,r}) \to 0 \neq \widetilde{e}_{r,\Theta}$, which gives again a contradiction. Thus, u_r does not vanish identically.

Since $\{u_{n,r}\}$ is a bounded minimization sequence for $\mathcal{I}_r(u_{n,r})|_{S_{r,\Theta}}$, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that for every $\varphi \in H^1(\Omega_r)$,

$$\int_{\Omega_r} \nabla u_{n,r} \cdot \nabla \varphi + \int_{\Omega_r} V u_{n,r} \varphi + \lambda_n u_{n,r} \varphi - \beta f(u_{n,r}) \varphi - |u_{n,r}|^{2^*-2} u_{n,r} \varphi = o(1)\|\varphi\| \tag{6.7}$$

as $n \to \infty$, by the Lagrange multipliers rule. Choosing $\varphi = u_{n,r}$, we deduce that $\{\lambda_n\}$ is bounded as well, and hence up to a subsequence $\lambda_n \to \lambda_r \in \mathbb{R}$. Moreover, passing to the limit in (6.7) by weak convergence, we obtain

$$-\Delta u_r + V u_r + \lambda_r u_r = |u_r|^{2^*-2} u_r + \beta f(u_r), \qquad x \in \Omega_r.$$

Recalling that $v_{n,r} = u_{n,r} - u_r \rightharpoonup 0$ in $H_0^1(\Omega_r)$, we know

$$\|\nabla u_{n,r}\|_2^2 = \|\nabla u_r\|_2^2 + \|\nabla v_{n,r}\|_2^2 + o(1).$$

By the Brézis–Lieb lemma [12], we have

$$\|u_{n,r}\|_{2^*}^{2^*} = \|u_r\|_{2^*}^{2^*} + \|v_{n,r}\|_{2^*}^{2^*} + o(1).$$

Moreover,

$$\|\nabla u_r\|_2^2 \leq \liminf_{n\to\infty} \|\nabla u_{n,r}\|_2^2 \leq \widetilde{T_\Theta}^2,$$

that is, $u_r \in \widetilde{\mathcal{V}}_{r,\Theta}$. Note that

$$\int_{\Omega_r} V u_{n,r}^2 dx \to \int_{\Omega_r} V u_r^2 dx \quad \text{as } n \to \infty,$$

hence

$$\|v_{n,r}\|_{2^*}^{2^*} = \|\nabla v_{n,r}\|_2^2 \to \ell$$

as well. Therefore, $\ell \geq S\ell^{\frac{2}{2^*}}$, and we deduce that either $\ell = 0$, or $\ell \geq S^{\frac{N}{2}}$. Let us suppose at first that $\ell \geq S^{N/2}$. Since $\mathcal{I}_r(v_{n,r}) \to 0$, we have that

$$\begin{aligned}
o(1) &= \mathcal{I}_r(v_{n,r}) \\
&= \frac{1}{2}\int_{\Omega_r} |\nabla v_{n,r}|^2\, dx + \frac{1}{2}\int_{\Omega_r} V v_{n,r}^2 dx - \frac{1}{2^*}\int_{\Omega_r} |v_{n,r}|^{2^*}\, dx - \beta \int_{\Omega_r} F(v_{n,r}) dx \\
&= \frac{1}{2}\|\nabla v_{n,r}\|_2^2 + o(1) = \frac{\ell}{N} + o(1),
\end{aligned}$$

which is not possible. If instead $\ell = 0$, we have that $u_{n,r} \to u_r$ in $H_0^1(\Omega_r)$, so $\mathcal{I}_r(u_r) < 0$. Therefore u is an interior point of $\widetilde{\mathcal{V}}_{r,\Theta}$ because $\mathcal{I}_r(u) \geq \widetilde{h}_1(\widetilde{T_\Theta}) > 0$ for any $u \in \partial\widetilde{\mathcal{V}}_{r,\Theta}$ by (6.4). The Lagrange multiplier theorem implies that there exists $\lambda_r \in \mathbb{R}$ such that (λ_r, u_r) is a solution of (6.1). Moreover,

$$\begin{aligned}
\lambda_r \Theta &= \int_{\Omega_r} |u_r|^{2^*}\, dx + \beta \int_{\Omega_r} f(u_r) u_r dx - \frac{2}{2^*}\int_{\Omega_r} |u_r|^{2^*} dx - 2\beta \int_{\Omega_r} F(u_r) dx - 2\mathcal{I}_r(u_r) \\
&= \frac{2^*-2}{2^*}\int_{\Omega_r} |u_r|^{2^*}\, dx + \beta \int_{\Omega_r} [f(u_r)u_r - 2F(u_r)] dx - 2\mathcal{I}_r(u_r) \\
&> -2\mathcal{I}_r(u_r) = -2\widetilde{e}_{r,\Theta}. \tag{6.8}
\end{aligned}$$

It follows from the definition of $\tilde{e}_{r,\Theta}$ that $\tilde{e}_{r,\Theta}$ is nonincreasing with respect to r. Hence, $\tilde{e}_{r,\Theta} \leq \tilde{e}_{r_\Theta,\Theta} < 0$ for any $r > r_\Theta$ and $0 < \Theta < \Theta_V$. In view of (6.8), we have

$$\liminf_{r\to\infty} \lambda_r > 0.$$

Finally, the strong maximum principle implies $u_r > 0$. $\qquad\square$

Proof of Theorem 1.9. The proof is a direct consequence of Lemma 6.1 and Lemma 3.6. $\qquad\square$

7 Proof of Theorem 1.10

In this subsection, we assume $\beta \leq 0$ and the assumptions of Theorem 1.10 hold. Consider the following equation

$$\begin{cases} -\Delta u + Vu + \lambda u = |u|^{2^*-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{7.1}$$

For $\frac{1}{2} \leq s \leq 1$, we define the functional $\mathcal{I}_{r,s} : S_{r,\Theta} \to \mathbb{R}$ by

$$\mathcal{I}_{r,s}(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} Vu^2 dx - \frac{s}{2^*}\int_{\Omega_r} |u|^{2^*} dx - \beta \int_{\Omega_r} F(u)dx. \tag{7.2}$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $\mathcal{I}_{r,s}$, then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the equation

$$\begin{cases} -\Delta u + Vu + \lambda u = s|u|^{2^*-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{7.3}$$

Lemma 7.1. *For any $\Theta > 0$, there exist $r_\Theta > 0$ and $u^0, u^1 \in S_{r_\Theta,\Theta}$ such that*

(i) $\mathcal{I}_{r,s}(u^1) \leq 0$ *for any $r > r_\Theta$ and $s \in \left[\frac{1}{2}, 1\right]$,*

$$\left\|\nabla u^0\right\|_2^2 < \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}} < \left\|\nabla u^1\right\|_2^2$$

and

$$\mathcal{I}_{r,s}\left(u^0\right) < \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}}.$$

(ii) *If $u \in S_{r,\Theta}$ satisfies*

$$\|\nabla u\|_2^2 = \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}},$$

then there holds

$$\mathcal{I}_{r,s}(u) \geq \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}}.$$

(iii) *Set*

$$\widetilde{m}_{r,s}(\Theta) = \inf_{\gamma \in \widetilde{\Gamma}_{r,\Theta}} \sup_{t \in [0,1]} \mathcal{I}_{r,s}(\gamma(t))$$

with

$$\widetilde{\Gamma}_{r,\Theta} = \left\{\gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1\right\}.$$

Then

$$\frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}} \leq \widetilde{m}_{r,s}(\Theta) \leq \mathbf{h}(\mathbf{h}_\Theta),$$

where $\mathbf{h}(\mathbf{h}_\Theta) = \max_{t \in \mathbb{R}^+} h(t)$, *the function* $\mathbf{h} : \mathbb{R}^+ \to \mathbb{R}$ *being defined by*

$$\mathbf{h}(t) = \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta\Theta - \alpha\beta C_{N,p_1} \Theta^{\frac{p_1}{2}} \theta^{\frac{N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{2}} - \frac{1}{2 \cdot 2^*} t^{2^*} \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}}.$$

Here θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω, and $|\Omega|$ is the volume of Ω.

Proof. (i) By the Hölder inequality,

$$\int_\Omega |v_1(x)|^{2^*} dx \geq \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}}. \tag{7.4}$$

For $x \in \Omega_{\frac{1}{t}}$ and $t > 0$, define $v_t(x) := t^{\frac{N}{2}} v_1(tx)$. Using (3.3), (7.4), (3.5) and $\frac{1}{2} \leq s \leq 1$, it holds

$$\begin{aligned}
\mathcal{I}_{\frac{1}{t},s}(v_t) &\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta\Theta - \alpha\beta C_{N,p_1} \Theta^{\frac{p_1}{2}} \theta^{\frac{N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{2}} \\
&\quad - \frac{1}{2 \cdot 2^*} t^{\frac{N(2^*-2)}{2}} \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}} \\
&=: \mathbf{h}(t). \tag{7.5}
\end{aligned}$$

Note that since $2 < p_1 < 2 + \frac{4}{N} < q = 2^*$ and $\beta \leq 0$ there exist $0 < \mathbf{h}_\Theta < t_0$ such that $\mathbf{h}(t_0) = 0, \mathbf{h}(t) < 0$ for any $t > t_0, \mathbf{h}(t) > 0$ for any $0 < t < t_0$ and $\mathbf{h}(\mathbf{h}_\Theta) = \max_{t \in \mathbb{R}^+} \mathbf{h}(t)$. As a consequence, there holds

$$\mathcal{I}_{r,s}(v_{t_0}) = \mathcal{I}_{\frac{1}{t_0},s}(v_{t_0}) \leq \mathbf{h}(t_0) = 0 \tag{7.6}$$

for any $r \geq \frac{1}{t_0}$ and $s \in \left[\frac{1}{2}, 1\right]$. Moreover, there exists $0 < t_1 < \mathbf{h}_\Theta$ such that

$$\mathbf{h}(t) < \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}} \tag{7.7}$$

for $t \in [0, t_1]$. On the other hand, it follows from the Sobolev inequality and the Hölder inequality that

$$\mathcal{I}_{r,s}(u) \geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) \int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{2^*}{2}}. \tag{7.8}$$

Define

$$\mathbf{g}(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} t^{\frac{2^*}{2}}$$

and

$$\widetilde{t} = \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}},$$

it is easy to see that $\mathbf{g}$ is increasing on $(0, \widetilde{t})$ and decreasing on $(\widetilde{t}, \infty)$, and

$$\mathbf{g}(\widetilde{t}) = \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}}.$$

For $r \geq \tilde{r}_\Theta := \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{\tilde{t}}}\right\}$ we have $v_{\frac{1}{\tilde{r}_\Theta}} \in S_{r,\Theta}$ and

$$\|\nabla v_{\frac{1}{\tilde{r}_\Theta}}\|_2^2 = \left(\frac{1}{\tilde{r}_\Theta}\right)^2 \|\nabla v_1\|_2^2 < \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}}. \tag{7.9}$$

Moreover, there holds

$$\mathcal{I}_{\tilde{r}_\Theta,s}\left(v_{\frac{1}{\tilde{r}_\Theta}}\right) \leq \mathbf{h}\left(\frac{1}{\tilde{r}_\Theta}\right) \leq \mathbf{h}(t_1). \tag{7.10}$$

Setting $u^0 = v_{\frac{1}{\tilde{r}_\Theta}}, u^1 = v_{t_0}$ and

$$r_\Theta = \max\left\{\frac{1}{t_0}, \tilde{r}_\Theta\right\}. \tag{7.11}$$

Since (7.6), (7.7), (7.8) and (7.9), then (i) holds.
    (ii) By (7.8) and a direct calculation, (ii) holds.
    (iii) Since $\mathcal{I}_{r,s}\left(u^1\right) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$, we have

$$\|\nabla\gamma(0)\|_2^2 < \tilde{t} < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (7.8) that

$$\max_{t\in[0,1]} \mathcal{I}_{r,s}(\gamma(t)) \geq \mathbf{g}(\tilde{t}) = \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}}$$

for any $\gamma \in \tilde{\Gamma}_{r,\Theta}$, hence the first inequality in (iii) holds. Now we define a path $\gamma \in \tilde{\Gamma}_{r,\Theta}$ by

$$\gamma(\tau)(x) = \left(\tau t_0 + (1-\tau)\frac{1}{\tilde{r}_\Theta}\right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\tilde{r}_\Theta}\right)x\right)$$

for $\tau \in [0,1]$ and $x \in \Omega_r$. Then by (7.5) we have $\tilde{m}_{r,s}(\Theta) \leq \mathbf{h}(\mathbf{h}_\Theta)$, where $\mathbf{h}(\mathbf{h}_\Theta) = \max_{t\in\mathbb{R}^+} \mathbf{h}(t)$. Note that $\mathbf{h}_\Theta$ is independent of r and s. $\qquad\square$

Using Proposition 3.2 to $\mathcal{I}_{r,s}$, it follows that

$$A(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} V(x)u^2 dx - \beta\int_{\Omega_r} F(u)dx \quad \text{and} \quad B(u) = \frac{1}{2^*}\int_{\Omega_r} |u|^{2^*} dx.$$

Hence, for almost every $s \in \left[\frac{1}{2}, 1\right]$, there exists a bounded Palais–Smale sequence $\{u_n\}$ satisfying

$$\mathcal{I}_{r,s}(u_n) \to \tilde{m}_{r,s}(\Theta) \quad \text{and} \quad \mathcal{I}'_{r,s}(u_n)\big|_{T_{u_n}S_{r,\Theta}} \to 0.$$

Next, we are devoted to proving compactness.

Lemma 7.2. *If $\beta \leq 0$ and the assumptions of Theorem 1.10 hold, then $\tilde{m}_{r,s}(\Theta) < \frac{\zeta}{N}S^{\frac{N}{2}}$, where $\zeta = s^{-\frac{2}{2^*-2}}$.*

Proof. Let U_ε be defined by $U_\varepsilon(x) := \left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{\frac{N-2}{2}}$ (up to a scalar factor, U_ε is the bubble centered in the origin, with concentration parameter $\varepsilon > 0$, defined in (1.3)). Let also $\varphi \in C_c^\infty(\Omega_r)$ be a radial cut-off function with $\varphi \equiv 1$ in $B_1, \varphi \equiv 0$ in B_2^c, and φ radially decreasing. We define

$$u_\varepsilon(x) := \varphi(x)U_\varepsilon(x), \quad \text{and} \quad v_\varepsilon(x) := \sqrt{\Theta}\frac{u_\varepsilon(x)}{\|u_\varepsilon\|_2}.$$

Notice that $u_\varepsilon \in C_c^\infty(\Omega_r)$, and $v_\varepsilon \in S_{r,\Theta}$. Let us recall the following useful estimates (see [27, Lemma A.1]):

$$\|\nabla u_\varepsilon\|_2^2 = K_1 + O\left(\varepsilon^{N-2}\right), \tag{7.12}$$

$$\|u_\varepsilon\|_{2^*}^2 = \begin{cases} K_2 + O\left(\varepsilon^N\right), & \text{if } N \geq 4, \\ K_2 + O\left(\varepsilon^2\right), & \text{if } N = 3, \end{cases} \tag{7.13}$$

$$\|u_\varepsilon\|_2^2 = \begin{cases} \varepsilon^2 K_3 + O\left(\varepsilon^{N-2}\right), & \text{if } N \geq 5, \\ \omega\varepsilon^2 |\log \varepsilon| + O\left(\varepsilon^2\right), & \text{if } N = 4, \\ \omega\left(\int_0^2 \varphi(r)dr\right)\varepsilon + O\left(\varepsilon^2\right), & \text{if } N = 3, \end{cases} \tag{7.14}$$

$$\|u_\varepsilon\|_q^q = \varepsilon^{N - \frac{N-2}{2}q}\left(K_4 + O\left(\varepsilon^{(N-2)q-N}\right)\right)$$

$$\text{if } N \geq 4 \text{ and } q \in (2, 2^*), \text{ and if } N = 3 \text{ and } q \in (3, 6). \tag{7.15}$$

as $\varepsilon \to 0$. Since U_ε is extremal for the Sobolev inequality, we have that $\frac{K_1}{K_2} = S$. Therefore, using $\frac{1}{2} \leq s \leq 1$, we have

$$\mathcal{I}_{r,s}(tv_\varepsilon) \leq \frac{t^2}{2}\int_{\Omega_r}|\nabla v_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\Omega_r}Vv_\varepsilon^2 dx - \frac{st^{2^*}}{2^*}\int_{\Omega_r}|v_\varepsilon|^{2^*}dx - \alpha\beta t^{p_1}\int_{\Omega_r}|v_\varepsilon|^{p_1}dx$$

$$=: h_3(t).$$

Clearly, $h_3(t) > 0$ for $t > 0$ small and $h_3(t) \to -\infty$ as $t \to \infty$, so $h_3(t)$ attains its maximum at some $t_\varepsilon > 0$ with $h_3'(t_\varepsilon) = 0$. Then, observing that the function

$$t \mapsto \frac{t^2}{2}\int_{\Omega_r}|\nabla v_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\Omega_r}Vv_\varepsilon^2 dx - \frac{st^{2^*}}{2^*}\int_{\Omega_r}|v_\varepsilon|^{2^*}dx - \alpha\beta t^{p_1}\int_{\Omega_r}|v_\varepsilon|^{p_1}dx$$

is increasing on the interval of

$$\left[0, \left[\frac{-2^*\alpha\beta(p_1-2)\|v_\varepsilon\|_{p_1}^{p_1}}{s(2^*-2)\|v_\varepsilon\|_{2^*}^{2^*}}\right]^{\frac{1}{2^*-p_1}}\right].$$

This fact combined with (7.12)-(7.15) implies that there exist $\delta_1, \delta_2 > 0$, independent of $\varepsilon > 0$, such that

$$\delta_1 \leq t_\varepsilon \leq \delta_2.$$

Moreover, observing that the function

$$t \mapsto \frac{t^2}{2}\int_{\Omega_r}|\nabla v_\varepsilon|^2 dx - \frac{st^{2^*}}{2^*}\int_{\Omega_r}|v_\varepsilon|^{2^*}dx$$

is increasing on the interval of

$$\left[0, \left(\frac{\|\nabla v_\varepsilon\|_2^2}{s\|v_\varepsilon\|_{2^*}^{2^*}}\right)^{\frac{1}{2^*-2}}\right].$$

Using (7.12)–(7.15) and the fact that $\frac{K_1}{K_2} = S$, if $N = 3$, the same estimate holds eventually, using $\|u_\varepsilon\|_{2^*}^2 = K_2 + O(\varepsilon^2)$ instead of $\|u_\varepsilon\|_{2^*}^2 = K_2 + O(\varepsilon^N)$. Therefore, the maximum level is

$$h_3(t_\varepsilon) \leq \frac{t_\varepsilon^2}{2}\|\nabla v_\varepsilon\|_2^2 + \frac{t_\varepsilon^2}{2}\int_{\Omega_r} Vv_\varepsilon^2 dx - \frac{st_\varepsilon^{2^*}}{2^*}\int_{\Omega_r}|v_\varepsilon|^{2^*}dx - \alpha\beta t_\varepsilon^{p_1}\int_{\Omega_r}|v_\varepsilon|^{p_1}dx$$

$$\leq \frac{1}{Ns^{\frac{2}{2^*-2}}}\left(\frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_{2^*}^2}\right)^{\frac{2^*}{2^*-2}} + \frac{\int_{\Omega_r}Vv_\varepsilon^2 dx}{2}\left(\frac{\|\nabla v_\varepsilon\|_2^2}{s\|v_\varepsilon\|_{2^*}^2}\right)^{\frac{2}{2^*-2}} - \alpha\beta\|v_\varepsilon\|_{p_1}^{p_1}\left(\frac{\|\nabla v_\varepsilon\|_2^2}{s\|v_\varepsilon\|_{2^*}^2}\right)^{\frac{p_1}{2^*-2}}$$

$$= \frac{1}{Ns^{\frac{2}{2^*-2}}}\left(\frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^2}\right)^{\frac{N}{2}} + \frac{\int_{\Omega_r}Vv_\varepsilon^2 dx}{2s^{\frac{2}{2^*-2}}}\left(\Theta^{\frac{2-2^*}{2}}\cdot\frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^2}\cdot\frac{\|u_\varepsilon\|_{2^*}^2}{\|u_\varepsilon\|_2^2}\right)^{\frac{2}{2^*-2}}$$

$$- \frac{\alpha\beta}{s^{\frac{p_1}{2^*-2}}}\left(\Theta^{\frac{2-2^*}{2}}\cdot\frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^2}\cdot\frac{\|u_\varepsilon\|_{2^*}^2}{\|u_\varepsilon\|_2^2}\right)^{\frac{p_1}{2^*-2}}\cdot\Theta^{\frac{p_1}{2}}\frac{\|u_\varepsilon\|_{p_1}^{p_1}}{\|u_\varepsilon\|_2^{p_1}}$$

$$\leq \frac{1}{Ns^{\frac{2}{2^*-2}}}\left[\frac{K_1 + O\left(\varepsilon^{N-2}\right)}{K_2 + O\left(\varepsilon^N\right)}\right]^{\frac{N}{2}} + \frac{\max\limits_{x\in\Omega_r}V(x)}{2s^{\frac{2}{2^*-2}}}\cdot\|u_\varepsilon\|_2^2\cdot\frac{\|\nabla u_\varepsilon\|_2^{\frac{4}{2^*-2}}}{\|u_\varepsilon\|_{2^*}^{\frac{2\cdot2^*}{2^*-2}}}$$

$$- \frac{\alpha\beta}{s^{\frac{p_1}{2^*-2}}}\cdot\|u_\varepsilon\|_{p_1}^{p_1}\cdot\frac{\|\nabla u_\varepsilon\|_2^{\frac{2p_1}{2^*-2}}}{\|u_\varepsilon\|_{2^*}^{\frac{p_1\cdot2^*}{2^*-2}}}$$

$$= \frac{1}{Ns^{\frac{2}{2^*-2}}}S^{\frac{N}{2}} + O\left(\varepsilon^{N-2}\right) + C_1\|u_\varepsilon\|_2^2 + C_2\|u_\varepsilon\|_{p_1}^{p_1}$$

$$= \frac{1}{Ns^{\frac{2}{2^*-2}}}S^{\frac{N}{2}}$$

as $\varepsilon \to 0$, where $\zeta = s^{-\frac{2}{2^*-2}}$ and $C_1 \geq 0$, $C_2 \geq 0$ because of $\beta \leq 0$. In the penultimate equal sign, we used

$$\frac{1}{N}\left[\frac{K_1 + O\left(\varepsilon^{N-2}\right)}{K_2 + O\left(\varepsilon^N\right)}\right]^{\frac{N}{2}} = \frac{1}{N}\left[\frac{K_1}{K_2} + O\left(\varepsilon^{N-2}\right)\right]^{\frac{N}{2}} = \frac{S^{\frac{N}{2}}}{N} + O\left(\varepsilon^{N-2}\right).$$

This completes the proof. $\qquad\square$

Lemma 7.3. *For any $\Theta > 0$, let $r > r_\Theta$, where r_Θ is defined in Lemma 7.1. Then problem (7.3) has a solution $(\lambda_{r,s}, u_{r,s})$ for almost every $s \in \left[\frac{1}{2}, 1\right]$. Moreover, $u_{r,s} \geq 0$ and $\mathcal{I}_{r,s}(u_{r,s}) = \widetilde{m}_{r,s}(\Theta)$.*

Proof. Based on the previous analysis, we know that, for almost every $s \in \left[\frac{1}{2}, 1\right]$, there exists a bounded Palais–Smale sequence $\{u_n\}$ satisfying

$$\mathcal{I}_{r,s}(u_n) \to \widetilde{m}_{r,s}(\Theta) \quad \text{and} \quad \mathcal{I}_{r,s}'(u_n)\big|_{T_{u_n}S_{r,\Theta}} \to 0. \tag{7.16}$$

Then

$$\lambda_n = -\frac{1}{\Theta}\left(\int_{\Omega_r}|\nabla u_n|^2\,dx + \int_{\Omega_r}V(x)u_n^2 dx - \beta\int_{\Omega_r}f(u_n)u_n dx - s\int_{\Omega_r}|u_n|^{2^*}dx\right)$$

is bounded and

$$\mathcal{I}_{r,s}'(u_n) + \lambda_n u_n \to 0 \quad \text{in } H^{-1}(\Omega_r). \tag{7.17}$$

Moreover, since $\{u_n\}$ is a bounded Palais–Smale sequence, there exist $u_0 \in H_0^1(\Omega_r)$ and $\lambda \in \mathbb{R}$ such that, up to a subsequence,

$$
\begin{aligned}
&\lambda_n \to \lambda \quad \text{in } \mathbb{R}, \\
&u_n \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega_r), \\
&u_n \to u_0 \quad \text{in } L^t(\Omega_r) \text{ for all } 2 \le t < 2^*,
\end{aligned}
\tag{7.18}
$$

where u_0 satisfies

$$
\begin{cases}
-\Delta u_0 + V u_0 + \lambda u_0 = s\,|u_0|^{2^*-2}\,u_0 + \beta f(u_0) & \text{in } \Omega_r, \\
u_0 \in H_0^1(\Omega_r), \quad \int_{\Omega_r} |u_0|^2\,dx = \Theta.
\end{cases}
$$

Using (7.17), we have

$$
\mathcal{I}'_{r,s}(u_n)\,u_0 + \lambda_n \int_{\Omega_r} u_n u_0\,dx \to 0 \quad \text{as } n \to \infty
$$

and

$$
I'_{r,s}(u_n)\,u_n + \lambda_n \Theta \to 0 \quad \text{as } n \to \infty.
$$

Note that

$$
\lim_{n\to\infty} \int_{\Omega_r} V(x) u_n^2\,dx = \int_{\Omega_r} V(x) u_0^2\,dx,
$$

$$
\lim_{n\to\infty} \int_{\Omega_r} f(u_n) u_n\,dx = \int_{\Omega_r} f(u_0) u_0\,dx,
$$

$$
\lim_{n\to\infty} \int_{\Omega_r} f(u_n) u_0\,dx = \int_{\Omega_r} f(u_0) u_0\,dx.
$$

Now, we show that $u_n \to u_0$ in $H_0^1(\Omega_r)$. Firstly, note that the weak limit u_0 does not vanish identically. Suppose by contradiction that $u_0 \equiv 0$. Since $\{u_n\}$ is bounded in $H^1(\Omega_r)$, up to a subsequence we have that $\|\nabla u_n\|_2^2 \to \ell \in \mathbb{R}$. Using (f_2), (7.17), (7.18), we have

$$
\langle \mathcal{I}'_{r,s}(u_n), u_n \rangle = \int_{\Omega_r} |\nabla u_n|^2\,dx + \int_{\Omega_r} V u_n^2\,dx - s\int_{\Omega_r} |u_n|^{2^*}\,dx - \beta \int_{\Omega_r} f(u_n) u_n\,dx
$$
$$
\to 0,
$$

hence

$$
s\|u_n\|_{2^*}^{2^*} = \|\nabla u_n\|_2^2 \to \ell
$$

as well. Therefore, $\ell \ge s^{-\frac{2}{2^*}} S \ell^{\frac{2}{2^*}}$, and we deduce that either $\ell = 0$, or $\ell \ge s^{-\frac{2}{2^*-2}} S^{\frac{N}{2}}$. Let us suppose at first that $\ell \ge s^{-\frac{2}{2^*-2}} S^{\frac{N}{2}}$. Since $\mathcal{I}_{r,s}(u_n) \to \widetilde{m}_{r,s}(\Theta) < \frac{\zeta}{N} S^{\frac{N}{2}}$, we have that

$$
\frac{\zeta}{N} S^{\frac{N}{2}} > \widetilde{m}_{r,s}(\Theta) \leftarrow \mathcal{I}_{r,s}(u_n) + o(1)
$$
$$
= \frac{1}{2} \int_{\Omega_r} |\nabla u_n|^2\,dx + \frac{1}{2} \int_{\Omega_r} V u_n^2\,dx - \frac{s}{2^*} \int_{\Omega_r} |u_n|^{2^*}\,dx - \beta \int_{\Omega_r} F(u_n)\,dx
$$
$$
= \frac{\ell}{N} \ge s^{-\frac{2}{2^*-2}} \frac{S^{\frac{N}{2}}}{N},
$$

which is not possible. If instead $\ell = 0$, we have $\|u_n\|_{2^*} \to 0$, $\|\nabla u_n\|_2 \to 0$ and $F(u_n) \to 0$. But then $I_{r,s}(u_n) \to 0 \ne \widetilde{m}_{r,s}(\Theta)$, which gives again a contradiction. Thus, u_r does not vanish identically.

Since $\{u_{n,r}\}$ is a bounded minimization sequence for $\mathcal{I}_r(u_{n,r})|_{S_{r,\Theta}}$, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that for every $\varphi \in H^1(\Omega_r)$,

$$\int_{\Omega_r} \nabla u_{n,r} \cdot \nabla \varphi + \int_{\Omega_r} V u_{n,r} \varphi + \lambda_n u_{n,r} \varphi - \beta f(u_{n,r}) \varphi - s |u_{n,r}|^{2^*-2} u_{n,r} \varphi = o(1) \|\varphi\| \qquad (7.19)$$

as $n \to \infty$, by the Lagrange multipliers rule. Choosing $\varphi = u_{n,r}$, we deduce that $\{\lambda_n\}$ is bounded as well, and hence up to a subsequence $\lambda_n \to \lambda_r \in \mathbb{R}$. Moreover, passing to the limit in (7.19) by weak convergence, we obtain

$$-\Delta u_r + V u_r + \lambda_r u_r = s|u_r|^{2^*-2} u_r + \beta f(u_r), \qquad x \in \Omega_r.$$

Recalling that $v_{n,r} = u_{n,r} - u_r \rightharpoonup 0$ in $H_0^1(\Omega_r)$, we know

$$\|\nabla u_{n,r}\|_2^2 = \|\nabla u_r\|_2^2 + \|\nabla v_{n,r}\|_2^2 + o(1).$$

By the Brézis–Lieb lemma [12], we have

$$\|u_{n,r}\|_{2^*}^{2^*} = \|u_r\|_{2^*}^{2^*} + \|v_{n,r}\|_{2^*}^{2^*} + o(1).$$

Note that

$$\int_{\Omega_r} V v_{n,r}^2 dx \to 0 \quad \text{as } n \to \infty,$$

hence

$$s\|v_{n,r}\|_{2^*}^{2^*} = \|\nabla v_{n,r}\|_2^2 \to \ell$$

as well. Therefore, by the Sobolev inequality $\ell \geq s^{-\frac{2}{2^*}} S \ell^{\frac{2}{2^*}}$, and we deduce that either $\ell = 0$, or $\ell \geq s^{-\frac{2}{2^*-2}} S^{\frac{N}{2}}$. Let us suppose at first that $\ell \geq s^{-\frac{2}{2^*-2}} S^{\frac{N}{2}}$. Since $\mathcal{I}_{r,s}(u_n) \to \widetilde{m}_{r,s}(\Theta) < \frac{\zeta}{N} S^{\frac{N}{2}}$, we have that

$$\frac{\zeta}{N} S^{\frac{N}{2}} > \widetilde{m}_{r,s}(\Theta) \leftarrow \mathcal{I}_{r,s}(v_n) + o(1)$$

$$= \frac{1}{2} \int_{\Omega_r} |\nabla v_n|^2 \, dx + \frac{1}{2} \int_{\Omega_r} V v_n^2 dx - \frac{s}{2^*} \int_{\Omega_r} |v_n|^{2^*} \, dx - \beta \int_{\Omega_r} F(v_n) dx$$

$$= \frac{\ell}{N} \geq s^{-\frac{2}{2^*-2}} \frac{S^{\frac{N}{2}}}{N},$$

which is not possible. If instead $\ell = 0$, we have that $u_{n,r} \to u_r$ in $H_0^1(\Omega_r)$, so $\mathcal{I}_r(u_r) > 0$.

Similar to the proof of Lemma 3.3, we also obtain that $u_{r,s} \geq 0$. $\qquad \square$

In order to obtain a solution of (7.1), we also need to prove a uniform estimate for the solutions of (7.3) established in Lemma 7.3. Similar to the proof of Lemma 3.4 and Lemma 3.5, we obtain the following lemmas.

Lemma 7.4. *If $(\lambda, u) \in \mathbb{R} \times S_{r,\Theta}$ is a solution of (7.3) established in Lemma 7.3 for some r and s, then*

$$\int_{\Omega_r} |\nabla u|^2 dx \leq \frac{4N}{N(2^*-2)-4} \left(\frac{2^*-2}{2} \mathbf{h}(\mathbf{h}_\Theta) + \Theta \left(\frac{1}{2N} \|\widetilde{V}\|_\infty + \frac{2^*-2}{4} \|V\|_\infty \right) \right),$$

where the constant $\mathbf{h}(\mathbf{h}_\Theta)$ is defined in (iii) of Lemma 7.1 and is independent of r and s.

Lemma 7.5. *For every $\Theta > 0$, problem (7.3) has a solution (λ_r, u_r) provided $r > r_\Theta$ where r_Θ is as in Lemma 7.1. Moreover, $u_r \geq 0$ in Ω_r.*

Proof of Theorem 1.10. The proof is an immediate consequence of Lemmas 7.5 and 3.6. $\qquad \square$

8   Proof of Theorem 1.11

In this subsection, we assume $\beta > 0$ and the assumptions of Theorem 1.10 hold. Consider the following equation

$$\begin{cases} -\Delta u + Vu + \lambda u = |u|^{2^*-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{8.1}$$

For $\frac{1}{2} \leq s \leq 1$, we define the functional $\mathcal{J}_{r,s} : S_{r,\Theta} \to \mathbb{R}$ by

$$\mathcal{J}_{r,s}(u) = \frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} Vu^2 dx - \frac{s}{2^*} \int_{\Omega_r} |u|^{2^*} dx - s\beta \int_{\Omega_r} F(u) dx. \tag{8.2}$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $\mathcal{J}_{r,s}$, then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the equation

$$\begin{cases} -\Delta u + Vu + \lambda u = s|u|^{2^*-2}u + s\beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{8.3}$$

Lemma 8.1. *For any $\Theta > 0$, there exist $\widehat{r}_\Theta > 0$ and $u^0, u^1 \in S_{r_\Theta,\Theta}$ such that*

(i) *For $r > \widehat{r}_\Theta$ and $s \in \left[\frac{1}{2}, 1\right]$ we have $\mathcal{J}_{r,s}\left(u^1\right) \leq 0$ and*

$$\mathcal{J}_{r,s}\left(u^0\right) < \widehat{A}^{-\frac{2}{2^*-2}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}} \left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right],$$

where

$$\widehat{A} = S^{-\frac{2^*}{2}}\left[\frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))} + \frac{1}{2^*}\right].$$

Moreover,

$$\left\|\nabla u^0\right\|_2^2 < \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^*\widehat{A}}\right]^{\frac{2}{2^*-2}}, \qquad \left\|\nabla u^1\right\|_2^2 > \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^*\widehat{A}}\right]^{\frac{2}{2^*-2}}.$$

(ii) *If $u \in S_{r,\Theta}$ satisfies*

$$\|\nabla u\|_2^2 = \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^*\widehat{A}}\right]^{\frac{2}{2^*-2}},$$

then there holds

$$\mathcal{J}_{r,s}(u) \geq \widehat{A}^{-\frac{2}{2^*-2}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}}\left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right].$$

(iii) *Let*

$$\widehat{m}_{r,s}(\Theta) = \inf_{\gamma \in \widehat{\Gamma}_{r,\Theta}} \sup_{t \in [0,1]} \mathcal{J}_{r,s}(\gamma(t)),$$

where

$$\widehat{\Gamma}_{r,\Theta} = \left\{ \gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1 \right\}.$$

Then

$$\widehat{m}_{r,s}(\Theta) \geq \widehat{A}^{-\frac{2}{2^*-2}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}} \left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right]$$

and

$$\widehat{m}_{r,s}(\Theta) \leq \frac{N(2^*-2)-4}{2} \left(\frac{\theta\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)}{N(2^*-2)}\right)^{\frac{N(2^*-2)}{N(2^*-2)-4}} (4 \cdot 2^*)^{\frac{4}{N(2^*-2)-4}} |\Omega|^{\frac{2(2^*-2)}{N(2^*-2)-4}}$$

$$\cdot \Theta^{\frac{N(2^*-2)-2\cdot 2^*}{N(2^*-2)-4}},$$

where θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω.

Proof. (i) By the Hölder inequality, we know

$$\int_{\Omega} |v_1(x)|^{2^*} dx \geq \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}}. \tag{8.4}$$

For $x \in \Omega_{\frac{1}{t}}$ and $t > 0$, define $v_t(x) := t^{\frac{N}{2}} v_1(tx)$. Using (5.2), (8.4), (4.4) and $\frac{1}{2} \leq s \leq 1$, it holds

$$\mathcal{J}_{\frac{1}{t},s}(v_t) \leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta \Theta - \frac{\beta}{2} \alpha_1 t^{\frac{N(p_2-2)}{2}} \int_{\Omega} |v_1|^{p_2} dx - \frac{1}{2 \cdot 2^*} t^{\frac{N(2^*-2)}{2}} \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}}$$

$$\leq \widehat{h}_2(t), \tag{8.5}$$

where

$$\widehat{h}_2(t) = \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta \Theta - \frac{1}{2 \cdot 2^*} t^{\frac{N(2^*-2)}{2}} \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}}.$$

A simple computation shows that $\widehat{h}_2(t_0) = 0$ for

$$t_0 := \left[\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) 2^* \theta \Theta^{\frac{2-2^*}{2}} |\Omega|^{\frac{2^*-2}{2}}\right]^{\frac{2}{N(2^*-2)-4}}$$

and $\widehat{h}_2(t) < 0$ for any $t > t_0, \widehat{h}_2(t) > 0$ for any $0 < t < t_0$. Moreover, $\widehat{h}_2(t)$ achieves its maximum at

$$t_\Theta = \left[\frac{4 \cdot 2^*\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)\theta}{N(2^*-2)} \Theta^{\frac{2-2^*}{2}} |\Omega|^{\frac{2^*-2}{2}}\right]^{\frac{2}{N(2^*-2)-4}}.$$

This implies

$$\mathcal{J}_{r,s}(v_{t_0}) = J_{\frac{1}{t_0},s}(v_{t_0}) \leq \widehat{h}_2(t_0) = 0 \tag{8.6}$$

for any $r \geq \frac{1}{t_0}$ and $s \in \left[\frac{1}{2}, 1\right]$. There exists $0 < t_1 < t_\Theta$ such that for any $t \in [0, t_1]$,

$$\widehat{h}_2(t) < A^{-\frac{2}{2^*-2}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}}\left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right]. \tag{8.7}$$

On the other hand, it follows from (3.5) that

$$\mathcal{J}_{r,s}(u) \geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) \int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}\left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{2^*}{2}}$$
$$- \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}.$$

Define

$$\widehat{g}_1(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}t^{\frac{2^*}{2}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{4}}$$
$$= t^{\frac{N(p_1-2)}{4}}\left[\frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t^{\frac{4-N(p_1-2)}{4}} - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}t^{\frac{2\cdot 2^*-N(p_1-2)}{4}}\right]$$
$$- \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{4}}.$$

In view of $2 < p < 2 + \frac{2}{N} < q < 2^*$ and the definition of $\widetilde{\Theta}_V$, there exist $0 < l_1 < l_M < l_2$ such that $\widehat{g}_1(t) < 0$ for any $0 < t < l_1$ and $t > l_2, \widehat{g}_1(t) > 0$ for $l_1 < t < l_2$ and $\widehat{g}_1(l_M) = \max_{t\in\mathbb{R}^+} \widehat{g}_1(t) > 0$. Let

$$t_2 = \left(\frac{\alpha\beta C_{N,p_1}S^{\frac{2^*}{2}}N(p_1-2)(4-N(p_1-2))}{4(2^*-2)}\right)^{\frac{4}{2\cdot 2^*-N(p_1-2)}}\Theta^{\frac{2p_1-N(p_1-2)}{2\cdot 2^*-N(p_1-2)}}.$$

Then by a direct calculation, we have $g_1''(t) \leq 0$ if and only if $t \geq t_2$. Hence

$$\max_{t\in\mathbb{R}^+}\widehat{g}_1(t) = \max_{t\in[t_2,\infty)}\widehat{g}_1(t).$$

Note that for any $t \geq t_2$,

$$g_1(t) = \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}t^{\frac{2^*}{2}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{4}}$$
$$= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\cdot t^{\frac{N(p_1-2)}{4}} - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}t^{\frac{2^*}{2}}$$
$$= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - \frac{4(2^*-2)}{S^{\frac{2^*}{2}}N(p_1-2)(4-N(p_1-2))}\cdot t_2^{\frac{2\cdot 2^*-N(p_1-2)}{4}}\cdot t^{\frac{N(p_1-2)}{4}}$$
$$- \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}t^{\frac{2^*}{2}}$$
$$\geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - S^{-\frac{2^*}{2}}\left[\frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))} + \frac{1}{2^*}\right]t^{\frac{2^*}{2}}$$
$$=: \widehat{g}_2(t). \tag{8.8}$$

Now, we will determine the value of $\widetilde{\Theta}_V$. In fact, $\widehat{g}_1(l_M) = \max_{t\in\mathbb{R}^+}\widehat{g}_1(t) > 0$ as long as

$\widehat{g}_2(t_2) > 0$, that is,

$$\widehat{g}_2(t_2) = \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t_2 - S^{-\frac{2^*}{2}}\left[\frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))} + \frac{1}{2^*}\right] t_2^{\frac{2^*}{2}}$$

$$= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}}}{A_{p_1}}\right)^{-\frac{4}{2\cdot 2^* - N(p_1-2)}} \Theta^{\frac{2p_1 - N(p_1-2)}{2\cdot 2^* - N(p_1-2)}}$$

$$- S^{-\frac{2^*}{2}}\left(A_{p_1} + \frac{1}{2^*}\right)\left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}}}{A_{p_1}}\right)^{-\frac{2\cdot 2^*}{2\cdot 2^* - N(p_1-2)}} \Theta^{\frac{2^*[2p_1 - N(p_1-2)]}{2[2\cdot 2^* - N(p_1-2)]}}$$

$$> 0,$$

where

$$A_{p_1} = \frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))}.$$

Hence, we take

$$\widetilde{\Theta}_V = \left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}}}{A_{p_1}}\right)^{-\frac{4}{2p_1 - N(p_1-2)}}\left[\frac{S^{\frac{2^*}{2}}}{2\cdot 2^*}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)(2^* A_{p_1} + 1)\right]^{\frac{2[2\cdot 2^* - N(p_1-2)]}{(2^*-2)[2p_1 - N(p_1-2)]}}.$$

Let

$$\widehat{A} = S^{-\frac{2^*}{2}}\left[\frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))} + \frac{1}{2^*}\right], \quad t_g = \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^* \widehat{A}}\right]^{\frac{2}{2^*-2}},$$

so that $t_g > t_2$ by the definition of $\widetilde{\Theta}_V$, $\max_{t\in[t_2,\infty)} \widehat{g}_2(t) = \widehat{g}_2(t_g)$ and

$$\max_{t\in\mathbb{R}^+} \widehat{g}_1(t) \geq \max_{t\in[t_2,\infty)} \widehat{g}_2(t) = \widehat{A}^{-\frac{2}{2^*-2}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}}\left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right].$$

Set $\bar{r}_\Theta = \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{t_g}}\right\}$, then $v_{\frac{1}{\bar{r}_\Theta}} \in S_{r,\Theta}$ for any $r > \bar{r}_\Theta$, and

$$\left\|\nabla v_{\frac{1}{\bar{r}_\Theta}}\right\|_2^2 = \left(\frac{1}{\bar{r}_\Theta}\right)^2 \|\nabla v_1\|_2^2 < t_g = \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^* \widehat{A}}\right]^{\frac{2}{2^*-2}}. \tag{8.9}$$

Moreover,

$$\mathcal{J}_{\bar{r}_\Theta,s}\left(v_{\frac{1}{\bar{r}_\Theta}}\right) \leq \widehat{h}\left(\frac{1}{\bar{r}_\Theta}\right) \leq \widehat{h}(t_1). \tag{8.10}$$

Let $u^0 = v_{\frac{1}{\bar{r}_\Theta}}, u^1 = v_{t_0}$ and

$$\widetilde{r}_\Theta = \max\left\{\frac{1}{t_0}, \bar{r}_\Theta\right\}.$$

Then the statement (i) holds by (8.6), (8.7), (8.9), (8.10).

(ii) holds by (8.8) and a direct calculation.

(iii) In view of $J_{r,s}(u^1) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$ and the definition of t_0, we have

$$\|\nabla\gamma(0)\|_2^2 < t_g < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (8.8) that

$$
\max_{t \in [0,1]} J_{r,s}(\gamma(t)) \;\geq\; g_2\left(t_g\right) = \widehat{A}^{-\frac{2}{2^*-2}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}} \left[\frac{2^*-2}{2} \left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}} \right]
$$

for any $\gamma \in \Gamma_{r,\Theta}$, hence the first inequality in (iii) holds. We define a path $\gamma : [0,1] \to S_{r,\Theta}$ by

$$
\gamma(t) : \Omega_r \to \mathbb{R}, \quad x \mapsto \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}}_\Theta\right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}}_\Theta\right)x\right).
$$

Then $\gamma \in \Gamma_{r,\Theta}$, and the second inequality in (iii) follows from (8.5). □

Using Proposition 3.2 to $\widetilde{J}_{r,s}$, it follows that

$$
A(u) = \frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V(x)u^2 dx \quad \text{and} \quad B(u) = \frac{1}{2^*} \int_{\Omega_r} |u|^{2^*} dx + \beta \int_{\Omega_r} F(u)dx.
$$

Hence, for almost every $s \in \left[\frac{1}{2}, 1\right]$, there exists a bounded Palais–Smale sequence $\{u_n\}$ satisfying

$$
\mathcal{J}_{r,s}(u_n) \to \widetilde{m}_{r,s}(\Theta) \quad \text{and} \quad \mathcal{J}'_{r,s}(u_n)\big|_{T_{u_n} S_{r,\Theta}} \to 0.
$$

Similar to the proof of Lemmas 7.2 and 7.3, we have the following lemmas.

Lemma 8.2. *If $\beta > 0$ and the assumptions of Theorem 1.11 hold, then $\widetilde{m}_{r,s}(\Theta) < \frac{\zeta}{N} S^{\frac{N}{2}}$, where $\zeta = s^{-\frac{2}{2^*-2}}$.*

Lemma 8.3. *Assume $0 < \Theta < \widetilde{\Theta}_V$ where $\widetilde{\Theta}_V$ is given in Theorem 1.11, let $r > r_\Theta$, where r_Θ is defined in Lemma 8.1. Then problem (8.3) has a solution $(\lambda_{r,s}, u_{r,s})$ for almost every $s \in \left[\frac{1}{2}, 1\right]$. Moreover, $u_{r,s} \geq 0$ and $\mathcal{J}_{r,s}(u_{r,s}) = \widetilde{m}_{r,s}(\Theta)$.*

In order to obtain a solution of (8.1), we also need to prove a uniform estimate for the solutions of (8.3) established in Lemma 8.3.

Lemma 8.4. *For fixed $\Theta > 0$ the set of solutions $u \in S_{r,\Theta}$ of (8.3) is bounded uniformly in s and r.*

Proof. Since u is a solution of (8.3), we have

$$
\int_{\Omega_r} |\nabla u|^2 dx + \int_{\Omega_r} V u^2 dx = s \int_{\Omega_r} |u|^{2^*} dx + s\beta \int_{\Omega_r} f(u)u\,dx - \lambda \int_{\Omega_r} |u|^2 dx.
$$

The Pohozaev identity implies

$$
\frac{N-2}{2N} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma + \frac{1}{2N} \int_{\Omega_r} \widetilde{V}(x)u^2 + \frac{1}{2} \int_{\Omega_r} V u^2 dx
$$
$$
= -\frac{\lambda}{2} \int_{\Omega_r} |u|^2 dx + \frac{s}{2^*} \int_{\Omega_r} |u|^{2^*} dx + s\beta \int_{\Omega_r} F(u)dx
$$

where $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Omega_r$. It then follows from $\beta > 0$ and (f_2) that

$$
\frac{1}{N} \int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma - \frac{1}{2N} \int_{\Omega_r} (\nabla V \cdot x) u^2 dx
$$
$$
\leq \frac{2^*-2}{2} \left(\frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V u^2 dx - \widehat{m}_{r,s}(\Theta) \right) + s\frac{\beta(p_2 - 2^*)}{2} \int_{\Omega_r} F(u)dx.
$$

Using Gagliardo–Nirenberg inequality, (3.5) and (iii) in Lemma 8.1, we have

$$\frac{2^* - 2}{2}\widehat{m}_{r,s}(\Theta) \geq \frac{N(p_2 - 2) - 4}{4N}\int_{\Omega_r}|\nabla u|^2 dx - \Theta\left(\frac{1}{2N}\|\nabla V \cdot x\|_\infty + \frac{p_2 - 2}{4}\|V\|_\infty\right)$$

$$+ \frac{s\alpha\beta(p_2 - 2^*)}{2}C_{N,p_1}\Theta^{\frac{2p_1 - N(p_1 - 2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2 dx\right)^{\frac{N(p_1 - 2)}{4}}.$$

Since $2 < p_1 < 2 + \frac{4}{N}$, we can bound $\int_{\Omega_r}|\nabla u|^2 dx$ uniformly in s and r. $\square$

Lemma 8.5. *Assume $0 < \Theta < \widetilde{\Theta}_V$, where $\widetilde{\Theta}_V$ is given in Theorem 1.11, and let $r > \widetilde{r}_\Theta$, where $\widetilde{r}_\Theta$ is defined in Lemma 8.1. Then equation (8.3) admits a solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ for every $r > \widetilde{r}_\Theta$ such that $u_{r,\Theta} > 0$ in Ω_r.*

Proof. The proof of lemma is similar to the Lemma 7.5. $\square$

Proof of Theorem 1.11. The proof is an immediate consequence of Lemmas 8.5 and 3.6. $\square$

9 Mass critical case

9.1 Proof of Theorem 1.12

This subsection considers the case of $p_1 = 2 + \frac{4}{N}$, so we need to modify the proof of Theorem 1.5.

Lemma 9.1. *For $0 < \Theta < \widetilde{\Theta}_V$ where $\widetilde{\Theta}_V$ is defined in Theorem 1.12, there exist $\widetilde{r}_\Theta > 0$ and $u^0, u^1 \in S_{r_\Theta, \Theta}$ such that*

(i) For $r > \widetilde{r}_\Theta$ and $s \in \left[\frac{1}{2}, 1\right]$ we have $J_{r,s}\left(u^1\right) \leq 0$ and

$$J_{r,s}\left(u^0\right) < \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1} - 2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}}\left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

Moreover,

$$\|\nabla u^0\|_2^2 < \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1} - 2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}}$$

and

$$\left\|\nabla u^1\right\|_2^2 > \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1} - 2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}}.$$

(ii) If $u \in S_{r,\Theta}$ satisfies

$$\|\nabla u\|_2^2 = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1} - 2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}},$$

then there holds

$$J_{r,s}(u) \geq \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1} - 2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}}\left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

(iii) Let

$$m_{r,s}(\Theta) = \inf_{\gamma \in \Gamma_{r,\Theta}} \sup_{t \in [0,1]} J_{r,s}(\gamma(t)),$$

where

$$\Gamma_{r,\Theta} = \left\{ \gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1 \right\}.$$

Then

$$m_{r,s}(\Theta) \geq \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}} \left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}$$

and

$$m_{r,s}(\Theta) \leq \frac{N(q-2)-4}{2}\left(\frac{\theta\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)}\right)^{\frac{N(q-2)}{N(q-2)-4}} (4q)^{\frac{4}{N(q-2)-4}} |\Omega|^{\frac{2(q-2)}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

where θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω.

Proof. We only need to modify the proof of Lemma 5.1. There exists $0 < t_1 < t_\Theta$ such that for any $t \in [0, t_1]$,

$$h_2(t) < \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}} \left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}. \tag{9.1}$$

On the other hand, it follows from (3.5), the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$J_{r,s}(u) = \left[\frac{1 - \|V_-\|_{\frac{N}{2}} S^{-1}}{2} - \alpha\beta C_N \Theta^{\frac{2}{N}}\right] \|\nabla u\|_2^2 - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} \|\nabla u\|_2^{\frac{N(q-2)}{2}}. \tag{9.2}$$

Define

$$g_1(t) := \left[\frac{1 - \|V_-\|_{\frac{N}{2}} S^{-1}}{2} - \alpha\beta C_N \Theta^{\frac{2}{N}}\right] t - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}}$$

and

$$t_g = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}},$$

it is easy to see that g_1 is increasing on $(0, t_g)$ and decreasing on (t_g, ∞), and

$$g_1(t_g) = \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}} \left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

Set $\bar{r}_\Theta = \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{t_g}}\right\}$, then $v_{\frac{1}{\bar{r}_\Theta}} \in S_{r,\Theta}$ for any $r > \bar{r}_\Theta$, and

$$\left\|\nabla v_{\frac{1}{\bar{r}_\Theta}}\right\|_2^2 = \left(\frac{1}{\bar{r}_\Theta}\right)^2 \|\nabla v_1\|_2^2$$

$$< t_g = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}}. \tag{9.3}$$

Then the statement (i) holds by (5.5), (9.1), (5.9), (9.3).

   (ii) holds by (9.2) and a direct calculation.

   (iii) In view of $J_{r,s}(u^1) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$ and the definition of t_0, we have

$$\|\nabla\gamma(0)\|_2^2 < t_g < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (9.2) that

$$\max_{t\in[0,1]} J_{r,s}(\gamma(t)) \geq g_1(t_g)$$

$$= \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}}S^{-1} - 2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}}\left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}$$

for any $\gamma \in \Gamma_{r,\Theta}$, hence the first inequality in (iii) holds. We define a path $\gamma : [0,1] \rightarrow S_{r,\Theta}$ by

$$\gamma(t) : \Omega_r \rightarrow \mathbb{R}, \qquad x \mapsto \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}}\Theta\right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}}\Theta\right)x\right).$$

Then $\gamma \in \Gamma_{r,\Theta}$, and the second inequality in (iii) follows from (5.4). □

Lemma 9.2. *For fixed $\Theta > 0$ the set of solutions $u \in S_{r,\Theta}$ of* (5.1) *is bounded uniformly in s and r.*

Proof. We only need to modify the proof of Lemma 5.3. Using the Gagliardo–Nirenberg inequality, (3.5) and (iii) in Lemma 5.1, we have

$$\frac{q-2}{2}m_{r,s}(\Theta) \geq \left[\frac{N(p_2-2)-4}{4N} - \frac{s\alpha\beta(q-p_2)}{2}C_N\Theta^{\frac{2}{N}}\right]\int_{\Omega_r}|\nabla u|^2dx$$

$$- \Theta\left(\frac{1}{2N}\|\nabla V \cdot x\|_\infty + \frac{p_2-2}{4}\|V\|_\infty\right).$$

Since $0 < \Theta < \widetilde{\Theta}_V$, we can bound $\int_{\Omega_r}|\nabla u|^2dx$ uniformly in s and r. □

Proof of Theorem 1.12. The proof is an immediate consequence of Lemmas 5.4 and 3.6. □

9.2 Proof of Theorem 1.13

Firstly, we modify the proof of Lemma 3.1. Using (3.3), (3.4), (3.5) and $\frac{1}{2} \leq s \leq 1$, it holds

$$I_{\frac{1}{t},s}(v_t) \leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}}S^{-1} - 2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)t^2\theta\Theta - \frac{1}{2q}t^{\frac{N(q-2)}{2}}\Theta^{\frac{q}{2}}\cdot|\Omega|^{\frac{2-q}{2}}$$

$$=: h(t).$$

Note that since $2 + \frac{4}{N} < q < 2^*$ and $\beta \leq 0$, there exist $0 < T_\Theta < t_0$ such that $h(t_0) = 0, h(t) < 0$ for any $t > t_0, h(t) > 0$ for any $0 < t < t_0$ and $h(T_\Theta) = \max_{t\in\mathbb{R}^+} h(t)$.

Lemma 9.3. *Let $(\lambda_{r,\Theta}, u_{r,\Theta})$ be the solution of* (1.1) *from Lemma 3.5. If $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, then there exists $\bar{\Theta} > 0$ such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad for\ 0 < \Theta < \bar{\Theta}.$$

Proof. We only need to modify the proof of Lemma 3.7. It follows from (3.5), (3.27), (3.28), (3.29), (f_2) and $2 + \frac{4}{N} < q < 2^*$ that

$$\left(\frac{1}{q} - \frac{1}{2}\right) \lambda_\Theta \int_{\mathbb{R}^N} u_\Theta^2 dx \leq \left[\frac{(N-2)q - 2N}{2Nq} - \frac{\beta(q - p_2)\alpha C_N}{q} \Theta^{\frac{2}{N}}\right] \int_{\mathbb{R}^N} |\nabla u_\Theta|^2 \, dx$$

$$+ \frac{\|\widetilde{V}\|_\infty}{2N} \Theta + \frac{(q-2)\|V\|_\infty}{2q} \Theta$$

$$\to -\infty \quad \text{as } \Theta \to 0,$$

since $0 < \Theta < \widehat{\Theta}_V$. $\square$

Proof of Theorem 1.13. The proof is an immediate consequence of Lemmas 3.5, 3.6 and 9.3. $\square$

9.3   Proof of Theorem 1.14

Similarly, we only need to modify the proof of Theorem 1.4. Since $\beta > 0$, it follows from the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$I_r(u) = \frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V(x) u^2 dx - \frac{N}{2N+4} \int_{\Omega_r} |u|^{2 + \frac{4}{N}} dx - \beta \int_{\Omega_r} F(u) dx$$

$$\geq \frac{1}{2} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1} - \frac{NC_N \Theta^{\frac{2}{N}}}{N+2}\right) \int_{\Omega_r} |\nabla u|^2 dx$$

$$- \alpha \beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1 - 2)}{4}} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(p_1 - 2)}{4}}$$

$$= h_1(t),$$

where

$$h_1(t) := \frac{1}{2} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1} - \frac{NC_N \Theta^{\frac{2}{N}}}{N+2}\right) t^2 - \alpha \beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1 - 2)}{4}} t^{\frac{N(p_1 - 2)}{2}}.$$

In view of $2 < p_1 < 2 + \frac{4}{N}$, there exists $T_\Theta > 0$ such that $h_1(t) < 0$ for $0 < t < T_\Theta$ and $h_1(t) > 0$ for $t > T_\Theta$.

Proof of Theorem 1.14. The proof is a direct consequence of Lemma 4.1 and Lemma 3.6. $\square$

10   Final comments

Some similar result (Theorems 1.3, 1.4, 1.5, but there are subtle changes in the assumptions) can be proved for the following class of problem

$$\begin{cases} -\Delta u + V(x)u + \lambda u = w(u) + \beta |u|^{p-2} u, & x \in \Omega, \\ \int_\Omega |u|^2 dx = \Theta, u \in H_0^1(\Omega), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is either all of $\mathbb{R}^N$ or a bounded smooth convex domain, $N \geq 3$, $2 < p < 2 + \frac{4}{N}$, the mass $\Theta > 0$ and the parameter $\beta \in \mathbb{R}$ are prescribed. Nonlinearity w satisfies:

(W_1) $w \in C^1(\mathbb{R}, \mathbb{R})$ and w is odd.

(W_2) There exists some $(p_1, p_2) \in \mathbb{R}_+^2$ satisfying $2 + \frac{4}{N} < p_2 \leq p_1 < 2^*$ such that

$$p_2 W(\tau) \leq w(\tau)\tau \leq p_1 W(\tau) \quad \text{with } W(\tau) = \int_0^\tau w(t) dt.$$

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References

[1] T. Akahori, S. Ibrahim, H. Kikuchi, H. Nawa, Existence of a ground state and blow-up problem for a nonlinear Schrödinger equation with critical growth, *Differential Integral Equations* **25**(2012), No. 3, 383–402. https://doi.org/10.57262/die/1356012740

[2] T. Akahori, S. Ibrahim, H. Kikuchi, H. Nawa, Existence of a ground state and scattering for a nonlinear Schrödinger equation with critical growth, *Sel. Math. New Ser.* **19**(2013), No. 2, 545–609. https://doi.org/10.1007/s00029-012-0103-5

[3] T. Aubin, Problémes isopérimétriques et espaces de Sobolev (in French), *J. Differential Geometry* **11**(1976), No. 4, 573–598. https://doi.org/10.4310/jdg/1214433725

[4] T. Bartsch, S. Qi, W. Zou, Normalized solutions to Schödinger equations with potential and inhomogeneous nonlinearities on large convex domains, *Math. Ann.*, published online on 30 April 2024, 47 pp. https://doi.org/10.1007/s00208-024-02857-1

[5] T. Bartsch, N. Soave, A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems, *J. Funct. Anal.* **272**(2017), No. 12, 4998–5037. https://doi.org/10.1016/j.jfa.2017.01.025

[6] T. Bartsch, N. Soave, Multiple normalized solutions for a competing system of Schrödinger equations, *Calc. Var. Partial Differential Equations* **58**(2019), 1–24. https://doi.org/10.1007/s00526-018-1476-x

[7] J. Bellazzini, L. Jeanjean, T. Luo, Existence and instability of standing waves with prescribed norm for a class of Schrödinger–Poisson equations, *Proc. Lond. Math. Soc.* **107**(2013), No. 2, 303–339. https://doi.org/10.1112/plms/pds072

[8] B. Bieganowski, J. Mederski, Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth, *J. Funct. Anal.* **280**(2021), No. 11, 108989. https://doi.org/10.1016/j.jfa.2021.108989

[9] P. B. Blakie, W. X. Wang, Bose Einstein condensation in an optical lattice, *Phys. Rev. A.* **76**(2007), No. 5, 053620. https://doi.org/10.1103/physreva.76.053620

[10] D. Bonheure, J. Casteras, T. Gou, L. Jeanjean, Normalized solutions to the mixed dispersion nonlinear Schrödinger equation in the mass critical and supercritical regime, *Trans. Amer. Math. Soc.* **372**(2019), No. 3, 2167–2212. https://doi.org/10.1090/tran/7769

[11] J. Borthwick, X. Chang, L. Jeanjean, N. Soave, Bounded Palais–Smale sequences with Morse type information for some constrained functionals, *Trans. Amer. Math. Soc.* **377**(2024), No. 6, 4481–4517. https://doi.org/10.1090/tran/9145

[12] H. Brézis, E. A. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Am. Math. Soc.* **88**(1983), No. 3, 486–490. https://doi.org/10.1090/s0002-9939-1983-0699419-3

[13] X. Chang, L. Jeanjean, N. Soave, Normalized solutions of L^2-supercritical NLS equations on compact metric graphs, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **41**(2024), No. 4, 933–959. https://doi.org/10.4171/aihpc/88

[14] Y. Ding, X. Zhong, Normalized solution to the Schrödinger equation with potential and general nonlinear term: mass super-critical case, *J. Differential Equations* **334**(2022), 194–215. https://doi.org/10.1016/j.jde.2022.06.013

[15] S. Dovetta, Existence of infinitely many stationary solutions of the L^2-subcritical and critical NLSE on compact metric graphs, *J. Differential Equations* **264**(2018), No. 7, 4806–4821. https://doi.org/10.1016/j.jde.2017.12.025

[16] L. Erdős, B. Schlein, H. Yau, Derivation of the Gross–Pitaevskii equation for the dynamics of Bose–Einstein condensate, *Ann. Math.* **172**(2010), 291–370. https://doi.org/10.4007/annals.2010.172.291

[17] M. Esteban, P. L. Lions, Existence and nonexistence results for semilinear elliptic problems in unbounded domains, *Proc. R. Soc. Edinb. Sect. A* **93**(1982), 1–14. https://doi.org/10.1017/s0308210500031607

[18] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1983. https://doi.org/10.1007/978-3-642-61798-0

[19] N. Ikoma, Y. Miyamoto, Stable standing waves of nonlinear Schrödinger equations with potentials and general nonlinearities, *Calc. Var. Partial Differential Equations* **59**(2020), No. 2, Art. No. 48. https://doi.org/10.1007/s00526-020-1703-0

[20] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A* **129**(1999), 787–809. https://doi.org/10.1017/S0308210500013147

[21] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.* **28**(1997), No. 10, 1633–1659. https://doi.org/10.1016/s0362-546x(96)00021-1

[22] L. Jeanjean, S. S. Lu, A mass supercritical problem revisited, *Calc. Var. Partial Differential Equations* **59**(2020), No. 5, 174. https://doi.org/10.1007/s00526-020-01828-z

[23] L. Jeanjean, T. Le, Multiple normalized solutions for a Sobolev critical Schrödinger equation, *Math. Ann.* **384**(2022), No. 1–2, 101–134. https://doi.org/10.1007/s00208-021-02228-0

[24] R. Killip, T. Oh, O. Pocovnicu, M. Visan, Solitons and scattering for the cubic-quintic nonlinear Schrödinger equation on $\mathbb{R}^3$, *Arch. Ration. Mech. Anal.* **225**(2017), No. 1, 469–548. https://doi.org/10.1007/s00205-017-1109-0

[25] C. Miao, G. Xu, L. Zhao, The dynamics of the 3D radial NLS with the combined terms, *Commun. Math. Phys.* **318**(2013), No. 3, 767–808. https://doi.org/10.1007/s00220-013-1677-2

[26] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities, *J. Differential Equations* **269**(2020), No. 9, 6941–6987. https://doi.org/10.1016/j.jde.2020.05.016

[27] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case, *J. Funct. Anal.* **279**(2020), No. 6, 108610. https://doi.org/10.1016/j.jfa.2020.108610

[28] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* **110**(1976), No. 4, 353–372. https://doi.org/10.1007/bf02418013

[29] J. Wei, Y. Wu, Normalized solutions for Schrödinger equations with critical Sobolev exponent and mixed nonlinearities, *J. Funct. Anal.* **283**(2022), No. 6, 109574. https://doi.org/10.1016/j.jfa.2022.109574