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Infinitely many solutions for an anisotropic differential inclusion on unbounded domains

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Abstract. We consider the differential inclusion problem given by

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\right) + V(x)|u(x)|^{p_N^o-2}u \in a(x)\partial F(x,u),$$

in $\mathbb{R}^N$. The problem deals with the anisotropic $p(x)$-Laplacian operator where p_i are Lipschitz continuous functions $2 \leq p_i(x) < N$ for all $x \in \mathbb{R}^N$ and $i \in \{1,\dots,N\}$. Assume $p_N^o(x) = \max_{1\leq i\leq N} p_i(x)$, $a \in L^1_+(\mathbb{R}^N) \cap L^{\frac{N}{p_N^o(x)-1}}(\mathbb{R}^N)$, $F(x,t)$ is locally Lipschitz in the t-variable integrand and $\partial F(x,t)$ is the subdifferential with respect to the t-variable in the sense of Clarke. By establishing the existence of infinitely many solutions, we achieve a first result within the anisotropic framework.

Keywords: anisotropic $p(x)$-Laplacian, differential inclusion problem, locally Lipschitz function, infinitely many solutions, variational method.

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1 Introduction

The Wulff shape, also known as the equilibrium crystal shape, is closely associated with a convex hypersurface in $\mathbb{R}^N$ based on a given norm. Wulff [36] introduced a variational problem concerning an anisotropic geometric functional in the context of crystal growth. He conjectured, without providing a proof, that the Wulff shape, among closed convex hypersurfaces with constant enclosed volume, minimizes the anisotropic surface energy. This seminal work by Wulff has spurred significant research in the field of phase transitions, particularly in scenarios involving anisotropic and nonhomogeneous media. Recent studies have considered the existence of solutions for anisotropic problems, (see [4, 7, 12–16, 29–34, 37] and the related literature for more details).

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In 2019, Ge and Rădulescu [19] proved the existence of infinitely many solutions to the differential inclusion problem involving the $p(x)$-Laplacian

$$-\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u \in a(x)\partial F(x,u), \quad \text{in } \mathbb{R}^N$$

via a combination the variational principle for locally Lipschitz functions with the properties of the generalized Lebesgue Sobolev space.

Here, with the inspiration of [19], we investigate the existence of infinitely many solutions of a differential inclusion problem involving the anisotropic $p(x)$-Laplacian

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) + V(x)|u(x)|^{p_N^o(x)-2}u \in a(x)\partial F(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where p_i are Lipschitz continuous functions such that $2 \le p_i(x) < N$ for all $x \in \mathbb{R}^N$ and $i \in \{1,\dots,N\}$, $p_N^o(x) = \max_{1\le i\le N} p_i(x)$, $a \in L_+^1(\mathbb{R}^N) \cap L^{\frac{N}{p_N^o(x)-1}}(\mathbb{R}^N)$, $F(x,t)$ is locally Lipschitz in the t-variable integrand and $\partial F(x,t)$ is the subdifferential with respect to the t-variable in the sense of Clarke [5]. Notice that $L_+^1(\mathbb{R}^N) := \{\eta \in L^1(\mathbb{R}^N) : \eta(x) > 0 \text{ for all } x \in \mathbb{R}^N\}$ and

$$C_+(\mathbb{R}^N) := \left\{h \in C(\mathbb{R}^N) : h(x) > 1 \text{ for all } x \in \mathbb{R}^N\right\}.$$

For any $h \in C_+(\mathbb{R}^N)$, we will denote

$$h^- := \inf_{x\in\mathbb{R}^N} h(x) \quad \text{and} \quad h^+ := \sup_{x\in\mathbb{R}^N} h(x).$$

Also we set the order $h_1 \ll h_2$ the fact that $\inf_{x\in\mathbb{R}^N}(h_2(x) - h_1(x)) > 0$.

For the potential function V, we make the following assumptions:

(V_1) $V \in C(\mathbb{R}^N)$, $0 < V^-$.

(V_2) There exists $r > 0$ such that for any $b > 0$,

$$\lim_{|y|\to\infty} \mu\left(\left\{x \in \mathbb{R}^N : V(x) \le b\right\} \cap B_r(y)\right) = 0,$$

where μ is the Lebesgue measure on $\mathbb{R}^N$.

For the nonlinearity F, suppose the function $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is such that $F(x,0) = 0$ a.e. on $\mathbb{R}^N$, and

(f_0) F is a Carathéodory function, that is, for all $t \in \mathbb{R}$, the mapping $x \mapsto F(x,t)$ is measurable and, for almost all $x \in \mathbb{R}^N$, the function $t \mapsto F(x,t)$ is locally Lipschitz.

(f_1) for almost all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}$ and all $w \in \partial F(x,t)$, we have

$$|w| \le c\left(1 + |t|^{p_N^o(x)-1}\right).$$

(f_2) there exist $\delta > 0$ and $\subseteq \in L_-^\infty(\mathbb{R}^N)$ such that, for almost all $x \in \mathbb{R}^N$, we have

$$\sup_{0<|t|<\delta} F(x,t) \le \subseteq(x) < 0,$$

where $L_-^\infty(\mathbb{R}^N) = \{\eta \in L^\infty(\mathbb{R}^N) : \eta(x) < 0 \text{ for all } x \in \mathbb{R}^N\}$.

(f_3) There exists $q \in C_+(\mathbb{R}^N)$ with

$$p_N^{o\,+} < q^- \leq q(x) < \frac{Np_N^o(x) - p_N^o(x)(p_N^o(x)-1)}{N - p_N^o(x)} < \kappa(x) \quad \text{for all } x \in \mathbb{R}^N$$

where $\kappa(x) = p_N^{o\,*}(x) = \frac{Np_N^o(x)}{N - p_N^o(x)}$, such that for almost all $x \in \mathbb{R}^N$ we have

$$\liminf_{|t|\to\infty} \frac{F(x,t)}{|t|^{q(x)}} > 0.$$

(f_4) $F(x,-t) = F(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Now we state the main result of this article.

Theorem 1.1. *Suppose that (f_0)–(f_4), (V_1) and (V_2) hold. Then problem (1.1) has infinitely many nontrivial solutions.*

The unique aspect of this paper lies in its capacity to extend the conclusions of Ge and Rădulescu [19] to a more generalized, anisotropic setting.

The subsequent sections of this paper are structured as follows. In Section 2, we will revisit the definitions and various properties associated with variable exponent Sobolev spaces. Moving on to Section 3, we will establish the existence of an infinite number of nontrivial solutions for the problem (1.1), with credit given to the essential ideas derived from [19].

2 Preliminaries

We recall some preliminary results on the theory of variable exponent Sobolev space. For more details see [8,10,11,17,21,23,24,26–28], where additional information and specifics can be found.

For $p(x) \in C_+(\mathbb{R}^N)$, we define the variable exponent Lebesgue space

$$L^{p(x)} := \left\{ u : u \text{ is measurable and } \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < +\infty \right\}$$

with the norm

$$|u|_{L^{p(x)}(\mathbb{R}^N)} := |u|_{p(x)} := \inf \left\{ \Lambda > 0 : \int_{\mathbb{R}^N} |\frac{u(x)}{\Lambda}|^{p(x)} dx \leq 1 \right\},$$

and we define the variable exponent Sobolev space

$$W^{1,p(x)}(\mathbb{R}^N) := \left\{ u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\},$$

with the norm $|u|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. We recall that spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Here, we introduce a natural generalization of the variable exponent Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^N)$ that will enable us to study problem (1.1) with sufficient accuracy (see [21,22]).

Let us denote by $\overrightarrow{p} : \mathbb{R}^N \to \mathbb{R}^N$ the vectorial function

$$\overrightarrow{p}(x) = (p_1(x), \dots, p_N(x)).$$

Also assume $1 < p_1(x), \dots, p_N(x) < \infty$.

One can define $\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$ by

$$\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N) := \left\{ u \in L^{p_N^o(\cdot)}(\mathbb{R}^N) : \partial_{x_i} u \in L^{p_i(x)}(\mathbb{R}^N), i = 1, \dots, N \right\}.$$

This is a reflexive Banach space with respect to the norm

$$\|u\|_{1,\overrightarrow{p}(\cdot)} := \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\mathbb{R}^N)}.$$

This space continuously embedded in $L^{p_N^o(\cdot)}(\mathbb{R}^N)$, where $p_N^o = \max_{1\le i\le N} p_i(x)$. We introduce $\overrightarrow{P}_+, \overrightarrow{P}_- \in \mathbb{R}^N$ by

$$\overrightarrow{P}_+ := (p_1^+, \dots, p_N^+), \qquad \overrightarrow{P}_- := (p_1^-, \dots, p_N^-)$$

and $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ by

$$P_+^+ = \max\{p_1^+, \dots, p_N^+\}, \qquad P_-^+ = \max\{p_1^-, \dots, p_N^-\}, \qquad P_-^- = \min\{p_1^-, \dots, p_N^-\}. \tag{2.1}$$

Throughout this paper we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1. \tag{2.2}$$

We define

$$p^*(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)} - 1}, \qquad p_\infty = \max\{P_-^+, p^{*-}\}, \tag{2.3}$$

where $p^{*-} = \inf_{x\in\mathbb{R}^N} p^*(x)$.

Define $J : \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) := \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} |\partial_{x_i} u(x)|^{p_i(x)} dx + \int_{\mathbb{R}^N} \frac{1}{p_N^o(x)} V(x) |u(x)|^{p_N^o(x)} dx$$

for all $u \in \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$. Then $J \in C^1(\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N, \mathbb{R})$. If we denote

$$A = J' : \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N) \to (\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N))^*,$$

then

$$\begin{aligned} \langle A(u), v\rangle = & \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} |\partial_{x_i} u(x)|^{p_i(x)-2} \partial_{x_i} u(x) \partial_{x_i} v(x) \\ & + \int_{\mathbb{R}^N} \frac{1}{p_N^o(x)} V(x) |u(x)|^{p_N^o(x)-2} uv dx \end{aligned}$$

for all $u, v \in \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $(\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N))^*$ and $\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$.

Definition 2.1. A mapping $f : X \to X^*$ is said to be of type $(S)_+$, if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle f(u_n), u_n - u\rangle \le 0$ implies $u_n \to u$ in X.

Similar to [3] we have the following proposition.

Proposition 2.2. *Suppose $G := \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$ and A is as above. Then $A : G \to G^*$ is*

(1) convex, bounded and strictly monotone operator;

(2) a mapping of type $(S)_+$;

(3) a homeomorphism.

Denote by $L^{q(x)}(\mathbb{R}^N)$ the conjugate space of $L^{p_N^o(x)}(\mathbb{R}^N)$ with $\frac{1}{p_N^o(x)} + \frac{1}{q(x)} = 1$. Then the Hölder type inequality

$$\int_{\mathbb{R}^N} |uv|dx \leq \left(\frac{1}{p_N^{o\,-}} + \frac{1}{q^-}\right)|u|_{L^{p_N^o(x)}(\mathbb{R}^N)}|v|_{L^{q(x)}(\mathbb{R}^N)}$$

holds for all $u \in L^{p_N^o(x)}(\mathbb{R}^N)$ and $v \in L^{q(x)}(\mathbb{R}^N)$. Furthermore, if we define the mapping $\rho : L^{p_N^o(x)}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p_N^o(x)}dx,$$

then the following relations hold:

$$|u|_{p_N^o(x)} = \mu \text{ for } u \neq 0 \iff \rho\left(\frac{u}{\mu}\right) = 1, \tag{2.4}$$

$$|u|_{p_N^o(x)} < 1(= 1, > 1) \iff \rho(u) < 1(= 1, > 1), \tag{2.5}$$

$$|u|_{p_N^o(x)} > 1 \implies |u|_{p_N^o(x)}^{p_N^{o\,-}} \leq \rho(u)|u|_{p_N^o(x)}^{p_N^{o\,+}}, \tag{2.6}$$

$$|u|_{p_N^o(x)} < 1 \implies |u|_{p_N^o(x)}^{p_N^{o\,+}} \leq \rho(u)|u|_{p_N^o(x)}^{p_N^{o\,-}}. \tag{2.7}$$

Proposition 2.3 ([9]). *Let $p_N^o(x), q(x)$ be measurable functions such that $p_N^o(x) \in L^\infty(\mathbb{R}^N)$ and $1 \leq p_N^o(x)q(x) \leq \infty$ almost everywhere in $\mathbb{R}^N$. Let $u \in L^{q(x)}(\mathbb{R}^N)$, $u \neq 0$. Then*

$$|u|_{p_N^o(x)q(x)} \geq 1 \implies |u|_{p_N^o(x)q(x)}^{p_N^{o\,-}} \leq ||u|^{p_N^o(x)}|_{q(x)} \leq |u|_{p_N^o(x)q(x)}^{p_N^{o\,+}},$$
$$|u|_{p_N^o(x)q(x)} \leq 1 \implies |u|_{p_N^o(x)q(x)}^{p_N^{o\,+}} \leq ||u|^{p_N^o(x)}|_{q(x)} \leq |u|_{p_N^o(x)q(x)}^{p_N^{o\,-}}.$$

In particular, if $p_N^o(x) = p_N^o$ is a constant, then $||u|^{p_N^o(x)}|_{q(x)} = |u|_{p_N^o q(x)}^{p_N^o}$.

Next we consider the case that V satisfies (V_1) and (V_2). We equip the linear subspace

$$E = \left\{ u \in \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N) : \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_{x_i} u|^{p_i(x)}dx + \int_{\mathbb{R}^N} V(x)|u|^{p_N^o(x)}dx < +\infty \right\}$$

with the norm

$$\|u\|_E = \inf\left\{ \lambda > 0 : \sum_{i=1}^N \int_{\mathbb{R}^N} |\frac{\partial_{x_i} u}{\lambda}|^{p_i(x)}dx + \int_{\mathbb{R}^N} V(x)|\frac{u}{\lambda}|^{p_N^o(x)}dx < +\infty \right\}.$$

Then $(E, \|\cdot\|_E)$ is continuously embedded into $\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E, \|\cdot\|_E)$ is also a separable reflexive Banach space. It is easy to see that with the norm $\|\cdot\|_E$, Proposition 2.2 remains valid, that is, the following properties hold true.

Proposition 2.4. *Set* $I(u)=\sum_{i=1}^{N}\int_{\mathbb{R}^N}|\partial_{x_i}u|^{p_i(x)}dx+\int_{\mathbb{R}^N}V(x)|u|^{p_N^o(x)}dx$. *If* $u(x)\in\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$, *then*

(i) *for* $u\neq 0$, $\|u\|_E=\lambda$ *if and only if* $I(\frac{u}{\lambda})=1$,

(ii) $\|u\|_E<1(=1,>1)$ *if and only if* $I(u)<1(=1,>1)$,

(iii) $\|u\|_E>1$ *implies* $\|u\|_E^{p_N^{o\,-}}\leq I(u)\leq\|u\|_E^{p_N^{o\,+}}$,

(iv) $\|u\|_E<1$ *implies* $\|u\|_E^{p_N^{o\,+}}\leq I(u)\leq\|u\|_E^{p_N^{o\,-}}$.

Here we recall the following theorem from [6, Theorem 2.4.] which implies there exits an embedding from $\mathcal{D}^{1,\overrightarrow{p}}(\mathbb{R}^N)$ into $L^{\overline{p}^\star}(\mathbb{R}^N)$, where $\overline{p}^\star:=\frac{N\overline{p}}{N-\overline{p}}$. This is a particular case of the result of Troisi [35].

Theorem 2.5. *Assume* $p_i\geq 1$ *for* $i=1,\ldots,N$. *Then there exists a continuous embedding* $\mathcal{D}^{1,\overrightarrow{p}}(\mathbb{R}^N)$ *into* $L^{\overline{p}^\star}(\mathbb{R}^N)$, *i.e.* $\mathcal{D}^{1,\overrightarrow{p}}(\mathbb{R}^N)\hookrightarrow L^{\overline{p}^\star}(\mathbb{R}^N)$.

This implies that we have a continuous embedding $E\hookrightarrow L^{p_N^o(x)}(\mathbb{R}^N)$.

Now by a similar argument as [1, Theorem 3.2] which is in the Heisenberg group setting (or by combining [25, Theorem 2.1] and [2, Lemma 4.4] which are in the Euclidean setting) we have the following proposition.

Proposition 2.6. *Assume* $\kappa(x)=p_N^{o\,*}(x)=\frac{Np_N^o(x)}{N-p_N^o(x)}$. *If* V *satisfies* (V_1) *and* (V_2), *then*

(i) *we have a compact embedding* $E\hookrightarrow L^{p_N^o(x)}(\mathbb{R}^N)$,

(ii) *for any measurable function* $q:\mathbb{R}^N\to\mathbb{R}$ *with* $p_N^o(x)<q(x)\ll\kappa(x)$, *we have a compact embedding* $E\hookrightarrow L^{q(x)}(\mathbb{R}^N)$.

Let $(Y,\|\cdot\|)$ be a real Banach space and Y^* its topological dual. A function $\phi:Y\to\mathbb{R}$ is called locally Lipschitz if each point $u\in Y$ possesses a neighborhood N_u such that $|f(u_1)-f(u_2)|\leq\|u_1-u_2\|$, for all $u_1,u_2\in N_u$, for a constant $L>0$ depending on N_u. The generalized directional derivative of ϕ at the point $u\in Y$ in the direction $h\in Y$ is

$$\phi^o(u,h)=\liminf_{\substack{w\to u\\ t\to 0^+}}\frac{\phi(w+th)-\phi(w)}{t}.$$

The generalized gradient of ϕ at $u\in Y$ is defined by

$$\partial\phi(u)=\{u^*\in Y^*:\langle u^*,h\rangle\leq\phi^o(u,h)\text{ for all }h\in Y\},$$

which is a nonempty, convex and w^*-compact subset of Y, where $\langle\cdot,\cdot\rangle$ is the duality pairing between Y^* and Y. We say that $u\in Y$ is a critical point of ϕ if $0\in\partial\phi(u)$, (see [3] for further details).

Definition 2.7. Let Y be a real Banach space, and $\phi:Y\to\mathbb{R}$ is a locally Lipschitz function. We say that ϕ satisfies the nonsmooth (PS_c) condition if any sequence $\{u_n\}\subset Y$ such that $\phi(u_n)\to c$ and $m(u_n)\to 0$ as $n\to\infty$ has a strongly convergent subsequence, where $m(u_n)=\inf\{\|u^*\|_{Y^*}:u^*\in\partial\phi(u_n)\}$. If this property holds at every level $c\in\mathbb{R}$, then we simply say that ϕ satisfies the nonsmooth (PS) condition.

We recall the following lemma [18, Theorem 2.1.7].

Lemma 2.8. *Assume that X is an infinite-dimensional Banach space, and let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function that satisfies the nonsmooth (PS_c) condition for every $c > 0$. Assume $\phi(u) = \phi(-u)$ for all $u \in X$ and $\phi(0) = 0$. Suppose $X = X_1 \bigoplus X_2$, where X_1 is finite-dimensional, and assume the following conditions:*

(a) $\alpha > 0, \delta > 0$ such that $\|u\| = \delta$ with $u \in X_2$ implies $\phi(u) \geq \alpha$.

(b) For any finite-dimensional subspace $W \subset X_1$, there is $R = R(W)$ such that $\phi(u) \leq 0$ for $u \in W$ with $\|u\| \geq R$.

Then ϕ possesses an unbounded sequence of critical values.

Before ending this section we can define the weighted variable exponent Lebesgue space $L^{q(x)}_{a(x)}(\mathbb{R}^N)$ (see [20]) as follows:

Assume $a \in L^1_+(\mathbb{R}^N)$, then $a(x)$ is a measurable, nonnegative real-valued function for $x \in \mathbb{R}^N$. Define

$$L^{q(x)}_{a(x)}(\mathbb{R}^N) = \left\{ u \text{ is measurable and } \int_{\mathbb{R}^N} a(x)|u(x)|^{q(x)}dx < +\infty \right\}$$

with the norm

$$|u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} = \inf\left\{ \sigma > 0 : \int_{\mathbb{R}^N} a(x)|\frac{u}{\sigma}|^{q(x)}dx \leq 1 \right\}.$$

Then $L^{q(x)}_{a(x)}(\mathbb{R}^N)$ is a Banach space.

Remark 2.9. The embedding $E \hookrightarrow W^{1,p^o_N(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}_{a(x)}(\mathbb{R}^N)$ is continuous.

Set $h(x) = q(x)\frac{N}{N-p^o_N(x)-1}$, where $q(x)$ is mentioned in (f_3). Then ${p^o_N}^+ < h^-$ and $p^o_N(x) < h(x) < \kappa(x)$ for all $x \in \mathbb{R}^N$, where $\kappa(x) = {p^o_N}^*(x) = \frac{Np^o_N(x)}{N-p^o_N(x)}$. Hence, by Proposition 2.6, there is a continuous embedding $E \hookrightarrow L^{h(x)}(\mathbb{R}^N)$. Thus, for $u \in E$, we have $|u(x)|^{q(x)} \in L^{\frac{N}{N-p^o_N(x)+1}}(\mathbb{R}^N)$. By the Hölder inequality,

$$\int_{\mathbb{R}^N} a(x)|u|^{q(x)}dx \leq 2|a|_{L^{\frac{N}{p^o_N(x)-1}}(\mathbb{R}^N)} ||u|^{q(x)}|_{L^{\frac{N}{N-p^o_N(x)+1}}(\mathbb{R}^N)} < +\infty. \tag{2.8}$$

It follows that $u \in L^{q(x)}_{a(x)}(\mathbb{R}^N)$, and hence the embedding $E \hookrightarrow W^{1,p^o_N(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}_{a(x)}(\mathbb{R}^N)$ is continuous.

3 Infinitely many nontrivial solutions

Here, we introduce the energy functional $\phi : E \to \mathbb{R}$ associated with problem (1.1) by

$$\phi(u) = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}dx + \int_{\mathbb{R}^N} \frac{1}{p^o_N(x)}V(x)|u|^{p^o_N(x)}dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

From the hypotheses on F, it is standard to check that ϕ is locally Lipschitz on E and $\partial\phi(u) \subset A(u) - \partial F(x,u)$ for all $u \in E$ (see [3]).

Definition 3.1. $u \in E$ is called a solution of (1.1) to which there corresponds a mapping $x \in \mathbb{R}^N \to w(x)$ with $w(x) \in \partial F(x,u)$ for almost every $x \in \mathbb{R}^N$ having the property $x \to w(x)h(x) \in L^1(\mathbb{R}^N)$ for every $h \in E$, and

$$\sum_{i=1}^{N} \int_{\mathbb{R}^N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} h dx + \int_{\mathbb{R}^N} V(x)|u|^{p_N^o(x)-2} uh\, dx = \int_{\mathbb{R}^N} w(x)h(x)dx.$$

Notice that weak solutions of problem (1.1) are exactly the critical points of the functional ϕ.

Lemma 3.2. *Assume that all conditions of Theorem 1.1 are satisfied. Then the energy functional ϕ satisfies the nonsmooth (PS) condition in E.*

Proof. Suppose that $\{u_n\} \subset E$ is a $(\mathrm{PS_c})$ sequence for ϕ, that is $\phi(u_n) \to c$ and $m(u_n) \to 0$ as $n \to \infty$, which shows that

$$c = \phi(u_n) + o(1), \qquad m(u_n) = o(1), \tag{3.1}$$

where $o(1) \to 0$ as $n \to +\infty$.

We claim that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded. Suppose that this is not the case. By passing to a subsequence if necessary, we may assume that $\|u_n\|_E \to +\infty$ as $n \to \infty$. Without loss of generality, we assume $\|u_n\|_E \geq 1$. Let $u_n^* \partial\phi(u_n)$ be such that $m(u_n) = \|u_n^*\|_{E^*}$, $n \in N$. We have $u_n^* = A(u_n) - w_n$, $w_n(x) \in \partial F(x, u_n(x))$, $w_n \in L^{p'(x)}(\mathbb{R}^N)$, where $\frac{1}{p_N^o(x)} + \frac{1}{p'(x)} = 1$. By (3.1), there is a constant $M_1 > 0$ such that

$$|\phi(u_n)| \leq M_1, \quad \text{for all } n \geq 1, \tag{3.2}$$

and there is a constant $C > 0$ such that

$$\begin{aligned} C\|u_n\|_E &\geq \langle u_n^*, u_n \rangle \\ &= \langle A(u_n), u_n \rangle - \int_{\mathbb{R}^N} a(x) w_n u_n dx \\ &= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} dx + \int_{\mathbb{R}^N} \frac{1}{p_N^o(x)} V(x)|u|^{p_N^o(x)} dx - \int_{\mathbb{R}^N} a(x) w_n u_n dx. \end{aligned} \tag{3.3}$$

Then by (3.2) and (3.3), we have

$$M_1 p_N^{o\,-} + C\|u_n\|_E \geq \int_{\mathbb{R}^N} a(x)(p_N^{o\,-} F(x,u_n) - w_n u_n) dx, \tag{3.4}$$

where $p_N^{o\,-}$ is given by (2.1).

Next we estimate (3.4). By virtue of (f_3), there exists $c_1 > 0$ and $M_2 > 0$ such that, for almost all $x \in \mathbb{R}^N$ and all $|t| \geq M_2$, we have $F(x,t) \geq c_1|t|^{q(x)}$. On the other hand, from (f_1), for almost all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$ such that $|t| < M_2$, we have $|F(x,t)| \leq C$, where $C = C(M_2) > 0$. Therefore, for almost all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$ the above two inequalities imply

$$F(x,t) \geq c_1|t|^{q(x)} - c_2, \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{3.5}$$

where $c_2 = C + \max\{M_2^{q^-}, M_2^{q^+}\}c_1$. Using (f_1) again, we deduce another estimate:

$$|wt| \leq c(|t| + |t|^{p_N^o(x)}) \leq 2c(1 + |t|^{p_N^o(x)}). \tag{3.6}$$

From (3.5) and (3.6), we have

$$p_N^{o\,-}F(x,t) - wt \geq p_N^{o\,-}c_1|t|^{q(x)} - p_N^{o\,-}c_2 - 2c(1+|t|^{p_N^o(x)}). \tag{3.7}$$

By (3.4) and (3.7), we get

$$\begin{aligned} p_N^{o\,-}c_1 \int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}dx - 2c\int_{\mathbb{R}^N} a(x)|u_n|^{p_N^o(x)}dx \\ \leq M_1 p_N^{o\,-} + C\|u_n\|_E + (p_N^{o\,-}c_2 + 2c)|a|_1. \end{aligned} \tag{3.8}$$

Note that $q^- > p_N^{o\,+}$. Then, applying Young's inequality with ϵ, we get

$$\begin{aligned} |u|^{p_N^o(x)} = 1 \times |u|^{p_N^o(x)} &= \epsilon^{-\frac{p_N^o(x)}{q(x)}} \times \epsilon^{\frac{p_N^o(x)}{q(x)}}|u|^{p_N^o(x)} \\ &\leq (\epsilon^{-\frac{p_N^o(x)}{q(x)}})^{\frac{q(x)}{q(x)-p_N^o(x)}} + \epsilon||u(x)|^{p_N^o(x)}|^{\frac{q(x)}{p_N^o(x)}} \\ &= \epsilon^{-\frac{p_N^o(x)}{q(x)-p_N^o(x)}} + \epsilon|u(x)|^{q(x)} \\ &\leq \epsilon^{-\frac{p_N^{o\,+}}{q^- - p_N^{o\,+}}} + \epsilon|u(x)|^{q(x)}. \end{aligned} \tag{3.9}$$

Hence, by (3.8) and (3.9), we obtain

$$(p_N^{o\,-}c_1 - 2c\epsilon)\int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}dx \leq M_1 p_N^{o\,-} + C\|u_n\|_E + (p_N^{o\,-}c_2 + 2c)|a|_1 + 2c\epsilon^{\frac{p_N^{o\,+}}{q^- - p_N^{o\,+}}}|a|_1.$$

Then, choosing ϵ_0 small enough such that $0 < \epsilon < \frac{p_N^{o\,-}c_1}{2c}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}dx \leq & \frac{M_1 p_N^{o\,-} + (p_N^{o\,-}c_2 + 2c)|a|_1 + 2c\epsilon^{\frac{p_N^{o\,+}}{q^- - p_N^{o\,+}}}|a|_1}{(p_N^{o\,-}c_1 - 2c\epsilon)} \\ & + \frac{C}{(p_N^{o\,-}c_1 - 2c\epsilon)}\|u_n\|_E, \end{aligned} \tag{3.10}$$

for all $n \geq 1$.

On the other hand, using (f_1) again, we deduce another estimate:

$$|F(x,u_n)| \leq 2c(1+|u_n|^{p(x)}). \tag{3.11}$$

Hence we obtain from (3.2), (3.11) and $p_N^{o\,+} < q(x)$ that

$$\begin{aligned} \frac{1}{P_+^+}\|u_n\|_E^{p_N^{o\,-}} &\leq \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}dx + \int_{\mathbb{R}^N} V(x)|u_n|^{p_N^o(x)}dx \\ &= \phi(u_n) + \int_{\mathbb{R}^N} a(x)F(x,u)dx \\ &\leq M_1 + 2c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)|u_n|^{p_N^o(x)}dx \\ &\leq M_1 + 2c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)(1+|u_n|^{q(x)})dx \\ &= M_1 + 4c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}dx. \end{aligned} \tag{3.12}$$

Therefore, combining (3.10) and (3.12), the boundedness of $\{u_n\}_{n=1}^{\infty}$ immediately follows, that is, there is constant $C > 0$ such that $\|u_n\|_E \leq C$. Thus, passing to a subsequence if necessary, we assume that $u_n \rightharpoonup u_0$ in E, so it follows from (3.1) that

$$\langle A(u_n), u_n - u_0 \rangle - \int_{\mathbb{R}^N} a(x) w_n (u_n - u_0) dx \leq \epsilon_n, \tag{3.13}$$

with $\epsilon_n \downarrow 0$, $w_n(x)\partial F(x, u_n(x))$.

Next we prove that $\int_{\mathbb{R}^N} a(x) w_n (u_n - u_0) dx$ as $n \to +\infty$. Clearly, by hypothesis (f_1), we have

$$\int_{\mathbb{R}^N} a(x)|w_n|\;|u_n - u_0| dx \leq c \int_{\mathbb{R}^N} a(x)|u_n - u_0| dx + \int_{\mathbb{R}^N} a(x)|u_n|^{p_N^o(x)-1}|u_n - u_0| dx. \tag{3.14}$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} a(x)|u_n|^{p_N^o(x)-1}|u_n - u_0| dx \\ &\leq 3|a|_{L^{\frac{N}{p_N^o(x)-1}}(\mathbb{R}^N)} \left||u_n|^{p_N^o(x)-1}\right|_{L^{\frac{\kappa(x)}{p_N^o(x)-1}}(\mathbb{R}^N)} |u_n - u_0|_{L^{p_N^o(x)}(\mathbb{R}^N)} \\ &\leq 3|a|_{L^{\frac{p_N^o(x)}{p_N^o(x)-r(x)}}(\mathbb{R}^N)} |u_n - u_0|_{L^{\frac{\kappa(x)}{p_N^o(x)}}(\mathbb{R}^N)} \left||u_n|^{p_N^o(x)-1}\right|_{L^{\frac{\kappa(x)}{p_N^o(x)-1}}(\mathbb{R}^N)}. \end{aligned} \tag{3.15}$$

where $\kappa(x) = {p_N^o}^*(x)$. We claim that

$$\left||u_n|^{p_N^o(x)-1}\right|_{L^{\frac{\kappa(x)}{p_N^o(x)-1}}(\mathbb{R}^N)} \leq |u_n|_{\kappa(x)}^{{p_N^o}^+-1} + 2. \tag{3.16}$$

Indeed, we have that

$$\text{if } |u_n|_{\kappa(x)} \geq 1, \text{ then } \left||u_n|^{p_N^o(x)-1}\right|_{L^{\frac{\kappa(x)}{p_N^o(x)-1}}(\mathbb{R}^N)} \leq |u_n|_{\kappa(x)}^{{p_N^o}^+-1}. \tag{I}$$

This is seen as follows: According to (2.4), to prove (I), this is equivalent to proving that $|u_n|_{\kappa(x)} \geq 1$ implies

$$\int_{\mathbb{R}^N} \frac{|u_n(x)|^{(p_N^o(x)-1)\frac{\kappa(x)}{p_N^o(x)-1}}}{|u_n(x)|_{\kappa(x)}^{({p_N^o}^+-1)\frac{\kappa(x)}{p_N^o(x)-1}}} dx = \int_{\mathbb{R}^N} \frac{|u_n(x)|^{\kappa(x)}}{|u_n(x)|_{\kappa(x)}^{({p_N^o}^+-1)\frac{\kappa(x)}{p_N^o(x)-1}}} dx \leq 1.$$

This inequality is justified as follows: since $|u_n|_{p_N^o(x)} \geq 1$ and

$$\begin{aligned} ({p_N^o}^+ - 1)\frac{\kappa(x)}{p_N^o(x)-1} - \kappa(x) &= {p_N^o}^+ \frac{\kappa(x)}{p_N^o(x)-1} - \left(\kappa(x) + \frac{\kappa(x)}{p_N^o(x)-1}\right) \\ &= {p_N^o}^+ \frac{\kappa(x)}{p_N^o(x)-1} - p_N^o(x)\frac{\kappa(x)}{p_N^o(x)-1} \\ &= \frac{\kappa(x)}{p_N^o(x)-1}({p_N^o}^+ - p_N^o(x)) \\ &\geq 0, \end{aligned}$$

we infer that

$$\frac{|u_n(x)|^{\kappa(x)}}{|u_n|_{\kappa(x)}^{({p_N^o}^+-1)\frac{\kappa(x)}{p_N^o(x)-1}}} = \frac{|u_n(x)|^{\kappa(x)}}{|u_n|_{\kappa(x)}^{\kappa(x)}} \frac{1}{|u_n|_{\kappa(x)}^{({p_N^o}^+-1)\frac{\kappa(x)}{p_N^o(x)-1}-\kappa(x)}} \leq \frac{|u_n(x)|^{\kappa(x)}}{|u_n|_{\kappa(x)}^{\kappa(x)}},$$

which implies

$$\int_{\mathbb{R}^N} \frac{|u_n(x)|^{(p_N^o(x)-1)p'(x)}}{|u_n|_{p_N^o(x)}^{(p_N^{o\,+}-1)p'(x)}} dx \le \int_{\mathbb{R}^N} \frac{|u_n(x)|^{\kappa(x)}}{|u_n|_{\kappa(x)}^{\kappa(x)}} dx = 1,$$

and the proof of (I) is complete.

$$\text{If } |u_n|_{\kappa(x)} < 1, \text{ then } ||u_n|^{p_N^o(x)-1}|_{\frac{\kappa(x)}{p_N^o(x)-1}} < 2. \tag{II}$$

Indeed, by $|u_n|_{\kappa(x)} < \int_{\mathbb{R}^N} |u_n|^{\kappa(x)} dx + 1$ and (2.7), we obtain

$$\begin{aligned} ||u_n|^{p_N^o(x)-1}|_{\frac{\kappa(x)}{p_N^o(x)-1}} &< \int_{\mathbb{R}^N} |u_n|^{(p_N^o(x)-1)\frac{\kappa(x)}{p_N^o(x)-1}} dx + 1 \\ &= \int_{\mathbb{R}^N} |u_n|^{\kappa(x)} dx + 1 < 1 + 1 = 2. \end{aligned}$$

Clearly, (3.6) is a consequence of (I) and (II).

Notice that the inclusion $E \hookrightarrow L^{\kappa(x)}(\mathbb{R}^N)$ is continuous, and hence there exists $C_1 > 0$ such that

$$|u_n|_{\kappa(x)} \le C_1 \|u_n\|_E \le CC_1. \tag{3.17}$$

By Proposition 2.6, the embedding $E \hookrightarrow L^{p_N^o(x)}(\mathbb{R}^N)$ is compact, and $u_n \rightharpoonup u$ in E implies $u_n \to u$ in $L^{p_N^o(x)}(\mathbb{R}^N)$. Hence using (3.15), (3.16) and (3.17), we have

$$\int_{\mathbb{R}^N} a(x)|u_n|_{p_N^o(x)-1}|u_n - u_0| dx \to 0 \quad \text{as } n \to +\infty. \tag{3.18}$$

Choose $\theta(x) = \frac{p_N^o(x)}{p_N^o(x)-1}$. Then $\theta \in C_+(\mathbb{R}^N), 1 < \theta(x) < \frac{N}{p_N^o(x)-1}$ for all $x \in \mathbb{R}^N$, and there exists $\lambda : \mathbb{R}^N \to (0,1)$ such that

$$\frac{1}{\theta(x)} = \frac{\lambda(x)}{1} + \frac{1-\lambda(x)}{\frac{N}{p_N^o(x)-1}} \quad \text{a.e. } x \in \mathbb{R}^N.$$

Then, for $x \in \mathbb{R}^N$, we have

$$s(x) = \frac{1}{\theta(x)\lambda(x)} > 1, \qquad t(x) = \frac{N}{\theta(x)(p(x)-1)(1-\lambda(x))} > 1.$$

Using $a \in L^1_+(\mathbb{R}^N) \cap L^{\frac{N}{p_N^o(x)-1}}(\mathbb{R}^N)$, we deduce

$$\begin{aligned} \int_{\mathbb{R}^N} |a|^{\theta(x)} dx &= \int_{\mathbb{R}^N} |a|^{\frac{1}{s(x)}} |a|^{\frac{\frac{N}{p_N^o(x)-1}}{t(x)}} dx \\ &\le 2\Big[\Big(\int_{\mathbb{R}^N} |a| dx\Big)^{\frac{1}{s^+}} + \Big(\int_{\mathbb{R}^N} |a| dx\Big)^{\frac{1}{s^-}}\Big] \\ &\quad \times \Big[\Big(\int_{\mathbb{R}^N} |a|^{\frac{N}{p(x)-1}} dx\Big)^{\frac{1}{t^+}} + \Big(\int_{\mathbb{R}^N} |a|^{\frac{N}{p(x)-1}} dx\Big)^{\frac{1}{t^-}}\Big]. \end{aligned} \tag{3.19}$$

This implies $a \in L^{\frac{p_N^o(x)}{p_N^o(x)-1}}(\mathbb{R}^N)$. Hence

$$\int_{\mathbb{R}^N} a(x)|u_n - u_0| dx \le 2|a|_{\frac{p(x)}{p(x)-1}} |u_n - u_0|_{p(x)} \to 0 \quad \text{as } n \to +\infty.$$

Combining (3.13), (3.14), (3.18) and (3.19), we get $\lim_{n\to+\infty}\langle A(u_n), u_n - u\rangle = 0$. By Proposition 2.2 (2), we get $u_n \to u_0$ in E. This proves that $\phi(u)$ satisfies the nonsmooth (PS) condition on E. □

Lemma 3.3. *Assume that all conditions of Theorem 1.1 are satisfied. Then there exist $\alpha > 0$ and $\nu > 0$ such that, for any $u \in E$ with $\|u\|_E = \nu$, we have $\phi(u) \geq \alpha$.*

Proof. Firstly, choose $q \in C_+(\mathbb{R}^N)$ (q is mentioned in (f_3)). Then

$$1 < \frac{\kappa(x)}{\kappa(x) - q(x)} = \frac{Np_N^o(x)}{Np_N^o(x) - q(x)(N - p_N^o(x))} < \frac{N}{p_N^o(x) - 1}, \qquad x \in \mathbb{R}^N. \tag{3.20}$$

By Proposition 2.6, the embedding $E \hookrightarrow L^{\kappa(x)}(\mathbb{R}^N)$ is continuous, and there is constant $c_5 > 0$ such that

$$|u|_{\kappa(x)} \leq c_5\|u\|_E \quad \text{for all } u \in E. \tag{3.21}$$

Now choose $\gamma > 0$ such that $\gamma < \min\{1, \frac{1}{c_5}\}$. Then, for such a fixed γ, we have

$$|u|_{\kappa(x)} \leq 1 \quad \text{for all } u \in E \text{ with } \|u\|_E = \gamma. \tag{3.22}$$

Moreover, by virtue of hypothesis (f_2), we obtain

$$F(x,t) \leq \vartheta(x), \tag{3.23}$$

for any $x \in \mathbb{R}^N$ and $0 < |t| < \delta$.

On the other hand, for all $x \in \mathbb{R}^N$ and all $|t| \geq \delta$, (f_1) implies

$$|F(x,t)| \leq c_6|t|^{p(x)}, \tag{3.24}$$

where $c_6 = (1 + \frac{1}{\tau})c$ and $\tau = \min\{|\delta|^{p_N^{o\,+}}, |\delta|^{p_N^{o\,-}}\}$.

From (3.23) and (3.24), for all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$, we have $F(x,t) \leq \vartheta(x) + c_7|t|^{p(x)}$, where $c_7 = c_6 + \frac{|\vartheta|_\infty}{\tau}$.

Thus, for all $u \in E$ with $\|u\|_E = \gamma$, we have

$$\begin{aligned}
\phi(u) &= \sum_{i=1}^{N}\int_{\mathbb{R}^N}\frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}dx + \int_{\mathbb{R}^N}\frac{1}{p_N^o(x)}V(x)|u|^{p_N^o(x)}dx - \int_{\mathbb{R}^N}a(x)F(x,u)dx \\
&\geq \frac{1}{p_N^{o\,+}}\|u\|_E^{p_N^{o\,+}} - c_7\int_{\mathbb{R}^N}a(x)|u|^{p_N^o(x)}dx - \int_{\mathbb{R}^N}a(x)\vartheta(x)dx \\
&\geq \frac{1}{p_N^{o\,+}}\|u\|_E^{p_N^{o\,+}} - c_7\int_{\mathbb{R}^N}a(x)|u|^{p_N^o(x)}dx + \int_{\mathbb{R}^N}a(x)\vartheta(x)dx.
\end{aligned} \tag{3.25}$$

Applying Young's inequality with ε, we get

$$\begin{aligned}
|u|^{p_N^o(x)} &= 1 \times |u|^{p_N^o(x)} \\
&\leq \varepsilon \times 1^{\frac{q(x)}{q(x)-p_N^o(x)}} + \varepsilon^{-\frac{q(x)-p_N^o(x)}{p_N^o(x)}}\left||u|^{p_N^o(x)}\right|^{\frac{q(x)}{p_N^o(x)}} \\
&= \varepsilon + \varepsilon^{-\frac{q(x)-p_N^o(x)}{p_N^o(x)}}|u|^{q(x)} \\
&\leq \varepsilon + \varepsilon^{-\frac{q^+-p_N^{o\,-}}{p_N^{o\,-}}}|u|^{q(x)}.
\end{aligned} \tag{3.26}$$

So, returning to (3.25) and using (3.26), for all $u \in E$ with $\|u\|_E = \gamma$, we obtain

$$\begin{aligned}\phi(u) \geq \frac{1}{p_N^{o\,+}}\|u\|^{p_N^{o\,+}} - \varepsilon^{-\frac{q^+ - p_N^{o\,-}}{p_N^{o\,-}}} c_7 \int_{\mathbb{R}^N} a(x)|u(x)|^{q(x)}dx \\ - c_7\varepsilon \int_{\mathbb{R}^N} a(x)dx - \int_{\mathbb{R}^N} a(x)\nu(x)dx.\end{aligned} \tag{3.27}$$

Since $\nu \in L^\infty(\mathbb{R}^N)$, there exists some $c_8 > 0$ such that $-\nu(x) > c_8$. We can choose and ε_0 small enough such that $c_8|a|_1 - \varepsilon_0 c_7 |a|_1 > 0$, and then (3.27) immediately implies

$$\phi(u) \geq \frac{1}{p_N^{o\,+}}\|u\|_E^{p_N^{o\,+}} - \varepsilon^{-\frac{q^+ - p_N^{o\,-}}{p_N^{o\,-}}} c_7 \int_{\mathbb{R}^N} a(x)|u(x)|^{q(x)}dx. \tag{3.28}$$

Similarly to the proof of (3.19), and combining inequality (3.20), we have $a(x) \in L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)$. Using Proposition 2.3, (3.21) and (3.22), for all $u \in E$ with $\|u\|_E = \gamma$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} a(x)|u|^{q(x)}dx \leq & |a|_{L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)} \, ||u|^{q(x)}|_{L^{\frac{\kappa(x)}{q(x)}}(\mathbb{R}^N)} \\ \leq & |a|_{L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)} \, |u|^{q^-}_{\kappa(x)} \\ \leq & |a|_{L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)} \, c_5^{q^-} \, \|u\|_E^{q^-}.\end{aligned} \tag{3.29}$$

Using (3.29) in (3.28), we see that, for any $u \in E$ with $\|u\|_E = \gamma$, we have

$$\phi(u) \geq \frac{1}{p_N^{o\,+}}\|u\|_E^{p_N^{o\,+}} - \varepsilon_0^{-\frac{q^+ - p_N^{o\,-}}{p_N^{o\,-}}} c_5^{q^-} |a|_{L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)} \|u\|_E^{q^-}.$$

which implies that there exist $\alpha > 0$ and $\nu > 0$ such that $\phi(u) \geq \alpha$ for any $u \in E$ with $\|u\|_E = \nu$. □

Lemma 3.4. *Assume that all conditions of Theorem 1.1 are satisfied. Then $\phi(u) \to -\infty$ as $\|u\|_E \to +\infty$ for all $u \in \mathcal{F}$, where $\mathcal{F}$ is an arbitrary finite-dimensional subspace of E.*

Proof. By virtue of hypothesis (f_3), we can find $M_4 > 0$ such that

$$F(x,t) \geq c_9|t|^{q(x)} \quad \text{for all } x \in \mathbb{R}^N, \ |t| \geq M_4. \tag{3.30}$$

In addition, from hypothesis (f_1), for almost all $x \in \mathbb{R}^N$ and $|t| < M_4$, we have

$$|F(x,t)| \geq c_3, \tag{3.31}$$

where $c_3 = (1 + M_4^{p_N^{o\,+}} + M_4^{p_N^{o\,-}})c$. Thus, using 3.30 and 3.31, we obtain

$$F(x,t) \geq c_9|t|^{q(x)} \quad \text{for all } x \in \mathbb{R}^N, \ t \in \mathbb{R}, \tag{3.32}$$

where $c_4 = c_3 + c_9 \max\{M_4^{p_N^{o\,+}}, M_4^{p_N^{o\,-}}\}$. Moreover, similar to (2.6) and (2.7), we get

$$\begin{aligned}|u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1 \Longrightarrow |u|^{q^-}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} a(x)|u|^{q(x)}dx \leq |u|^{q^+}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \\ |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} < 1 \Longrightarrow |u|^{q^+}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} a(x)|u|^{q(x)}dx \leq |u|^{q^-}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)}.\end{aligned} \tag{3.33}$$

Because W is a finite-dimensional subspace of E, all norms are equivalent, so we can find $0 < C = C(\mathcal{F}) < 1$ such that

$$C\|u\|_E \le |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \le \frac{1}{C}\|u\|_E \quad \text{for all } u \in \mathcal{F}. \tag{3.34}$$

Taking into account (3.32), (3.33) and (3.34), for every $u \in \mathcal{F}$ with $\|u\|_E > 1$ and $|u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1$, we have

$$\begin{aligned}
\phi(u) &= \sum_{i=1}^{N}\int_{\mathbb{R}^N}\frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}dx + \int_{\mathbb{R}^N}\frac{1}{p_N^o(x)}V(x)|u|^{p_N^o(x)}dx - \int_{\mathbb{R}^N}a(x)F(x,u)dx \\
&\le \frac{1}{p_N^{o\,-}}\|u\|^{p_N^{o\,+}} - c_9\int_{\mathbb{R}^N}a(x)|u|^{q(x)}dx + c_4\int_{\mathbb{R}^N}a(x)dx \\
&\le \begin{cases} \frac{1}{p_N^{o\,-}}\|u\|^{p_N^{o\,+}} + c_4|a|_1 - c_9C^{q^-}\|u\|_E^{q^-} & \text{if } |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1, \\ \frac{1}{p_N^{o\,-}}\|u\|^{p_N^{o\,+}} + c_4|a|_1 - c_9C^{q^+}\|u\|_E^{q^+} & \text{if } |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} < 1, \end{cases}
\end{aligned} \tag{3.35}$$

Because of $q^+ \ge q^- > p_N^{o\,+}$, we see that $\phi(u) \to -\infty$ as $\|u\|_E \to +\infty$. □

Here is the proof of Theorem 1.1.

Proof. It is obvious that ϕ is even and $\phi(0) = 0$. Besides, Lemmas 3.2, 3.3 and 3.4 permit the application of Lemma 2.8 with $X = E, X_1 = \mathcal{F}$ (see Lemma 3.4) and $X_2 = E \oplus \mathcal{F}$ (see Lemma 3.3). Therefore, we obtain that the functional ϕ has an unbounded sequence of critical values, so problem (1.1) possesses infinitely many nontrivial solutions. □

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