Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **54**, 1–14; https://doi.org/10.14232/ejqtde.2024.1.54 www.math.u-szeged.hu/ejqtde/

Positive solutions for concave-convex type problems for the one-dimensional ϕ-Laplacian

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Received 7 June 2024, appeared 14 September 2024
Communicated by Paul Eloe

Abstract. Let $\Omega = (a,b) \subset \mathbb{R}$, $0 \leq m,n \in L^1(\Omega)$, $\lambda, \mu > 0$ be real parameters, and $\phi : \mathbb{R} \to \mathbb{R}$ be an odd increasing homeomorphism. In this paper we consider the existence of positive solutions for problems of the form

$$\begin{cases} -\phi(u')' = \lambda m(x) f(u) + \mu n(x) g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f, g : [0, \infty) \to [0, \infty)$ are continuous functions which are, roughly speaking, sublinear and superlinear with respect to ϕ, respectively. Our assumptions on ϕ, m and n are substantially weaker than the ones imposed in previous works. The approach used here combines the Guo–Krasnoselskiĭ fixed-point theorem and the sub-supersolutions method with some estimates on related nonlinear problems.

Keywords: elliptic one-dimensional problems, ϕ-Laplacian, positive solutions.

2020 Mathematics Subject Classification: 34B15, 34B18.

1 Introduction

Let $\Omega = (a,b) \subset \mathbb{R}$, $m, n \in L^1(\Omega)$ and $\lambda, \mu > 0$ be a real parameters. In this article we consider problems of the form

$$\begin{cases} -\phi(u')' = \lambda m(x) f(u) + \mu n(x) g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism and $f, g : [0, \infty) \to [0, \infty)$ are continuous functions which are, roughly speaking, sublinear and superlinear with respect to ϕ, respectively. When the nonlinearities f and g are concave and convex, the problem (1.1) with

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$\phi(x) = x$ was first studied by Ambrosetti, Brezis and Cerami in their celebrated paper [1]. More precisely, in that article the authors studied the N-dimensional problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

with $0 < q < 1 < p$ and Ω a bounded domain in $\mathbb{R}^N$. They proved the following facts: there exists $\Lambda > 0$ such that: if $\lambda \in (0, \Lambda)$ then (1.2) has at least two positive solutions, if $\lambda = \Lambda$ there is at least one positive solution, and if $\lambda > \Lambda$ then there are no positive solutions.

Several authors have studied generalizations of (1.2), see for instance [2, 11, 14] and their references, where the corresponding problem for the p-Laplacian is considered. Also, in [12] the authors have treated the N-dimensional problem for the ϕ-Laplacian operator.

Regarding the one-dimensional ϕ-Laplacian problem that we will deal with in this article, Wang in [15, Theorem 1.2] and [16, Theorem 1.2] studied the case $m = n \geq 0$, $m \not\equiv 0$ on any subinterval in Ω, $m \in C(\overline{\Omega})$ and $\lambda = \mu$. In these papers it is proved that there exist $\lambda_0, \lambda_1 > 0$ such that if $\lambda \in (0, \lambda_0)$, then (1.1) has at least two positive solutions; and if $\lambda > \lambda_1$, then there are no positive solutions. Let us note that the hypothesis on ϕ imposed in [15, 16] are much stronger than the ones that we shall require here. More precisely, Wang assumes

(Φ) There exist increasing homeomorphisms $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ such that $\psi_1(t)\,\phi(x) \leq \phi(tx) \leq \psi_2(t)\,\phi(x)$ for all $t, x > 0$.

On other hand, (1.1) is also considered in [8] with $m = n \geq 0$, $m \not\equiv 0$ on any subinterval in Ω and $\lambda = \mu$ like in [15, 16]. However, the regularity assumptions for m allow some $m \in L^1_{loc}(\Omega)$. Regarding the hypothesis on ϕ they require that

(Φ') There exist an increasing homeomorphism $\psi_1 : [0, \infty) \to [0, \infty)$ and a function $\psi_2 : [0, \infty) \to [0, \infty)$ such that $\psi_1(t)\,\phi(x) \leq \phi(tx) \leq \psi_2(t)\,\phi(x)$ for all $t, x > 0$.

The authors prove that there exist $\lambda_1 \geq \lambda_0 > 0$ such that (1.1) has at least two positive solutions for $\lambda \in (0, \lambda_0)$, one positive solution for $\lambda \in [\lambda_0, \lambda_1]$, and no positive solution for $\lambda > \lambda_1$.

In this article, employing the method of sub and supersolutions and the Guo–Krasnoselskiĭ fixed-point theorem along with some estimates for related problems, we shall prove that there are at least two positive solutions for $\lambda \approx 0$, under much weaker assumptions on ϕ, m and n. Moreover, as a consequence of Theorem 4.4 we shall see that (Φ) and (Φ') are in fact equivalent.

To be more precise, let us introduce the following hypothesis.

(F) There exist $c_0, t_0, q > 0$ such that

$$f(t) \geq c_0 t^q \text{ for all } t \in [0, t_0] \quad \text{and} \quad \lim_{t \to 0^+} \frac{t^q}{\phi(t)} = \infty. \tag{1.3}$$

(G1) There exist $c_1, t_1, r_1 > 0$ such that

$$g(t) \leq c_1 t^{r_1} \text{ for all } t \in [0, t_1] \quad \text{and} \quad \lim_{t \to 0^+} \frac{t^{r_1}}{\phi(t)} = 0. \tag{1.4}$$

(G2) There exist $c_2, t_2, r_2 > 0$ such that

$$g(t) \geq c_2 t^{r_2} \text{ for all } t \geq t_2 \quad \text{and} \quad \lim_{t\to\infty} \frac{t^{r_2}}{\phi(t)} = \infty. \tag{1.5}$$

Note that when $\phi(t) = |t|^{p-2}t$, $f(u) = u^q$ and $g(u) = u^r$, the limits in (F) and (G1) are satisfied if and only if $0 < q < p-1 < r$. Let us set $\mathcal{C}_0^1(\overline{\Omega}) := \{u \in \mathcal{C}^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ and

$$\mathcal{P}^\circ := \left\{ u \in \mathcal{C}_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega \text{ and } u'(b) < 0 < u'(a) \right\}.$$

Our main result is the following theorem:

Theorem 1.1. *Let $0 \leq m, n \in L^1(\Omega)$.*

(I) Assume that $m \not\equiv 0$ and (F) and (G1) hold. Then for all $\mu > 0$ there exists $\lambda_0(\mu) > 0$ such that (1.1) has a solution $u_\lambda \in \mathcal{P}^\circ$ for all $0 < \lambda < \lambda_0(\mu)$. Moreover, the solutions u_λ can be chosen such that

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0. \tag{1.6}$$

(II) Assume that $n \not\equiv 0$ and (G1) and (G2) hold. Then for all $\mu > 0$ there exists $\lambda_1(\mu) > 0$ such that (1.1) has a solution $v_\lambda \in \mathcal{P}^\circ$ for all $0 < \lambda < \lambda_1(\mu)$. Furthermore, there exists $\rho > 0$ such that $\|v_\lambda\|_\infty > \rho$ for all $0 < \lambda < \lambda_1(\mu)$.

(III) Assume that $\{\lambda > 0 : (1.1) \text{ has a solution in } \mathcal{P}^\circ\} \neq \emptyset$ and (F) holds for all $t_0 > 0$. Let

$$\Lambda := \sup\{\lambda > 0 : (1.1) \text{ has a solution in } \mathcal{P}^\circ\}.$$

Then, for $0 < \lambda < \Lambda$ (1.1) has at least one solution in $\mathcal{P}^\circ$.

As an immediate consequence of the above theorem we have the following

Corollary 1.2. *Let $\mu > 0$ and $0 \leq m, n \in L^1(\Omega)$ with $m, n \not\equiv 0$. Assume that (F), (G1) and (G2) hold. Then (1.1) has at least two solutions in $\mathcal{P}^\circ$ for $\lambda \approx 0$.*

The rest of the paper is organized as follows. In the next section we state some necessary facts about nonlinear problems involving the ϕ-Laplacian, and in Section 3 we prove our main results. Finally, in Section 4 we introduce some concepts about Orlicz spaces indices which we use to prove Theorem 4.4 (and, in particular, the equivalence of (Φ) and (Φ')), and at the end of the section we give several examples of functions ϕ illustrating our conditions and their relations with the ones used in the previous works. Let us mention that all the ϕ's constructed in Example (e) satisfy conditions (F), (G1) and (G2) but do not fulfill condition (Φ).

2 Preliminaries

Let $\phi : \mathbb{R} \to \mathbb{R}$ be an odd increasing homeomorphism. We start considering problems of the form

$$\begin{cases} -\phi(v')' = h(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is well known that for all $h \in L^1(\Omega)$, (2.1) possesses a unique solution $v \in C_0^1(\overline{\Omega})$ such that $\phi(v')$ is absolutely continuous and that the equation holds pointwise *a.e.* $x \in \Omega$. Furthermore, the solution operator $\mathcal{S}_\phi \colon L^1(\Omega) \to C^1(\overline{\Omega})$ is completely continuous and nondecreasing, see [3, Lemma 2.1] and [6, Lemma 2.2].

We need now to introduce some notation. For $0 \leq h \in L^1(\Omega)$ with $h \not\equiv 0$, set

$$\mathcal{A}_h := \{x \in \Omega : h(y) = 0 \ a.e.\ y \in (a,x)\},$$
$$\mathcal{B}_h := \{x \in \Omega : h(y) = 0 \ a.e.\ y \in (x,b)\},$$

and

$$\alpha_h := \begin{cases} \sup \mathcal{A}_h & \text{if } \mathcal{A}_h \neq \emptyset, \\ a & \text{if } \mathcal{A}_h = \emptyset, \end{cases} \qquad \beta_h := \begin{cases} \inf \mathcal{B}_h & \text{if } \mathcal{B}_h \neq \emptyset, \\ b & \text{if } \mathcal{B}_h = \emptyset, \end{cases}$$
$$\underline{\theta}_h := \min\left\{\frac{1}{\beta_h - a}, \frac{1}{b - \alpha_h}\right\}, \qquad \overline{\theta}_h := \frac{\alpha_h + \beta_h}{2}. \tag{2.2}$$

We observe that $\underline{\theta}_h$ is well defined because $h \not\equiv 0$, and $\alpha_h < \beta_h$ (and so, $\overline{\theta}_h \in (\alpha_h, \beta_h)$). We also write

$$\delta_\Omega(x) := \operatorname{dist}(x, \partial\Omega) = \min(x - a, b - x).$$

We shall utilize the following estimates on several occasions in the sequel. For the proof, see [6, Lemma 2.3 and (2.6)] and [7, Corollary 2.2].

Lemma 2.1. *Let $0 \leq h \in L^1(\Omega)$ with $h \not\equiv 0$.*

(i) In $\overline{\Omega}$ it holds that

$$\underline{\theta}_h \min\left\{\int_a^{\overline{\theta}_h} \phi^{-1}\left(\int_y^{\overline{\theta}_h} h\right) dy, \int_{\overline{\theta}_h}^b \phi^{-1}\left(\int_{\overline{\theta}_h}^y h\right) dy\right\} \delta_\Omega \leq \mathcal{S}_\phi(h) \leq \phi^{-1}\left(\int_a^b h\right) \delta_\Omega. \tag{2.3}$$

(ii) In $\overline{\Omega}$ it holds that

$$\mathcal{S}_\phi(h) \geq \underline{\theta}_h \left\|\mathcal{S}_\phi(h)\right\|_\infty \delta_\Omega. \tag{2.4}$$

(iii) For $M > 0$ there exists $c > 0$ not depending on M such that it holds that

$$\min\left\{\int_a^{\overline{\theta}_h} \phi^{-1}\left(\int_y^{\overline{\theta}_h} Mh\right) dy, \int_{\overline{\theta}_h}^b \phi^{-1}\left(\int_{\overline{\theta}_h}^y Mh\right) dy\right\} \geq c\phi^{-1}(cM). \tag{2.5}$$

Observe that, since $\overline{\theta}_h \in (\alpha_h, \beta_h)$, the constant that appears in the first term of the inequalities in (2.3) is strictly positive. Note also that, since $\underline{\theta}_h \left\|\delta_\Omega\right\|_\infty \geq 1/2$, using the lower bound of (2.3) and taking into account the monotonicity of the infinite norm we get

$$\frac{1}{2} \min\left\{\int_a^{\overline{\theta}_h} \phi^{-1}\left(\int_y^{\overline{\theta}_h} h\right) dy, \int_{\overline{\theta}_h}^b \phi^{-1}\left(\int_{\overline{\theta}_h}^y h\right) dy\right\} \leq \left\|\mathcal{S}_\phi(h)\right\|_\infty. \tag{2.6}$$

Observe also that for h as in Lemma 2.1 $\mathcal{S}_\phi(h) \in \mathcal{P}^\circ$.

Let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (that is, $h(x, \cdot)$ is continuous for *a.e.* $x \in \Omega$ and $h(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}$). We now consider problems of the form

$$\begin{cases} -\phi(u')' = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

We shall say that $v \in C(\overline{\Omega})$ is a *subsolution* of (2.7) if there exists a finite set $\Sigma \subset \Omega$ such that $\phi(v') \in AC_{loc}(\overline{\Omega} \setminus \Sigma)$, $v'(\tau^+) := \lim_{x \to \tau^+} v'(x) \in \mathbb{R}$, $v'(\tau^-) := \lim_{x \to \tau^-} v'(x) \in \mathbb{R}$ for each $\tau \in \Sigma$, and

$$\begin{cases} -\phi\left(v'\right)' \leq h\left(x, v\left(x\right)\right) & a.e.\ x \in \Omega, \\ v \leq 0 \text{ on } \partial\Omega, & v'(\tau^-) < v'(\tau^+) \text{ for each } \tau \in \Sigma. \end{cases} \tag{2.8}$$

If the inequalities in (2.8) are inverted, we shall say that v is a *supersolution* of (2.7).

For the sake of completeness, we state an existence result in the presence of well-ordered sub and supersolutions, and a particular case of the well-known Guo–Krasnoselskiĭ fixed-point theorem (for a proof, see e.g. [13, Theorem 7.16] and [4, Theorem 2.3.4], respectively).

Lemma 2.2. *Let v and w be sub and supersolutions respectively of (2.7) such that $v \leq w$ in Ω. Suppose there exists $g \in L^1(\Omega)$ such that*

$$\left|h\left(x, \xi\right)\right| \leq g\left(x\right) \quad \textit{for a.e. } x \in \Omega \textit{ and all } \xi \in \left[v\left(x\right), w\left(x\right)\right].$$

Then there exists $u \in C_0^1(\overline{\Omega})$ solution of (2.7) with $v \leq u \leq w$ in Ω.

Lemma 2.3. *Let X be a Banach space and let K be a cone in X. Let $\Omega_1, \Omega_2 \subset X$ be two open sets with $0 \in \Omega_1$ and $\Omega_1 \subset \Omega_2$. Suppose that $T : K \cap (\Omega_2 \setminus \Omega_1) \to K$ is a completely continuous operator and*

$$\begin{aligned} \|Tv\| &\geq \|v\|, \quad \textit{for } v \in K \cap \partial\Omega_2, \\ \|Tv\| &\leq \|v\|, \quad \textit{for } v \in K \cap \partial\Omega_1. \end{aligned}$$

Then, T has a fixed point in $K \cap (\Omega_2 \setminus \Omega_1)$.

3 Proof of the main results

3.1 Proof of item (I)

We start this section with two lemmas concerning sub and supersolutions that shall be used to prove item (I) of Theorem 1.1.

Lemma 3.1. *Let $m, n \in L^1(\Omega)$ such that $0 \not\equiv m + n \geq 0$. Assume that (G1) holds. Then for all $\mu > 0$ there exists $\lambda_0(\mu) > 0$ such that for each $0 < \lambda < \lambda_0(\mu)$ there exists $w_\lambda \in \mathcal{P}^\circ$ supersolution of (1.1). Moreover,*

$$\lim_{\lambda \to 0^+} \|w_\lambda\|_\infty = 0. \tag{3.1}$$

Proof. Let c_1, t_1, r_1 be given by (G1). Let us define $c_\Omega := \max_{\overline{\Omega}} \delta_\Omega$. By the continuity of ϕ^{-1} and the fact that $\phi^{-1}(0) = 0$, there exists $K_0 > 0$ such that

$$\phi^{-1}\left(\kappa \int_a^b m(s) + n(s) ds\right) \leq \frac{t_1}{c_\Omega} \quad \text{for all } \kappa \leq K_0. \tag{3.2}$$

We observe that by the second condition on (1.4), for $\rho > 0$ fixed we have

$$\lim_{t \to 0^+} \frac{[\phi^{-1}(\rho t)]^{r_1}}{t} = 0. \tag{3.3}$$

We now define

$$\epsilon := \frac{1}{c_1 \mu c_\Omega^{r_1}}, \qquad \rho := \int_a^b m(s) + n(s) ds.$$

We can deduce from (3.3) that there exists $K_1 = K_1(\epsilon, \rho) > 0$ such that

$$[\phi^{-1}(\kappa\rho)]^{r_1} \le \kappa\epsilon \quad \text{for all } \kappa \le K_1. \tag{3.4}$$

Let $C = \max_{[0,t_1]} f(t)$ and choose $\lambda_0 > 0$ such that

$$\lambda_0 C \le \min\{K_0, K_1\}. \tag{3.5}$$

Also, for each $0 < \lambda < \lambda_0$, pick κ_λ such that

$$\lambda C \le \kappa_\lambda \le \min\{K_0, K_1\}, \tag{3.6}$$

and for such κ_λ define $w_\lambda := \mathcal{S}_\phi(\kappa_\lambda(m+n))$. Since $\kappa_\lambda \le K_0$, the upper bound in (2.3) and (3.2) tell us that $\|w_\lambda\|_\infty \le t_1$. Taking into account (3.4), (3.5) and (3.6), employing (G1) and the upper bound in (2.3) we deduce that

$$\begin{aligned}
\lambda m(x) f(w_\lambda) + \mu n(x) g(w_\lambda) &\le \lambda m(x) C + c_1 \mu n(x) w_\lambda^{r_1} \\
&\le \kappa_\lambda m(x) + c_1 \mu n(x) \left[\phi^{-1}(\kappa_\lambda \int_a^b m(s) + n(s) ds) \delta_\Omega\right]^{r_1} \\
&\le \kappa_\lambda (m(x) + n(x)) = -\phi(w_\lambda')' \quad \text{in } \Omega,
\end{aligned}$$

and hence w_λ is a supersolution of (1.1).

In order to prove (3.1), we choose κ_λ satisfying (3.6) and such that $\kappa_\lambda \to 0$ when $\lambda \to 0^+$. Hence, using the second inequality (2.3) we get that

$$0 \le w_\lambda(x) = \mathcal{S}_\phi(\kappa_\lambda(m+n)) \le \phi^{-1}\left(\int_a^b \kappa_\lambda(m+n)\right) \delta_\Omega(x) \to 0$$

uniformly in $\overline{\Omega}$ when $\lambda \to 0^+$. Thus, $\lim_{\lambda \to 0^+} \|w_\lambda\|_\infty = 0$. □

Lemma 3.2. *Let $0 \le m, n \in L^1(\Omega)$ with $m \not\equiv 0$. Assume that (F) holds. Then for all $\lambda, \mu > 0$ (1.1) has a subsolution $v \in \mathcal{P}^\circ$.*

Proof. Let $\lambda, \mu > 0$ and let c_0, t_0, q be given by (F). Recall that $c_\Omega := \max_{\overline{\Omega}} \delta_\Omega$. Since ϕ^{-1} is continuous and $\phi^{-1}(0) = 0$, there exists $\varepsilon_0 > 0$ such that

$$\phi^{-1}\left(\varepsilon \int_a^b m(s) \delta_\Omega^q(s) ds\right) \le \frac{t_0}{c_\Omega} \quad \text{for all } \varepsilon \le \varepsilon_0. \tag{3.7}$$

By the second condition in (1.3), for $\rho > 0$ fixed

$$\lim_{t \to 0^+} \frac{[\phi^{-1}(\rho t)]^q}{t} = \infty. \tag{3.8}$$

Let us define

$$M := \frac{1}{\lambda c_0 c^q},$$

where c is the constant in (2.5) with $h = m\delta_\Omega^q$. It follows from (3.8) that there exists $\varepsilon_1 = \varepsilon_1(M, \rho)$ such that

$$[\phi^{-1}(\varepsilon\rho)]^q \ge M\varepsilon \quad \text{for all } \varepsilon \le \varepsilon_1. \tag{3.9}$$

Let us choose

$$0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\} \tag{3.10}$$

and for such ε define $v := \mathcal{S}_\phi(\varepsilon m \delta_\Omega^q)$. Since $\varepsilon \le \varepsilon_0$, the upper bound of Lemma 2.1 and (3.7) tell us that $\|v\|_\infty \le t_0$. Consequently, taking into account (3.9) and (3.10), employing (F) and (2.5) we deduce that

$$\lambda m(x) f(v) + \mu n(x) g(v) \ge \lambda c_0 m(x) v^q \ge \lambda c_0 m(x) [c\phi^{-1}(c\varepsilon)\delta_\Omega]^q \ge \varepsilon m(x) \delta_\Omega^q \quad \text{in } \Omega.$$

In other words, v is a subsolution of (1.1). □

Proof of Theorem 1.1 (I). Given $\mu > 0$, let $\lambda_0(\mu)$ be as in Lemma 3.1. For $0 < \lambda < \lambda_0(\mu)$, let $w_\lambda \in \mathcal{P}^\circ$ be a supersolution provided by the aforementioned lemma, and let $v_\lambda \in \mathcal{P}^\circ$ be a subsolution given by Lemma 3.2 with ε_λ chosen such that $\varepsilon_\lambda m(x) \delta_\Omega^q(x) \le \kappa_\lambda (m(x) + n(x))$ for $a.e. x \in \Omega$. It follows that v_λ, w_λ are a pair of well-ordered sub and supersolutions of (1.1). Hence, Lemma 2.2 gives a solution of (1.1) $u_\lambda \in \mathcal{P}^\circ$. Moreover, (1.6) follows from (3.1). □

3.2 Proof of item (II)

Proof of Theorem 1.1 (II). We shall use Lemma 2.3 with the operator

$$Tv := \mathcal{S}_\phi(\lambda m(x) f(v) + \mu n(x) g(v)),$$

the cone

$$\mathcal{K} := \{v \in C(\overline{\Omega}) : v \ge \underline{\theta}_n \|v\|_\infty \delta_\Omega\}$$

($\underline{\theta}_n$ as in (2.2)) and the open balls $B_R(0), B_\rho(0) \subset C(\overline{\Omega})$ with $0 < \rho < R$. Observe that $C_0^1(\overline{\Omega}) \cap (\mathcal{K} \setminus \{0\}) \subset \mathcal{P}^\circ$ and that any fixed point of T belongs to $C_0^1(\overline{\Omega})$.

Let c_2, t_2 and r_2 be given by (G2). We consider the function $h := c_2 \mu (\underline{\theta}_n)^{r_2} n \delta_\Omega^{r_2}$. Taking into account (2.5), we can find $c = c(\mu) > 0$ such that for all $M > 0$

$$\min\left\{\int_a^{\overline{\theta}_n} \phi^{-1}(M \int_y^{\overline{\theta}_n} h) dy, \int_{\overline{\theta}_n}^b \phi^{-1}(M \int_{\overline{\theta}_n}^y h) dy\right\} \ge c\phi^{-1}(cM). \tag{3.11}$$

On other hand, the second condition in (G2) is equivalent to

$$\lim_{t\to\infty} \frac{\phi^{-1}(\rho t^{r_2})}{t} = \infty$$

for all fixed $\rho > 0$, and then there exists $\bar{t} > 0$ such that

$$\phi^{-1}(ct^{q_2}) \ge \frac{2t}{c} \quad \text{for all } t \ge \bar{t}. \tag{3.12}$$

Let us fix $R > \max\{t_2, \bar{t}\}$. Taking into account that $\mathcal{S}_\phi$ and ϕ^{-1} are nondecreasing, the inequality (2.6), (G2), (3.11) and (3.12) we obtain that for $v \in \mathcal{K} \cap \partial B_R(0)$,

$$\begin{aligned}
\|Tv\|_\infty &= \|\mathcal{S}_\phi(\lambda m(x) f(v)) + \mu n(x) g(v))\| \ge \|S_\phi(\mu n(x) g(v))\|_\infty \\
&\ge \frac{1}{2} \min\left\{\int_a^{\overline{\theta}_n} \phi^{-1}\left(\int_y^{\overline{\theta}_n} \mu n g(v)\right) dy, \int_{\overline{\theta}_n}^b \phi^{-1}\left(\int_{\overline{\theta}_n}^y \mu n g(v)\right) dy\right\} \\
&\ge \frac{1}{2} \min\left\{\int_a^{\overline{\theta}_n} \phi^{-1}\left(c_2 \mu \int_y^{\overline{\theta}_n} n v^{r_2}\right) dy, \int_{\overline{\theta}_n}^b \phi^{-1}\left(c_2 \mu \int_{\overline{\theta}_n}^y n v^{r_2}\right) dy\right\} \\
&\ge \frac{1}{2} \min\left\{\int_a^{\overline{\theta}_n} \phi^{-1}\left(c_2 \mu (\underline{\theta}_n \|v\|_\infty)^{r_2} \int_y^{\overline{\theta}_n} n \delta_\Omega^{r_2}\right) dy, \int_{\overline{\theta}_n}^b \phi^{-1}\left(c_2 \mu (\underline{\theta}_n \|v\|_\infty)^{r_2} \int_{\overline{\theta}_n}^y n \delta_\Omega^{r_2}\right) dy\right\} \\
&\ge \frac{1}{2} c\phi^{-1}(c \|v\|_\infty^{r_2}) \\
&\ge \|v\|_\infty .
\end{aligned}$$

That is, $\|Tv\|_\infty \geq \|v\|_\infty$ for such v.

On other side, let $N := c_1 \int_a^b n$. The second condition in (G1) implies that there exists $\underline{t} > 0$ such that $\phi(t/c_\Omega) > \mu N t^{r_1}$ for all $t \in (0, \underline{t})$. Set $C := \max_{[0,R]} f(t)$ and $M := \int_a^b m$. Let $0 < \rho < \min\{\underline{t}, R/2, t_1\}$ be fixed and define

$$\lambda_1 := \frac{\phi(\rho/c_\Omega) - \mu N \rho^{r_1}}{MC}. \tag{3.13}$$

Note that $\lambda_1 > 0$ by our election of $\underline{t}$.

Now, taking into account (2.3), (G1), (3.13) and the monotonicity of ϕ^{-1} we see for $0 < \lambda \leq \lambda_1$ and all $v \in \mathcal{K} \cap \partial B_\rho(0)$,

$$\begin{aligned} Tv &\leq \phi^{-1}\left(\int_a^b \lambda m(x) f(v) + \mu n(x) g(v) dx\right)\delta_\Omega \\ &\leq \phi^{-1}\left(\lambda C \int_a^b m(x)dx + c_1\mu \int_a^b n(x) v^{r_1} dx\right)\delta_\Omega \\ &\leq \phi^{-1}\left(\lambda_1 MC + \mu N \rho^{r_1}\right)\delta_\Omega \\ &\leq \rho \text{ in } \Omega. \end{aligned}$$

This tells us that $\|Tv\|_\infty \leq \rho = \|v\|_\infty$ for all $v \in \mathcal{K} \cap \partial B_\rho(0)$.

Thus, Lemma 2.3 says that T has a fixed point in $\mathcal{K} \cap (\overline{B_R(0)} \setminus B_\rho(0))$. □

3.3 Proof of item (III)

Proof of Theorem 1.1 (III). In order to prove (III) we combine Lemma 3.2 and the inequality (2.4). Let $0 < \lambda < \Lambda$. By the definition of Λ there exists $\overline{\lambda} \in (\lambda, \Lambda]$ and $u_{\overline{\lambda}} \in \mathcal{P}^\circ$ solution of (1.1) associated to $\overline{\lambda}$. Since $\lambda < \overline{\lambda}$ it follows that $u_{\overline{\lambda}}$ is a supersolution (1.1) associated to λ. Now, thanks to Lemma 3.2 there exists $\varepsilon > 0$ such that $v = \mathcal{S}_\phi(\varepsilon m \delta_\Omega^q)$ is a subsolution of (1.1) associated to λ. Moreover, taking ε smaller if necessary, we get that $v \leq u_{\overline{\lambda}}$. Now, (III) follows from Lemma 2.2. □

4 Comments about the hypothesis

Let us introduce some concepts about Orlicz spaces indices. Given a nonbounded, increasing, continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$, we define

$$M(t, \phi) := \sup_{x>0} \frac{\phi(tx)}{\phi(x)}.$$

This function is nondecreasing and submultiplicative with $M(1, \phi) = 1$. Then, thanks to e.g. [9, Chapter 11], the following limits exist:

$$\alpha_\phi := \lim_{t \to 0^+} \frac{\ln M(t,\phi)}{\ln t}, \qquad \beta_\phi := \lim_{t \to \infty} \frac{\ln M(t,\phi)}{\ln t},$$

and moreover, $0 \leq \alpha_\phi \leq \beta_\phi \leq \infty$. These numbers are called **Orlicz space indices** or **Matuszewska–Orlicz's indices**, who introduced them in [10].

As usual, we say that ϕ satisfies the Δ_2 condition if there exists $k > 0$ such that

$$\phi(2x) \leq k\phi(x) \quad \text{for all } x \geq 0.$$

Remark 4.1.

(i) For $\varepsilon > 0$, there exists $t_1 > 0$ such that $\phi(tx) \le t^{\alpha_\phi - \varepsilon}\phi(x)$ for all $x > 0$ and $t \in [0, t_1]$.

(ii) Suppose that $\beta_\phi < \infty$. Then, for $\varepsilon > 0$, there exists $t_2 > 0$ such that $\phi(tx) \le t^{\beta_\phi + \varepsilon}\phi(x)$ for all $x > 0$ and $t \in [t_2, \infty)$. So, if $\beta_\phi < \infty$ then ϕ satisfies the Δ_2 condition.

(iii) If $x^{-p}\phi(x)$ is nondecreasing for all $x > 0$, then $\alpha_\phi \ge p$.

(iv) If $x^{-p}\phi(x)$ is nonincreasing for all $x > 0$, then $\beta_\phi \le p$.

(v) The following relationships between the Orlicz space indices of ϕ and ϕ^{-1} hold:

$$\beta_\phi = \frac{1}{\alpha_{\phi^{-1}}} \quad \text{and} \quad \alpha_\phi = \frac{1}{\beta_{\phi^{-1}}}.$$

As usual, we set $1/0 = \infty$ and $1/\infty = 0$.

We shall need the next two useful lemmas to prove Theorem 4.4 below.

Lemma 4.2 ([5, page 34])**.** *If $0 < \alpha_\phi \le \beta_\phi < \infty$ then there exist $C, p, q > 0$ such that*

$$C^{-1}\min\{t^p, t^q\}\phi(x) \le \phi(tx) \le C\max\{t^p, t^q\}\phi(x) \quad \textit{for all } t, x \ge 0.$$

Lemma 4.3 ([9, Theorem 11.7])**.** *The function ϕ satisfies the Δ_2 condition if and only if the constant β_ϕ is finite.*

Theorem 4.4. *The following hypothesis for ϕ are equivalent:*

(i) $0 < \alpha_\phi \le \beta_\phi < \infty$.

(ii) (Φ).

(iii) (Φ').

Proof. It is obvious that (ii) implies (iii), and Lemma 4.2 shows that (i) implies (ii). Let us prove that (iii) implies (i).

Since $\alpha_\phi = 1/\beta_{\phi^{-1}}$, Lemma 4.3 and Remark 4.1 (v) tell us that $\alpha_\phi > 0$ if and only if ϕ^{-1} satisfies Δ_2. Let us check that the first inequality in (Φ') implies that ϕ^{-1} satisfies Δ_2. Indeed, taking into account that

$$\psi_1(t)\phi(x) \le \phi(xt) \quad \text{for all } t, x > 0,$$

setting $y = \phi(x)$ and $s = \psi(t)$ we get that

$$sy \le \phi(\psi_1^{-1}(s)\phi^{-1}(y)) \quad \text{for all } s, y > 0.$$

Since ϕ^{-1} is increasing its follows that

$$\phi^{-1}(sy) \le \psi_1^{-1}(s)\phi^{-1}(y) \quad \text{for all } s, y > 0.$$

This implies that ϕ^{-1} satisfies Δ_2. Thus, $\alpha_\phi > 0$. Moreover, the second inequality in (Φ') implies that ϕ satisfies Δ_2. Then, $\beta_\phi < \infty$. □

The following two lemmas will be useful to compare the indices α_ϕ and β_ϕ with our hypotheses (F), (G1) and (G2) stated in Section 1.

Lemma 4.5. *Let $q > 0$.*

(i) If $\lim_{t\to 0^+} \dfrac{t^q}{\phi(t)} = 0$ *then* $\alpha_\phi \le q$.

(ii) If $\lim_{t\to \infty} \dfrac{t^q}{\phi(t)} = 0$ *then* $\beta_\phi \ge q$.

(iii) If $\lim_{t\to 0^+} \dfrac{t^q}{\phi(t)} = \infty$ *then* $\beta_\phi \ge q$.

(iv) If $\lim_{t\to \infty} \dfrac{t^q}{\phi(t)} = \infty$ *then* $\alpha_\phi \le q$.

Proof. We start proving (i). If $\alpha_\phi > q$, by Remark 4.1 (i) there exists $t_1 > 0$ such that

$$\phi(tx) \le t^q \phi(x) \quad \text{for all } x > 0 \text{ and } t \in (0, t_1).$$

Let us set $C = \phi(1)^{-1}$ and fix $x = 1$. Using the above inequality we have that $C \le \frac{t^q}{\phi(t)}$ for all $t \in (0, t_1)$, which contradicts that $\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = 0$. Therefore, we must have $\alpha_\phi \le q$. Item (ii) follows similarly. Indeed, if $\beta_\phi < q$, by Remark 4.1 (ii) we have that there exists $t_1 > 0$ such that

$$\phi(tx) \le t^q \phi(x) \quad \text{for all } x > 0 \text{ and } t > t_1.$$

We now again define $C = \phi(1)^{-1}$ and fix $x = 1$. Employing the above inequality we have that $C \le \frac{t^q}{\phi(t)}$ for all $t > t_1$, contradicting that $\lim_{t\to\infty} \frac{t^q}{\phi(t)} = 0$. Thus, $\beta_\phi \ge q$.

We prove (iii). We notice first that

$$\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = \infty \quad \text{if and only if} \quad \lim_{t\to 0^+} \frac{t^{1/q}}{\phi^{-1}(t)} = 0. \tag{4.1}$$

Indeed, the first limit is true if for every sequence $\{t_k\}$ with $0 < t_k \to 0$, it holds that $\frac{t_k^q}{\phi(t_k)} \to \infty$. Thus, taking $s_k = \phi(t_k)$ we have that $0 < s_k \to 0$ and $\frac{[\phi^{-1}(s_k)]^q}{s_k} \to \infty$. Since $h(t) = t^{1/q}$ is continuous and converges to ∞ as $t \to \infty$, it follows that $\frac{\phi^{-1}(s_k)}{s_k^{1/q}} \to \infty$, which is equivalent to $\frac{s_k^{1/q}}{\phi^{-1}(s_k)} \to 0$. Since $0 \le \frac{t^{1/q}}{\phi^{-1}(t)}$ for all $t > 0$ it follows that $\lim_{t\to 0^+} \frac{t^{1/q}}{\phi^{-1}(t)} = 0$. Now, from (4.1) and item (i) we deduce that $\alpha_{\phi^{-1}} \le 1/q$, and recalling Remark 4.1 (v) we get that $\beta_\phi \ge q$, and (iii) holds. Analogously, (iv) follows from (ii), taking into account that

$$\lim_{t\to \infty} \frac{t^q}{\phi(t)} = \infty \quad \text{if and only if} \quad \lim_{t\to \infty} \frac{t^{1/q}}{\phi^{-1}(t)} = 0,$$

and using again Remark 4.1 (v). □

Lemma 4.6. *Let $\phi : [0,\infty) \to [0,\infty)$ be a nonbounded, increasing, continuous function with $\phi(0) = 0$.*

(i) If $q < \alpha_\phi$ *then* $\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = \infty$.

(ii) If $q > \beta_\phi$ *then* $\lim_{t\to \infty} \frac{t^q}{\phi(t)} = \infty$.

(iii) If $q < \alpha_\phi$ *then* $\lim_{t\to \infty} \frac{t^q}{\phi(t)} = 0$.

(iv) If $q > \beta_\phi$ then $\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = 0$.

Let us note that the reciprocals of items (i) and (ii) of the above lemma *are not true*, see Example (e.1) below.

Proof. Let us begin by proving (i). Let $\varepsilon > 0$ such that $\alpha_\phi - \varepsilon > q$. By Remark 4.1 (i) there exists $t_1 > 0$ such that $\phi(tx) \leq t^{\alpha_\phi - \varepsilon}\phi(x)$ for all $x > 0$ and $t < t_1$. Taking $x = 1$ we get that $\frac{1}{t^{\alpha_\phi-\varepsilon}} \leq \frac{\phi(1)}{\phi(t)}$ for $t < t_1$. Multiplying by t^q on both sides and taking limit as $t \to 0^+$ it follows that

$$\lim_{t\to 0^+} \frac{t^q}{t^{\alpha_\phi - \varepsilon}} \leq \lim_{t\to 0^+} \frac{\phi(1)t^q}{\phi(t)}.$$

Since $q < \alpha_\phi - \varepsilon$, the first limit is infinite, and so also the second one. Thus, (i) is proved.

Analogously, let $\varepsilon > 0$ such that $\beta_\phi + \varepsilon < q$. By Remark 4.1 (ii) there exists $t_1 > 0$ such that $\phi(tx) \leq t^{\beta_\phi - \varepsilon}\phi(x)$ for all $x > 0$ and $t > t_1$. Taking $x = 1$ we have $\frac{1}{t^{\beta_\phi-\varepsilon}} \leq \frac{\phi(1)}{\phi(t)}$ for $t < t_1$. Multiplying by t^q on both sides and taking limit as $t \to \infty$ we get

$$\lim_{t\to \infty} \frac{t^q}{t^{\beta_\phi + \varepsilon}} \leq \lim_{t\to \infty} \frac{\phi(1)t^q}{\phi(t)}.$$

Since $q > \beta_\phi + \varepsilon$, the first limit is infinite, and thus also the second one.

On other hand, (iii) follows from (ii) noting that

$$\lim_{t\to\infty} \frac{t^q}{\phi(t)} = 0 \quad \text{if and only if} \quad \lim_{t\to\infty} \frac{t^{1/q}}{\phi^{-1}(t)} = \infty,$$

and taking into account that $\alpha_\phi > q$ if and only if $\beta_{\phi^{-1}} < 1/q$. Similarly, (iv) follows from (i) noting that

$$\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = 0 \quad \text{if and only if} \quad \lim_{t\to 0^+} \frac{t^{1/q}}{\phi^{-1}(t)} = \infty,$$

and recalling that $\beta_\phi < q$ if and only if $\alpha_{\phi^{-1}} > 1/q$. □

Corollary 4.7. *Let q, r_1 and r_2 be given by (F), (G1) and (G2) respectively.*

1. *Suppose that α_ϕ is positive.*

   (a) If $q < \alpha_\phi$ then the limit in (F) holds.

2. *Suppose that β_ϕ is finite.*

   (a) If $r_1 > \beta_\phi$ then the limit in (G1) holds.
   (b) If $r_2 > \beta_\phi$ then the limit in (G2) holds.

4.1 Examples

Let us conclude the article with some examples of functions ϕ. We suppose $x \geq 0$ and we extend the function oddly.

a. Let

$$\phi(x) = x^{p_1} + x^{p_2}, \quad \text{with } p_1 \geq p_2 > 0.$$

Since $\phi(x)/x^{p_1}$ is nonincreasing and $\phi(x)/x^{p_2}$ is nondecreasing, we see that $\beta_\phi < \infty$ and $\alpha_\phi > 0$.

b. Let
$$\phi(x) = \frac{x^{p_1}}{1+x^{p_2}}, \quad \text{with } p_1 > p_2 > 0.$$
Since $\phi(x)/x^{p_1}$ is nonincreasing and $\phi(x)/x^{p_1-p_2}$ is nondecreasing, we get that $\beta_\phi < \infty$ and $\alpha_\phi > 0$.

c. Let
$$\phi(x) = x\left(|\ln x| + 1\right).$$
We have that $\phi(x)/x^2$ is nonincreasing. Then, $\beta_\phi < \infty$. Furthermore, given $p \in (0,1)$ there exists $T > 0$ such that
$$\phi(tx) \leq t^p \phi(x) \quad \text{for } t \in [0,T] \text{ and all } x \geq 0.$$
This inequality implies that $\alpha_\phi \geq 1$.

d. Let
$$\phi(x) := x - \ln(x+1).$$
As in the above example, $\phi(x)/x^2$ is nonincreasing and then $\beta_\phi < \infty$. Also, there exist $C, T > 0$ such that
$$\phi(tx) \leq \; Ct\phi(x) \quad \text{for } t \in [0,T] \text{ and all } x \geq 0.$$
The above inequality implies that $\alpha_\phi \geq 1$. Moreover, since
$$\lim_{t\to\infty} \frac{t^q}{\phi(t)} = \infty \quad \text{for all } q > 1,$$
thanks to Lemma 4.5 (iv) we deduce that $\alpha_\phi = 1$.

e. Let $h : (0,\infty) \to (1,\infty)$ be an increasing differentiable function such that $\lim_{t\to 0^+} h(t) = 1$,
$$\lim_{t\to\infty} \frac{qt^{q-1}h(t)}{h'(t)} = \infty \quad \text{for all } q > 0, \tag{4.2}$$
and there exists $p_1 > 0$ such that
$$\lim_{t\to 0^+} \frac{qt^{q-1}h(t)}{h'(t)} = \begin{cases} 0 & \text{if } q > p_1, \\ \infty & \text{if } q < p_1. \end{cases} \tag{4.3}$$
Define
$$\phi(x) := (\ln(h(x)))^p, \quad \text{with } p > 0.$$
By (4.2), ϕ satisfies the limit in (G2). Moreover, from Lemma 4.5 (iv) we can deduce that $\alpha_\phi = 0$. Then ϕ *does not satisfy* the hypothesis (Φ) (and (Φ')) at the introduction. And since (4.3) holds it follows that
$$\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = \begin{cases} 0 & \text{if } q > pp_1. \\ \infty & \text{if } q < pp_1. \end{cases}$$
Therefore, ϕ satisfies the limits in (F) and (G1). Let us exhibit next a few particular cases.

e.1 Let

$$\phi(x) := (\ln(x+1))^p, \quad \text{with } p > 0.$$

A few computations show that $h(x) = x + 1$ satisfies (4.2) and (4.3). Moreover, we can see that $\phi(x)/x^p$ is nonincreasing and thus $\beta_\phi \leq p$, and since

$$\lim_{t \to 0^+} \frac{t^q}{\ln(t+1)} = \infty \quad \text{for all } q < 1,$$

by Lemma 4.5 it follows that $\beta_\phi = p$. This shows that the reciprocals of the items (i) and (ii) in Lemma 4.6 *are not true*.

e.2 Let

$$\phi(x) := \operatorname{arcsinh}(x) = \ln\left(\sqrt{x^2+1} + x\right).$$

One can see that $h(x) = \sqrt{x^2+1} + x$ satisfies (4.2) and (4.3).

e.3 Let

$$\phi(x) := \ln(\ln(x+1) + 1).$$

One can verify that $h(x) = \ln(x+1) + 1$ satisfies (4.2) and (4.3).

Acknowledgements

This work was supported in part by Secyt-UNC 33620180100016CB.

References

[1] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122**(1994), 519–543. https://doi.org/10.1006/jfan.1994.1078; MR1276168; Zbl 0805.35028

[2] A. Ambrosetti, J. Garcia Azorero, I. Peral, Multiplicity results for some nonlinear elliptic equations, *J. Funct. Anal.* **137**(1996), 219–242. https://doi.org/10.1006/jfan.1996.0045; MR1383017; Zbl 0852.35045

[3] H. Dang, S. Oppenheimer, Existence and uniqueness results for some nonlinear boundary problems, *J. Math. Anal. Appl.* **198**(1996), 35–48. https://doi.org/10.1006/jmaa.1996.0066; MR1373525; Zbl 0855.34021

[4] D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Notes and Reports in Mathematics in Science and Engineering, Vol. 5, Academic Press, Inc., Boston, MA, 1988. Zbl 0661.47045

[5] J. Gustavsson, J. Peetre, Interpolation of Orlicz spaces, *Studia Math.* **60**(1977), 33–59. https://doi.org/10.4064/sm-60-1-33-59; MR438102; Zbl 0353.46019

[6] U. Kaufmann, L. Milne, Positive solutions for nonlinear problems involving the one-dimensional ϕ-Laplacian, *J. Math. Anal. Appl.* **461**(2018), 24–37. https://doi.org/10.1016/j.jmaa.2017.12.063; MR3759527; Zbl 1456.34021

[7] U. Kaufmann, L. Milne, Positive solutions of generalized nonlinear logistic equations via sub-super solutions, *J. Math. Anal. Appl.* **471**(2019), 653–670. https://doi.org/10.1016/j.jmaa.2018.11.001; MR3906345; Zbl 1404.35190

[8] Y.-H. Lee, X. Xu, Existence and multiplicity results for generalized Laplacian problems with a parameter, *Bull. Malays. Math. Sci. Soc.* **43**(2020), 403–424. https://doi.org/10.1007/s40840-018-0691-0; MR4044894; Zbl 1491.34039

[9] L. Maligranda, *Orlicz spaces and interpolation*, Seminários de Matemática, Vol. 5, CUniversidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989. MR2264389, Zbl 0874.46022

[10] W. Matuszewska, W. Orlicz, On certain properties of Φ functions, *Bull. Acad. Polon. Sci.* **8**(1960), 439–443. Zbl 0101.09001

[11] N. Papageorgiou, G. Smyrlis, Positive solutions for parametric p-Laplacian equations, *Commun. Pure Appl. Anal.* **15**(2016), 1545–1570. https://doi.org/10.3934/cpaa.2016002; MR3538869; Zbl 1351.35035

[12] N. Papageorgiou, P. Winkert, Positive solutions for nonlinear nonhomogeneous Dirichlet problems with concave-convex nonlinearities, *Positivity* **20**(2016), 945–979. https://doi.org/10.1007/s11117-015-0395-8; MR3568178; Zbl 1359.3507

[13] I. Rachůnková, S. Staněk, M. Tvrdý, *Solvability of nonlinear singular problems for ordinary differential equations*, Contemporary Mathematics and its Applications, Vol. 5, Hindawi Publishing Corporation, New York, 2008. MR2572243; Zbl 1228.34003

[14] J. Sánchez, P. Ubilla, One-dimensional elliptic equation with concave and convex nonlinearities, *Electron. J. Differential Equations* **2000**, No. 50, 1–9. MR1772735; Zbl 0955.34013

[15] H. Wang, On the number of positive solutions of nonlinar systems, *J. Math. Anal. Appl.* **281**(2003), 287–306. https://doi.org/10.1016/S0022-247X(03)00100-8; MR1980092; Zbl 1036.34032

[16] H. Wang, On the structure of positive radial solutions for quasilinear equations in annular domains, *Adv. Differential Equations* **8**(2003), 111–128. MR1946560; Zbl 1042.34052