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Structures and evolution of bifurcation diagrams for a multiparameter p-Laplacian Dirichlet problem

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Abstract. We study the multiparameter p-Laplacian Dirichlet problem

$$\begin{cases} \left(\varphi_p(u'(x))\right)' + \lambda(ku^{p-1} + \sum_{i=1}^m a_i u^{q_i}) - \mu \sum_{j=1}^n b_j u^{r_j} = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$, $\left(\varphi_p(u')\right)'$ is the one-dimensional p-Laplacian, $\lambda > 0$ and $\mu \geq 0$ are two bifurcation parameters. We assume that $k \geq 0$, $0 < p-1 < q_1 < q_2 < \cdots < q_m < r_1 < r_2 < \cdots < r_n$, $m, n \geq 1$, $a_1 = 1$, $a_i > 0$ for $i = 1, 2, \ldots, m$ and $b_1 = 1$, $b_j > 0$ for $j = 1, 2, \ldots, n$. We mainly prove that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram consists of a strictly decreasing curve for $\mu = 0$, and always consists of a $\subset$-shaped curve for fixed $\mu > 0$. We then study the structures and evolution of the bifurcation diagrams with varying $\mu \geq 0$.

Keywords: bifurcation diagram, evolution, positive solution, p-Laplacian, $\subset$-shaped bifurcation curve, time map.

2020 Mathematics Subject Classification: 34B18, 74G35.

1 Introduction

In this paper we study the structures and evolution of bifurcation diagrams for the multiparameter p-Laplacian Dirichlet problem

$$\begin{cases} \left(\varphi_p(u'(x))\right)' + \lambda(ku^{p-1} + \sum_{i=1}^m a_i u^{q_i}) - \mu \sum_{j=1}^n b_j u^{r_j} = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$, $\left(\varphi_p(u')\right)'$ is the one-dimensional p-Laplacian, and $\lambda > 0$ and $\mu \geq 0$ are two bifurcation parameters. We assume that the nonlinearity

$$f_{k,\mu,\lambda}(u) \equiv \lambda(ku^{p-1} + \sum_{i=1}^m a_i u^{q_i}) - \mu \sum_{j=1}^n b_j u^{r_j} \tag{1.2}$$

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is a generalized polynomial (see [9]) satisfying

$$\begin{cases} k \geq 0,\ 0 < p-1 < q_1 < q_2 < \cdots < q_m < r_1 < r_2 < \cdots < r_n,\ m,n \geq 1, \\ a_1 = 1,\ a_i > 0 \text{ for } i = 1,2,\ldots,m \text{ and } b_1 = 1,\ b_j > 0 \text{ for } j = 1,2,\ldots,n. \end{cases} \tag{1.3}$$

This problem arises in the study of non-Newtonian fluids, nonlinear diffusion problems, and population dynamics of one species. The quantity p is a characteristic of the medium. Media with $1 < p < 2$ are called pseudoplastics fluids and those with $p > 2$ are called dilatant. If $p = 2$, they are Newtonian fluids (see, e.g., Díaz [3,4] and their bibliographies). In population dynamics, in (1.1), the one-dimensional p-Laplacian operator $\left(\varphi_p(u')\right)'$ acts as the diffusive mechanism describing the migration of u throughout the habitat $(-1,1)$ which is assumed to be surrounded by a completely hostile boundary $\{\pm 1\}$. In (1.1), the reaction term $\lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}) - \mu \sum_{j=1}^{n} b_j u^{r_j}$ is the growth rate of the population, which consists of a source term $\lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i})$ and an absorption term $\mu \sum_{j=1}^{n} b_j u^{r_j}$. Note that, by (1.3), if $\mu > 0$, the absorption term $\mu \sum_{j=1}^{n} b_j u^{r_j}$ is dominated by the source term when u near 0^+ and dominate the source term when u is large enough, and the domination of the absorption term over the source term is assumed to be strictly increasing on $(0,\infty)$. Murray [11] suggested using diffusion of the form p in the study of diffusion-kinetic enzymes problems. By a positive solution to p-Laplacian problem (1.1) with general $p > 1$, we mean a positive function $u \in C^1[-1,1]$ with $\varphi_p(u') \in C^1[-1,1]$ satisfying (1.1). Let $Z = \{x \in [-1,1] : u'(x) = 0\}$. We note that it is easy to show that, if u is a positive solution of (1.1), then $u \in C^2[-1,1]$ if $1 < p \leq 2$ and $u \in C^2([-1,1] \setminus Z)$ if $p > 2$. For the proof we refer to [1, Lemma 6].

To study bifurcation diagrams of positive solutions of (1.1), (1.3), it is important to study the shape of nonlinearity $f_{k,\mu,\lambda}(u)$ on $(0,\infty)$ in the beginning. We show that there exist three positive numbers $\beta_{\mu,\lambda} > \zeta_{\mu,\lambda} > \gamma_{\mu,\lambda}$ such that $f_{k,\mu,\lambda}(u)$ with $\lambda, \mu > 0$ satisfies (1.4), (1.9), and (1.11) stated behind. That is, positive numbers $\beta_{\mu,\lambda} > \zeta_{\mu,\lambda} > \gamma_{\mu,\lambda}$ are the unique positive zero, critical point, and p-inflection point of $f_{k,\mu,\lambda}(u)$ on $(0,\infty)$, respectively. First, we easily observe that, for $f_{k,\mu,\lambda}(u)$ with $\lambda, \mu > 0$ satisfying (1.3), the number of sign changes in the sequence of coefficients for the *generalized polynomial* $f_{k,\mu,\lambda}(u)$

$$(\lambda k, \lambda a_1, \lambda a_2, \ldots, \lambda a_m, -\mu b_1, -\mu b_2, \ldots, -\mu b_n)$$

is 1. Applying Laguerre's Theorem [10] (see also [9, Theorem 4.7]) on the number of positive zeros to the generalized polynomial $f_{k,\mu,\lambda}(u)$, we obtain that there exists a unique positive number $\beta_{\mu,\lambda}$ such that

$$\begin{cases} f_{k,\mu,\lambda}(u) > 0 \quad \text{on } (0, \beta_{\mu,\lambda}), \\ f_{k,\mu,\lambda}(0) = f_{k,\mu,\lambda}(\beta_{\mu,\lambda}) = 0, \\ f_{k,\mu,\lambda}(u) < 0 \quad \text{on } (\beta_{\mu,\lambda}, \infty). \end{cases} \tag{1.4}$$

We set $\beta_{\mu=0,\lambda} = \infty$ if $\mu = 0$. Notice that, by (1.3), it is easy to see that, for fixed $\lambda > 0$,

$$\lim_{\mu \to \infty} \beta_{\mu,\lambda} = 0. \tag{1.5}$$

In addition,

$$\text{for fixed } \mu > 0,\ \beta_{\mu,\lambda} \text{ is a continuous, strictly increasing function of } \lambda \text{ on } (0,\infty) \tag{1.6}$$

and

$$\text{for fixed } \lambda > 0,\ \beta_{\mu,\lambda} \text{ is a continuous, strictly decreasing function of } \mu \text{ on } (0,\infty). \tag{1.7}$$

Secondly, we compute that

$$f'_{k,\mu,\lambda}(u) = \lambda \left[(p-1)ku^{p-2} + \sum_{i=1}^{m} a_i q_i u^{q_i - 1} \right] - \mu \sum_{j=1}^{n} b_j r_j u^{r_j - 1}. \tag{1.8}$$

Thus again, similarly, applying (1.3) and Laguerre's Theorem [10] on the number of positive zeros to the *generalized polynomial* $f'_{k,\mu,\lambda}(u)$ in (1.8), we obtain that there exists a unique positive number $\zeta_{\mu,\lambda} < \beta_{\mu,\lambda}$ such that

$$\begin{cases} f'_{k,\mu,\lambda}(u) > 0 \text{ on } (0, \zeta_{\mu,\lambda}), \\ f'_{k,\mu,\lambda}(u)(\zeta_{\mu,\lambda}) = 0, \\ f'_{k,\mu,\lambda}(u) < 0 \text{ on } (\zeta_{\mu,\lambda}, \beta_{\mu,\lambda}). \end{cases} \tag{1.9}$$

So $f_{k,\mu,\lambda}(u)$ with $\lambda, \mu > 0$ is increasing-decreasing on $(0, \beta_{\mu,\lambda})$. Thirdly, we compute that

$$(p-2)f'_{k,\mu,\lambda}(u) - uf''_{k,\mu,\lambda}(u) = \lambda \sum_{i=1}^{m} a_i q_i (p-1-q_i) u^{q_i-1} - \mu \sum_{j=1}^{n} b_j r_j (p-1-r_j) u^{r_j-1}, \tag{1.10}$$

in which $p-1-q_i < 0$ for $i = 1, 2, \ldots, m$ and $p-1-r_j < 0$ for $j = 1, 2, \ldots, n$. Thus again, applying (1.3) and Laguerre's Theorem [10] on the number of positive zeros to the *generalized polynomial* $(p-2)f'_{k,\mu,\lambda}(u) - uf''_{k,\mu,\lambda}(u)$ in (1.10), we obtain that there exists a unique positive number $\gamma_{\mu,\lambda} < \zeta_{\mu,\lambda}$ such that

$$\begin{cases} (p-2)f'_{k,\mu,\lambda}(u) - uf''_{k,\mu,\lambda}(u) < 0 \text{ on } (0, \gamma_{\mu,\lambda}), \\ (p-2)f'_{k,\mu,\lambda}(\gamma_{\mu,\lambda}) - uf''_{k,\mu,\lambda}(\gamma_{\mu,\lambda}) = 0, \\ (p-2)f'_{k,\mu,\lambda}(u) - uf''_{k,\mu,\lambda}(u) > 0 \text{ on } (\gamma_{\mu,\lambda}, \beta_{\mu,\lambda}). \end{cases} \tag{1.11}$$

In this case $f_{k,\mu,\lambda}(u)$ with $\lambda, \mu > 0$ is said to be p-convex-concave on $(0, \beta_{\mu,\lambda})$.

Note that, in (1.1), $\lambda k u^{p-1}$ is the p-linear term for generalized polynomial nonlinearity $f_{k,\mu,\lambda}$ if bifurcation parameter $k > 0$. If $k = 0$, then $f_{k,\mu,\lambda}$ has no p-linear term. In this paper we are concerned only with positive solutions u of (1.1), (1.3) satisfying

$$0 < \|u\|_\infty < \beta_{\mu,\lambda} \begin{cases} = \infty & \text{if } \mu = 0, \\ < \infty & \text{if } \mu > 0. \end{cases} \tag{1.12}$$

Positive solutions u of (1.1), (1.3) satisfying (1.12) are called *classical* positive solutions. Note that positive solutions u of (1.1), (1.3) satisfying $\|u\|_\infty = \beta_{\mu,\lambda}$ are called *flat-core* positive solutions.

For problem (1.1), (1.3), we study evolutionary bifurcation diagrams $S_{p,k,\mu}$ on the $(\lambda, \|u\|_\infty)$-plane defined by:

$$S_{p,k,\mu} = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a (classical) positive solution of (1.1), (1.3)}\}, \ \mu \geq 0. \tag{1.13}$$

First, when $\mu = 0$ and $f_{k,\mu=0,\lambda}(u) \equiv \lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i})$, we study $S_{p,k,\mu=0}$ on the $(\lambda, \|u\|_\infty)$-plane in the next proposition. We let

$$\bar{\lambda} \equiv \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \begin{cases} < \infty & \text{if } k > 0, \\ = \infty & \text{if } k = 0. \end{cases} \tag{1.14}$$

Proposition 1.1 (See Figs. 2.1–2.2 depicted behind)**.** *Let $p > 1$. Consider p-Laplacian problem (1.1), (1.3) with $\mu = 0$ and $f_{k,\mu=0,\lambda}(u) = \lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}) > 0$ on $(0, \infty)$. Then the bifurcation diagram $S_{p,k,\mu=0}$ satisfies the following assertions (i)–(ii):*

(i) On the $(\lambda, \|u\|_\infty)$-plane, $S_{p,k,\mu=0}$ emanates from the positive $\|u\|_\infty$-axis as $\lambda \to 0^+$, tends to the point $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ if $k > 0$ and tends to the positive λ-axis as $\lambda \to \infty$ if $k = 0$, and consists of a continuous, strictly decreasing curve.

(ii) Moreover, if $k = 0$, $m = 1$, $q \equiv q_1 > p - 1$ and $f_{k=0,\mu=0,\lambda}(u) \equiv \lambda u^q > 0$ on $(0, \infty)$, then

$$S_{p,k=0,\mu=0} = \left\{(\lambda, \|u_\lambda\|_\infty) = (c_{p,q}\alpha^{p-q-1}, \alpha),\ \alpha = \|u_\lambda\|_\infty > 0\right\},$$

where

$$c_{p,q} \equiv \left(\frac{p-1}{p}\right)(q+1)^{1-p}\left[\frac{\Gamma(\frac{p-1}{p})\Gamma(\frac{1}{q+1})}{\Gamma(\frac{pq+2p-q-1}{p(q+1)})}\right]^p > 0, \tag{1.15}$$

and $\Gamma(t) \equiv \int_0^\infty x^{t-1}e^{-x}dx$ is the usual gamma function.

Proof. (I) We prove part (i). To study $S_{p,k,\mu=0}$ for p-Laplacian problem (1.1), (1.3) with $\mu = 0$, we apply the time-map method for which the time-map formula takes the form as follows:

$$\lambda^{1/p} = \left(\frac{p-1}{p}\right)^{1/p}\int_0^\alpha \frac{1}{[\bar{F}(\alpha) - \bar{F}(u)]^{1/p}}du \equiv T_{\bar{f}}(\alpha) \quad \text{for } \alpha = \|u\|_\infty > 0, \tag{1.16}$$

where

$$\bar{f}(u) \equiv f_{k,\mu=0,\lambda=1}(u) = ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}$$

and $\bar{F}(u) \equiv \int_0^u \bar{f}(t)dt$; see, e.g., [2, Lemmas 2.1 and 2.2] for the derivation of the time map formula $T(\alpha)$ in (1.16). We have that positive solution $u_\lambda(x)$ of p-Laplacian problem (1.1), (1.3) with $\mu = 0$ corresponds to $\|u_\lambda\|_\infty = \alpha > 0$ satisfying (1.16), e.g., [13, p. 382]. It is easy to compute that, by (1.3),

$$\lim_{u\to 0^+}\frac{\bar{f}(u)}{u^{p-1}} = \frac{ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}}{u^{p-1}} = k \geq 0, \qquad \lim_{u\to\infty}\frac{\bar{f}(u)}{u^{p-1}} = \frac{ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}}{u^{p-1}} = \infty,$$

and

$$(p-1)\bar{f}(u) - u\bar{f}'(u) = \sum_{i=1}^{m} a_i(p-1-q_i)u^{q_i} < 0 \quad \text{on } (0, \infty).$$

Thus, by [13, (1.7), (1.9) and (4.4)], we have that $\lim_{\alpha\to 0^+} T_{\bar{f}}(\alpha) = \left(\frac{p-1}{k}\right)^{1/p}\frac{\pi}{p}\csc\frac{\pi}{p} \in (0, \infty]$, $\lim_{\alpha\to\infty} T_{\bar{f}}(\alpha) = 0$, and $T_{\bar{f}}(\alpha)$ is a strictly decreasing function on $(0, \infty)$. So part (i) directly follows from (1.13) and (1.16).

(II) We prove part (ii). We have that $\bar{f}(u) = u^q$, $q > p - 1 > 0$ and $\bar{F}(u) \equiv \int_0^u \bar{f}(t)dt = \frac{1}{q+1}u^{q+1}$. It can be computed that

$$\begin{aligned} T_{\bar{f}}(\alpha) &= \left(\frac{p-1}{p}\right)^{1/p}\int_0^\alpha \frac{1}{[\bar{F}(\alpha) - \bar{F}(u)]^{1/p}}du \\ &= \left(\frac{p-1}{p}\right)^{1/p}(q+1)^{1/p}\int_0^\alpha \frac{1}{[\alpha^{q+1} - u^{q+1}]^{1/p}}du \\ &= \left(\frac{p-1}{p}\right)^{1/p}(q+1)^{(1-p)/p}\left[\frac{\Gamma(\frac{p-1}{p})\Gamma(\frac{1}{q+1})}{\Gamma(\frac{pq+2p-q-1}{p(q+1)})}\right]\alpha^{\frac{p-q-1}{p}} \end{aligned}$$

by [6, p. 212, formula 855.42] or using symbolic manipulator *Mathematica 11.0*. Thus by (1.15) and (1.16), we obtain that

$$\begin{aligned}\lambda &= \left[T_{\bar{f}}(\alpha)\right]^p = \left(\frac{p-1}{p}\right)(q+1)^{1-p}\left[\frac{\Gamma(\frac{p-1}{p})\Gamma(\frac{1}{q+1})}{\Gamma(\frac{pq+2p-q-1}{p(q+1)})}\right]^p \alpha^{p-q-1} \\ &= c_{p,q}\alpha^{p-q-1}. \end{aligned} \tag{1.17}$$

So part (ii) holds.

The proof of Proposition 1.1 is now complete. □

2 Main results

The main results in this paper are next Theorem 2.1 and Theorem 2.2 for problem (1.1), (1.3) with $1 < p \leq 2$ and $p > 2$, respectively. In Theorems 2.1–2.2 with any fixed $\mu > 0$, we prove that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram $S_{p,k,\mu}$ always consists of a continuous, $\subset$-shaped curve with exactly one (right) turning point at some point $(\lambda^*, \|u_{\lambda^*}\|_\infty)$. While the upper branch of each $\subset$-shaped bifurcation diagram $S_{p,k,\mu}$ is *unbounded* if $1 < p \leq 2$ and is *bounded* if $p > 2$. We then study the structures and evolution of bifurcation diagrams $S_{p,k,\mu}$ with varying $\mu \geq 0$; see Fig. 2.1 with $1 < p \leq 2$ and Fig. 2.2 with $p > 2$. Theorem 2.1 and Theorem 2.2 substantially improve [14, Corollary 2.2] and [14, Corollary 2.4], respectively. Cf. [14, Corollary 2.2] with $1 < p \leq 2$ and [14, Corollary 2.4] with $p > 2$ for details. Also see Remark 3.2 stated behind.

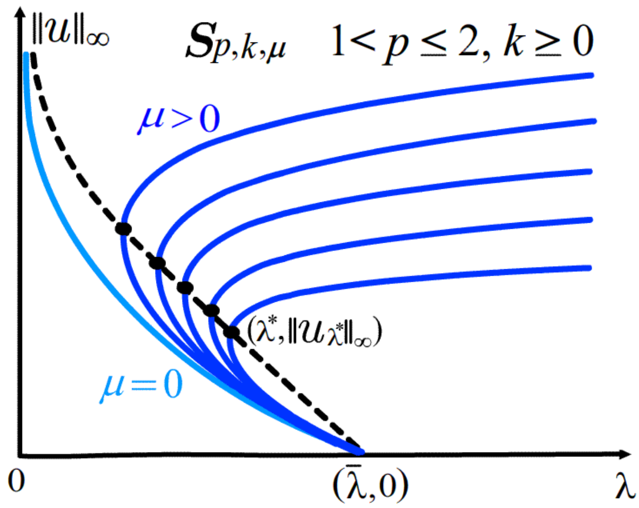

Figure 2.1: Evolutionary bifurcation diagrams $S_{p,k,\mu}$ for (1.1), (1.3) with fixed $p \in (1,2]$, $k \geq 0$ and varying $\mu \geq 0$.

Theorem 2.1 (See Fig. 2.1)**.** *Let $1 < p \leq 2$ and $k \geq 0$. Consider p-Laplacian problem (1.1), (1.3) with varying $\mu \geq 0$. Then the bifurcation diagram $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane and the following assertions (i)–(v) hold:*

(i) *For $\mu = 0$, $S_{p,k,\mu=0}$ emanates from the positive $\|u\|_\infty$-axis as $\lambda \to 0^+$, tends to the point $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ if $k > 0$ and tends to the positive λ-axis as $\lambda \to \infty$ if $k = 0$, and consists of a strictly decreasing curve.*

(ii) *For any fixed* $\mu > 0$, $S_{p,k,\mu}$ *always starts at the point* $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ *if* $k > 0$ *and emanates from the positive* λ*-axis as* $\lambda \to \infty$ *if* $k = 0$ *(that is,* $(\bar{\lambda}, 0) = (\infty, 0)$ *if* $k = 0$*).* $S_{p,k,\mu}$ *is a* $\subset$*-shaped curve with exactly one turning point at some point* $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ *satisfying*

$$0 < \lambda^* < \bar{\lambda} \text{ and } 0 < \|u_{\lambda^*}\|_\infty < \beta_{\mu,\lambda^*}.$$

In addition, the upper branch of $S_{p,k,\mu}$ *tends to infinity when* $\lambda \to \infty$*. Thus, (1.1), (1.3) has exactly two (classical) positive solutions for* $\lambda^* < \lambda < \bar{\lambda}$*, exactly one (classical) positive solution for* $\lambda = \lambda^*$ *and* $\lambda \geq \bar{\lambda}$*, and no (classical) positive solution for* $0 < \lambda < \lambda^*$*.*

(iii) *For any nonnegative* $\mu_1 < \mu_2$, S_{p,k,μ_2} *lies on the right hand side of* S_{p,k,μ_1}*. (So* S_{p,k,μ_1} *and* S_{p,k,μ_2} *do not intersect.)*

(iv) *For the turning points* $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ *of* $S_{p,k,\mu}$ *with* $\mu > 0$, λ^* *is a continuous, strictly increasing function of* $\mu > 0$, $\|u_{\lambda^*}\|_\infty$ *is a continuous function of* $\mu > 0$,

$$\lim_{\mu \to 0^+} (\lambda^*, \|u_{\lambda^*}\|_\infty) = (0, \infty) \text{ and } \lim_{\mu \to \infty} (\lambda^*, \|u_{\lambda^*}\|_\infty) = (\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right).$$

In particular, when $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, *and* $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$*, then* $\|u_{\lambda^*}\|_\infty$ *is a strictly decreasing function of* $\mu > 0$*.*

(v) *When* $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, *and* $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$*, then all points* $(\lambda, \|u_\lambda\|_\infty) \in S_{p,k=0,\mu}$ *satisfy*

$$0 < \left(\frac{c_{p,q}}{\lambda}\right)^{\frac{1}{q-p+1}} < \|u_\lambda\|_\infty < \beta_{\mu,\lambda} = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{r-q}}, \tag{2.1}$$

where $c_{p,q}$ *is defined in (1.15).*

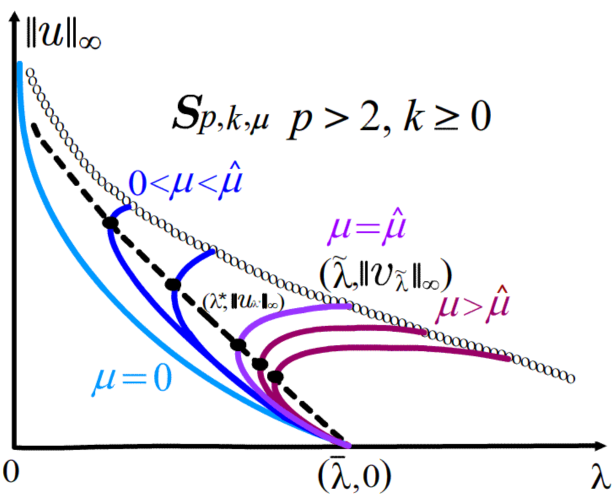

Figure 2.2: Evolutionary bifurcation diagrams $S_{p,k,\mu}$ for (1.1), (1.3) with fixed $p > 2$, $k \geq 0$ and varying $\mu \geq 0$.

Theorem 2.2 (See Fig. 2.2)**.** *Let* $p > 2$ *and* $k \geq 0$*. Consider one-dimensional* p*-Laplacian problem (1.1), (1.3) with varying* $\mu \geq 0$*. Then the bifurcation diagram* $S_{p,k,\mu}$ *consists of a continuous curve on the* $(\lambda, \|u\|_\infty)$*-plane and the following assertions (i)–(vi) hold:*

(i) *For* $\mu = 0$, $S_{p,k,\mu=0}$ *emanates from the positive* $\|u\|_\infty$*-axis as* $\lambda \to 0^+$, *tends to the point* $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ *if* $k > 0$ *and tends to the positive* λ*-axis as* $\lambda \to \infty$ *if* $k = 0$, *and consists of a strictly decreasing curve.*

(ii) *For any fixed* $\mu > 0$, $S_{p,k,\mu}$ *starts at the same point* $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ *if* $k > 0$ *and emanates from the positive* $\|u\|_\infty$*-axis as* $\lambda \to 0^+$ *if* $k = 0$ *(that is,* $(\bar{\lambda}, 0) = (\infty, 0)$ *if* $k = 0$*), ends at some point* $(\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty)$ *satisfying* $0 < \tilde{\lambda} < \infty$ *and* $0 < \|v_{\tilde{\lambda}}\|_\infty = v_{\tilde{\lambda}}(0) = \beta_{\mu,\tilde{\lambda}}$ *satisfying* $f_{k,\mu,\tilde{\lambda}}(\beta_{\mu,\tilde{\lambda}}) = 0$ *(that is,* $v_{\tilde{\lambda}}(x) \equiv \lim_{\lambda \to \tilde{\lambda}^-} v_\lambda(x)$ *is a* flat-core *positive solution of problem* (1.1), (1.3), *see part (a) stated below for* (classical) *positive solutions* $v_\lambda(x)$ *with* $\lambda^* < \lambda < \tilde{\lambda}$*). Moreover,* $S_{p,k,\mu}$ *is a* $\subset$*-shaped curve with exactly one turning point at some point* $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ *satisfying*

$$0 < \lambda^* < \min(\bar{\lambda}, \tilde{\lambda}) \quad \textit{and} \quad 0 < \|u_{\lambda^*}\|_\infty < \|v_{\tilde{\lambda}}\|_\infty = \beta_{\mu,\tilde{\lambda}}.$$

Moreover, there exists a unique positive $\hat{\mu} = \hat{\mu}(p, k, q_i, r_j, a_i, b_j) < \infty$ *if* $k > 0$ *and* $\hat{\mu} = \infty$ *if* $k = 0$ *such that:*

(a) *If* $0 < \mu < \hat{\mu}$, *then* $(\lambda^* <)\ \tilde{\lambda} < \bar{\lambda}$ *such that problem* (1.1), (1.3) *has exactly two (classical) positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *satisfying* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda^* < \lambda < \tilde{\lambda}$, *exactly one (classical) positive solution* u_λ *satisfying* $\|u_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda = \lambda^*$ *and* $\tilde{\lambda} \le \lambda < \bar{\lambda}$, *and no (classical) positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \ge \bar{\lambda}$. *In addition,* $\lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \tilde{\lambda}^-} \|v_\lambda\|_\infty = \|v_{\tilde{\lambda}}\|_\infty = \beta_{\mu,\tilde{\lambda}}$.

(b) *If* $\mu = \hat{\mu}$, *then* $(\lambda^* <)\ \tilde{\lambda} = \bar{\lambda}$ *such that problem* (1.1), (1.3) *has exactly two (classical) positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *satisfying* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda^* < \lambda < \bar{\lambda}$, *exactly one (classical) positive solution* u_λ *satisfying* $\|u_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda = \lambda^*$, *and no (classical) positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \ge \bar{\lambda}$. *In addition,* $\lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \tilde{\lambda}^-} \|v_\lambda\|_\infty = \|v_{\tilde{\lambda}}\|_\infty = \beta_{\mu,\tilde{\lambda}}$.

(c) *If* $\mu > \hat{\mu}$, *then* $(\lambda^* <)\ \bar{\lambda} < \tilde{\lambda}$ *such that problem* (1.1), (1.3) *has exactly two (classical) positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *satisfying* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda^* < \lambda < \bar{\lambda}$, *exactly one (classical) positive solution* u_λ *satisfying* $\|u_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda = \lambda^*$ *and exactly one (classical) positive solution* v_λ *satisfying* $\|v_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\bar{\lambda} \le \lambda < \tilde{\lambda}$, *and no (classical) positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \ge \tilde{\lambda}$. *In addition,* $\lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \tilde{\lambda}^-} \|v_\lambda\|_\infty = \|v_{\tilde{\lambda}}\|_\infty = \beta_{\mu,\tilde{\lambda}}$.

(iii) *For any nonnegative* $\mu_1 < \mu_2$, S_{p,k,μ_2} *lies on the right hand side of* S_{p,k,μ_1}. *(So* S_{p,k,μ_1} *and* S_{p,k,μ_2} *do not intersect.)*

(iv) *For the ending points* $(\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty)$ *of* $S_{p,k,\mu}$ *with* $\mu > 0$, $\tilde{\lambda}$ *is a continuous, strictly increasing function of* $\mu > 0$, $\|v_{\tilde{\lambda}}\|_\infty$ *is a continuous, strictly decreasing function of* $\mu > 0$,

$$\lim_{\mu \to 0^+} (\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty) = (0, \infty) \ \textit{and} \ \lim_{\mu \to \infty} (\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty) = (\infty, 0). \tag{2.2}$$

(v) *For the turning points* $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ *of* $S_{p,k,\mu}$ *with* $\mu > 0$, λ^* *is a continuous, strictly increasing function of* $\mu > 0$, $\|u_{\lambda^*}\|_\infty$ *is a continuous function of* $\mu > 0$,

$$\lim_{\mu \to 0^+} (\lambda^*, \|u_{\lambda^*}\|_\infty) = (0, \infty) \ \textit{and} \ \lim_{\mu \to \infty} (\lambda^*, \|u_{\lambda^*}\|_\infty) = (\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right).$$

(vi) *When* $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, *and* $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, *then all points* $(\lambda, \|u_\lambda\|_\infty) \in S_{p,k=0,\mu}$ *satisfy* (2.1).

Remark 2.3 (See Fig. 2.2)**.** By Theorem 2.2, for fixed $p \in (2, \infty)$ and $k > 0$, it is easy to see that, when $\mu \to \infty$, $S_{p,k,\mu}$ converges to the half-line $[\bar{\lambda}, \infty)$ on the positive λ-axis.

3 Lemmas

To prove Theorems 2.1–2.2 for p-Laplacian problem (1.1), (1.3), we need the following Lemmas 3.1 and 3.3–3.12. In particular, Theorems 2.1–2.2 is based on Lemma 3.1 which is due to Wang and Yeh [14]. Wang and Yeh [14] considered the p-Laplacian Dirichlet problem with one parameter λ:

$$\begin{cases} (\varphi_p(u'(x)))' + f_\lambda(u(x)) = 0, \ -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{3.1}$$

They assumed that the nonlinearity

$$f_\lambda(u) \equiv \lambda g(u) - h(u), \tag{3.2}$$

where functions $g, h \in C[0, \infty) \cap C^2(0, \infty)$ satisfy hypotheses (H1)–(H4) if $1 < p \leq 2$ and satisfy hypotheses (H1)–(H5) if $p > 2$:

(H1) $g(0) = h(0) = 0$, $g(u), h(u) > 0$ on $(0, \infty)$, and

$$0 = \lim_{u \to 0^+} \frac{h(u)}{u^{p-1}} \leq m_0^g \equiv \lim_{u \to 0^+} \frac{g(u)}{u^{p-1}} < \infty.$$

(H2) The positive function $h(u)/g(u)$ is strictly increasing on $(0, \infty)$, and

$$\lim_{u \to 0^+} \frac{h(u)}{g(u)} = 0, \ \ \lim_{u \to \infty} \frac{h(u)}{g(u)} = \infty.$$

(H3) $(p-2)g'(u) - ug''(u) < 0$ on $(0, \infty)$ and $(p-2)h'(u) - uh''(u) < 0$ on $(0, \infty)$.

(H4) The positive function $[(p-2)h'(u) - uh''(u)] \,/\, [(p-2)g'(u) - ug''(u)]$ is strictly increasing on $(0, \infty)$, and

$$\lim_{u \to 0^+} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = 0, \qquad \lim_{u \to \infty} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \infty.$$

(H5) There exists a positive number $p^* > p - 1$ such that $g(u)/u^{p^*}$ is strictly decreasing on $(0, \infty)$ and $h(u)/u^{p^*}$ is strictly increasing on $(0, \infty)$. In addition, for each fixed $s \in (0, 1)$,

$$\frac{h(su)}{u^{p-1}} \left(\frac{h(u)g(su)}{g(u)h(su)} - 1 \right)$$

is a strictly increasing function of u on $(0, \infty)$, and

$$\lim_{u \to \infty} \frac{h(u)g(su)}{g(u)h(su)} \in (1, \infty].$$

Notice that, for p-Laplacian problem (3.1), hypotheses (H1)–(H2) imply that, for each fixed $\lambda > 0$, there exists a unique positive number β_λ such that

$$\begin{cases} f_\lambda(u) = \lambda g(u) - h(u) > 0 \text{ on } (0, \beta_\lambda), \\ f_\lambda(0) = \lambda g(0) - h(0) = 0 \text{ and } f_\lambda(\beta_\lambda) = \lambda g(\beta_\lambda) - h(\beta_\lambda) = 0, \\ f_\lambda(u) = \lambda g(u) - h(u) < 0 \text{ on } (\beta_\lambda, \infty). \end{cases}$$

Moreover, the number β_λ is a continuous, strictly increasing function of λ on $(0, \infty)$, $\lim_{\lambda \to 0^+} \beta_\lambda = 0$ and $\lim_{\lambda \to \infty} \beta_\lambda = \infty$. See [14, (1.4)–(1.5)]. Also, hypotheses (H1)–(H4) imply that, for each fixed $\lambda > 0$, the function $f_\lambda(u)$ with $\lambda > 0$ is p-convex-concave on $(0, \beta_\lambda)$. More precisely, there exists a unique positive number $\gamma_\lambda < \beta_\lambda$ such that

$$\begin{cases} (p-2) f_\lambda'(u) - u f_\lambda''(u) < 0 \text{ on } (0, \gamma_\lambda), \\ (p-2) f_\lambda'(\gamma_\lambda) - \gamma_\lambda f_\lambda''(\gamma_\lambda) = 0, \\ (p-2) f_\lambda'(u) - u f_\lambda''(u) > 0 \text{ on } (\gamma_\lambda, \beta_\lambda). \end{cases}$$

See [14, (1.6)]. In [14], Wang and Yeh are concerned only with positive solutions u of (3.1) satisfying $0 < \|u\|_\infty < \beta_\lambda$. Let

$$\hat{\lambda} \equiv \left(\frac{p-1}{m_0^g} \right) \left(\frac{\pi}{p} \csc \frac{\pi}{p} \right)^p \begin{cases} < \infty & \text{if } m_0^g > 0, \\ = \infty & \text{if } m_0^g = 0. \end{cases} \tag{3.3}$$

For $f_\lambda(u) = \lambda g(u) - h(u)$ in (3.2), we define $F_\lambda(u) = \int_0^u f_\lambda(t) dt$ and

$$T_\lambda(\alpha) = (\frac{p-1}{p})^{1/p} \int_0^\alpha [F_\lambda(\alpha) - F_\lambda(u)]^{-1/p} du \quad \text{for } 0 < \alpha < \beta_\lambda.$$

Lemma 3.1. *Consider p-Laplacian problem* (3.1) *with* $p > 1$. *Then the following assertions (i)–(ii) hold:*

(i) ([14, Theorem 2.1 and Fig. 1]) Let $1 < p \leq 2$. *If* $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$ *satisfy (H1)–(H4). Then the bifurcation diagram consists of a continuous, ⊂-shaped curve on the* $(\lambda, \|u\|_\infty)$*-plane. More precisely, the there exists a positive number* $\lambda^* < \hat{\lambda}$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \hat{\lambda}$, *exactly one positive solution* v_λ *for* $\lambda = \lambda^*$ *and* $\lambda \geq \hat{\lambda}$, *and no positive solution for* $0 < \lambda < \lambda^*$. *Moreover,* $\lim_{\lambda \to \hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \infty} \|v_\lambda\|_\infty = \infty$.

(ii) ([14, Theorem 2.3 and Fig. 3]) Let $p > 2$. *If* $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$ *satisfy (H1)–(H5). Then the bifurcation diagram consists of a continuous, ⊂-shaped curve on the* $(\lambda, \|u\|_\infty)$*-plane. More precisely, there exist three positive numbers* $\lambda^* < \tilde{\lambda}$ *and* $\beta_{\tilde{\lambda}}$ *satisfying* $\lambda^* < \hat{\lambda}$ $(\leq \infty)$ *and* $f_{\tilde{\lambda}}(\beta_{\tilde{\lambda}}) = 0$ *and* $\lim_{\alpha \to \beta_{\tilde{\lambda}}^-} T_{\tilde{\lambda}}(\alpha) = 1$ *such that:*

(a) (See [14, Fig. 3(a)–(b)]) If $\tilde{\lambda} < \hat{\lambda}$ $(\leq \infty)$, *then* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \tilde{\lambda}$, *exactly one positive solution* u_λ *for* $\lambda = \lambda^*$ *and* $\tilde{\lambda} \leq \lambda < \hat{\lambda}$, *and no positive solution for* $0 < \lambda < \lambda^*$ *and for* $\lambda \geq \hat{\lambda}$ *(if* $\hat{\lambda} < \infty$*).*

(b) (See [14, Fig.3(c)–(d)]) If $\hat{\lambda} \leq \tilde{\lambda}$, *then* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \hat{\lambda}$, *exactly one positive solution* v_λ *for* $\lambda = \lambda^*$ *and for* $\hat{\lambda} \leq \lambda < \tilde{\lambda}$ *(if* $\tilde{\lambda} > \hat{\lambda}$*), and no positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \geq \tilde{\lambda}$. *Moreover,* $\lim_{\lambda \to \hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \tilde{\lambda}^-} \|v_\lambda\|_\infty = \beta_{\tilde{\lambda}}$.

Remark 3.2. To Lemma 3.1(i)–(ii), Wang and Yeh [14, Corollaries 2.2 and 2.4] gave examples of generalized polynomial nonlinearities for

$$f_\lambda(u) = \lambda g(u) - h(u) = \lambda(ku^{p-1} + u^q) - u^r$$

satisfying $r > q > p-1 > 0$ and $k \geq 0$, which is a special case of

$$f_{k,\mu,\lambda}(u) = \lambda \left(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i} \right) - \mu \sum_{j=1}^{n} b_j u^{r_j}$$

defined in (1.2) satisfying (1.3).

For p-Laplacian problem (1.1), (1.3) with two parameters μ and λ and $f_{k,\mu,\lambda}(u)$ defined in (1.2), we define the time map formula as follows:

$$T_{\mu,\lambda}(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \frac{du}{\left[F_{k,\mu,\lambda}(\alpha) - F_{k,\mu,\lambda}(u)\right]^{1/p}} \quad \text{for } 0 < \alpha < \beta_{\mu,\lambda}, \tag{3.4}$$

where $\beta_{\mu,\lambda}$ is defined in (1.4) and

$$F_{k,\mu,\lambda}(u) = \int_0^u f_{k,\mu,\lambda}(t)dt. \tag{3.5}$$

We define $f_{k,\mu,\lambda}(u) = \lambda g(u) - \mu\tilde{h}(u)$ where $g(u) = ku^{p-1} + \sum_{i=1}^m a_i u^{q_i}$, $\tilde{h}(u) = \sum_{j=1}^n b_j u^{r_j}$, $G(u) = \int_0^u g(t)dt$ and $\tilde{H}(u) = \int_0^u \tilde{h}(t)dt$.

We suppose that $u_{\mu,\lambda}(x)$ is a (classical) positive solution of p-Laplacian problem (1.1), (1.3) satisfying (1.12). Then (classical) positive solution $u_{\mu,\lambda}(x)$ corresponds to $\|u_{\mu,\lambda}\|_\infty = \alpha$ and

$$T_{\mu,\lambda}(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \frac{du}{\left[F_{k,\mu,\lambda}(\alpha) - F_{k,\mu,\lambda}(u)\right]^{1/p}} = 1 \quad \text{for } 0 < \alpha < \beta_{\mu,\lambda}. \tag{3.6}$$

See, e.g., [8, (3.9)].

Recall the number $\bar{\lambda} = \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p} \csc \frac{\pi}{p}\right)^p$ defined in (1.14).

Lemma 3.3. *Consider p-Laplacian problem* (1.1), (1.3) *with* $p > 1$, $\lambda > 0$ *and* $\mu > 0$. *Then the following assertions (i)–(ii) hold:*

(i) $\lim_{\alpha \to 0^+} T_{\mu,\lambda}(\alpha) = (\frac{\bar{\lambda}}{\lambda})^{1/p}$ *and* $T_{\mu,\lambda}(\alpha)$ *has exactly an critical point at some* $\alpha^*_{\mu,\lambda}$, *a minimum, on* $(0, \beta_{\mu,\lambda})$. *Moreover,*

$$\lim_{\alpha \to \beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = \infty \quad \textit{if } 1 \leq p < 2.$$

(ii) There exist two positive numbers $C < D < \beta_{\mu,\lambda}$ *such that* $C < \alpha^*_{\mu,\lambda} < D$ *where* $C = C(k,\mu,\lambda)$, $D = D(k,\mu,\lambda)$ *satisfy*

$$(p-1)f_{k,\mu,\lambda(\mu)}(C) - Cf'_{k,\mu,\lambda(\mu)}(C) = 0 \quad \textit{and} \quad pF_{k,\mu,\lambda(\mu)}(D) - Df_{k,\mu,\lambda(\mu)}(D) = 0, \tag{3.7}$$

respectively. Cf. [14, *(3.10) and (3.11)*]. *Then* $T'_{\mu,\lambda}(\alpha) < 0$ *for* $\alpha \in (0, C]$ *and* $T'_{\mu,\lambda}(\alpha) > 0$ *for* $\alpha \in [D, \beta_{\mu,\lambda})$.

Proof. Parts (i) and (ii) simply follow by (1.4), (1.11), and slight modification of the proofs of [14, Lemmas 3.1 and 3.2]. We omit the detailed proofs here. □

We show comparison results for $T_{\mu,\lambda}(\alpha)$ in the next lemma; cf. [7, Lemma 3.3(i)–(ii)]. Notice that, for any fixed $\mu > 0$ and $0 < \lambda_1 < \lambda_2$, $\beta_{\mu,\lambda_1} < \beta_{\mu,\lambda_2}$ by (1.6), and for any fixed $\lambda > 0$ and $0 < \mu_1 < \mu_2$, $\beta_{\mu_2,\lambda} < \beta_{\mu_1,\lambda}$ by (1.7).

Lemma 3.4. *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu > 0$ and $0 < \lambda_1 < \lambda_2$, $T_{\mu,\lambda_1}(\alpha) > T_{\mu,\lambda_2}(\alpha)$ for $0 < \alpha < \beta_{\mu,\lambda_1}$.

(ii) For any fixed $\lambda > 0$ and $0 < \mu_1 < \mu_2$, $T_{\mu_1,\lambda}(\alpha) < T_{\mu_2,\lambda}(\alpha)$ for $0 < \alpha < \beta_{\mu_2,\lambda}$.

Proof. The proofs of parts (i)–(ii) follow by modification of those of [7, Lemma 3.3(i)–(ii)]. We omit them here. □

Lemma 3.5. *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu \geq 0$ and $0 < \lambda_1 < \lambda_2$, $T_{\mu,\lambda}(\alpha)$ is a continuous function of $\lambda \in [\lambda_1, \lambda_2]$ for $0 < \alpha < \beta_{\mu,\lambda_1}$.

(ii) For any fixed $\lambda > 0$ and $0 \leq \mu_1 < \mu_2$, $T_{\mu,\lambda}(\alpha)$ is a continuous function of $\mu \in [\mu_1, \mu_2]$ for $0 < \alpha < \beta_{\mu_2,\lambda}$.

Proof. The proofs of parts (i)–(ii) follow by modification of those of [7, Lemma 3.4(i)–(ii)]. We omit them here. □

By Lemma 3.3(i), $T_{\mu,\lambda}(\alpha)$ has exactly one critical point at some $\alpha^*_{\mu,\lambda}$, a minimum, on $(0, \beta_{\mu,\lambda})$. Let

$$m(\mu,\lambda) \equiv T_{\mu,\lambda}(\alpha^*_{\mu,\lambda}) = \min_{\alpha \in (0,\beta_{\mu,\lambda})} T_{\mu,\lambda}(\alpha).$$

Lemma 3.6. *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu \in (0,\infty)$, there exists a unique $\lambda^* > 0$ such that $m(\mu,\lambda^*) = 1$.

(ii) For any fixed $\lambda \in (0,\bar{\lambda})$, there exists a unique $\mu^* > 0$ such that $m(\mu^*,\lambda) = 1$.

Moreover, for any fixed $\lambda \geq \bar{\lambda}$ and $\mu > 0$, $m(\mu,\lambda) < 1$.

Proof. (I) We prove part (ii). We have that $\lim_{\alpha\to\infty} T_{\mu=0,\lambda}(\alpha) = 0$, which follows from Proposition 1.1(i) and since $\lim_{\mu\to 0^+} T_{\mu,\lambda}(\alpha) = T_{\mu=0,\lambda}(\alpha)$. So we can find a number $\mu_1 > 0$ such that $m(\mu_1,\lambda) = \min_{\alpha\in(0,\beta_{\mu_1,\lambda})} T_{\mu_1,\lambda}(\alpha) < 1$. In addition, we have that $\lim_{\alpha\to 0^+} T_{\mu,\lambda}(\alpha) > 1$, which follows from Lemma 3.3(i) for $0 < \lambda < \bar{\lambda}$. By (3.7), we compute and obtain that

$$\frac{\lambda}{\mu} = \frac{\sum_{j=1}^{n} b_j (r_j - p + 1) C^{r_j}}{\sum_{i=1}^{m} a_i (q_i - p + 1) C^{q_i}} = \frac{\sum_{j=1}^{n} b_j \frac{r_j - p + 1}{r_j + 1} D^{r_j}}{\sum_{i=1}^{m} a_i \frac{q_i - p + 1}{q_i + 1} D^{q_i}}.$$

So, for fixed $\lambda \in (0,\bar{\lambda})$, we have that $\lim_{\mu\to\infty} \alpha^*_{\mu,\lambda} = 0$ since $\lim_{\mu\to\infty} C = \lim_{\mu\to\infty} D = 0$ and $\alpha^*_{\mu,\lambda} \in (C, D)$ by Lemma 3.3(ii). Hence we can find a number $\mu_2 > 0$ such that $m(\mu_2,\lambda) = \min_{\alpha\in(0,\beta_{\mu_2,\lambda})} T_{\mu_2,\lambda}(\alpha) > 1$.

Next, we set positive numbers $\alpha_1 \equiv \inf_{\mu\in[\mu_1,\mu_2]} \alpha^*_{\mu,\lambda}$ and $\alpha_2 \equiv \sup_{\mu\in[\mu_1,\mu_2]} \alpha^*_{\mu,\lambda} \geq \alpha_1$. If $\alpha_1 = \alpha_2$, then $\alpha^*_{\mu,\lambda} = \alpha_1 = \alpha_2$ for all $\mu \in [\mu_1, \mu_2]$. Thus $T_{\mu_2,\lambda}(\alpha_1) = m(\mu_2, \lambda) > 1$ and $T_{\mu_1,\lambda}(\alpha_1) = m(\mu_1, \lambda) < 1$. So, by Lemma 3.5(ii) and the Intermediate Value Theorem, there exists $\mu^* \in (\mu_1, \mu_2)$ such that

$$m(\mu^*, \lambda) = T_{\mu^*,\lambda}(\alpha_1) = 1.$$

By Lemma 3.4(ii), $m(\mu, \lambda) = T_{\mu,\lambda}(\alpha_1)$ is strictly increasing in $\mu \in [\mu_1, \mu_2]$, and hence μ^* is unique.

While if $\alpha_1 < \alpha_2$, we first show that $m(\mu, \lambda)$ is a continuous function of μ on $[\mu_1, \mu_2]$ as follows. By Lemma 3.4(ii) and Lemma 3.5(ii), for each $\mu_1 < \mu_2$ and fixed $\alpha \in (0, \beta_{\mu_2,\lambda})$, $T_{\mu,\lambda}(\alpha)$ is a continuous, strictly increasing function of μ on (μ_1, μ_2). So for any fixed $\check{\mu} \in [\mu_1, \mu_2]$, by the Dini Theorem [12, p. 195], it is easy to see that

$$\lim_{\mu\to\check{\mu}} \left(\min_{\alpha\in[\alpha_1,\alpha_2]} T_{\mu,\lambda}(\alpha) \right) = \min_{\alpha\in[\alpha_1,\alpha_2]} T_{\check{\mu},\lambda}(\alpha). \tag{3.8}$$

Since for any $\mu \in [\mu_1, \mu_2]$, the minimum of $T_{\mu,\lambda}(\alpha)$ occurs at $\alpha^*_{\mu,\lambda} \in [\alpha_1, \alpha_2]$. So

$$m(\mu, \lambda) = \min_{\alpha\in(0,\beta_{\mu,\lambda})} T_{\mu,\lambda}(\alpha) = \min_{\alpha\in[\alpha_1,\alpha_2]} T_{\mu,\lambda}(\alpha) \quad \text{for } \mu \in [\mu_1, \mu_2]. \tag{3.9}$$

By (3.8) and (3.9), $\lim_{\mu\to\check{\mu}} m(\mu, \lambda) = m(\check{\mu}, \lambda)$. Hence $m(\mu, \lambda)$ is a continuous function of μ on $[\mu_1, \mu_2]$. By the Intermediate Value Theorem, there exists $\mu^* \in (\mu_1, \mu_2)$ such that $m(\mu^*, \lambda) = 1$. Moreover, since $m(\mu, \lambda)$ is strictly increasing in $\mu \in [\mu_1, \mu_2]$, we obtain that μ^* is unique.

For any fixed $\lambda \geq \bar{\lambda}$ and $\mu > 0$. By Lemma 3.3, we have $\lim_{\alpha\to0^+} T_{\mu,\lambda}(\alpha) \leq 1$ and $m(\mu, \lambda) < 1$.

By above, part (ii) holds.

(II) The proof of part (i) is similar to that of part (ii). We omit it here.

The proof of Lemma 3.6 is now complete. □

Lemma 3.7 (See Theorems 2.1–2.2 and Figs. 2.1–2.2)**.** *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$. Then, for the turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ of $S_{p,k,\mu}$ with varying $\mu > 0$, λ^* is a continuous, strictly increasing function of $\mu > 0$. Moreover, $\lim_{\mu\to0^+} \lambda^* = 0$ and $\lim_{\mu\to\infty} \lambda^* = \bar{\lambda}$.*

Proof. For fixed $\mu_2 > \mu_1 > 0$, by Lemma 3.6(i), there exists $\lambda^*_2(\mu_2) > 0$ (resp. $\lambda^*_1(\mu_1) > 0$) such that $T_{\mu_2,\lambda^*_2}(\alpha)$ (resp. $T_{\mu_1,\lambda^*_1}(\alpha)$) has exactly one minimum point $\alpha_{\mu_2,\lambda^*_2} \in (0, \beta_{\mu_2,\lambda^*_2})$ (resp. $\alpha_{\mu_1,\lambda^*_1} \in (0, \beta_{\mu_1,\lambda^*_1})$) satisfying $T_{\mu_2,\lambda^*_2}(\alpha_{\mu_2,\lambda^*_2}) = 1$ (resp. $T_{\mu_1,\lambda^*_1}(\alpha_{\mu_1,\lambda^*_1}) = 1$). Observe that $\alpha_{\mu_2,\lambda^*_2} \in (0, \beta_{\mu_2,\lambda^*_2}) \subsetneq (0, \beta_{\mu_1,\lambda^*_2})$. So we obtain that

$$\begin{aligned} T_{\mu_1,\lambda^*_2}(\alpha_{\mu_2,\lambda^*_2}) &= \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\alpha_{\mu_2,\lambda^*_2}} [\lambda^*_2(G(\alpha_{\mu_2,\lambda^*_2}) - G(u)) - \mu_1(\tilde{H}(\alpha_{\mu_2,\lambda^*_2}) - \tilde{H}(u))]^{-1/p} du \\ &< \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\alpha_{\mu_2,\lambda^*_2}} [\lambda^*_2(G(\alpha_{\mu_2,\lambda^*_2}) - G(u)) - \mu_2(\tilde{H}(\alpha_{\mu_2,\lambda^*_2}) - \tilde{H}(u))]^{-1/p} du \\ &= T_{\mu_2,\lambda^*_2}(\alpha_{\mu_2,\lambda^*_2}) \\ &= 1. \end{aligned}$$

So

$$\min_{\alpha\in(0,\beta_{\mu_1,\lambda^*_2})} T_{\mu_1,\lambda^*_2}(\alpha) \leq T_{\mu_1,\lambda^*_2}(\alpha_{\mu_2,\lambda^*_2}) < 1. \tag{3.10}$$

For any $\alpha \in (0, \beta_{\mu_1,\lambda_1^*})$, we have that

$$T_{\mu_1,\lambda_1^*}(\alpha) \geq \min_{\alpha\in(0,\beta_{\mu_1,\lambda_1^*})} T_{\mu_1,\lambda_1^*}(\alpha) = T_{\mu_1,\lambda_1^*}(\alpha_{\mu_1,\lambda_1^*}) = 1. \tag{3.11}$$

By (3.10), (3.11) and Lemma 3.4(i), we obtain $\lambda_1^*(\mu_1) < \lambda_2^*(\mu_2)$. By Lemma 3.6, we obtain $\lambda^*((0,\infty)) = (0,\bar{\lambda})$. Hence $\lambda^*(\mu) : (0,\infty) \to (0,\bar{\lambda})$ is a continuous, strictly increasing function. Moreover, $\lim_{\mu\to 0^+} \lambda^*(\mu) = 0$ and $\lim_{\mu\to\infty} \lambda^*(\mu) = \bar{\lambda}$.

The proof of Lemma 3.7 is now complete. □

Lemma 3.8 (See Theorems 2.1–2.2 and Figs. 2.1–2.2)**.** *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$ and $k \geq 0$. Then, for the turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ of $S_{p,k,\mu}$ with varying $\mu > 0$, $\frac{\lambda^*(\mu)}{\mu}$ is a continuous, strictly decreasing function of $\mu > 0$. Moreover, $\lim_{\mu\to 0^+} \frac{\lambda^*(\mu)}{\mu} = \infty$ and $\lim_{\mu\to\infty} \frac{\lambda^*(\mu)}{\mu} = 0$.*

Proof. For fixed $\mu_2 > \mu_1 > 0$, we let $N_1 \equiv \frac{\lambda^*(\mu_1)}{\mu_1}$ and $N_2 \equiv \frac{\lambda^*(\mu_2)}{\mu_2}$. Then

$$\begin{aligned}
&T_{\mu_2,\lambda^*(\mu_2)}(\|u_{\lambda^*(\mu_2)}\|_\infty) = 1 = T_{\mu_1,\lambda^*(\mu_1)}(\|u_{\lambda^*(\mu_1)}\|_\infty)\\
&\quad = T_{\mu_1,N_1\mu_1}(\|u_{\lambda^*(\mu_1)}\|_\infty)\\
&\quad = \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\|u_{\lambda^*(\mu_1)}\|_\infty} [N_1\mu_1(G(\|u_{\lambda^*(\mu_1)}\|_\infty) - G(u)) - \mu_1(\tilde{H}(\|u_{\lambda^*(\mu_1)}\|_\infty) - \tilde{H}(u))]^{-1/p} du\\
&\quad = \left(\frac{\mu_2}{\mu_1}\right)^{1/p} \left(\frac{p-1}{p}\right)^{1/p}\\
&\qquad \times \int_0^{\|u_{\lambda^*(\mu_1)}\|_\infty} [N_1\mu_2(G(\|u_{\lambda^*(\mu_1)}\|_\infty) - G(u)) - \mu_2(\tilde{H}(\|u_{\lambda^*(\mu_1)}\|_\infty) - \tilde{H}(u))]^{-1/p} du\\
&\quad = \left(\frac{\mu_2}{\mu_1}\right)^{1/p} T_{\mu_2,N_1\mu_2}(\|u_{\lambda^*(\mu_1)}\|_\infty)\\
&\quad > T_{\mu_2,N_1\mu_2}(\|u_{\lambda^*(\mu_1)}\|_\infty)
\end{aligned}$$

since $\mu_2 > \mu_1 > 0$. If $\lambda^*(\mu_2) \geq N_1\mu_2$, then $T_{\mu_2,N_1\mu_2}(\|u_{\lambda^*(\mu_1)}\|_\infty) \geq T_{\mu_2,\lambda^*(\mu_2)}(\|u_{\lambda^*(\mu_1)}\|_\infty) \geq T_{\mu_2,\lambda^*(\mu_2)}(\|u_{\lambda^*(\mu_2)}\|_\infty)$ by Lemma 3.4(i) and Lemma 3.3(i), which leads to a contradiction since $T_{\mu_2,\lambda^*(\mu_2)}(\|u_{\lambda^*(\mu_2)}\|_\infty) > T_{\mu_2,N_1\mu_2}(\|u_{\lambda^*(\mu_1)}\|_\infty)$. Hence we obtain $\lambda^*(\mu_2) < N_1\mu_2$. So

$$N_2 = \frac{\lambda^*(\mu_2)}{\mu_2} < N_1 = \frac{\lambda^*(\mu_1)}{\mu_1}.$$

By Lemma 3.7, we obtain that $\lambda^*(\mu)$ is a continuous function of $\mu > 0$. Hence $\frac{\lambda^*(\mu)}{\mu}$ is a continuous, strictly decreasing function of $\mu > 0$.

We prove $\lim_{\mu\to 0^+} \frac{\lambda^*(\mu)}{\mu} = \infty$ by method of contradiction. Suppose $\lim_{\mu\to 0^+} \frac{\lambda^*(\mu)}{\mu} < \infty$, then there exists a positive $N_3 > \lim_{\mu\to 0^+} \frac{\lambda^*(\mu)}{\mu}$ such that

$$\begin{aligned}
&\lim_{\mu\to 0^+} \left(\frac{1}{\mu}\right)^{1/p} T_{\mu=1,\lambda=N_3}\left(\|u_{\lambda^*(\mu)}\|_\infty\right)\\
&\quad = \lim_{\mu\to 0^+} \left(\frac{1}{\mu}\right)^{1/p} \left(\frac{p-1}{p}\right)^{1/p}\\
&\qquad \times \int_0^{\|u_{\lambda^*(\mu)}\|_\infty} [N_3(G(\|u_{\lambda^*(\mu)}\|_\infty) - G(u)) - (\tilde{H}(\|u_{\lambda^*(\mu)}\|_\infty) - \tilde{H}(u))]^{-1/p} du
\end{aligned}$$

$$
\begin{aligned}
&< \lim_{\mu\to 0^+}\left(\frac{1}{\mu}\right)^{1/p}\left(\frac{p-1}{p}\right)^{1/p} \\
&\quad\times \int_0^{\|u_{\lambda^*(\mu)}\|_\infty}\left[\frac{\lambda^*(\mu)}{\mu}(G(\|u_{\lambda^*(\mu)}\|_\infty)-G(u))-(\tilde{H}(\|u_{\lambda^*(\mu)}\|_\infty)-\tilde{H}(u))\right]^{-1/p}du \\
&= \lim_{\mu\to 0^+}\left(\frac{p-1}{p}\right)^{1/p} \\
&\quad\times \int_0^{\|u_{\lambda^*(\mu)}\|_\infty}[\lambda^*(\mu)(G(\|u_{\lambda^*(\mu)}\|_\infty)-G(u))-\mu(\tilde{H}(\|u_{\lambda^*(\mu)}\|_\infty)-\tilde{H}(u))]^{-1/p}du \\
&= \lim_{\mu\to 0^+} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty).
\end{aligned}
$$

By Lemma 3.3(i), there exist two fixed positive numbers α_{1,N_3} and L_{1,N_3} such that $T_{\mu=1,\lambda=N_3}(\alpha_{1,N_3})=L_{1,N_3}$ is an absolute minimum on $(0,\beta_{1,N_3})$. So

$$
1=\lim_{\mu\to 0^+} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty) > \lim_{\mu\to 0^+}\left(\frac{1}{\mu}\right)^{1/p} T_{\mu=1,\lambda=N_3}(\|u_{\lambda^*(\mu)}\|_\infty) \geq \lim_{\mu\to 0^+}\left(\frac{1}{\mu}\right)^{1/p} L_{1,N_3}=\infty,
$$

which leads to a contradiction. Hence $\lim_{\mu\to 0^+}\frac{\lambda^*(\mu)}{\mu}=\infty$.

Similarly, we prove $\lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu}=0$ by method of contradiction. Suppose $\lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu}>0$, then there exists a positive N_4 such that $N_4<\lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu}$. By Lemma 3.3(i), there exist two fixed positive numbers α_{1,N_4} and L_{1,N_4} such that $T_{\mu=1,\lambda=N_4}(\alpha_{1,N_4})=L_{1,N_4}$ is an absolute minimum on $(0,\beta_{1,N_4})$. We then need the next claim.

Claim A. $\lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\alpha_{1,N_4}) \geq \lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty)$.

Proof of Claim A. Since $N_4<\lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu}$, there exists a positive number μ_0 such that, for all $\mu>\mu_0$, $N_4<\frac{\lambda^*(\mu)}{\mu}$. By (1.4) and (1.6), for all $\mu>\mu_0$, we have $\beta_{1,N_4}<\beta_{1,\frac{\lambda^*(\mu)}{\mu}}$. For $\mu>0$, by (1.4), we have $\beta_{1,\frac{\lambda^*(\mu)}{\mu}}=\beta_{\mu,\lambda^*(\mu)}$ since $f_{k,1,\frac{\lambda^*(\mu)}{\mu}}(\beta_{\mu,\lambda^*(\mu)})=\frac{1}{\mu}f_{k,\mu,\lambda^*(\mu)}(\beta_{\mu,\lambda^*(\mu)})=0$. Hence, for all $\mu>\mu_0$, $0<\alpha_{1,N_4}<\beta_{1,N_4}<\beta_{\mu,\lambda^*(\mu)}$. By Lemma 3.3(i), Lemma 3.6(i) and (3.6), for all $\mu>\mu_0$, we have $T_{\mu,\lambda^*(\mu)}(\alpha_{1,N_4})\geq T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty)$. Moreover, $\lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\alpha_{1,N_4})\geq \lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty)$. So Claim A holds.

We thus have that

$$
\begin{aligned}
&\lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p} T_{\mu=1,\lambda=N_4}(\alpha_{1,N_4}) \\
&= \lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p}\left(\frac{p-1}{p}\right)^{1/p}\int_0^{\alpha_{1,N_4}}[N_4(G(\alpha_{1,N_4})-G(u))-(\tilde{H}(\alpha_{1,N_4})-\tilde{H}(u))]^{-1/p}du \\
&> \lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p}\left(\frac{p-1}{p}\right)^{1/p}\int_0^{\alpha_{1,N_4}}\left[\frac{\lambda^*(\mu)}{\mu}(G(\alpha_{1,N_4})-G(u))-(\tilde{H}(\alpha_{1,N_4})-\tilde{H}(u))\right]^{-1/p}du \\
&= \lim_{\mu\to\infty}\left(\frac{p-1}{p}\right)^{1/p}\int_0^{\alpha_{1,N_4}}[\lambda^*(\mu)(G(\alpha_{1,N_4})-G(u))-\mu(\tilde{H}(\alpha_{1,N_4})-\tilde{H}(u))]^{-1/p}du \\
&= \lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\alpha_{1,N_4}) \\
&\geq \lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty) \quad \text{(by Claim A)} \\
&= 1,
\end{aligned}
$$

which leads to a contradiction since

$$\lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p} T_{\mu=1,\lambda=N_4}(\alpha_{1,N_4}) = \lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p} L_{1,N4} = 0.$$

Hence $\lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu} = 0$.

The proof of Lemma 3.8 is now complete. □

Lemma 3.9 (See Theorems 2.1–2.2 and Figs. 2.1–2.2)**.** *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$ and $k \geq 0$. Then, for the turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ of $S_{p,k,\mu}$ with varying $\mu > 0$,*

$$C < \|u_{\lambda^*}\|_\infty < D, \tag{3.12}$$

where $C = C(k,\mu,\lambda^*(\mu))$, $D = D(k,\mu,\lambda^*(\mu))$ satisfy

$$(p-1)f_{k,\mu,\lambda^*(\mu)}(C) - Cf'_{k,\mu,\lambda^*(\mu)}(C) = 0 \quad \text{and} \quad pF_{k,\mu,\lambda^*(\mu)}(D) - Df_{k,\mu,\lambda^*(\mu)}(D) = 0, \tag{3.13}$$

respectively. Moreover, $\lim_{\mu\to0^+}\|u_{\lambda^*}\|_\infty = \infty$ and $\lim_{\mu\to\infty}\|u_{\lambda^*}\|_\infty = 0$.

Proof. By Lemma 3.3(ii) and (3.6), it is easy to see that $C < \|u_{\lambda^*}\|_\infty < D$. Equations in (3.13) follow by Eqs. (3.7) directly. By (3.13), (1.2) and (3.5), we then compute and obtain that

$$\frac{\lambda^*(\mu)}{\mu} = \frac{\sum_{j=1}^n b_j(r_j-p+1)C^{r_j}}{\sum_{i=1}^m a_i(q_i-p+1)C^{q_i}} = \frac{\sum_{j=1}^n b_j\frac{r_j-p+1}{r_j+1}D^{r_j}}{\sum_{i=1}^m a_i\frac{q_i-p+1}{q_i+1}D^{q_i}},$$

in which $r_j - p + 1 > 0$ for $j = 1,2,\ldots,n$ and $q_i - p + 1 > 0$ for $i = 1,2,\ldots,m$. Thus, by applying Lemma 3.8 and (1.3), we have that $\lim_{\mu\to0^+} C = \lim_{\mu\to0^+} D = \infty$ and $\lim_{\mu\to\infty} C = \lim_{\mu\to\infty} D = 0$. Hence $\lim_{\mu\to0^+}\|u_{\lambda^*}\|_\infty = \infty$ and $\lim_{\mu\to\infty}\|u_{\lambda^*}\|_\infty = 0$ by (3.12).

The proof of Lemma 3.9 is now complete. □

Lemma 3.10. *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 2$ and $k \geq 0$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu > 0$, $\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha)$ is a continuous, strictly decreasing function of λ on $(0,\infty)$. Moreover,

$$\lim_{\lambda\to0^+}\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = \infty \quad \text{and} \quad \lim_{\lambda\to\infty}\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = 0.$$

(ii) For any fixed $\lambda > 0$, $\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha)$ is a continuous, strictly increasing function of μ on $(0,\infty)$. Moreover,

$$\lim_{\mu\to0^+}\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = 0 \quad \text{and} \quad \lim_{\mu\to\infty}\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = \infty.$$

Proof. (I) We prove part (i) where

$$g(u) = ku^{p-1} + \sum_{i=1}^m a_i u^{q_i} \quad \text{and} \quad h(u) = \mu\sum_{j=1}^n b_j u^{r_j} \quad (\text{with } m,n \geq 1 \text{ and } \mu > 0)$$

satisfy (1.3). We take $p^* = \frac{q_m + r_1}{2} > p - 1$. Then it is easy to see that $\frac{g(u)}{u^{p^*}}$ is strictly decreasing on $(0, \infty)$ and $\frac{h(u)}{u^{p^*}}$ is strictly increasing on $(0, \infty)$. For each fixed $s \in (0, 1)$,

$$\begin{aligned} \frac{h(su)}{u^{p-1}}\left[\frac{h(u)g(su)}{g(u)h(su)} - 1\right] &= \frac{h(u)g(su) - h(su)g(u)}{u^{p-1}g(u)} \\ &= \frac{\mu \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} a_i b_j (s^{q_i} - s^{r_j}) u^{q_i + r_j} + \mu k u^{p-1} \sum_{j=1}^{n} b_j (s^{p-1} - s^{r_j}) u^{r_j}}{u^{p-1}(k u^{p-1} + \sum_{i=1}^{m} a_i u^{q_i})} \end{aligned}$$

is a strictly increasing function of u on $(0, \infty)$ and $\lim_{u\to\infty} \frac{h(u)g(su)}{g(u)h(su)} = s^{q_m - r_n} \in (1, \infty)$ since $s \in (0, 1)$ and $q_m < r_n$. So g, h satisfy (H5). For $\lambda \in (0, \infty)$, by (3.4), we obtain that

$$\begin{aligned} &\lim_{\alpha \to \beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) \\ &\quad = \lim_{\alpha \to \beta_{\mu,\lambda}^-} \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha f_{k,\mu,\lambda}(t)dt\right]^{-1/p} du \\ &\quad = \lim_{\alpha \to \beta_{\mu,\lambda}^-} \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha \frac{h(\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})} g(t) - h(t)dt\right]^{-1/p} du \\ &\qquad \text{(since } f_{k,\mu,\lambda}(u) = \lambda g(u) - h(u) \text{ and by (1.4))} \\ &\quad = \lim_{\alpha \to \beta_{\mu,\lambda}^-} \left(\frac{p-1}{p}\right)^{1/p} \alpha^{(p-1)/p} \int_0^1 \left[\int_v^1 \frac{h(\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})} g(s\alpha) - h(s\alpha)ds\right]^{-1/p} dv \\ &\qquad \text{(let } u = \alpha v \text{ and } t = \alpha s \text{)} \\ &\quad = \left(\frac{p-1}{p}\right)^{1/p} \beta_{\mu,\lambda}^{(p-1-p^*)/p} \lim_{\alpha \to \beta_{\mu,\lambda}^-} \int_0^1 \left[\int_v^1 s^{p^*} \left(\frac{h(\beta_{\mu,\lambda})g(s\alpha)}{g(\beta_{\mu,\lambda})\,(s\alpha)^{p^*}} - \frac{h(s\alpha)}{(s\alpha)^{p^*}}\right) ds\right]^{-1/p} dv \\ &\quad = \left(\frac{p-1}{p}\right)^{1/p} \beta_{\mu,\lambda}^{(p-1-p^*)/p} \int_0^1 \left[\int_v^1 \lim_{\alpha \to \beta_{\mu,\lambda}^-} s^{p^*} \left(\frac{h(\beta_{\mu,\lambda})g(s\alpha)}{g(\beta_{\mu,\lambda})\,(s\alpha)^{p^*}} - \frac{h(s\alpha)}{(s\alpha)^{p^*}}\right) ds\right]^{-1/p} dv \\ &\qquad \text{(by (H5) and the Monotone Convergence Theorem)} \\ &\quad = \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}} \left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})} - 1\right) ds\right]^{-1/p} dv. \qquad (3.14) \end{aligned}$$

By (H5) and since $\beta_{\mu,\lambda}$ is strictly increasing in $\lambda \in (0, \infty)$, we obtain that $\lim_{\alpha \to \beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha)$ is a strictly decreasing function of λ on $(0, \infty)$.

For any number $\lambda_0 \in (0,\infty)$, by (3.14), we obtain that

$$\begin{aligned}
\lim_{\lambda\to\lambda_0}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) &= \lim_{\lambda\to\lambda_0}\left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p}dv\\
&= \left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \lim_{\lambda\to\lambda_0}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p}dv\\
&\qquad \text{(by (H5) and the Monotone Convergence Theorem)}\\
&= \left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \frac{h(s\beta_{\mu,\lambda_0})}{\beta_{\mu,\lambda_0}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda_0})g(s\beta_{\mu,\lambda_0})}{g(\beta_{\mu,\lambda_0})h(s\beta_{\mu,\lambda_0})}-1\right)ds\right]^{-1/p}dv\\
&= \lim_{\alpha\to\beta_{\mu,\lambda_0}^-} T_{\mu,\lambda_0}(\alpha).
\end{aligned}$$

So we obtain that $\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha)$ is a continuous function of λ on $(0,\infty)$.

Finally, we prove $\lim_{\lambda\to 0^+}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) = \infty$ and $\lim_{\lambda\to\infty}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) = 0$. By (3.14), we obtain that

$$\begin{aligned}
\lim_{\lambda\to 0^+}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) &= \lim_{\lambda\to 0^+}\left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p}dv\\
&= \left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \lim_{\lambda\to 0^+}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p}dv
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\lambda\to \infty}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) &= \lim_{\lambda\to \infty}\left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p}dv\\
&= \left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \lim_{\lambda\to \infty}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p}dv
\end{aligned}$$

by (H5) and the Monotone Convergence Theorem. For each fixed $s \in (0,1)$, we have that, by (H5),

$$\lim_{\lambda\to 0^+}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right) = \lim_{\beta_{\mu,\lambda}\to 0^+}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right) = 0$$

and

$$\lim_{\lambda\to \infty}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right) = \lim_{\beta_{\mu,\lambda}\to \infty}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right) = \infty.$$

So $\lim_{\lambda\to 0^+}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) = \infty$ and $\lim_{\lambda\to\infty}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) = 0$.

(II) The proof of part (ii) is similar to that of part (i). We omit it here.

The proof of Lemma 3.10 is now complete. □

Lemma 3.11. *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 2$ and $k \geq 0$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu \in (0, \infty)$, there exists a unique $\tilde{\lambda} > 0$ such that $\lim_{\alpha \to \beta^-_{\mu,\tilde{\lambda}}} T_{\mu,\tilde{\lambda}}(\alpha) = 1$.

(ii) For any fixed $\lambda \in (0, \infty)$, there exists a unique $\tilde{\mu} > 0$ such that $\lim_{\alpha \to \beta^-_{\tilde{\mu},\lambda}} T_{\tilde{\mu},\lambda}(\alpha) = 1$.

Proof. (I) We prove part (i). For any fixed $\mu > 0$, by Lemma 3.10(i) and the Intermediate Value Theorem, there exists a unique $\tilde{\lambda} > 0$ such that $\lim_{\alpha \to \beta^-_{\mu,\tilde{\lambda}}} T_{\mu,\tilde{\lambda}}(\alpha) = 1$.

(II) We prove part (ii). For any fixed $\lambda > 0$, by Lemma 3.10(ii) and the Intermediate Value Theorem, there exists a unique $\tilde{\mu} > 0$ such that $\lim_{\alpha \to \beta^-_{\tilde{\mu},\lambda}} T_{\tilde{\mu},\lambda}(\alpha) = 1$.

The proof of Lemma 3.11 is now complete. □

Lemma 3.12 (See Theorem 2.2 and Fig. 2.2)**.** *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 2$ and $k \geq 0$. Then, for the ending points $(\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty)$ of $S_{p,k,\mu}$ with varying $\mu > 0$, $\tilde{\lambda}$ is a continuous, strictly increasing function of $\mu > 0$. Moreover, $\lim_{\mu \to 0^+} \tilde{\lambda} = 0$ and $\lim_{\mu \to \infty} \tilde{\lambda} = \infty$.*

Proof. For fixed $\mu_2 > \mu_1 > 0$, by Lemma 3.10(ii), $\lim_{\alpha \to \beta^-_{\mu_2 \tilde{\lambda}_1}} T_{\mu_2,\tilde{\lambda}_1}(\alpha) > \lim_{\alpha \to \beta^-_{\mu_1,\tilde{\lambda}_1}} T_{\mu_1,\tilde{\lambda}_1}(\alpha) = 1$. If $\tilde{\lambda}_1 \geq \tilde{\lambda}_2$, then by Lemma 3.10(i), $\lim_{\alpha \to \beta^-_{\mu_2,\tilde{\lambda}_1}} T_{\mu_2,\tilde{\lambda}_1}(\alpha) \leq \lim_{\alpha \to \beta^-_{\mu_2,\tilde{\lambda}_2}} T_{\mu_2,\tilde{\lambda}_2}(\alpha) = 1$, which leads to a contradiction. So we obtain $\tilde{\lambda}_1 < \tilde{\lambda}_2$. By Lemma 3.11, we obtain $\tilde{\lambda}((0, \infty)) = (0, \infty)$. Hence $\tilde{\lambda}(\mu) : (0, \infty) \to (0, \infty)$ is a continuous, strictly increasing function of $\mu > 0$. Moreover, $\lim_{\mu \to 0^+} \tilde{\lambda} = 0$ and $\lim_{\mu \to \infty} \tilde{\lambda} = \infty$.

The proof of Lemma 3.12 is now complete. □

4 Proofs of main results

Proof of Theorem 2.1. Let $1 < p \leq 2$ and $k \geq 0$.

(I)(a) We prove that, for any fixed $\mu > 0$, the bifurcation diagram $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane. For any fixed $\mu > 0$ and $\lambda > 0$, it is easy to see that $T_{\mu,\lambda}(\alpha)$ defined in (3.4) is a continuous function of $\alpha \in (0, \beta_{\mu,\lambda})$. By Lemma 3.4(i) and Lemma 3.6(i), we have that, for any fixed $\mu > 0$, the set $\{\alpha \in (0, \beta_{\mu,\lambda}) : T_{\mu,\lambda}(\alpha) = 1 \text{ for all } \lambda > 0\}$ is connected. Thus, by Lemma 3.5(i), for any fixed $\mu > 0$, $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane.

(I)(b) Part (i) follows from Proposition 1.1(i).

(II) We prove part (ii) where nonlinearities

$$g(u) = ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i} \quad \text{and} \quad h(u) = \mu \sum_{j=1}^{n} b_j u^{r_j} \quad (\text{with } \mu > 0)$$

satisfy (1.3). We prove that g, h satisfy (H1)–(H4). It is first easy to see that $g, h \in C[0, \infty) \cap C^2(0, \infty)$ satisfy (H1) with $m_0^g \equiv \lim_{u \to 0^+} \frac{g(u)}{u^{p-1}} = k \geq 0$. Hence, by (3.3), $\hat{\lambda} = \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p} \csc \frac{\pi}{p}\right)^p = \bar{\lambda}$. By (1.3), it is easy to see that the function

$$\frac{h(u)}{g(u)} = \frac{\mu \sum_{j=1}^{n} b_j u^{r_j}}{ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}} \quad (\mu > 0)$$

is positive and strictly increasing on $(0, \infty)$ and satisfies that

$$\lim_{u \to 0^+} \frac{h(u)}{g(u)} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{h(u)}{g(u)} = \infty.$$

Thus g, h satisfy (H2). It is clear that, by (1.3),

$$(p-2)g'(u) - ug''(u) = \sum_{i=1}^{m} a_i q_i (p-1-q_i) u^{q_i - 1} < 0 \quad \text{on } (0,\infty),$$

$$(p-2)h'(u) - uh''(u) = \mu \sum_{j=1}^{n} b_j r_j (p-1-r_j) u^{r_j - 1} < 0 \quad \text{on } (0,\infty).$$

Thus g, h satisfy (H3). Finally, by (1.3), we compute that

$$\begin{aligned}\frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} &= \frac{\mu \sum_{j=1}^{n} b_j r_j (p-1-r_j) u^{r_j - 1}}{\sum_{i=1}^{m} a_i q_i (p-1-q_i) u^{q_i - 1}} \quad (\mu > 0)\\ &= \frac{\mu \sum_{j=1}^{n} b_j r_j (p-1-r_j) u^{r_j}}{\sum_{i=1}^{m} a_i q_i (p-1-q_i) u^{q_i}}\end{aligned}$$

which is positive and strictly increasing on $(0,\infty)$ and satisfies that

$$\lim_{u\to 0^+} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = 0 \quad \text{and} \quad \lim_{u\to \infty} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \infty.$$

So g, h satisfy (H4). By above, we conclude that g, h satisfy (H1)–(H4). So part (ii) follows from Lemma 3.1(i).

(III) We prove part (iii). Consider any nonnegative $\mu_1 < \mu_2$. If, on the $(\lambda, \|u\|_\infty)$-plane, bifurcation diagrams S_{p,k,μ_1} and S_{p,k,μ_2} attain a fixed number $\|u\|_\infty = \bar{\alpha}$ for any feasible $\bar{\alpha} > 0$ at $\lambda = \lambda_1 > 0$ and $\lambda = \lambda_2 > 0$, respectively. Then by (3.6), we have the following equalities:

$$T_{\mu_1,\lambda_1}(\bar{\alpha}) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\bar{\alpha}} [\lambda_1(G(\bar{\alpha}) - G(u)) - \mu_1(\tilde{H}(\bar{\alpha}) - \tilde{H}(u))]^{-1/p} du = 1, \tag{4.1}$$

$$T_{\mu_2,\lambda_2}(\bar{\alpha}) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\bar{\alpha}} [\lambda_2(G(\bar{\alpha}) - G(u)) - \mu_2(\tilde{H}(\bar{\alpha}) - \tilde{H}(u))]^{-1/p} du = 1. \tag{4.2}$$

Suppose that $\lambda_1 \geq \lambda_2$, since $0 < \mu_1 < \mu_2$ and $\lambda_1 \geq \lambda_2$, we have that

$$\lambda_1(G(\bar{\alpha}) - G(u)) - \mu_1(\tilde{H}(\bar{\alpha}) - \tilde{H}(u)) > \lambda_2(G(\bar{\alpha}) - G(u)) - \mu_2(\tilde{H}(\bar{\alpha}) - \tilde{H}(u)).$$

Thus $T_{\mu_1,\lambda_1}(\bar{\alpha}) < T_{\mu_2,\lambda_2}(\bar{\alpha})$, which leads to a contradiction since the above equality (4.1) for $T_{\mu_1,\lambda_1}(\bar{\alpha})$ and equality (4.2) for $T_{\mu_2,\lambda_2}(\bar{\alpha})$ are both equal to 1. So $\lambda_1 < \lambda_2$. Hence, for any nonnegative $\mu_1 < \mu_2$, on the $(\lambda, \|u\|_\infty)$-plane, S_{p,k,μ_2} lies on the right hand side of S_{p,k,μ_1}.

(IV) We prove part (iv). By Lemma 3.7, $\lambda^*(\mu) : (0,\infty) \to (0,\bar{\lambda})$ is a continuous, strictly increasing function. Moreover, $\lim_{\mu\to 0^+} \lambda^*(\mu) = 0$ and $\lim_{\mu\to\infty} \lambda^*(\mu) = \bar{\lambda}$. It is easy to show that $\|u_{\lambda^*}\|_\infty$ $(= \|u_{\mu,\lambda^*}\|_\infty)$ is a continuous function of $\mu > 0$. By Lemma 3.9, we have that $\lim_{\mu\to 0^+} \|u_{\lambda^*}\|_\infty = \infty$ and $\lim_{\mu\to\infty} \|u_{\lambda^*}\|_\infty = 0$.

In particular, when $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p-1$, $r \equiv r_1 > q$, and $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, we let $u = u_{\mu,\lambda}$ be a (classical) positive solution of (1.1), (1.3). Then the change of variables

$$u_{\mu,\lambda}(x) = \left(\frac{\lambda}{\mu}\right)^{1/(r-q)} v(\mu^{\frac{p-q-1}{p(r-q)}} \lambda^{\frac{r-p+1}{p(r-q)}} x)$$

transforms $u_{\mu,\lambda}$ into a solution v of

$$\begin{cases} (\varphi_p(v'(x)))' + v^q - v^r = 0, & -L < x < L, \\ v(-L) = v(L) = 0, \end{cases} \tag{4.3}$$

with

$$L \equiv \mu^{\frac{p-q-1}{p(r-q)}} \lambda^{\frac{r-p+1}{p(r-q)}}. \tag{4.4}$$

Cf. [5, p. 463]. For p-Laplacian problem (4.3) with $1 < p \leq 2$ and $0 < p-1 < q < r$, we define the time map formula as follows:

$$\tilde{T}(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \frac{dv}{\left[\tilde{F}(\alpha) - \tilde{F}(v)\right]^{1/p}} \quad \text{for } 0 < \alpha < 1, \tag{4.5}$$

where $\tilde{F}(v) \equiv \int_0^v \tilde{f}(t)dt$ and $\tilde{f}(v) \equiv v^q - v^r$. By [14, Lemma 3.1], there exist two fixed positive numbers $\|v^*\|_\infty$ and L^* such that $\tilde{T}(\alpha)$ has exactly one critical point, an absolute minimum $\tilde{T}(\|v^*\|_\infty) = L^*$, on $(0,1)$. Thus, by (4.4),

$$L^* = \mu^{\frac{p-q-1}{p(r-q)}} (\lambda^*)^{\frac{r-p+1}{p(r-q)}},$$

and hence

$$\left(\frac{\lambda^*}{\mu}\right)^{1/(r-q)} = (L^*)^{\frac{p}{r-p+1}} \mu^{\frac{1}{p-1-r}}.$$

So we get that

$$\|u_{\lambda^*}\|_\infty = \left(\frac{\lambda^*}{\mu}\right)^{1/(r-q)} \|v^*\|_\infty = (L^*)^{\frac{p}{r-p+1}} \mu^{\frac{1}{p-1-r}} \|v^*\|_\infty.$$

Since $r > p-1$, $\|u_{\lambda^*}\|_\infty$ is a continuous, strictly decreasing function of $\mu > 0$.

(V) We prove part (v). We consider $0 < \alpha < \beta_{\mu,\lambda}$ and have that

$$\begin{aligned} T_{\mu=0,\lambda}(\alpha) &= \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha [\lambda(G(\alpha) - G(u))]^{-1/p} du \\ &< \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha [\lambda(G(\alpha) - G(u)) - \mu(\tilde{H}(\alpha) - \tilde{H}(u))]^{-1/p} du = T_{\mu,\lambda}(\alpha). \end{aligned}$$

By (3.4), (1.16) and (1.17), we obtain that

$$\lambda^{1/p} T_{\mu=0,\lambda}(\alpha) = T_{\bar{f}}(\alpha) = (c_{p,q} \alpha^{p-1-q})^{1/p}.$$

If $T_{\mu=0,\lambda}(\alpha) = 1$, then $\alpha = \left(\frac{c_{p,q}}{\lambda}\right)^{\frac{1}{q-p+1}}$. Then by (3.6) and Proposition 1.1(ii), we obtain that

$$0 < \left(\frac{c_{p,q}}{\lambda}\right)^{\frac{1}{q-p+1}} < \|u_\lambda\|_\infty < \beta_{\mu,\lambda}$$

since $T_{\bar{f}}(\alpha)$ is a strictly decreasing function on $(0,\infty)$ and $T_{\mu=0,\lambda}\left(\left(\frac{c_{p,q}}{\lambda}\right)^{\frac{1}{q-p+1}}\right) = 1$. So (2.1) holds.

The proof of Theorem 2.1 is now complete. □

***Proof of Theorem* 2.2.** Let $p > 2$ and $k \geq 0$.

(I)(a) We have that, for any fixed $\mu > 0$, the bifurcation diagram $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane. The proof is exactly the same as that given in part (I)(a) of the proof of Theorem 2.1 with $1 < p \leq 2$. So we omit it here.

(I)(b) Part (i) follows from Proposition 1.1(i).

(II)(a) We prove part (ii) where

$$g(u) = ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i} \quad \text{and} \quad h(u) = \mu \sum_{j=1}^{n} b_j u^{r_j} \quad (\text{with } \mu > 0)$$

satisfy (1.3). We prove that g, h satisfy (H1)–(H5). We first notice that the proofs of g, h satisfying (H1)–(H4) when $p > 2$ are exactly the same as those of g, h satisfying (H1)–(H4) when $1 < p \leq 2$, given in part (II) of the proof of Theorem 2.1. So we omit them here. We then show that g, h satisfy (H5). We take the number $p^* = \frac{q_m + r_1}{2} > p - 1$ by (1.3). It is easy to see that

$$\frac{g(u)}{u^{p^*}} = ku^{p-1-p^*} + \sum_{i=1}^{m} a_i u^{q_i - p^*} \quad \text{is strictly decreasing on } (0, \infty)$$

and

$$\frac{h(u)}{u^{p^*}} = \mu \sum_{j=1}^{n} b_j u^{r_j - p^*} \quad \text{is strictly increasing on } (0, \infty).$$

For each fixed $s \in (0, 1)$, we compute that

$$\begin{aligned} \frac{h(su)}{u^{p-1}} \left[\frac{h(u)g(su)}{g(u)h(su)} - 1 \right] &= \frac{h(u)g(su) - h(su)g(u)}{u^{p-1}g(u)} \\ &= \frac{\mu \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j (s^{q_i} - s^{r_j}) u^{q_i + r_j} + \mu k u^{p-1} \sum_{j=1}^{n} b_j (s^{p-1} - s^{r_j}) u^{r_j}}{u^{p-1}(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i})} \end{aligned}$$

which is a strictly increasing function of u on $(0, \infty)$ and satisfies that

$$\lim_{u \to \infty} \frac{h(u)g(su)}{g(u)h(su)} = s^{q_m - r_n} \in (1, \infty)$$

since $s \in (0, 1)$ and $q_m < r_n$. So g, h satisfy (H5). By above, we conclude that g, h satisfy (H1)–(H5). So part (ii) follows from Lemma 3.1(ii).

(II)(b) By Lemma 3.10(ii), we have that

$$\lim_{\mu \to 0^+} \lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) = 0, \qquad \lim_{\mu \to \infty} \lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) = \infty$$

and $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha)$ is a continuous, strictly increasing function of μ on $(0, \infty)$. So by the Intermediate Value Theorem, there exists a positive number $\hat{\mu}$ such that $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) < 1$ if $0 < \mu < \hat{\mu}$, $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) = 1$ if $\mu = \hat{\mu}$, and $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) > 1$ if $\mu > \hat{\mu}$.

For each fixed $k > 0$ and $\mu > 0$, $\lim_{\alpha \to \beta^-_{\mu,\tilde{\lambda}}} T_{\mu,\tilde{\lambda}}(\alpha) = 1$ by Lemma 3.1(ii) and $\lim_{\alpha \to \beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha)$ is a continuous, strictly decreasing function of λ on $(0, \infty)$ by Lemma 3.10(i). Hence we obtain that $\tilde{\lambda} < \bar{\lambda}$ if $0 < \mu < \hat{\mu}$, $\tilde{\lambda} = \bar{\lambda}$ if $\mu = \hat{\mu}$, and $\bar{\lambda} < \tilde{\lambda}$ if $\mu > \hat{\mu}$.

For each $\mu > 0$ and $k = 0$, we have $\bar{\lambda} = \infty$. By Lemma 3.10(i), $\lim_{\lambda \to \infty} \lim_{\alpha \to \beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = 0$. Hence $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) = 0$. Since $\lim_{\alpha \to \beta^-_{\mu,\tilde{\lambda}}} T_{\mu,\tilde{\lambda}}(\alpha) = 1$ by Lemma 3.1(ii) and $\lim_{\alpha \to \beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha)$ is a continuous, strictly decreasing function of λ on $(0, \infty)$ by Lemma 3.10(i), we obtain that $\tilde{\lambda} < \bar{\lambda}$ and $\hat{\mu} = \infty$.

(III) The proof of part (iii) of Theorem 2.2 is exactly the same as that of part (iii) of Theorem 2.1. So we omit it here.

(IV) We prove part (iv). By Lemma 3.12, $\tilde{\lambda}(\mu) : (0,\infty) \to (0,\infty)$ is a continuous, strictly increasing function. Moreover, $\lim_{\mu\to 0^+} \tilde{\lambda} = 0$ and $\lim_{\mu\to\infty} \tilde{\lambda} = \infty$. We let $u = u_{\mu,\lambda}$ be a (classical) positive solution of (1.1), (1.3). Then the change of variables

$$u_{\mu,\lambda}(x) = v(\mu^{\frac{1}{p}} x)$$

transforms $u_{\mu,\lambda}$ into a solution v of

$$\begin{cases} \left(\varphi_p(v'(x))\right)' + \frac{\lambda}{\mu}(kv^{p-1} + \sum_{i=1}^m a_i v^{q_i}) - \sum_{j=1}^n b_j v^{r_j}, & -L < x < L, \\ v(-L) = v(L) = 0, \end{cases} \tag{4.6}$$

with

$$L \equiv \mu^{\frac{1}{p}}.$$

Cf. [5, p. 463]. By Lemma 3.11(i), for any fixed $\mu \in (0,\infty)$, there exists a unique $\tilde{\lambda}(\mu) > 0$ such that $\lim_{\alpha\to\beta^-_{\mu,\tilde{\lambda}(\mu)}} T_{\mu,\tilde{\lambda}(\mu)}(\alpha) = 1$. Then there exists a unique $\eta = \frac{\tilde{\lambda}(\mu)}{\mu} > 0$ such that $\lim_{\alpha\to\beta^-_{\mu,\tilde{\lambda}(\mu)}} T_{1,\eta}(\alpha) = \mu^{\frac{1}{p}}$ and $\beta_{\mu,\tilde{\lambda}(\mu)} = \beta_{1,\eta}$. For $0 < \mu_1 < \mu_2$,

$$\lim_{\alpha\to\beta^-_{\mu_1,\tilde{\lambda}(\mu_1)}} T_{1,\eta_1}(\alpha) = \mu_1^{\frac{1}{p}} < \mu_2^{\frac{1}{p}} = \lim_{\alpha\to\beta^-_{\mu_2,\tilde{\lambda}(\mu_2)}} T_{1,\eta_2}(\alpha).$$

Hence $\eta_1 > \eta_2$ by Lemma 3.10(i). Similarly, for any fixed $\eta \in (0,\infty)$, there exists a unique $\mu > 0$ such that $\lim_{\alpha\to\beta^-_{1,\eta}} T_{1,\eta}(\alpha) = \mu^{\frac{1}{p}}$ and $\beta_{\mu,\tilde{\lambda}(\mu)} = \beta_{1,\eta}$. It is clear that $\beta_{1,\eta}$ is a continuous, strictly increasing function of η on $(0,\infty)$. Hence

$$\|v_{\tilde{\lambda}(\mu_1)}\|_\infty = \beta_{\mu_1,\tilde{\lambda}(\mu_1)} = \beta_{1,\eta_1} > \beta_{1,\eta_2} = \beta_{\mu_2,\tilde{\lambda}(\mu_2)} = \|v_{\tilde{\lambda}(\mu_2)}\|_\infty.$$

So $\|v_{\tilde{\lambda}}\|_\infty$ is a continuous, strictly decreasing function of $\mu > 0$. By Lemma 3.10(i), $\lim_{\mu\to 0^+} \eta(\mu) = \infty$ and $\lim_{\mu\to\infty} \eta(\mu) = 0$ since $\lim_{\alpha\to\beta^-_{1,\eta}} T_{1,\eta}(\alpha) = \mu^{\frac{1}{p}}$. Hence

$$\lim_{\mu\to 0^+} \|v_{\tilde{\lambda}(\mu)}\|_\infty = \lim_{\mu\to 0^+} \beta_{\mu,\tilde{\lambda}(\mu)} = \lim_{\eta\to\infty} \beta_{1,\eta} = \infty$$

and

$$\lim_{\mu\to\infty} \|v_{\tilde{\lambda}(\mu)}\|_\infty = \lim_{\mu\to\infty} \beta_{\mu,\tilde{\lambda}(\mu)} = \lim_{\eta\to 0^+} \beta_{1,\eta} = 0.$$

(V) The proof of part (v) of Theorem 2.2 is exactly the same as that of part (iv) of Theorem 2.1. So we omit it here.

(VI) The proof of part (vi) of Theorem 2.2 is exactly the same as that of part (v) of Theorem 2.1. So we omit it here.

The proof of Theorem 2.2 is now complete. □

5 A final remark

For evolutionary bifurcation diagram $S_{p,k,\mu}$ on the $(\lambda, \|u\|_\infty)$-plane studied in Theorems 2.1–2.2, analogically, we also study evolutionary bifurcation diagrams $\Sigma_{p,k,\lambda}$ on the $(\mu, \|u\|_\infty)$-plane defined by:

$$\Sigma_{p,k,\lambda} = \left\{ (\mu, \|u_\mu\|_\infty) : \mu > 0 \text{ and } u_\mu \text{ is a (classical) positive solution of (1.1), (1.3)} \right\}, \ \lambda > 0.$$

Applying Theorems 2.1–2.2 and by modified analytic techniques used in the proof of [7, Theorem 2.2], we obtain the following Theorem 5.1 and Fig. 5.1 with $1 < p \le 2$ and Theorem 5.2 and Fig. 5.2 with $p > 2$ for evolutionary bifurcation diagrams $\Sigma_{p,k,\lambda}$ on the $(\mu, \|u\|_\infty)$-plane. We omit the proofs here.

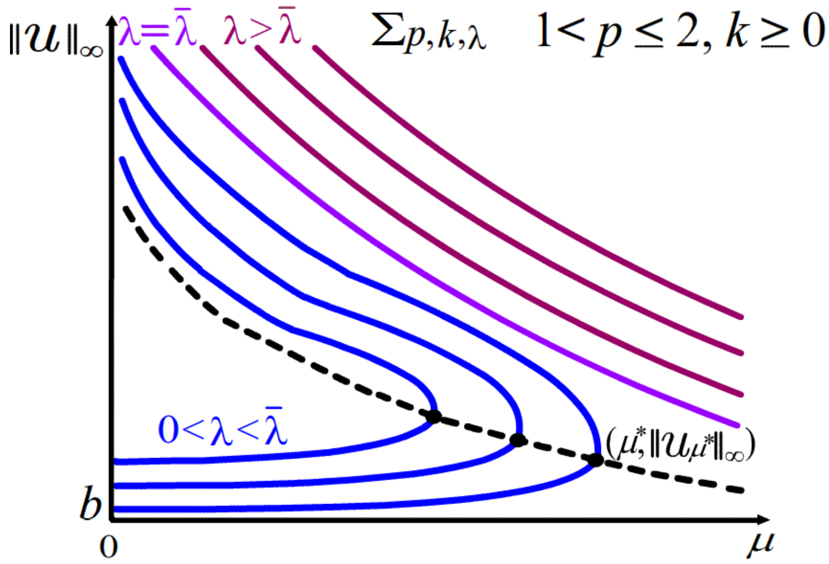

Figure 5.1: Evolutionary bifurcation diagrams $\Sigma_{p,k,\lambda}$ for (1.1), (1.3) with fixed $p \in (1, 2]$, $k \ge 0$ and varying $\lambda > 0$.

Theorem 5.1 (See Fig. 5.1). *Let $1 < p \le 2$ and $k \ge 0$. Consider p-Laplacian problem (1.1), (1.3) with varying $\lambda > 0$. Then the bifurcation diagram $\Sigma_{p,k,\lambda}$ consists of a continuous curve on the $(\mu, \|u\|_\infty)$-plane and the following assertions (i)–(iii) hold:*

(i) If

$$0 < \lambda < \bar{\lambda} = \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \begin{cases} < \infty & \text{if } k > 0, \\ = \infty & \text{if } k = 0, \end{cases}$$

then:

(a) $\Sigma_{p,k,\lambda}$ starts at some point $(0, b)$ where $b > 0$, tends to the positive $\|u\|_\infty$-axis as $\mu \to 0^+$, and is a reversed $\subset$-shaped curve with exactly one turning point at some point $(\mu^*, \|u_{\mu^*}\|_\infty)$ satisfying $\mu^* > 0$ and $\|u_{\mu^*}\|_\infty > b$. More precisely, problem (1.1), (1.3) has exactly two (classical) positive solutions u_μ, v_μ with $u_\mu < v_\mu$ for $0 < \mu < \mu^*$, exactly one (classical) positive solution u_{μ^*} for $\mu = \mu^*$, and no (classical) positive solution for $\mu > \mu^*$. In addition, $\lim_{\mu\to 0^+} \|u_\mu\|_\infty = b$ and $\lim_{\mu\to 0^+} \|v_\mu\|_\infty = \infty$.

(b) For the starting points $(0, b)$ of $\Sigma_{p,k,\lambda}$ with $0 < \lambda < \bar{\lambda}$, $b = b(\lambda)$ is a continuous, strictly decreasing function of $\lambda \in (0, \bar{\lambda})$, $\lim_{\lambda\to 0^+}(0, b) = (0, \infty)$ and $\lim_{\lambda\to \bar{\lambda}^-}(0, b) = (0, 0)$.

(c) For the turning points $(\mu^*, \|u_{\mu^*}\|_\infty)$ of $\Sigma_{p,k,\lambda}$ with $0 < \lambda < \bar{\lambda}$, μ^* is a continuous, strictly increasing function of $\lambda \in (0, \bar{\lambda})$, $\|u_{\mu^*}\|_\infty$ is a continuous function of $\lambda \in (0, \bar{\lambda})$,

$$\lim_{\lambda\to 0^+} (\mu^*, \|u_{\mu^*}\|_\infty) = (0, \infty) \quad \text{and} \quad \lim_{\lambda\to \bar{\lambda}^-} (\mu^*, \|u_{\mu^*}\|_\infty) = (\infty, 0).$$

In particular, when $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, and $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, then $\|u_{\mu^*}\|_\infty$ is a strictly decreasing function of $\lambda \in (0, \bar{\lambda})$.

(ii) *If $\lambda \geq \bar{\lambda}$, then $\Sigma_{p,k,\lambda}$ emanates from the positive $\|u\|_\infty$-axis as $\mu \to 0^+$, tends to the positive μ-axis as $\mu \to \infty$, and is a strictly monotone curve. More precisely, problem* (1.1), (1.3) *has exactly one (classical) positive solution for $\mu > 0$.*

(iii) *For any positive $\lambda_2 > \lambda_1$, Σ_{p,k,λ_2} lies on the right hand side of Σ_{p,k,λ_1}. (So Σ_{p,k,λ_1} and Σ_{p,k,λ_2} do not intersect.)*

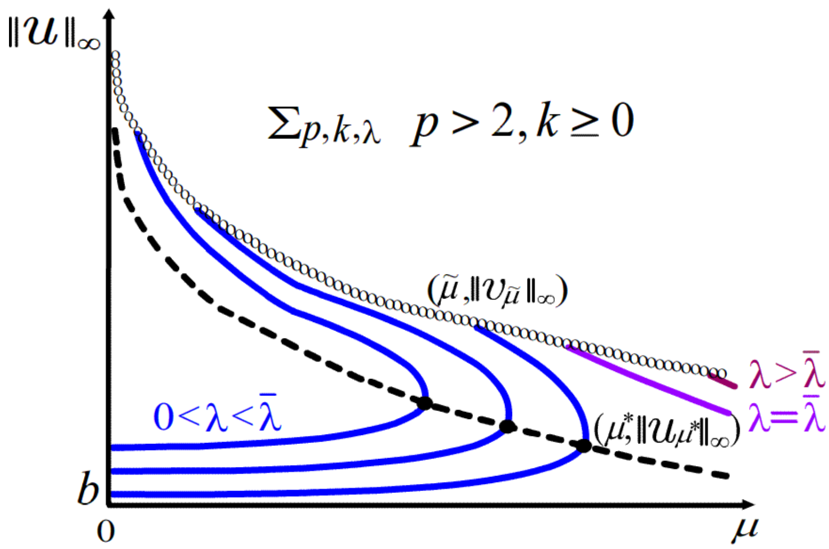

Figure 5.2: Evolutionary bifurcation diagrams $\Sigma_{p,k,\lambda}$ for (1.1), (1.3) with fixed $p > 2$, $k \geq 0$ and varying $\mu \geq 0$.

Theorem 5.2 (See Fig. 5.2)**.** *Let $p > 2$ and $k \geq 0$. Consider p-Laplacian problem (1.1), (1.3) with varying $\lambda > 0$. Then the bifurcation diagram $\Sigma_{p,k,\lambda}$ consists of a continuous curve on the $(\mu, \|u\|_\infty)$-plane and the following assertions (i)–(iv) hold:*

(i) *If*

$$0 < \lambda < \bar{\lambda} = \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \begin{cases} < \infty & \text{if } k > 0, \\ = \infty & \text{if } k = 0, \end{cases}$$

then:

(a) *$\Sigma_{p,k,\lambda}$ starts at some point $(0, b)$ where $b > 0$, ends at some point $(\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty)$ satisfying $0 < \tilde{\mu} < \infty$ and $0 < \|v_{\tilde{\mu}}\|_\infty = v_{\tilde{\mu}}(0) = \beta_{\tilde{\mu},\lambda}$ satisfying $f_{k,\tilde{\mu},\lambda}(\beta_{\tilde{\mu},\lambda}) = 0$ (that is, $v_{\tilde{\mu}}(x) \equiv \lim_{\lambda \to \tilde{\lambda}^-} v_\lambda(x)$ is a* flat-core *positive solution of* (1.1), (1.3), *see below for* (classical) *positive solutions $v_\lambda(x)$ with $\tilde{\mu} < \mu < \mu^*$). Moreover, $\Sigma_{p,k,\lambda}$ is a reverse $\subset$-shaped curve with exactly one turning point at some point $(\mu^*, \|u_{\mu^*}\|_\infty)$ satisfying*

$$0 < \tilde{\mu} < \mu^* \quad \textit{and} \quad 0 < \|u_{\mu^*}\|_\infty < \|v_{\tilde{\mu}}\|_\infty = \beta_{\tilde{\mu},\lambda}.$$

More precisely, problem (1.1), (1.3) *has exactly two (classical) positive solutions u_μ, v_μ with $u_\mu < v_\mu$ for $\tilde{\mu} < \mu < \mu^*$, exactly one (classical) positive solution u_{μ^*} for $\mu = \mu^*$ and $0 < \mu \leq \tilde{\mu}$, and no (classical) positive solution for $\mu > \mu^*$. In addition, $\lim_{\mu \to 0^+} \|u_\mu\|_\infty = b$ and $\lim_{\mu \to \tilde{\mu}^-} \|v_\mu\|_\infty = \|v_{\tilde{\mu}}\|_\infty = \beta_{\tilde{\mu},\lambda}$.*

(b) *For the starting points* $(0, b)$ *of* $\Sigma_{p,k,\lambda}$ *with* $0 < \lambda < \bar{\lambda}$, $b = b(\lambda)$ *is a continuous, strictly decreasing function of* $\lambda \in (0, \bar{\lambda})$, $\lim_{\lambda\to 0^+}(0, b) = (0, \infty)$ *and* $\lim_{\lambda\to\bar{\lambda}^-}(0, b) = (0, 0)$.

(c) *For the turning points* $(\mu^*, \|u_{\mu^*}\|_\infty)$ *of* $\Sigma_{p,k,\lambda}$ *with* $0 < \lambda < \bar{\lambda}$, μ^* *is a continuous, strictly increasing function of* $\lambda \in (0, \bar{\lambda})$, $\|u_{\mu^*}\|_\infty$ *is a continuous function of* $\lambda \in (0, \bar{\lambda})$,

$$\lim_{\lambda\to 0^+}(\mu^*, \|u_{\mu^*}\|_\infty) = (0, \infty) \quad \textit{and} \quad \lim_{\lambda\to\bar{\lambda}^-}(\mu^*, \|u_{\mu^*}\|_\infty) = (\infty, 0).$$

In particular, when $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, *and* $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, *then* $\|u_{\mu^*}\|_\infty$ *is a strictly decreasing function of* $\lambda \in (0, \bar{\lambda})$.

(ii) *If* $\lambda \geq \bar{\lambda}$, *then* $\Sigma_{p,k,\lambda}$ *emanates from the positive* μ*-axis as* $\mu \to \infty$, *and ends at some point* $(\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty)$ *in which* $v_{\tilde{\mu}}$ *is a* flat-core *positive solution. Moreover,* $\Sigma_{p,k,\lambda}$ *is a strictly monotone curve. More precisely, problem* (1.1), (1.3) *has exactly one (classical) positive solution for* $\mu > \tilde{\mu}$.

(iii) *For any positive* $\lambda_2 > \lambda_1$, Σ_{p,k,λ_2} *lies on the right hand side of* Σ_{p,k,λ_1}. *(So* Σ_{p,k,λ_1} *and* Σ_{p,k,λ_2} *do not intersect.)*

(iv) *For the ending points* $(\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty)$ *of* $\Sigma_{p,k,\lambda}$ *with* $\lambda > 0$, $\tilde{\mu}$ *is a continuous, strictly increasing function of* $\lambda > 0$, $\|v_{\tilde{\mu}}\|_\infty$ *is a continuous, strictly decreasing function of* $\lambda > 0$,

$$\lim_{\lambda\to 0^+}(\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty) = (0, \infty) \quad \textit{and} \quad \lim_{\lambda\to\infty}(\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty) = (\infty, 0).$$

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