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Existence of solutions for multi-point boundary value problem of fractional q-difference equation★

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Abstract: This paper is mainly concerned with the existence of solutions for a multi-point boundary value problem of nonlinear fractional q-difference equations by means of the Banach contraction principle and Krasnoselskii's fixed point theorem. Further, an example is presented to illustrate the main results.

Keywords: Fractional q-difference equations; Multi-point condition; Fixed point theorem.
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1 Introduction

Fractional differential calculus have recently been addressed by many researchers of various fields of science and engineering such as physics, chemistry, biology, economics, control theory, and biophysics, etc. [1-4]. In particular, the existence of solutions to fractional boundary value problems is under strong research recently, see [5-7] and references therein.

The fractional q-difference calculus had its origin in the works by Al-Salam [8] and Agarwal [9]. More recently, perhaps due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus were made, specifically, q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula [10,11], just to mention some.

The question of the existence of solutions for fractional q-difference boundary value problems is in its infancy, being few results available in the literature.

Ferreira [12] considered the existence of positive solutions to nonlinear q-difference boundary value problem:

$$(D_q^\alpha u)(t) = -f(t, u(t)), \quad 0 < t < 1, \ 1 < \alpha \leq 2$$
$$u(0) = u(1) = 0.$$

In other paper, Ferreira [13] studied the existence of positive solutions to nonlinear q-difference boundary value problem:

$$(D_q^\alpha u)(t) = -f(t, u(t)), \quad 0 < t < 1, \ 2 < \alpha \leq 3$$
$$u(0) = (D_q u)(0) = 0, (D_q u)(1) = \beta \geq 0.$$

M.El-Shahed and Farah M.Al-Askar [15] studied the existence of positive solutions to nonlinear q-difference equation:

$$({}_C D_q^\alpha u)(t) + a(t) f(u(t)) = 0, \quad 0 \leq t \leq 1, \ 2 < \alpha \leq 3$$
$$u(0) = (D_q^2 u)(0) = 0, \ \gamma D_q u(1) + \beta D_q^2 u(1) = 0.$$

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In this paper, we investigate the existence of solutions for nonlinear q-difference boundary value problem of the form

$$\begin{aligned} &({}_CD_q^\alpha u)(t) = -f\left(t, u(t), (\phi u)(t), (\psi u)(t)\right), \quad 1 < t < 2, 1 < \alpha \le 2, \\ &u(0) = u_0 + g(u),\ D_q u(1) = u_1 + \sum_{i=1}^{m-2} b_i D_q u(\xi_i), \end{aligned} \tag{1.1}$$

where $0 < \xi_i < 1\,(i = 1, 2, \ldots, m-2)\,, b_i \ge 0$ with $\rho = \sum_{i=1}^{m-2} b_i < 1$ and (${}_CD_q^\alpha$ represents the standard Caputo fractional q-derivative), $f : [0,1] \times X \times X \times X \to X$ is continuous, for $\gamma, \delta : [0,1] \times [0,1] \to [0,\infty)$,

$$(\phi u)(t) = \int_0^t \gamma(t,s) u(s) d_q s, \quad (\psi u)(t) = \int_0^t \delta(t,s) u(s) d_q s.$$

Here, $(X, \|\cdot\|)$ is a Banach space and $C = C([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

2 Preliminaries on fractional q-calculus

Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q},\ a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)^0 = 1,\ (a-b)^n = \prod_{k=0}^{n-1} (a - bq^k),\ n \in \mathbb{N},\ a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{i=0}^{\infty} \frac{a - bq^i}{a - bq^{\alpha+i}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}},\ x \in \mathbb{R} \setminus \{0, -1, -2 \ldots\}, 0 < q < 1,$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function $f(x)$ is here defined by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x} (x \ne 0),\ (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^n f)(x) = \begin{cases} f(x) & if\ n = 0, \\ D_q D_q^{n-1} f(x) & if\ n \in \mathbb{N}. \end{cases}$$

The q-integral of a function $f(x)$ defined in the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n,\ 0 \le |q| < 1,\ x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

If $c_j = bq^j$, for $j \in \{0, 1, \dots, n\}$, $a = c_n = bq^n$, $0 < q < 1$.
The restricted q-integral of a function $f(x)$ defined by

$$\int_a^b f(t)d_qt = \int_{bq^n}^b f(t)d_qt = (1-q)b\sum_{j=0}^{n-1} q^j f(bq^j) = (1-q)\sum_{j=0}^{n-1} c_j f(c_j),\ 0<q<1,\ b>0,\ n\in\mathbb{Z}^+.$$

Note that the restricted integral $\int_a^b f(t)d_qt$ is just a finite sum, so no questions about convergence arise.

Obviously, if $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(t)d_qt \geq \int_a^b g(t)d_qt$. If $0 < k < n$, then

$$\int_a^b f(t)d_qt = \int_a^{c_k} f(t)d_qt + \int_{c_k}^b f(t)d_qt.$$

The usual Riemann integral can be considered as a limit of the restricted definite q-integral in the following way. Since $a = bq^n$, $q = \left(\frac{a}{b}\right)^{\frac{1}{n}}$. Fix a and b and let $n \to \infty$(hence, $q \to 1$). Then, $\int_a^b f(t)d_qt \to \int_a^b f(t)dt$ assuming that $f(t)$ is Riemann integrable on $[a, b]$. The above formulas were proved by the author [17].

Similarly as done for derivatives, it can be defined an operator I_q^n, namely,

$$(I_q^n f)(x) = \begin{cases} f(x) & if\ n = 0, \\ I_q I_q^{n-1} f(x) & if\ n \in \mathbb{N}. \end{cases}$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

and, more generally

$$(D_q^n I_q^n f)(x) = f(x), n \in \mathbb{N},$$

$$(I_q^n D_q^n f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{x^k}{\Gamma_q(k+1)}(D_q^k f)(0), n \in \mathbb{N}.$$

Basic properties of the two operators can be found in the book [14]. We point out here five formulas that will be used later, namely, the integration by parts formula

$$\int_0^x f(t)(D_q g)td_qt = [f(t)g(t)]_{t=0}^{t=x} - \int_0^x (D_q f)g(qt)td_qt,$$

and (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)},$$

$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$

$${}_sD_q(t-s)^{(\alpha)} = -[\alpha]_q (t-qs)^{(\alpha-1)},$$

$$({}_xD_q \int_0^x f(x,t)d_qt)(x) = \int_0^x {}_xD_q f(x,t)d_qt + f(qx, x).$$

Remark 2.1. We note that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$ [12].

The following definition was considered first in [9].

Definition 2.2. Let $\alpha \geq 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t) d_q t, \alpha > 0, x \in [0,1].$$

Definition 2.3. The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Definition 2.4. [16] The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$({}_C D_q^\alpha f)(x) = (I_q^{m-\alpha} D_q^m f)(x), \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Next, let us list some properties that are already know in the literature.

Lemma 2.5. Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then, the next formulas hold:

1.$(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,

2.$(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

The next result is important in the sequel. It was proved in a recent work by the author [12].

Theorem 2.1. [12] Let $\alpha > 0$ and $n \in \mathbb{N}$. Then, the following equality holds:

$$(I_q^\alpha D_q^n f)(x) = D_q^n I_q^\alpha f(x) - \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D_q^k f)(0).$$

Theorem 2.2. [16] Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, the following equality holds:

$$(I_q^\alpha {}_C D_q^\alpha f)(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^k}{\Gamma_q(k+1)} (D_q^k f)(0).$$

where m is the smallest integer greater than or equal to α.

Lemma 2.6. For a given $\sigma \in C[0,1]$ and $1 < \alpha \leq 2$, the unique solution of

$$\begin{aligned} &({}_C D_q^\alpha u)(t) = -\sigma(t), \\ &u(0) = u_0 + g(u), \ D_q u(1) = u_1 + \sum_{i=1}^{m-2} b_i D_q u(\xi_i), \end{aligned} \tag{2.1}$$

is given by

$$\begin{aligned} u(t) = &- \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \sigma(s) d_q s + \frac{1}{1-\rho} \int_0^1 \frac{(1-qs)^{(\alpha-2)} t}{\Gamma_q(\alpha-1)} \sigma(s) d_q s \\ &- \frac{1}{1-\rho} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-2)} t}{\Gamma_q(\alpha-1)} \sigma(s) d_q s + u_0 + g(u) + \frac{1}{1-\rho} u_1 t. \end{aligned}$$

Proof. Let us put $m = 2$. In view of Lemma 2.5 and Theorem 2.2, we have

$$\begin{aligned} ({}_C D_q^\alpha u)(t) = -\sigma(t) &\Longleftrightarrow (I_q^\alpha I_q^{2-\alpha} {}_C D_q^2 u)(t) = -I_q^\alpha \sigma(t) \\ &\Longleftrightarrow u(t) = - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \sigma(s) d_q s + c_0 + c_1 t, \end{aligned} \tag{2.2}$$

for some constants $c_0, c_1 \in \mathbb{R}$. Using the boundary condition $u(0) = u_0 + g(u)$, gives $c_0 = u_0 + g(u)$.

Furthermore, differentiation of (2.2) with respect to t produces

$$D_q u(x) = -\int_0^t \frac{[\alpha-1]_q (t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} \sigma(s)\, d_q s + c_1.$$

Using the boundary conditions $D_q u(1) = \sum_{i=1}^{m-2} b_i D_q u(\xi_i)$, we get

$$c_1 = \frac{1}{1-\rho} \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \sigma(s)\, d_q s - \frac{1}{1-\rho} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \sigma(s)\, d_q s + \frac{u_1}{1-\rho}.$$

Now, substitution of c_0 and c_1 into (2.1) gives

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \sigma(s)\, d_q s + \frac{1}{1-\rho} \int_0^1 \frac{(1-qs)^{(\alpha-2)} t}{\Gamma_q(\alpha-1)} \sigma(s)\, d_q s \\ &- \frac{1}{1-\rho} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-2)} t}{\Gamma_q(\alpha-1)} \sigma(s)\, d_q s + \frac{u_1 t}{1-\rho} + u_0 + g(u). \end{aligned}$$

The proof is complete. □

3 Main results

Define an operator $T : C \to C$ by

$$\begin{aligned} (Tu)(t) = &u_0 + g(u) + \frac{u_1 t}{1-\rho} - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), (\phi u)(s), (\psi u)(s)) d_q s \\ &+ \frac{t}{1-\rho} \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s, u(s), (\phi u)(s), (\psi u)(s)) d_q s \\ &- \frac{t}{1-\rho} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s, u(s), (\phi u)(s), (\psi u)(s)) d_q s, \quad t \in [0,1]. \end{aligned} \tag{3.1}$$

Clearly, the fixed points of the operator T are solutions of problem (1.1).

To establish the main results, we need the following assumptions:

(H_1) There exist positive functions $L_1(t), L_2(t), L_3(t)$ such that

$$\begin{aligned} &\| f(t, u(t), (\phi u)(t), (\psi u)(t)) - f(t, v(t), (\phi v)(t), (\psi v)(t)) \| \\ &\quad \le L_1(t) \|u - v\| + L_2(t) \|\phi u - \phi v\| + L_3(t) \|\psi u - \psi v\|, \quad t \in [0,1], u, v \in X \end{aligned}$$

(H_2) $g : C \to X$ is continuous and there exists a constant b such that

$$\|g(u) - g(v)\| \le b \|u - v\|, \ u, v \in C$$

Further,

$$\gamma_0 = \sup_{t\in[0,1]} \left| \int_0^t \gamma(t,s) d_q s \right|, \quad \delta_0 = \sup_{t\in[0,1]} \left| \int_0^t \delta(t,s) d_q s \right|,$$

$$I_q^\alpha L = \max \left\{ \sup_{t\in[0,1]} \left| I_q^\alpha L_1(t) \right|, \sup_{t\in[0,1]} \left| I_q^\alpha L_2(t) \right|, \sup_{t\in[0,1]} \left| I_q^\alpha L_3(t) \right| \right\},$$

$$I_q^{\alpha-1} L(1) = \max \left\{ |I_q^{\alpha-1} L_1(1)|, |I_q^{\alpha-1} L_2(1)|, |I_q^{\alpha-1} L_3(1)| \right\},$$

$$I_q^{\alpha-1} L(\xi_i) = \max \left\{ \left|I_q^{\alpha-1} L_1(\xi_i)\right|, \left|I_q^{\alpha-1} L_2(\xi_i)\right|, \left|I_q^{\alpha-1} L_3(\xi_i)\right| \right\}, \xi_i = 1, 2, \ldots, m-2.$$

(H_3) There exists a number $\wedge \leq \kappa < 1$, where

$$\wedge = (1+\gamma_0+\delta_0)\left\{ I_q^{\alpha} L + \frac{1}{1-\rho}\left(I_q^{\alpha-1} L\left(1\right) + \sum_{i=1}^{m-2} b_i I_q^{\alpha-1} L\left(\xi_i\right) \right) \right\}.$$

(H_4) There exists a number Δ, where

$$\Delta = b + (1+\gamma_0+\delta_0)\left\{ I_q^{\alpha} L + \frac{1}{1-\rho}\left(I_q^{\alpha-1} L\left(1\right) + \sum_{i=1}^{m-2} b_i I_q^{\alpha-1} L\left(\xi_i\right) \right) \right\}.$$

(H_5) $\| f(t,u\left(t\right),\left(\phi u\right)(t),\left(\psi u\right)(t)) \| \leq \mu\left(t\right), \forall (t,u(t),\left(\phi u\right),\left(\psi u\right)) \in [0,1]\times X\times X\times X, \mu \in L^1([0,1],R^+)$.

Theorem 3.1. Assume that $f:[0,1]\times X\times X\times X \to X$ is jointly continuous function and satisfies the assumptions $(H_1)-(H_4)$, then problem (1.1) has a unique solution provided $\wedge < 1$, where $\wedge$ is given in the assumption (H_3).

Proof. Let us set $\sup_{t\in[0,1]}|f(t,0,0,0)| = M$, and choose

$$r \geq \frac{1}{1-\wedge}\left\{ \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + M\left[\frac{1}{\Gamma_q\left(\alpha+1\right)} + \frac{1}{\left(1-\rho\right)\Gamma_q\left(\alpha\right)}\left(1+\sum_{i=1}^{m-2} b_i \xi_i^{(\alpha-1)}\right)\right]\right\},$$

where λ is such that $\wedge \leq \lambda < 1$. We show that $TB_r \subset B_r$, where $B_r = \left\{x \in C : \| u \| \leq r\right\}$. So let $u \in B_r$ and set $G = \sup_{u\in C}\left\|g\left(u\right)\right\|$. Then we have

$$\begin{aligned}
&\left\|\left(Tu\right)\left(t\right)\right\| \\
&\leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left\|f(s,u(s),(\phi u)(s),(\psi u)(s))\right\| d_q s \\
&\quad + \frac{1}{1-\rho}\left\{ \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left\|f(s,u(s),(\phi u)(s),(\psi u)(s))\right\| d_q s \right. \\
&\quad \left. + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left\|f(s,u(s),(\phi u)(s),(\psi u)(s))\right\| d_q s \right\} \\
&\leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(\left\|f(s,u(s),\left(\phi u\right)\left(s\right),\left(\psi u\right)\left(s\right)) - f\left(s,0,0,0\right)\right\| + \left\|f\left(s,0,0,0\right)\right\|\right) d_q s \\
&\quad + \frac{1}{1-\rho}\left\{ \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\left\|f(s,u(s),\left(\phi u\right)\left(s\right),\left(\psi u\right)\left(s\right)) - f\left(s,0,0,0\right)\right\| + \left\|f\left(s,0,0,0\right)\right\|\right) d_q s \right. \\
&\quad \left. + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\left\|f(s,u(s),\left(\phi u\right)\left(s\right),\left(\psi u\right)\left(s\right)) - f\left(s,0,0,0\right)\right\| + \left\|f\left(s,0,0,0\right)\right\|\right) d_q s \right\} \\
&\leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(L_1\left(s\right)\left\|u\left(s\right)\right\| + L_2\left(s\right)\left\|\left(\phi u\right)\left(s\right)\right\| + L_3\left(s\right)\left\|\left(\psi u\right)\left(s\right)\right\| + M\right) d_q s \\
&\quad + \frac{1}{1-\rho}\left\{ \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(L_1\left(s\right)\left\|u\left(s\right)\right\| + L_2\left(s\right)\left\|\left(\phi u\right)\left(s\right)\right\| + L_3\left(s\right)\left\|\left(\psi u\right)\left(s\right)\right\| + M\right) d_q s \right. \\
&\quad \left. + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(L_1\left(s\right)\left\|u\left(s\right)\right\| + L_2\left(s\right)\left\|\left(\phi u\right)\left(s\right)\right\| + L_3\left(s\right)\left\|\left(\psi u\right)\left(s\right)\right\| + M\right) d_q s \right\}
\end{aligned}$$

$$\leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(L_1(s)\|u(s)\| + \gamma_0 L_2(s)\|u(s)\| + \delta_0 L_3(s)\|u(s)\| + M\right) d_q s$$
$$+ \frac{1}{1-\rho} \left\{ \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(L_1(s)\|u(s)\| + \gamma_0 L_2(s)\|u(s)\| + \delta_0 L_3(s)\|u(s)\| + M\right) d_q s \right.$$
$$\left. + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(L_1(s)\|u(s)\| + \gamma_0 L_2(s)\|u(s)\| + \delta_0 L_3(s)\|u(s)\| + M\right) d_q s \right\}$$
$$\leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + \left(I_q^\alpha L_1(t) + \gamma_0 I_q^\alpha L_2(t) + \delta_0 I_q^\alpha L_3(t)\right) r + \frac{Mt^{(\alpha)}}{\Gamma_q(\alpha+1)}$$
$$+ \frac{1}{1-\rho} \left\{ \left(I_q^{\alpha-1} L_1(1) + \gamma_0 I_q^{\alpha-1} L_2(1) + \delta_0 I_q^{\alpha-1} L_3(1)\right) r + \frac{M}{\Gamma_q(\alpha)} \right.$$
$$\left. + \sum_{i=1}^{m-2} b_i \left(\left(I_q^{\alpha-1} L_1(\xi_i) + \gamma_0 I_q^{\alpha-1} L_2(\xi_i) + \delta_0 I_q^{\alpha-1} L_3(\xi_i)\right) r + \frac{M\xi_i^{(\alpha-1)}}{\Gamma_q(\alpha)} \right) \right\}$$
$$\leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + I_q^\alpha L(1+\gamma_0+\delta_0) r + \frac{M}{\Gamma_q(\alpha+1)}$$
$$+ \frac{1}{1-\rho} \left\{ I_q^{\alpha-1} L(1)(1+\gamma_0+\delta_0) r + \frac{M}{\Gamma_q(\alpha)} \right.$$
$$\left. + \sum_{i=1}^{m-2} b_i \left(I_q^{\alpha-1} L(\xi_i)(1+\gamma_0+\delta_0) r + \frac{M\xi_i^{(\alpha-1)}}{\Gamma_q(\alpha)} \right) \right\}$$
$$\leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + (1+\gamma_0+\delta_0) \left\{ I_q^\alpha L + \frac{1}{1-\rho} \left(I_q^{\alpha-1} L(1) + \sum_{i=1}^{m-2} b_i I_q^{\alpha-1} L(\xi_i) \right) \right\} r$$
$$+ M \left\{ \frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{1-\rho} \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\sum_{i=1}^{m-2} b_i \xi_i^{(\alpha-1)}}{\Gamma_q(\alpha)} \right) \right\}$$
$$\leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + \wedge r + M \left\{ \frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{(1-\rho)\Gamma_q(\alpha)} \left(1 + \sum_{i=1}^{m-2} b_i \xi_i^{(\alpha-1)} \right) \right\} \leq r$$

Now, for $u, v \in C$ and for each $t \in [0,1]$, we obtain

$$\|(Tu)(t) - (Tv)(t)\|$$
$$\leq \|g(u) - g(v)\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|f(s,u(s),(\phi u)(s),(\psi u)(s)) - f(s,v(s),(\phi v)(s),(\psi v)(s))\| d_q s$$
$$+ \frac{1}{1-\rho} \left\{ \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \|f(s,u(s),(\phi u)(s),(\psi u)(s)) - f(s,v(s),(\phi v)(s),(\psi v)(s))\| d_q s \right.$$
$$\left. + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \|f(s,u(s),(\phi u)(s),(\psi u)(s)) - f(s,v(s),(\phi v)(s),(\psi v)(s))\| d_q s \right\}$$
$$\leq \|g(u) - g(v)\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(L_1(s)\|u-v\| + L_2(s)\|\phi u - \phi v\| + L_3(s)\|\psi u - \psi v\|\right) d_q s$$
$$+ \frac{1}{1-\rho} \left\{ \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(L_1(s)\|u-v\| + L_2(s)\|\phi u - \phi v\| + L_3(s)\|\psi u - \psi v\|\right) d_q s \right.$$
$$\left. + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(L_1(s)\|u-v\| + L_2(s)\|\phi u - \phi v\| + L_3(s)\|\psi u - \psi v\|\right) d_q s \right\}$$

$$
\begin{aligned}
&\leq \|g(u)-g(v)\| + \left(I_q^{\alpha}L_1(t) + \gamma_0 I_q^{\alpha}L_2(t) + \delta_0 I_q^{\alpha}L_3(t)\right)\|u-v\| \\
&\quad + \frac{1}{1-\rho}\left\{I_q^{\alpha-1}L_1(1) + \gamma_0 I_q^{\alpha-1}L_2(1) + \delta_0 I_q^{\alpha-1}L_3(1)\right. \\
&\quad \left. + \sum_{i=1}^{m-2} b_i\left(I_q^{\alpha-1}L_1(\xi_i) + \gamma_0 I_q^{\alpha-1}L_2(\xi_i) + \delta_0 I_q^{\alpha-1}L_3(\xi_i)\right)\right\}\|u-v\| \\
&\leq b\|u-v\| + (1+\gamma_0+\delta_0)\left\{I_q^{\alpha}L + \frac{1}{1-\rho}\left(I_q^{\alpha-1}L(1) + \sum_{i=1}^{m-2} b_i I_q^{\alpha-1}L(\xi_i)\right)\right\}\|u-v\| \\
&= \Delta\|u-v\|
\end{aligned}
$$

where $\wedge$ is given in the assumption (H_4). As $\Delta < 1$, therefore T is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle.

Now, we state Krasnoselskii's fixed point theorem which is needed to prove the existence of at least one solution of (1.1). $\square$

Theorem 3.2. Let K be a closed convex and nonempty subset of a Banach space X. Let T, S be the operators such that (i) $Tu + Sv \in K$ whenever $u, v \in K$; (ii) T is compact and continuous; (iii) S is a contraction mapping. Then there exists $z \in K$ such that $z = Tz + Sz$.

Theorem 3.3. Suppose that $f : [0,1] \times X \times X \times X \to X$ is jointly continuous and the assumptions $(H_1) - (H_2)$ and (H_5) hold with

$$
\wedge_1 = b + (1+\gamma_0+\delta_0)\left\{\frac{1}{1-\rho}\left(I_q^{\alpha-1}L(1) + \sum_{i=1}^{m-2} b_i I_q^{\alpha-1}L(\xi_i)\right)\right\} < 1.
$$

Then there exists at least one solution of the boundary value problem (1.1) on $[0,1]$.

Proof. Choose

$$
r \geq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + \|\mu\|_{L^1}\left\{\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{(1-\rho)\Gamma_q(\alpha)}\left(1 + \sum_{i=1}^{m-2} b_i \xi_i^{(\alpha-1)}\right)\right\}.
$$

and consider $\Omega_r = \{u \in C : \|u\| \leq r\}$. We define the operators T_1 and T_2 on Ω_r as

$$
\begin{aligned}
(T_1u)(t) &= u_0 - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), (\phi u)(s), (\psi u)(s))\, d_q s, \\
(T_2u)(t) &= g(u) + \frac{u_1 t}{1-\rho} + \frac{1}{1-\rho}\left\{\int_0^1 \frac{(1-qs)^{(\alpha-2)}\, t}{\Gamma_q(\alpha-1)} f(s, u(s), (\phi u)(s), (\psi u)(s)) d_q s\right. \\
&\qquad \left. - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-2)}\, t}{\Gamma_q(\alpha-1)} f(s, u(s), (\phi u)(s), (\psi u)(s)) d_q s\right\}.
\end{aligned}
$$

Let's observe that if $u, v \in \Omega_r$ then $T_1u + T_2v \in \Omega_r$. Indeed it is easy to check the inequality

$$
\|T_1u + T_2v\| \leq \|u_0\| + G + \frac{\|u_1\|}{1-\rho} + \|\mu\|_{L^1}\left\{\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{(1-\rho)\Gamma_q(\alpha)}\left(1 + \sum_{i=1}^{m-2} b_i \xi_i^{(\alpha-1)}\right)\right\} \leq r.
$$

By (H_1), it is also that T_2 is a contraction mapping for $\wedge_1 < 1$.

Since f is continuous, then $(T_1u)(t)$ is continuous. Let's now note that T_1 is uniformly bounded on Ω_r. This follows from the inequality

$$\|(T_1u)(t)\| \leq \|u_0\| + \frac{\|u_1\|}{1-\rho} + \frac{\|\mu\|_{L^1}}{\Gamma_q(\alpha+1)}.$$

Now,we show that $T_1(\Omega_r)$ is equicontinuous. The functions T_1u, $u \in \Omega_r$ are equicontinuous at $t=0$. For $t_1, t_2 \in \{q^n : n \in \mathbb{N}_{\not\vdash}\}$, and $t_1 < t_2$. Using the fact that f is bounded on the compact set $[0,1]\times\Omega_r\times\Omega_r\times\Omega_r$, therefore, we define $\sup_{(t,u,\phi u,\psi u)\in[0,1]\times\Omega_r\times\Omega_r\times\Omega_r} \|f(t,u,\phi u,\psi u)\| = f_{\max} < \infty$. We have

$$\begin{aligned}
\|(T_1u)(t_2) - (T_1u)(t_1)\| &= \Bigg\| \int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),(\phi u)(s),(\psi u)(s))\, d_qs \\
&\quad - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),(\phi u)(s),(\psi u)(s))\, d_qs \Bigg\| \\
&= \Bigg\| \int_0^{t_1} \frac{(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),(\phi u)(s),(\psi u)(s))\, d_qs \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),(\phi u)(s),(\psi u)(s))\, d_qs \Bigg\| \\
&\leq \frac{f_{\max}}{\Gamma_q(\alpha+1)} \left| t_1^{(\alpha)} - t_2^{(\alpha)} \right|.
\end{aligned}$$

which is independent of $u \in \Omega_r$ when $t_1 \to t_2$. Indeed, let $u \in \Omega_r$, we have

$$\lim_{t_1\to t_2} \frac{f_{max}}{\Gamma_q(\alpha+1)} \left| t_1^{(\alpha)} - t_2^{(\alpha)} \right| \to 0.$$

Therefore, $T_1(\Omega_r)$ is relatively compact on Ω_r. Hence, by Arzela-Ascoli's Theorem, T_1 is compact on Ω_r. Thus all the assumptions of Theorem 3.2 are satisfied and the conclusion of Theorem 3.2 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

4 Examples

Example 4.1. Consider the following boundary value problem

$$\begin{cases} {}_CD_{0.5}^{1.5}u(t) = \dfrac{t}{8}\dfrac{|u|}{1+|u|} + \dfrac{1}{5}\displaystyle\int_0^t \frac{e^{-(s-t)}}{5} u(s)\, ds + \frac{1}{5}\int_0^t \frac{e^{-\frac{s-t}{2}}}{5} u(s)\, ds, \ t \in [0,1], \\ u(0) = 0, D_{0.5}u(1) = b_1 D_{0.5}u(\xi_1), \end{cases} \tag{4.1}$$

Here, $\gamma(t,s) = \frac{e^{-(s-t)}}{5}$, $\delta(t,s) = \frac{e^{-\frac{s-t}{2}}}{5}$, $G = \frac{1}{10}$, $b_1 = \frac{1}{10}$, $\xi = \frac{1}{2}$. With

$$\gamma_0 = \frac{e-1}{5}, \quad \delta_0 = \frac{2(\sqrt{e}-1)}{5},$$

$$\sup_{t\in[0,1]} I_{0.5}^{1.5}L(t) \approx 0.167983, \quad I_{0.5}^{0.5}L(1) \approx 0.217185, \quad I_{0.5}^{0.5}L(1/2) \approx 0.141421,$$

we find that

$$\wedge = \frac{1}{10} + \left(1 + \frac{e-1}{5} + \frac{2(\sqrt{e}-1)}{5}\right)\left[0.167983 + \frac{1}{1-1/10}(0.217185 + 0.1\times 0.141421)\right] \approx 0.781358 < 1.$$

Thus, by Theorem 3.1, the boundary value problem (4.1) has a unique solution on $[0,1]$.

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