ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A QUASILINEAR HYPERBOLIC EQUATION WITH NONLINEAR DAMPING

Mohammed Aassila

Institut de Recherche Mathématique Avancée,
Université Louis Pasteur et C.N.R.S.,
7 rue René Descartes, 67084 Strasbourg Cédex, France.
E-mail: aassila@math.u-strasbg.fr

Abstract: We prove the existence and uniqueness of a global solution of a damped quasilinear hyperbolic equation. Key point to our proof is the use of the Yosida approximation. Furthermore, we apply a method based on a specific integral inequality to prove that the solution decays exponentially to zero when the time t goes to infinity.

Key words and phrases: nonlinear damping, global existence, Yosida approximation, integral inequality, exponential decay.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary Γ. In this paper we are concerned with the global existence and asymptotic behavior of solutions to the mixed problem

$$(P) \qquad \begin{cases} u'' - f(\|\nabla u\|_2^2)\Delta u + g(u') = h(x,t) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega. \end{cases}$$

Here $f(\cdot)$ is a C^1-class function satisfying $f(s) \geq m_0 > 0$ for $s \geq 0$ with m_0 constant, $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, $\Gamma_1 \neq \emptyset$, and g is a continuous increasing odd function such that $g'(x) \geq \tau > 0$.

Physically, the problem (P) occurs in the study of vibrations of flexible structures in a bounded domain. The motivation for incorporating internal material damping in the quasilinear wave equation as in the first equation of (P) arises from the fact that inherent small material damping is present in real materials. Hence from the physical point of view we say that internal structural damping force will appear so long as the system vibrates.

The problem (P) with $h = 0$, $g(x) = \delta x\,(\delta > 0)$ and $\Gamma_1 = \emptyset$ was studied by De Brito [3]. She has shown the existence and uniqueness of global solutions for sufficiently small initial data by using a Galerkin method. When $g(x) = \delta x$ and $\Gamma_0 = \emptyset$, Ikehata [4] has shown the existence of global solutions by a Galerkin method, the key point of his proof is to restrict (P) to the range of $-\Delta$ on which $-\Delta$ is positive definite. In fact the restricted problem can be solved by a Galerkin method exactly as in De Brito [3]. When g is nonlinear, Ikehata's approach seems to be very difficult. The author in [1] has been successful in proving the global existence and establishing the precise decay rate of solutions when $\Gamma_1 = \emptyset$, g is nonlinear without any smallness conditions on the initial data and without the assumption $g'(x) \geq \tau > 0$.

Our study is motivated by Ikehata and Okazawa's work [5] where global existence was proved when $g(x) = \delta x\,(\delta > 0)$ and Dirichlet or Neumann boundary condition by using the Yosida approximation method together with compactness arguments. In our work, the feedback g is nonlinear, and furthermore we study the asymptotic behavior of the global solution when $h = 0$.

The contents of this paper are as follows. In section 2, we give our main results. In section 3, we establish the existence of global solutions. In section 4, we study the asymptotic behavior of solutions of (P) with $h = 0$.

2. Statement of the main theorems

We define the energy of the solution u to problem (P) by the formula

$$E(t) = \tfrac{1}{2}\|u'(t)\|_2^2 + \bar{f}(\|\nabla u\|_2^2) \tag{2.1}$$

where $\bar{f} = \int_0^s f(t)\,dt$ and $\|\cdot\|_n$ denotes the usual norm of $L^n(\Omega)$. Our main results are

Theorem 2.1

For any $(u_0, u_1) \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega)$ and $h \in L^1(0,\infty; H^1(\Omega)) \cap L^\infty(0,\infty; L^2(\Omega))$ satisfying

$$\frac{B_1 C_1}{m_0}\left[(|\Delta u_0|^2 + \frac{1}{m_0}|\nabla u_1|^2)^{\frac{1}{2}} + \frac{1}{\sqrt{m_0}}\int_0^\infty |\nabla h|\,dt\right] < \tau \tag{2.2}$$

there exists a unique solution $u(t)$ on $[0,\infty)$ to problem (P) such that

$$u \in L^\infty(0,\infty; H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \cap BC([0,\infty), H^1_{\Gamma_0}(\Omega)),$$

$$u' \in L^\infty(0,\infty; H^1_{\Gamma_0}(\Omega)) \cap BC([0,\infty), L^2(\Omega)),$$

$$u'' \in L^\infty(0,\infty; L^2(\Omega)),$$

$$\begin{cases} u'' - f(\|\nabla u\|_2^2)\Delta u + g(u') = h \quad in \quad L^2(\Omega) \quad a.e.\ on\,(0,\infty) \\ u = 0 \quad on \quad \Gamma_0 \times \mathbb{R}_+, \\ \frac{\partial u}{\partial \nu} = 0 \quad on \quad \Gamma_1 \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad in \quad \Omega. \end{cases}$$

Here B_1, C_1 are positive constants defined by

$$C_1 = \sqrt{2E(0)} + \int_0^\infty |h|\,dt \tag{2.3}$$

$$B_1 = \max_{0 \le s \le \frac{C_1^2}{m_0}} |f'(s)|$$

and $BC([0,\infty); L^2(\Omega))$ denotes the set of all $L^2(\Omega)$- bounded continuous functions on $[0,\infty)$.

Theorem 2.2

In addition to the conditions in theorem 2.1, we assume that f is non-decreasing, $h = 0$ and

$$|g(x)| \le C_2|x| \quad if \quad |x| \le 1, \tag{2.5}$$

$$|g(x)| \le 1 + C_3|x|^q \quad if \quad |x| > 1 \ (q \ge 1) \tag{2.6}$$

then we have the decay property

$$E(t) \le E(0)e^{1-t/C_4} \quad for\ all \quad t \ge 0$$

where $C_4 = C(\Omega)(1 + E(0)^{\frac{q-1}{2q}}$.

Before giving the proofs, we recall the:

Lemma 2.3 ([5] Lemma 3.1)

Let A be a nonnegative selfadjoint operator in a Hilbert space H with the norm $|\cdot|$, A_λ its Yosida approximation and $(A_\lambda)^{\frac{1}{2}}$ the square root of A_λ $(\lambda > 0)$. Then

$$\|(A_\lambda)^{\frac{1}{2}}\| \le \frac{1}{\sqrt{\lambda}} \quad (\lambda > 0), \tag{2.8}$$

$$|v - J_\lambda^{\frac{1}{2}} v| \le \sqrt{\lambda}|(A_\lambda)^{\frac{1}{2}} v|, \quad v \in H, \tag{2.9}$$

here $J_\lambda = (I + \lambda A)^{-1}$ $(\lambda > 0)$.

Lemma 2.4 ([6] Theorem 8.1)

Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-increasing function and assume that there exists a constant* $T > 0$ *such that*

$$\int_t^\infty E(s)\, ds \le TE(t) \quad \forall t \in \mathbb{R}_+,$$

then

$$E(t) \le E(0)e^{1-\frac{t}{T}} \quad \forall t \ge T.$$

Lemma 2.5 ([5] Lemma 3.2)

Let F *and* G *be nonnegative continuous functions on* $[0, T]$. *If*

$$F(t)^2 \le C + \int_0^t F(s)G(s)\, ds \quad \textit{on} \quad [0, T],$$

then

$$F(t) \le \sqrt{C} + \frac{1}{2}\int_0^t G(s)\, ds \quad \textit{on} \quad [0, T],$$

where $C > 0$ *is a constant.*

Lemma 2.5 is a special case of an inequality that can be found in Bihari [2].

3. Global existence

Let $-\Delta_\lambda$ $(\lambda > 0)$ be the Yosida approximation of $-\Delta$ and $(\nabla_\lambda)^{1/2}$ be the square root of $-\Delta_\lambda$, that is $(\nabla_\lambda)^{1/2} = (-\Delta_\lambda)^{1/2}(I + \lambda\Delta)^{-1/2}$.

First, we solve the approximate problem

$$(P_\lambda) \qquad \begin{cases} u''_\lambda - f(\|\nabla_\lambda u_\lambda\|_2^2)\Delta_\lambda u_\lambda + g(u'_\lambda) = h \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u_\lambda(0) = u_0 \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega),\ u'_\lambda(0) = u_1 \in H^1_{\Gamma_0}(\Omega). \end{cases}$$

Problem (P_λ) can be easily solved by successive approximation method. Hence problem (P_λ) has a unique local solution $u_\lambda \in C^1([0, T_\lambda), L^2(\Omega))$ on some interval $[0, T_\lambda)$. We shall see that $u_\lambda(t)$ can be extended to $[0, \infty)$.

Lemma 3.1

Let C_1 *be the constant defined by (2.3). Then the following inequality holds*

$$|u'_\lambda|^2 + m_0|\nabla_\lambda u_\lambda|^2 + 2\tau\int_0^t |u'_\lambda|^2\, ds \le C_1^2. \tag{3.1}$$

Proof

Multiplying both sides of the first equation in (P_λ) by $2u'_\lambda$, we have

$$\frac{d}{dt}|u'_\lambda|^2 + f(|\nabla_\lambda u_\lambda|^2)\frac{d}{dt}|\nabla_\lambda u_\lambda|^2 + 2(g(u'_\lambda), u'_\lambda) = 2(h, u'_\lambda) \text{ a.e. on } [0, T_\lambda).$$

After integration on $[0, t]$, we see that

$$|u'_\lambda(t)|^2 + \bar{f}(|\nabla_\lambda u_\lambda|^2) + 2\tau \int_0^t |u'_\lambda(s)|^2\, ds \le 2E(0) + 2\int_0^t |h(s)||u'_\lambda(s)|\, ds.$$

It follows from lemma 2.5 that

$$|u'_\lambda|^2 + m_0|\nabla_\lambda u_\lambda|^2 + 2\tau\int_0^t |u'_\lambda(s)|^2\, ds \le \left(\sqrt{2E(0)} + \int_0^\infty |h(s)|\, ds\right)^2 := C_1^2.$$

Lemma 3.2

Set

$$Z_\lambda(t) = |\Delta_\lambda u_\lambda(t)|^2 + \frac{|\nabla_\lambda u'_\lambda(t)|^2}{f(|\nabla_\lambda u_\lambda(t)|^2)}.$$

Assume that on $[0, T_\lambda)$

$$\left|\frac{d}{dt} f(|\nabla_\lambda u_\lambda(t)|^2)\right| \le 2\tau f(|\nabla_\lambda u_\lambda(t)|^2) \tag{3.3}$$

then for $t \in [0, T_\lambda)$ *we have*

$$Z_\lambda(t)^{1/2} \le \left(|\Delta u_0|^2 + \frac{1}{m_0}|\nabla u_1|^2\right)^{1/2} + \frac{1}{\sqrt{m_0}}\int_0^\infty |\nabla h(s)|\, ds. \tag{3.4}$$

Proof

Multiplying the both sides of the first equation in (P_λ) by $-2\Delta_\lambda u'_\lambda(t)$, we have

$$\frac{d}{dt}|\nabla_\lambda u'_\lambda(t)|^2 + f(|\nabla_\lambda u_\lambda(t)|^2)\frac{d}{dt}|\Delta_\lambda u_\lambda(t)|^2 = 2(h, -\Delta_\lambda u'_\lambda) - 2g'(u'_{lambda})|\nabla_\lambda u'_\lambda(t)|^2.$$

It follows that

$$f(|\nabla_\lambda u_\lambda(t)|^2) Z'_\lambda(t) \le 2|\nabla_\lambda h||\nabla_\lambda u'_\lambda(t)| - \left[\frac{\frac{d}{dt} f(|\nabla_\lambda u_\lambda|^2)}{f(|\nabla_\lambda u_\lambda|^2)} + 2\tau\right]|\nabla_\lambda u'_\lambda(t)|^2.$$

By (3.3) we obtain

$$Z'_\lambda(t) \le \frac{2}{\sqrt{m_0}}|\nabla_\lambda h| Z_\lambda(t)^{1/2}.$$

Integrating this inequality on $[0,t]$, we have

$$Z_\lambda(t) \le Z_\lambda(0) + \frac{2}{\sqrt{m_0}} \int_0^t |\nabla_\lambda h| Z_\lambda(s)^{1/2}\, ds.$$

Since $Z_\lambda(0) \le |\Delta u_0| + \frac{1}{m_0}|\nabla u_1|^2$, it follows from lemma 2.5 that

$$Z_\lambda(t)^{1/2} \le (|\Delta u_0|^2 + \frac{1}{m_0}|\nabla u_1|^2)^{1/2} + \frac{1}{\sqrt{m_0}} \int_0^t |\nabla_\lambda h|\, ds,$$

then we obtain (3.4).

Lemma 3.3

Set

$$\alpha_\lambda(t) = \frac{1}{f(|\nabla_\lambda u_\lambda|^2)} \left| \frac{d}{dt} f(|\nabla_\lambda u_\lambda|^2) \right|.$$

If (2.2) is satisfied, then we have

$$\alpha_\lambda(t) < 2\tau \quad on \quad [0, T_\lambda).$$

Proof

First we show that

$$\alpha_\lambda(0) < 2\tau.$$

Since

$$\left| \frac{d}{dt} f(|\nabla_\lambda u_\lambda|^2) \right| \le 2B_1 |u_\lambda'||\Delta_\lambda u_\lambda| \le 2B_1 C_1 Z_\lambda(t)^{1/2},$$

we have by definition

$$\alpha_\lambda(t) \le 2B_1 C_1 \frac{Z_\lambda(t)^{1/2}}{f(|\nabla_\lambda u_\lambda|^2)} \le \frac{2}{m_0} B_1 C_1 Z_\lambda(t)^{1/2}. \tag{3.8}$$

Setting $t = 0$ in (3.7), we see from (2.2) that $\alpha_\lambda(0) < 2\tau$. Now suppose that (3.6) does not hold on $[0, T_\lambda)$. Since $\alpha_\lambda(t)$ is continuous, (3.7) implies that there is a $t^* > 0$ such that $\alpha_\lambda(t) < 2\tau$ on $[0, t^*)$, and

$$\alpha_\lambda(t^*) = 2\tau \tag{3.9}$$

i.e. (3.3) is satisfied on $[0, t^*]$. Therefore, it follows from lemma 3.2 and (2.2) that

$$Z_\lambda(t^*)^{1/2} < \frac{m_0}{B_1 C_1}\tau. \tag{3.10}$$

Combining (3.10) with (3.8), we obtain $\alpha_\lambda(t^*) < 2\tau$. This contradicts (3.9).

Lemma 3.4

Set

$$C_5 = (|\Delta u_0|^2 + \frac{1}{m_0}|\nabla u_1|^2)^{1/2} + \frac{1}{\sqrt{m_0}}\int_0^\infty |\nabla h|.$$

If (2.2) is satisfied, $B_1C_1C_5 < m_0\tau$*, then*

$$|\Delta_\lambda u_\lambda(t)|^2 + \frac{1}{B_0}|\nabla_\lambda u'_\lambda(t)|^2 \leq C_5^2 \quad on \quad [0, T_\lambda),$$

where we set $B_0 = \max_{0\leq s\leq \frac{C_1^2}{m_0}} M(s)$*, and hence*

$$|u''_\lambda(t)| \leq B_0C_5 + C_6 + ess\ sup\{h(s) : 0 \leq s < \infty\} \tag{3.12}$$

where $C_6 = \max_{0\leq x\leq C_1} |g(x)|$.

Proof

It follows from lemmas 3.2 and 3.3 that $Z_\lambda(t) \leq C_5^2$, so we obtain (3.11). Next, multiplying the both sides of the first equation in (P_λ) by $u''_\lambda(t)$, we have

$$|u''_\lambda(t)|^2 = (h - g(u'_\lambda) - f(|\nabla_\lambda u_\lambda|^2)\Delta_\lambda u_\lambda, u''_\lambda) \quad \text{a.e. on} \quad (0, T_\lambda).$$

Therefore, (3.12) follows from lemma 3.1 and lemma 3.4.

Lemma 3.5

Assume that (2.2) is satisfied. Then for any $\lambda > 0$ *there exists a unique global solution* $u_\lambda \in C^1([0,\infty), L^2(\Omega))$ *of the approximate problem* (P_λ) *such that* $u'_\lambda(\cdot)$ *is locally absolutely continuous on* $[0,\infty)$ *and the first equation in* (P_λ) *holds a.e. on* $[0,\infty)$.

Proof

Let $u_\lambda(t)$ be a solution of (P_λ) on $[0, T_\lambda)$. Since $u'_\lambda(t)$ and $u''_\lambda(t)$ are uniformly bounded in $L^2(\Omega)$, $u_\lambda(T_\lambda)$ and $u'_\lambda(T_\lambda)$ exist and we can choose them as new initial values. Moreover, since $u_\lambda(t)$ is uniformly bounded, the local Lipschitz continuity of the mapping $u \to f(|\nabla_\lambda u|^2)\Delta_\lambda u$ is always verified. Therefore, $u_\lambda(t)$ can be extended onto the semi-infinite interval $[0,\infty)$.

Lemma 3.6

There is a subsequence $\{u_{\lambda_n}(\cdot)\}$ *of* $\{u_\lambda(\cdot)\}$ *and* $u(\cdot) \in BC([0,\infty), L^2(\Omega))$ *such that for any* $T > 0$

$$u_{\lambda_n}(\cdot) \to u(\cdot) \quad in \quad C([0,T], L^2(\Omega)) \quad as \quad n \to \infty, \tag{3.13}$$

where $\lambda_n > 0$ $(n \in \mathbb{N})$ and $\lambda_n \to 0\,(n \to \infty)$, $BC([0,\infty), L^2(\Omega))$ is the set of all L^2-valued bounded continuous functions on $[0,\infty)$.

Proof

By the fact that $\|u_\lambda\|_2$ is bounded on $[0, T_\lambda)$ and lemma 3.1, it follows that $J_\lambda^{1/2} u_\lambda$ and $\nabla_\lambda u_\lambda$ belong to $BC([0,\infty), L^2(\Omega))$. By the definition of ∇_λ we have

$$J_\lambda^{1/2} u_\lambda(t) = (I + \nabla)^{-1}(J_\lambda^{1/2} u_\lambda(t) + \nabla_\lambda u_\lambda(t)),$$

this implies that for each $t > 0$, $\{J_\lambda^{1/2} u_\lambda(t)\}$ is bounded in $H^1_{\Gamma_0}(\Omega)$, and then relatively compact in $L^2(\Omega)$. As $\{J_\lambda^{1/2} u_\lambda(\cdot)\}$ is equicontinuous, we can apply the Ascoli-Arzela theorem to $\{J_\lambda^{1/2} u_\lambda(\cdot)\}$ in $C([0,T], L^2)$ for any $T > 0$. Thus, there exist a subsequence $\{J_{\lambda_n}^{1/2} u_{\lambda_n}(\cdot)\}$ and $u(\cdot) \in BC([0,\infty), L^2)$ such that for any $T > 0$

$$J_{\lambda_n}^{1/2} u_{\lambda_n}(\cdot) \to u(\cdot) \quad \text{in} \quad C([0,T], L^2) \quad \text{as} \quad n \to \infty. \tag{3.14}$$

By (2.9) we conclude that for any $T > 0$

$$u_{\lambda_n}(\cdot) \to u(\cdot) \quad \text{in} \quad C([0,T], L^2) \quad \text{as} \quad n \to \infty.$$

Lemma 3.7

Let $\{\lambda_n\}$ and $u(\cdot)$ be as in lemma 3.6. Assume that (2.2) is satisfied. Then $u(\cdot) \in BC^1([0,\infty), L^2)$ and there is a subsequence $\{\mu_n\}$ of $\{\lambda_n\}$ such that for any $T > 0$

$$u'_{\mu_n}(\cdot) \to u'(\cdot) \quad \textit{in} \quad C([0,T], L^2) \quad \textit{as} \quad n \to \infty. \tag{3.15}$$

Furthermore, $u(\cdot) \in L^\infty(0,\infty; H^2 \cap H^1_{\Gamma_0})$, $u'(\cdot) \in L^\infty(0,\infty; H^1_{\Gamma_0})$ and

$$\Delta_{\lambda_n} u_{\lambda_n} \to \Delta u \quad \textit{weakly in} \quad L^2 \quad \textit{as} \quad n \to \infty, \tag{3.16}$$

$$\nabla_{\mu_n} u'_{\mu_n} \to \nabla u' \quad \textit{weakly in} \quad L^2 \quad \textit{as} \quad n \to \infty. \tag{3.17}$$

Here $BC^1([0,\infty), L^2) := \{u \in BC([0,\infty), L^2);\ u' \in BC([0,\infty), L^2)\}$.

Proof

As $J_{\lambda_n}^{1/2} u'_{\lambda_n}$ and $\nabla_{\lambda_n} u'_{\lambda_n}$ belong to $BC([0,\infty), L^2)$ and that $J_{\lambda_n}^{1/2} u''_{\lambda_n} \in L^\infty(0,T; L^2)$, (3.15) can be proved in the same way as in the proof of lemma 3.6, in fact we have

$$u(t) = u_0 + \int_0^t v(s)\,ds \quad \text{with} \quad v(s) = \lim_{n\to\infty} u'_{\mu_n}(s).$$

Since $\Delta_{\lambda_n} u_{\lambda_n}$ and $\nabla_{\mu_n} u'_{\mu_n}$ belong to $BC([0,\infty), L^2)$, (3.16) and (3.17) follow from (3.13) and (3.15) respectively. Therfore we have

$$|\Delta u| \le \liminf_{n\to\infty} |\Delta_{\lambda_n} u_{\lambda_n}| \le C_5, \tag{3.18}$$

$$|\nabla u'| \le \liminf_{n\to\infty} |\nabla_{\mu_n} u'_{\mu_n}| \le \sqrt{B_0} C_5 \tag{3.19}$$

i.e. $u \in L^\infty(0,\infty; H^2 \cap H^1_{\Gamma_0})$ and $u' \in L^\infty(0,\infty; H^1_{\Gamma_0})$.

Lemma 3.8

Let u and $\{\lambda_n\}$ be as in lemma 3.6. Assume that (2.2) is satisfied. Then $u \in BC([0,\infty); H^1_{\Gamma_0})$ and for any $T > 0$

$$\nabla_{\lambda_n} u_{\lambda_n} \to \nabla u \quad in \quad C([0,T], L^2) \quad as \quad n \to \infty, \tag{3.20}$$

and hence

$$f(|\nabla u|^2)\Delta u = weak \lim_{n\to\infty} f(|\nabla_{\lambda_n} u_{\lambda_n}|^2)\Delta_{\lambda_n} u_{\lambda_n}. \tag{3.21}$$

Proof

We have

$$|\nabla_{\lambda_n} u_{\lambda_n} - \nabla u|^2 = |\nabla_{\lambda_n} u_{\lambda_n}|^2 - |\nabla u|^2 + 2(u - J^{1/2}_{\lambda_n} u_{\lambda_n}, -\Delta u).$$

By (3.14) and (3.18)-(3.19) it suffices to show that

$$|\nabla_{\lambda_n} u_{\lambda_n}|^2 \to |\nabla u|^2 \quad \text{in} \quad C([0,T]) \quad \text{as} \quad n \to \infty, \tag{3.22}$$

which is equivalent to

$$(u_{\lambda_n}, -\Delta_{\lambda_n} u_{\lambda_n}) \to (u, -\Delta u) \quad \text{in} \quad C([0,T]) \quad \text{as} \quad n \to \infty.$$

But since

$$(u,-\Delta u) - (u_{\lambda_n}, -\Delta_{\lambda_n} u_{\lambda_n}) = (u - J_{\lambda_n} u_{\lambda_n}, -\Delta u) + (u_{\lambda_n}, -\Delta_{\lambda_n} u + \Delta_{\lambda_n} u_{\lambda_n})$$
$$= (u - J_{\lambda_n} u, -\Delta u) + (u - u_{\lambda_n}, -\Delta_{\lambda_n} u) + (-\Delta_{\lambda_n} u_{\lambda_n}, u - u_{\lambda_n}),$$

we have

$$|(u,-\Delta u) - (u_{\lambda_n}, -\Delta_{\lambda_n} u_{\lambda_n})| \le \lambda_n |-\Delta u|^2 + (|-\Delta u| + |-\Delta_{\lambda_n} u_{\lambda_n}|)|u - u_{\lambda_n}|.$$

Hence (3.22) follows from (3.13) and (3.18)-(3.19). Thus we obtain (3.20). Furthermore as

$$|\nabla u| = \lim_{n\to\infty} |\nabla_{\lambda_n} u_{\lambda_n}| \le \frac{C_1}{\sqrt{m_0}},$$

we see that $u \in BC([0,\infty), H^1_{\Gamma_0})$. For the proof of (3.21), since $f(\cdot)$ is of class C^1, we can use the mean value theorem, and so the proof follows from (3.1), (3.16) and (3.22).

Lemma 3.9

Let u and $\{\mu_n\}$ be as in lemma 3.7, then u' has a (strong) derivative $u'' \in L^\infty(0,\infty; L^2)$ and

$$u''_{\mu_n} \to u'' \quad \textit{weakly in} \quad L^2 \quad \textit{as} \quad n \to \infty \; a.e. \tag{3.23}$$

and hence

$$u'' - f(\|\nabla u\|_2^2)\Delta u + g(u') = h \quad a.e. \tag{3.24}$$

Proof

¿From (3.12), we note that u' is Lipschitz continuous. Therefore u' is differentiable a.e. on $(0,\infty)$ with $u'' \in L^\infty(0,\infty; L^2)$. It follows from the previous lemma that

$$u''_{\mu_n} \to w \quad \text{weakly in} \quad L^2 \; (n \to \infty),$$

where $w = h - g(u') + f(\|\nabla u\|_2^2)\Delta u$. So we see from the Banach-Steinhauss theorem that

$$\int_t^{t+h} (w(s), z)\, ds = \lim_{n\to\infty} \int_t^{t+h} (u''_{\mu_n}, z)\, ds \quad z \in L^2.$$

It then follows from (3.15) that

$$\frac{1}{h}\int_t^{t+h} (w(s), z)\, ds = \frac{(u'(t+h) - u'(t), z)}{h}.$$

Passing to the limit $h \to 0$, we obtain $w = u''$ a.e. on $(0,\infty)$.

Lemma 3.10

Let u be as in lemma 3.6. Assume that (2.2) is satisfied, then u is the unique solution to problem (P).

The proof follows immediately from the Gronwall's lemma.

4. Asymptotic behavior

In this section we consider the problem

$$(P1)\qquad \begin{cases} u'' - f(\|\nabla u\|_2^2)\Delta u + g(u') = 0 \quad \text{in} \quad \Omega\times\mathbb{R}_+, \\ u = 0 \quad \text{on} \quad \Gamma_0\times\mathbb{R}_+, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1\times\mathbb{R}_+, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega. \end{cases}$$

The energy defined by (2.1) is such that

$$E'(t) = -\int_\Omega u' g(u')\,dx \le 0,$$

hence the energy is non-increasing.

Multiplying the first equation in (P1) with u and integrating by parts, we obtain

$$(4.1)\qquad 2\int_0^T E(t)\,dt = -\left[\int_\Omega uu'\,dx\right]_0^T + 2\int_0^T\int_\Omega (u'^2 - ug(u'))\,dxdt +$$

$$+\int_0^T\int_\Omega \left(\int_0^{|\nabla u|^2} f(s)\,ds\right)dxdt - \int_0^T\int_\Omega f(|\nabla u|^2)|\nabla u|^2\,dxdt$$

for all $0 < T < +\infty$.

Whence, since f is non-decreasing, we obtain

$$(4.2)\qquad 2\int_0^T E(t)\,dt \le -\left[\int_\Omega uu'\,dx\right]_0^T + 2\int_0^T\int_\Omega (u'^2 - ug(u'))\,dxdt.$$

¿From now on, we shall denote by $c(\Omega)$ different positive constants which depend only on Ω. It is easy to verify that

$$(4.3)\qquad -\left[\int_\Omega uu'\,dx\right]_0^T + 2\int_0^T\int_\Omega u'^2\,dxdt \le c(\Omega)E(0).$$

By hypotheses (2.5)-(2.6) we have

$$(4.4)\qquad \left|\int_\Omega ug(u')\,dx\right| \le c(\Omega)E^{1/2}|E'|^{1/2} + c(\Omega)E^{1/2}|E'|^{q/(q+1)}.$$

We apply the Young inequality to the two terms of the RHS of (4.4), we obtain

$$(4.5)\qquad c(\Omega)E^{1/2}|E'|^{1/2} \le c(\Omega)|E'| + \frac{1}{3}E,$$

and

$$c(\Omega)E^{1/2}|E'|^{q/(q+1)} = c(\Omega)(|E'|^{\frac{q}{q+1}}E^{\frac{q-1}{2(q+1)}})(E^{\frac{1}{q+1}}) \tag{4.6}$$

$$\leq c(\Omega)E(0)^{\frac{q-1}{2q}}|E'| + \frac{1}{3}E.$$

Therfore, we conclude that

$$\int_0^T E(t)\,dt \leq c(\Omega)(1 + E(0)^{\frac{q-1}{2q}})E(0), \tag{4.7}$$

that is

$$\int_0^{+\infty} E(t)\,dt \leq c(\Omega)(1 + E(0)^{\frac{q-1}{2q}})E(0),$$

and by lemma 2.4 we arrive at

$$E(t) \leq E(0)e^{1-\frac{t}{\gamma}} \quad \forall t \geq 0$$

with $\gamma = c(\Omega)(1 + E(0)^{\frac{q-1}{2q}})$.

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