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On the existence of mild solutions for nonconvex fractional semilinear differential inclusions

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Abstract

We establish some Filippov type existence theorems for solutions of fractional semilinear differential inclusions involving Caputo's fractional derivative in Banach spaces.

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1 Introduction

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations of fractional order ([21, 22, 24] etc.). The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim ([16]). Very recently several qualitative results for fractional differential inclusions were obtained in [1, 3, 7-12,

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19, 23] etc.. Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced in [5] and afterwards adopted in the theory of linear visco elasticity, satisfies this demand. For a consistent bibliography on this topic, historical remarks and examples we refer to [1].

The study of theory of abstract differential equations with fractional derivatives in infinite dimensional spaces is also very recent. The main problem consists in how to introduce new concepts of mild solutions. One of the first paper on this topic is [15]. In [20] it is showed that several papers on fractional differential equations in Banach spaces were incorrect and used an approach to treat these equations based on the theory of resolvent operators for integral equations. A suitable definition of mild solutions based on Laplace transform and probability density functions may be found in [26-29].

In this paper we study fractional semilinear differential inclusions of the form

$$D_c^r x(t) \in Ax(t) + F(t, x(t)) \quad t \in I, \quad x(0) = x_0 \tag{1.1}$$

where $I = [0, T]$, X is a separable Banach space, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$, $F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map and D_c^r is the Caputo fractional derivative of order $r \in (0, 1]$.

The aim of the present paper is twofold. On one hand, we show that Filippov's ideas ([17]) can be suitably adapted in order to obtain the existence of a solution of problem (1.1). We recall that for a first order differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([17]) consists in proving the existence of o solution starting from a given "almost" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. On the other hand, we prove the existence of solutions continuously depending on a parameter for problem (1.1). This result may be interpreted as a continuous variant of Filippov's theorem for problem (1.1). The key tool in the proof of this theorem is a result of Bressan and Colombo ([4]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values. This result allows to obtain a continuous selection of the solution set of the problem considered.

Our results may be interpreted as extensions of previous results of Frankowska ([18]) and Staicu ([25]) obtained for "classical" semilinear differential inclusions.

The paper is organized as follows: in Section 2 we briefly recall some preliminary results that we will use in the sequel and in Section 3 we prove the main results of the paper.

2 Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (Y, d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset Y$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B);\, a \in A\}$, where $d(x, B) = \inf\{d(x, y); y \in B\}$. With $cl(A)$ we denote the closure of the set $A \subset X$.

Let I be the interval $[0, T]$, $T > 0$, denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. Denote by B the closed unit ball in X. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0, 1\}$ denotes the characteristic function of A.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|dt$.

Recall that a subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$. We denote by $\mathcal{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a set-valued map. Recall that $F(.,.)$ is called $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable if for any closed subset $C \subset X$ we have $\{(t, x) \in I \times X; F(t, x) \cap C \neq \emptyset\} \in \mathcal{L}(I) \otimes \mathcal{B}(X)$.

We recall next the following definitions. For more details, we refer to [21].

Definition 2.1. a) The fractional integral of order $r > 0$ of a Lebesgue integrable function $f : (0, \infty) \to \mathbf{R}$ is defined by

$$I^r f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(r)} f(s)ds, \quad t > 0, r > 0$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

b) The Riemann-Liouville derivative of order r of $f(.) \in L^1(I, \mathbf{R})$ is defined by

$$D_L^r f(t) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{r+1-n}} ds, \quad t > 0, \quad n-1 < r < n.$$

c) The Caputo fractional derivative of order r of $f(.) \in L^1(I, \mathbf{R})$ is defined by

$$D_c^r f(t) = D_L^r (f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)) \quad t > 0, \quad n-1 < r < n.$$

Remark 2.2. a) If $f(.) \in C^n([0, \infty), \mathbf{R})$ then $D_c^r f(t) = I^{n-r} f^{(n)}(t)$, $t > 0$, $n-1 < r < n$.

b) The Caputo derivative of a constant is equal to zero.

c) If $f : I \to X$, with X a Banach space, then integrals which appears in Definition 2.1 are taken in Bochner's sense.

Consider $A : D(A) \to X$ the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ and let $M \geq 0$ be such that $\sup_{t \in I} |T(t)| \leq M$.

Definition 2.3. A continuous function $x(.) \in C(I, X)$ is called a *mild solution* of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I, X)$ such that $f(t) \in F(t, x(t))$ *a.e.* (I) and

$$x(t) = S_1(t)x_0 + \int_0^t (t-u)^{r-1} S_2(t-u) f(u) du \quad \forall t \in I, \tag{2.1}$$

where

$$S_1(t) = \int_0^\infty \xi_r(\theta) T(t^r \theta) d\theta, \quad S_2(t) = r \int_0^\infty \theta \xi_r(\theta) T(t^r \theta) d\theta,$$

$$\xi_r(\theta) = \frac{1}{r} \theta^{-1-\frac{1}{r}} \omega_r(\theta^{-\frac{1}{r}}) \geq 0,$$

$$\omega_r(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-rn-1} \frac{\Gamma(nr+1)}{n!} sin(n\pi r), \quad \theta > 0$$

and ξ_r is a probability density function defined on $(0, \infty)$, i.e. $\xi_r(\theta) \geq 0$, $\theta \in (0, \infty)$ and $\int_0^\infty \xi_r(\theta) d\theta = 1$.

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) and we denote by $\mathcal{S}(x_0)$ the solution set of problem (1.1).

The results summarized in the next lemmas will be used in the proof of our main results.

Lemma 2.4. ([28,29]) *a) For any fixed* $t \geq 0$, $S_1(t)$ *and* $S_2(t)$ *are linear and bounded operators, i.e. for any* $x \in X$

$$|S_1(t)x| \leq M|x|, \quad |S_2(t)x| \leq \frac{M}{\Gamma(r)}|x|.$$

b)$\{S_1(t), t \geq 0\}$ *and* $\{S_2(t), t \geq 0\}$ *are strongly continuous.*

c) If $T(t), t \geq 0$ *is compact, then* $S_1(t), t \geq 0$ *and* $S_2(t), t \geq 0$ *are also compact operators.*

Lemma 2.5. ([18]) *Let* X *be a separable Banach space, let* $H : I \to \mathcal{P}(X)$ *be a measurable set-valued map with nonempty closed values and* $g, h : I \to X, L : I \to (0, \infty)$ *measurable functions. Then one has*

i) The function $t \to d(h(t), H(t)$ *is measurable.*

ii) If $H(t) \cap (g(t) + L(t)B) \neq \emptyset$ *a.e.* (I) *then the set-valued map* $t \to H(t) \cap (g(t) + L(t)B)$ *has a measurable selection.*

Moreover, if $F(.,.) : I \times X \to \mathcal{P}(X)$ *has nonempty closed values,* $F(., x)$ *is measurable for any* $x \in X$ *and* $x(.) \in C(I, X)$ *then the set-valued map* $t \to F(t, x(t))$ *is measurable.*

Next (S, d) is a separable metric space; we recall that a multifunction $G(\cdot) : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed.

Lemma 2.6. ([4]) *Let* $F^*(.,.) : I \times S \to \mathcal{P}(X)$ *be a closed-valued* $\mathcal{L}(I) \otimes \mathcal{B}(S)$*-measurable multifunction such that* $F^*(t,.)$ *is l.s.c. for any* $t \in I$.

Then the multifunction $G(.) : S \to \mathcal{D}(I, X)$ *defined by*

$$G(s) = \{v \in L^1(I, X); \quad v(t) \in F^*(t, s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p(.) : S \to L^1(I, \mathbf{R})$ *such that*

$$d(0, F^*(t, s)) \leq p(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 2.7. ([4]) *Let* $G(.) : S \to \mathcal{D}(I, X)$ *be a l.s.c. multifunction with closed decomposable values and let* $\phi(.) : S \to L^1(I, X)$, $\psi(.) : S \to L^1(I, \mathbf{R})$

be continuous such that the multifunction $H(.) : S \to \mathcal{D}(I, X)$ defined by

$$H(s) = cl\{v(.) \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e.\ (I)\}$$

has nonempty values.

Then $H(.)$ has a continuous selection, i.e. there exists a continuous mapping $h(.) : S \to L^1(I, X)$ such that $h(s) \in H(s)$ $\forall s \in S$.

3 The main results

In order to obtain a Filippov type existence result for problem (1.1) one need the following assumptions.

Hypothesis 3.1. i) The operator A generates a strongly continuous semigroup $\{T(t), t \geq 0\}$ on a real separable Banach space X and there exists a constant $M \geq 1$ such that $\sup_{t \in I} |T(t)| \leq M$.

ii) $F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map with non-empty closed values and for all $x \in X$, $F(., x)$ is measurable.

iii) There exists $l(.) \in L^1(I, \mathbf{R}_+)$ with $L := \sup_{t \in I} I^r l(t) < +\infty$ and for almost all $t \in I$, $F(t, \cdot)$ is $l(t)$ - Lipschitz in the sense that

$$d_H(F(t, x_1), F(t, x_2)) \leq l(t)|x_1 - x_2|, \quad \forall x_1, x_2 \in X.$$

In what follows $g(.) \in L^1(I, X)$ is given such that there exists $\lambda(.) \in L^1(I, \mathbf{R}_+)$ with $\Lambda := \sup_{t \in I} I^r \lambda(t) < +\infty$ which satisfies

$$d(g(t), F(t, y(t))) \leq \lambda(t) \quad a.e.\ (I),$$

where $y(.)$ is a solution of the fractional semilinear differential equation

$$D_c^r y(t) = Ay(t) + g(t) \quad t \in I, \quad y(0) = y_0,$$

with $y_0 \in X$.

Theorem 3.2. *Let Hypothesis 3.1 be satisfied, $ML < 1$ and consider $g(.)$, $\lambda(.)$, $y(.)$ y_0 as above.*

Then for any $\varepsilon > 0$ there exists $(x(.), f(.))$ a trajectory-selection pair of problem (1.1) such that

$$|x(t) - y(t)| \leq \frac{M(|x_0 - y_0| + \Lambda + \frac{T^r}{\Gamma(r+1)}\varepsilon)}{1 - ML}, \quad \forall t \in I, \tag{3.1}$$

$$|f(t) - g(t)| \leq \frac{l(t)M(|x_0 - y_0| + \Lambda + \frac{T^r}{\Gamma(r+1)}\varepsilon)}{1 - ML} + \lambda(t) + \varepsilon, \quad a.e.\ (I). \tag{3.2}$$

Proof. Let $\varepsilon > 0$, $m_0 = M(|x_0 - y_0| + \Lambda + \frac{T^r}{\Gamma(r+1)}\varepsilon)$.

We claim that is enough to construct the sequences $x_n(.) \in C(I, X)$, $f_n(.) \in L^1(I, X)$, $n \geq 1$ with the following properties below

$$x_n(t) = S_1(t)x_0 + \int_0^t (t-s)^{r-1} S_2(t-s) f_n(s) ds, \quad \forall t \in I, \tag{3.3}$$

$$|x_1(t) - x_0(t)| \leq m_0 \quad \forall t \in I, \tag{3.4}$$

$$|f_1(t) - f_0(t)| \leq \lambda(t) + \varepsilon \quad a.e.\ (I), \tag{3.5}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad a.e.\ (I),\ n \geq 1, \tag{3.6}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \quad a.e.\ (I), n \geq 1. \tag{3.7}$$

Indeed, from (3.3), (3.4) and (3.7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^t (t - t_1)^{r-1} |S_2(t - t_1)|.|f_{n+1}(t_1) - f_n(t_1)| dt_1 \leq$$

$$\frac{M}{\Gamma(r)} \int_0^t (t - t_1)^{r-1} |f_{n+1}(t_1) - f_n(t_1)| dt_1 \leq \frac{M}{\Gamma(r)} \int_0^t (t - t_1)^{r-1} l(t_1)|x_n(t_1)$$

$$-x_{n-1}(t_1)| dt_1 \leq \frac{M}{\Gamma(r)} \int_0^t (t - t_1)^{r-1} l(t_1) \frac{M}{\Gamma(r)} \int_0^{t_1} (t_1 - t_2)^{r-1} l(t_2)$$

$$|f_n(t_1) - f_{n-1}(t_1)| dt_2 dt_1 \leq M^n \frac{1}{\Gamma(r)} \int_0^t (t - t_1)^{r-1} l(t_1) \frac{1}{\Gamma(r)} \int_0^{t_1} (t_1 - t_2)^{r-1} l(t_2)$$

$$... \frac{1}{\Gamma(r)} \int_0^{t_{n-1}} (t_{n-1} - t_n)^{r-1} l(t_n) m_0 dt_n ... dt_1 \leq M^n m_0 L^n = (ML)^n m_0.$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I, X)$. Thus, from (3.7) for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in X. Moreover, from (3.4) and the last inequality we have

$$\begin{aligned} &|x_n(t) - y(t)| \leq |x_1(t) - y(t)| + \textstyle\sum_{i=2}^{n-1} |x_{i+1}(t) - x_i(t)| \\ &\leq m_0(1 + ML + (ML)^2 + ...) = \tfrac{m_0}{1-ML}. \end{aligned} \tag{3.8}$$

On the other hand, from (3.5), (3.7) and (3.8) we obtain for almost all $t \in I$

$$\begin{aligned} &|f_n(t) - g(t)| \leq \textstyle\sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - g(t)| \leq \\ &l(t) \textstyle\sum_{i=1}^{n-2} |x_i(t) - x_{i-1}(t)| + \lambda(t) + \varepsilon \leq l(t) \tfrac{m_0}{1-ML} + \lambda(t) + \varepsilon. \end{aligned} \tag{3.9}$$

Let $x(.) \in C(I, X)$ be the limit of the Cauchy sequence $x_n(.)$. From (3.9) the sequence $f_n(.)$ is integrably bounded and we have already proved that for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in X. Take $f(.) \in L^1(I, X)$ with $f(t) = \lim_{n\to\infty} f_n(t)$.

Using the fact that the values of $F(.,.)$ are closed we get that $f(t) \in F(t, x(t))$ a.e. (I).

One may write successively,

$$|\int_0^t (t-s)^{r-1} S_2(t-s) f_n(s)ds - \int_0^t (t-s)^{r-1} S_2(t-s) f(s)ds| \leq$$

$$\frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} |f_n(s) - f(s)|ds \leq \frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} l(s)|x_{n-1}(s)$$

$$-x(s)|ds \leq \frac{M}{\Gamma(r)} L|x_{n-1}(.) - x(.)|_C.$$

Therefore, we may pass to the limit in (3.1) and we obtain that $x(.)$ is a solution of problem (1.1)

Finally, passing to the limit in (3.8) and (3.9) we obtained the desired estimations.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.3)-(3.7). The construction will be done by induction.

We apply, first, Lemma 2.5 and we have that the set-valued map $t \to F(t, y(t))$ is measurable with closed values and

$$F(t, y(t)) \cap \{g(t) + (\lambda(t) + \varepsilon)B\} \neq \emptyset \quad a.e.\ (I).$$

From Lemma 2.5 we find $f_1(.)$ a measurable selection of the set-valued map $H_1(t) := F(t, y(t)) \cap \{g(t) + (\lambda(t) + \varepsilon)B\}$. Obviously, $f_1(.)$ satisfy (3.5). Define $x_1(.)$ as in (3.3) with $n = 1$. Therefore, we have

$$|x_1(t) - y(t)| \leq |S_1(t)(x_0 - y_0)| + |\int_0^t (t-s)^{r-1} S_2(t-s)(f_1(s) - g(s))ds|$$

$$\leq M|x_0 - y_0| + \frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} (\lambda(s) + \varepsilon)ds \leq$$

$$\leq M|x_0 - y_0| + M\Lambda + \frac{M}{\Gamma(r+1)} \varepsilon T^r = m_0.$$

Assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I,X)$ and $f_n(.) \in L^1(I,X), n = 1,2,...N$ satisfying (3.3)-(3.7). We define the set-valued map

$$H_{N+1}(t) := F(t, x_N(t)) \cap \{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|B\}, \quad t \in I.$$

From Lemma 2.5 the set-valued map $t \to F(t, x_N(t))$ is measurable and from the lipschitzianity of $F(t,.)$ we have that for almost all $t \in I$ $H_{N+1}(t) \neq \emptyset$. We apply Lemma 2.5 and find a measurable selection $f_{N+1}(.)$ of $F(., x_N(.))$ such that

$$|f_{N+1}(t) - f_N(t)| \leq L(t)|x_N(t) - x_{N-1}(t)| \quad a.e.\ (I)$$

We define $x_{N+1}(.)$ as in (3.3) with $n = N+1$ and the proof is complete.

If in Theorem 3.2 we take $g = 0$, $y = 0$, $y_0 = 0$, $\lambda = l$ and $\varepsilon = \frac{\Gamma(r+1)}{MT^r}\varepsilon$ then we obtain the following existence result for solutions of problem (1.1).

Corollary 3.3. *Let Hypothesis 3.1 be satisfied, $ML < 1$ and assume that $d(0, F(t,0)) \leq l(t)$ $\forall (t) \in I$.*

Then there exists $x(.) \in C(I,X)$ a solution of problem (1.1) such that

$$|x(t)| \leq \frac{ML + \varepsilon}{1 - ML}, \quad \forall (t) \in I.$$

Next we obtain a continuous version of Theorem 3.1. This result allows to provide a continuous selection of the solution set of problem (1.1).

Hypothesis 3.4. i) The operator A generates a strongly continuous semigroup $\{T(t), t \geq 0\}$ on a real separable Banach space X and there exists a constant $M \geq 1$ such that $\sup_{t\in I} |T(t)| \leq M$.

ii) $F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed values, $F(.,.)$ is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable and there exists $l(.) \in L^1(I, \mathbf{R}_+)$ with $L := \sup_{t\in I} I^r l(t) < +\infty$ such that, for almost all $t \in I, F(t,.)$ is $l(t)$-Lipschitz.

Hypothesis 3.5. i) S is a separable metric space, $a(.) : S \to X$, $\varepsilon(.) : S \to (0, \infty)$ are continuous mappings.

ii) There exists the continuous mappings $g(.) : S \to L^1(I,X)$,$\lambda(.) : S \to L^1(I, \mathbf{R}_+)$, $y(.) : S \to C(I,X)$ such that

$$D_c^r(y(s))(t) = Ay(s)(t) + g(s)(t) \quad \forall s \in S, t \in I,$$

$$d(g(s)(t), F(t, y(s)(t))) \leq \lambda(s)(t) \quad a.e.\ (I),\ \forall\ s \in S$$

and the mapping $s \to \Lambda(s) := \sup_{t\in I}(I^r\lambda(s))(t)$ is continuous.

Next we use the notation $b(s) := \sup_{t\in I} |a(s) - y(s)(0)|$.

Theorem 3.6. *Assume that Hypotheses 3.4 and 3.5 are satisfied and* $ML < 1$.

Then there exists the continuous mapping $x(.) : S \to C(I, X)$ *such that for any* $s \in S$, $x(s)(.)$ *is a mild solution of the problem*

$$D_c^r x(t) \in Ax(t) + F(t, x(t)), \quad x(0) = a(s)$$

and

$$|x(s)(t) - y(s)(t)| \leq \frac{M}{1 - ML}(b(s) + \varepsilon(s) + \Lambda(s)) \quad \forall (t, s) \in I \times S.$$

Proof. Set $x_0(.) = y(.)$, $\lambda_p(s) := (ML)^{p-1}M(b(s) + \varepsilon(s) + \Lambda(s))$, $p \geq 1$.

We consider the set-valued maps $G_0(.), H_0(.)$ defined, respectively, by

$$G_0(s) = \{v \in L^1(I, X); \quad v(t) \in F(t, y(s)(t)) \quad a.e.\ (I)\},$$

$$H_0(s) = cl\{v \in G_0(s); |v(t) - g(s)(t)| < \lambda(s)(t) + \frac{\Gamma(r+1)}{T^r}\varepsilon(s)\}.$$

Since $d(g(s)(t), F(t, y(s)(t)) \leq \lambda(s)(t) < \lambda(s)(t) + \frac{\Gamma(r+1)}{T^r}\varepsilon(s)$ the set $H_0(s)$ is not empty.

Set $F_0^*(t, s) = F(t, y(s)(t))$ and note that

$$d(0, F_0^*(t, s)) \leq |g(s)(t)| + \lambda(s)(t) =: \lambda^*(s)(t)$$

and $\lambda^*(.) : S \to L^1(I, \mathbf{R})$ is continuous.

Applying now Lemmas 2.6 and 2.7 we obtain the existence of a continuous selection f_0 of H_0 such that $\forall s \in S,\ t \in I$,

$$f_0(s)(t) \in F(t, y(s)(t)) \quad a.e.\ (I),\ \forall s \in S,$$

$$|f_0(s)(t) - g(s)(t)| \leq \lambda_0(s)(t) = \lambda(s)(t) + \frac{\Gamma(r+1)}{T^r}\varepsilon(s).$$

We define

$$x_1(s)(t) = S_1(t)a(s) + \int_0^t (t - u)^{r-1} S_2(t - u) f_0(s)(u) du$$

and one has

$$|x_1(s)(t) - x_0(s)(t)| \leq Mb(s) + \frac{M}{\Gamma(r)} \int_0^t (t-u)^{r-1} |f_0(s)(u) - g(s)(u)| du$$

$$\leq Mb(s) + \frac{M}{\Gamma(r)} \int_0^t (t-u)^{r-1} (\lambda(s)(u) + \frac{\Gamma(r+1)}{T^r} \varepsilon(s)) ds \leq$$
$$\leq Mb(s) + M\Lambda(s) + M\varepsilon(s) =: \lambda_1(s) \quad t \in I, s \in S.$$

We shall construct, using the same idea as in [14], two sequences of approximations $f_p(.) : S \to L^1(I, X)$, $x_p(.) : S \to C(I, X)$ with the following properties

a) $f_p(.) : S \to L^1(I, X)$, $x_p(.) : S \to C(I, X)$ are continuous,

b) $f_p(s)(t) \in F(t, x_p(s)(t))$ a.e. (I), $s \in S$,

c) $|f_p(s)(t) - f_{p-1}(s)(t)| \leq l(t)\lambda_p(s)$ a.e. (I), $s \in S$.

d) $x_{p+1}(s)(t) = S_1(t)a(s) + \int_0^t (t-u)^{r-1} S_2(t-u) f_p(s)(u) du$, $t \in I, s \in S$.

Suppose we have already constructed $f_i(.), x_i(.)$ satisfying a)-c) and define $x_{p+1}(.)$ as in d). From c) and d) one has

$$|x_{p+1}(s)(t) - x_p(s)(t)| \leq \tfrac{M}{\Gamma(r)} \int_0^t (t-u)^{r-1} |f_p(s)(u) - f_{p-1}(s)(u)| du \leq \tfrac{M}{\Gamma(r)} \int_0^t (t-u)^{r-1} l(u) \lambda_p(s) du < ML\lambda_p(s) =: \lambda_{p+1}(s). \tag{3.10}$$

On the other hand,

$$d(f_p(s)(t), F(t, x_{p+1}(s)(t)) \leq l(t)|x_{p+1}(s)(t) - x_p(s)(t)| < < l(t)\lambda_{p+1}(s). \tag{3.11}$$

For any $s \in S$ we define the set-valued maps

$$G_{p+1}(s) = \{v \in L^1(I, X); \quad v(t) \in F(t, x_{p+1}(s)(t)) \quad a.e.\, (I)\},$$

$$H_{p+1}(s) = cl\{v \in G_{p+1}(s); \quad |v(t) - f_p(s)(t)| < l(t)\lambda_{p+1}(s)\}.$$

We note that from (3.11) the set $H_{p+1}(s)$ is not empty.

Set $F^*_{p+1}(t, s) = F(t, x_{p+1}(s)(t))$ and note that

$$d(0, F^*_{p+1}(t, s)) \leq |f_p(s)(t)| + l(t)\lambda_{p+1}(s) =: \lambda^*_{p+1}(s)(t)$$

and $\lambda^*_{p+1}(.) : S \to L^1(I, \mathbf{R})$ is continuous.

By Lemmas 2.6 and 2.7 we obtain the existence of a continuous function $f_{p+1}(.) : S \to L^1(I,X)$ such that

$$f_{p+1}(s)(t) \in F(t, x_{p+1}(s)(t)) \quad a.e.\,(I),\ \forall s \in S,$$

$$|f_{p+1}(s)(t) - f_p(s)(t)| \le l(t)\lambda_{p+1}(s) \quad \forall s \in S,\ (t) \in I.$$

From (3.10), c) and d) we obtain

$$|x_{p+1}(s)(.) - x_p(s)(.)|_C \le \lambda_{p+1}(s) = (ML)^p M(b(s) + \varepsilon(s) + \Lambda(s)), \quad (3.12)$$

$$|f_{p+1}(s)(.) - f_p(s)(.)|_1 \le |l|_1\lambda_p(s) = (ML)^{p-1}M|l|_1(b(s)+\varepsilon(s)+\Lambda(s)). \quad (3.13)$$

Therefore $f_p(s)(.)$, $u_p(s)(.)$ are Cauchy sequences in the Banach space $L^1(I,X)$ and $C(I,X)$, respectively. Let $f(.) : S \to L^1(I,X)$, $x(.) : S \to C(I,X)$ be their limits. The function $s \to b(s) + \varepsilon(s) + \Lambda(s)$ is continuous, hence locally bounded. Therefore (3.13) implies that for every $s' \in S$ the sequence $f_p(s')(.)$ satisfies the Cauchy condition uniformly with respect to s' on some neighborhood of s. Hence, $s \to f(s)(.)$ is continuous from S into $L^1(I,X)$.

From (3.12), as before, $x_p(s)(.)$ is Cauchy in $C(I,X)$ locally uniformly with respect to s. So, $s \to x(s)(.)$ is continuous from S into $C(I,X)$. On the other hand, since $x_p(s)(.)$ converges uniformly to $x(s)(.)$ and

$$d(f_p(s)(t), F(t, x(s)(t)) \le l(t)|x_p(s)(t) - x(s)(t)| \quad a.e.\,(I), \quad \forall s \in S$$

passing to the limit along a subsequence of $f_p(s)(.)$ converging pointwise to $f(s)(.)$ we obtain

$$f(s)(t) \in F(t, x(s)(t)) \quad a.e.\ (I),\ \forall s \in S.$$

One may write successively,

$$|\int_0^t (t-u)^{r-1} S_2(t-u) f_p(s)(u)du - \int_0^t (t-u)^{r-1} S_2(t-u) f(s)(u)du| \le$$
$$\frac{M}{\Gamma(r)}\int_0^t (t-u)^{r-1}|f_p(s)(u) - f(s)(u)|du \le \frac{M}{\Gamma(r)}\int_0^t (t-u)^{r-1} l(u)|x_{p-1}(s)(u)$$
$$-x(s)(u)|du \le ML|x_{p-1}(s)(.) - x(s)(.)|_C.$$

Therefore one may pass to the limit in d) and we get $\forall t \in I, s \in S$

$$x(s)(t) = S_1(t)a(s) + \int_0^t (t-u)^{r-1} S_2(t-u) f(s)(u)du,$$

i.e., $x(s)(.)$ is the desired solution.

Moreover, by adding inequalities (3.10) for all $p \geq 1$ we get

$$|x_{p+1}(s)(t) - y(s)(t)| \leq \sum_{l=1}^{p+1} \lambda_l(s) \leq \frac{M(b(s) + \varepsilon(s) + \Lambda(s))}{1 - ML}. \tag{3.14}$$

Passing to the limit in (3.14) we obtain the conclusion of the theorem.

Hypothesis 3.7. Hypothesis 3.4 is satisfied and there exists $q(.) \in L^1(I, \mathbf{R}_+)$ with $\sup_{t \in I} I^r q(t) < \infty$ such that $d(0, F(t, 0)) \leq q(t)$ a.e. (I).

Corollary 3.8. *Assume that Hypothesis 3.7 is satisfied.*
Then there exists a function $x(.,.) : I \times X \to X$ such that
a) $x(., \xi) \in \mathcal{S}(\xi)$, $\forall \xi \in X$.
b) $\xi \to x(., \xi)$ is continuous from X into $C(I, X)$.

Proof. We take $S = X$, $a(\xi) = \xi$ $\forall \xi \in X$, $\varepsilon(.) : X \to (0, \infty)$ an arbitrary continuous function, $g(.) = 0$, $y(.) = 0$, $\lambda(s)(t) \equiv q(t)$ $\forall \xi \in X$, $t \in I$ and we apply Theorem 3.6 in order to obtain the conclusion of the corollary.

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