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Nonlocal Boundary Value Problem for Strongly Singular Higher-Order Linear Functional-Differential Equations

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Abstract

For strongly singular higher-order differential equations with deviating arguments, under nonlocal boundary conditions, Agarwal-Kiguradze type theorems are established, which guarantee the presence of the Fredholm property for the problems considered. We also provide easily verifiable conditions that guarantee the existence of a unique solution of the problem.

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1 Statement of the main results

1.1 Statement of the problems and the basic notation

Consider the differential equations with deviating arguments

$$u^{(2m+1)}(t) = \sum_{j=0}^{m} p_j(t)u^{(j)}(\tau_j(t)) + q(t) \quad \text{for} \quad a < t < b, \tag{1.1}$$

with the boundary conditions

$$\begin{gathered} \int_a^b u(s)d\varphi(s) = 0 \qquad \text{where} \qquad \varphi(b) - \varphi(a) \neq 0, \\ u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \qquad (i = 1, \dots, m). \end{gathered} \tag{1.2}$$

Here $m \in N$, $-\infty < a < b < +\infty$, $p_j, q \in L_{loc}(]a,\, b[)$ $(j = 0, \dots, m)$, $\varphi : [a, b] \to R$ is a function of bounded variation, and $\tau_j :]a,\, b[\to]a,\, b[$ are measurable functions. By $u^{(i)}(a)$ (resp., $u^{(i)}(b)$), we denote the right (resp., left) limit of the function $u^{(i)}$ at the point a (resp., b). Problem (1.1), (1.2) is said to be singular if some or all the coefficients of (1.1) are non-integrable on $[a, b]$, having singularities at the end-points of this segment.

The first step in studying the linear ordinary differential equations

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t) u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for} \quad a < t < b, \tag{1.3}$$

where m is the integer part of $n/2$, under two-point conjugated boundary conditions, in the case when the functions p_j and q have strong singularities at the points a and b, i.e.

$$\begin{gathered}
\int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m} p_1(s)]_+ ds < +\infty, \\
\int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)| ds < +\infty \quad (j = 1, \dots, m), \\
\int_a^b (s-a)^{n-m-1/2}(b-s)^{m-1/2}|q(s)| ds < +\infty,
\end{gathered} \tag{1.4}$$

are not fulfilled, was made by R. P. Agarwal and I. Kiguradze in the article [3].

In this paper, Agarwal-Kiguradze type theorems are proved which guarantee the Fredholm property for problem (1.1), (1.2), when for the coefficients p_j $(j = 1, \dots, m)$, conditions (1.4), with $n = 2m$, are not satisfied. Throughout the paper we use the following notation.

$R^+ = [0, +\infty[$;

$[x]_+$ is the positive part of a number x, that is $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a, b[)$ is the space of functions $y :]a,\, b[\to R$, which are integrable on $[a+\varepsilon,\, b-\varepsilon]$ for arbitrary small $\varepsilon > 0$;

$L_{\alpha,\beta}(]a, b[)$ $(L^2_{\alpha,\beta}(]a, b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y :]a,\, b[\to R$, with the norm

$$||y||_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha(b-s)^\beta |y(s)| ds \quad \Big(||y||_{L^2_{\alpha,\beta}} = \Big(\int_a^b (s-a)^\alpha(b-s)^\beta y^2(s) ds \Big)^{1/2} \Big);$$

$L([a,b]) = L_{0,0}(]a,b[),\ L^2([a,b]) = L^2_{0,0}(]a,b[)$;
$M(]a,b[)$ is the set of measurable functions $\tau :]a,b[\to]a,b[$;
$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ is the Banach space of functions $y \in L_{loc}(]a,b[)$ such that

$$||y||_{\widetilde{L}^2_{\alpha,\beta}} := \max\Big\{\Big[\int_a^t (s-a)^\alpha \Big(\int_s^t y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : a \le t \le \frac{a+b}{2}\Big\} +$$

$$+ \max\Big\{\Big[\int_t^b (b-s)^\beta \Big(\int_t^s y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : \frac{a+b}{2} \le t \le b\Big\} < +\infty.$$

$\widetilde{C}^n_{loc}(]a,\, b[)$ is the space of functions $y :]a,b[\to R$ which are absolutely continuous together with $y',\, y'',\, \dots,\, y^{(n)}$ on $[a+\varepsilon, b-\varepsilon]$ for an arbitrarily small $\varepsilon > 0$.
$\widetilde{C}^{n,\,m}(]a,\, b[)$ $(m \le n)$ is the space of functions $y \in \widetilde{C}^n_{loc}(]a,\, b[)$, satisfying

$$\int_a^b |y^{(m)}(s)|^2 ds < +\infty. \tag{1.5}$$

When problem (1.1), (1.2) is discussed, we assume that the conditions

$$p_j \in L_{loc}(]a,\, b[) \ \ (j = 0, \dots, m) \tag{1.6}$$

are fulfilled.
A solution of problem (1.1), (1.2) is sought for in the space $\widetilde{C}^{2m,\,m+1}(]a,\, b[)$.
By $h_j :]a,\, b[\times]a,\, b[\to R_+$ and $f_j : R \times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1, \dots, m)$ we denote the functions and, respectively, the operators defined by the equalities

$$\begin{aligned} h_1(t,s) &= \Big|\int_s^t [(-1)^m p_1(\xi)]_+ d\xi\Big|, \\ h_j(t,s) &= \Big|\int_s^t p_j(\xi)d\xi\Big| \quad (j = 2, \dots, m), \end{aligned} \tag{1.7}$$

and,

$$f_j(c,\tau_j)(t,s) = \Big|\int_s^t |p_j(\xi)| \Big|\int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)} d\xi_1\Big|^{1/2} d\xi\Big| \quad (j = 1, \dots, m), \tag{1.8}$$

and also we put that

$$f_0(t,s)=\Big|\int\limits_s^t |p_0(\xi)|d\xi\Big|.$$

Let $m = 2k+1$, then

$$m!! = \begin{cases} 1 & \text{for } \ m \le 0 \\ 1\cdot 3\cdot 5\cdot \dots \cdot m & \text{for } \ m \ge 1 \end{cases}.$$

1.2 Fredholm type theorems

Along with (1.1), we consider the homogeneous equation

$$v^{(2m+1)}(t) = \sum_{j=0}^{m} p_j(t)v^{(j)}(\tau_j(t)) \quad \text{for} \quad a < t < b. \tag{1.1$_0$}$$

Definition 1.1. We will say that problem (1.1), (1.2) has the Fredholm property in the space $\widetilde{C}^{2m,m+1}(]a,\, b[)$ if the unique solvability of the corresponding homogeneous problem (1.1_0), (1.2) in that space implies the unique solvability of problem (1.1), (1.2) for every $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$.

In the case where conditions (1.4) for $n = 2m$ are violated, the question on the presence of the Fredholm property for problem (1.1), (1.2) in some subspace of the space $\widetilde{C}^{2m}_{loc}(]a,\, b[)$ remains so far open. This question is answered in Theorem 1.1 formulated below which contains conditions guaranteeing the Fredholm property for problem (1.1), (1.2) in the space $\widetilde{C}^{2m,m+1}(]a,\, b[)$.

Theorem 1.1. *Let there exist $a_0 \in]a,\, b[$, $\ b_0 \in]a_0,\, b[$, numbers $l_{kj} > 0$, $\gamma_{k0} > 0$, $\gamma_{kj} > 0$ $(k = 0, 1,\ j = 1, \dots, m)$ such that*

$$\begin{gathered} (t-a)^{2m-j} h_j(t,s) \le l_{0j} \quad (j = 1, \dots, m) \quad \textit{for} \quad a < t \le s \le a_0, \\ \limsup_{t\to a} (t-a)^{m-\frac{1}{2}-\gamma_{00}} f_0(t,s) < +\infty, \\ \limsup_{t\to a} (t-a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(a,\tau_j)(t,s) < +\infty \quad (j = 1, \dots, m), \end{gathered} \tag{1.9}$$

$$\begin{gathered} (b-t)^{2m-j} h_j(t,s) \le l_{1j} \quad (j = 1, \dots, m) \quad \textit{for} \quad b_0 \le s \le t < b, \\ \limsup_{t\to b} (b-t)^{m-\frac{1}{2}-\gamma_{10}} f_0(t,s) < +\infty, \\ \limsup_{t\to b} (b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(b,\tau_j)(t,s) < +\infty \quad (j = 1, \dots, m), \end{gathered} \tag{1.10}$$

and
$$\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\, l_{kj} < 1 \quad (k=0,1). \tag{1.11}$$
Let, moreover, the homogeneous problem (1.1_0), (1.2) *have only the trivial solution in the space* $\widetilde{C}^{2m,m+1}(]a,\, b[)$. *Then problem* (1.1), (1.2) *has a unique solution* u *for an arbitrary* $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\; b[)$, *and there exists a constant* r, *independent of* q, *such that*
$$||u^{(m+1)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}. \tag{1.12}$$

Corollary 1.1. *Let numbers* $\kappa_{kj}, \nu_{kj} \in R^+$ *be such that*
$$\nu_{k1} > 4m+2, \quad \nu_{kj} > 2 \quad (k=0,1;\; j=2,\dots,m), \tag{1.13}$$
$$\limsup_{t\to a} \frac{|\tau_j(t)-t|}{(t-a)^{\nu_{0j}}} < +\infty, \;\; \limsup_{t\to b} \frac{|\tau_j(t)-t|}{(b-t)^{\nu_{1j}}} < +\infty \quad (j=1,\dots,m), \tag{1.14}$$
and
$$\sum_{j=1}^{m} \frac{2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\, \kappa_{kj} < 1 \; (k=0,1). \tag{1.15}$$
Moreover, let $\kappa \in R^+$, $p_{00} \in L_{m-1,\,m-1}(]a,b[;R^+)$, $p_{0j} \in L_{2m-j,\,2m-j}(]a,b[;R^+)$, *and*
$$-\frac{\kappa}{[(t-a)(b-t)]^{2m}} - p_{01}(t) \le (-1)^m p_1(t) \le \frac{\kappa_{01}}{(t-a)^{2m}} + \frac{\kappa_{11}}{(b-t)^{2m}} + p_{01}(t), \tag{1.16}$$
$$\begin{gathered} |p_0(t)| \le \frac{\kappa_{00}}{(t-a)^m} + \frac{\kappa_{10}}{(b-t)^m} + p_{00}(t) \\ |p_j(t)| \le \frac{\kappa_{0j}}{(t-a)^{2m-j+1}} + \frac{\kappa_{1j}}{(b-t)^{2m-j+1}} + p_{0j}(t) \quad (j=2,\dots,m). \end{gathered} \tag{1.17}$$
Let, moreover, the homogeneous problem (1.1_0), (1.2) *have only the trivial solution in the space* $\widetilde{C}^{2m,m+1}(]a,\, b[)$. *Then problem* (1.1), (1.2) *has a unique solution* u *for an arbitrary* $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\; b[)$, *and there exists a constant* r, *independent of* q, *such that* (1.12) *holds.*

1.3 Existence and uniqueness theorems

Theorem 1.2. *Let there exist numbers $t^* \in]a, b[$, $l_{k0} > 0$, $l_{kj} > 0$, $\bar{l}_{kj} \geq 0$, and $\gamma_{k0} > 0$, $\gamma_{kj} > 0$ $(k = 0, 1;\ \ j = 1, \dots, m)$ such that along with*

$$B_0 \equiv$$

$$\equiv \bar{l}_{00}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(t^*-a)^{\gamma_{00}}}{\sqrt{2\gamma_{00}}} \int\limits_a^b \frac{|\varphi(\xi)-\varphi(a)| + |\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi + \\ + \sum_{j=1}^m \Big(\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \Big) < \frac{1}{2}, \tag{1.18}$$

$$B_1 \equiv$$

$$\equiv \bar{l}_{10}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(b-t^*)^{\gamma_{10}}}{\sqrt{2\gamma_{10}}} \int\limits_a^b \frac{|\varphi(\xi)-\varphi(a)| + |\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi + \\ + \sum_{j=1}^m \Big(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < \frac{1}{2}, \tag{1.19}$$

the conditions

$$(t-a)^{m-\gamma_{00}-1/2} f_0(t,s) \leq \bar{l}_{00}, \\ (t-a)^{2m-j} h_j(t,s) \leq l_{0j},\ \ (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \leq \bar{l}_{0j} \tag{1.20}$$

for $a < t \leq s \leq t^*$ and

$$(b-t)^{m-\gamma_{10}-1/2} f_0(t,s) \leq \bar{l}_{10}, \\ (b-t)^{2m-j} h_j(t,s) \leq l_{1j},\ (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \leq \bar{l}_{1j} \tag{1.21}$$

for $t^* \leq s \leq t < b$ hold with any $j = 1, \dots, m$. Then problem (1.1), (1.2) *is uniquely solvable in the space $\widetilde{C}^{2m,\,m+1}(]a, b[)$ for every $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a, b[)$.*

Remark 1.1. Let all the conditions of Theorem 1.2 be satisfied. Then the unique solution u of problem (1.1), (1.2) for every $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a, b[)$ admits the estimate

$$||u^{(m+1)}||_{L^2} \leq r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}, \tag{1.22}$$

with

$$r = \frac{2^m}{(1-2\max\{B_0,\, B_1\})(2m-1)!!},$$

and thus the constant $r > 0$ depends only on the numbers l_{kj}, $\overline{l}_{k0}$, $\overline{l}_{kj}$, γ_{k0}, γ_{kj} $(k = 0,1;\ j = 0,\dots,m)$, and a, b, t^*.

To illustrate this theorem, we consider the third order differential equation with a deviating argument

$$u^{(3)}(t) = p_0(t)u(\tau_0(t)) + p_1(t)u'(\tau_1(t)) + q(t), \tag{1.23}$$

under the boundary conditions

$$\int_a^b u(s)ds = 0, \quad u(a) = 0,\ u(b) = 0. \tag{1.24}$$

As a corollary of Theorem 1.2 with $m = 1$, $t^* = (a+b)/2$, $\gamma_{00} = \gamma_{10} = 1/4$, $\gamma_{01} = \gamma_{11} = 1/2$, $\overline{l}_{00} = \overline{l}_{10} = 8\frac{2^{1/4}\kappa}{(b-a)^{5/4}}$, $l_{01} = l_{11} = \kappa_0$, $\overline{l}_{01} = \overline{l}_{11} = \frac{\sqrt{2}\kappa_1}{\sqrt{b-a}}$, we obtain the following statement.

Corollary 1.2. *Let function $\tau_1 \in M(]a,b[)$ be such that*

$$\begin{gathered} 0 \le \tau_1(t) - t \le \frac{2^6}{(b-a)^6}(t-a)^7 \quad \textit{for} \quad a < t \le \frac{a+b}{2}, \\ -\frac{2^6}{(b-a)^6}(b-t)^7 \le t - \tau_1(t) \le 0 \quad \textit{for} \quad \frac{a+b}{2} \le t < b. \end{gathered} \tag{1.25}$$

Moreover, let function $p :]a,b[\to R$ and constants κ_0, κ_1 be such that

$$\begin{gathered} |p_0(t)| \le \frac{\kappa}{[(b-t)(t-a)]^{5/4}} \quad \textit{for} \quad a < t < b \\ -\frac{2^{-2}(b-a)^2\kappa_0}{[(b-t)(t-a)]^2} \le p_1(t) \le \frac{2^{-7}(b-a)^6\kappa_1}{[(b-t)(t-a)]^4} \quad \textit{for} \quad a < t < b \end{gathered} \tag{1.26}$$

and

$$8\kappa\sqrt{2(b-a)} + 4\kappa_0 + \kappa_1 < \frac{1}{2}. \tag{1.27}$$

Then problem (1.23), (1.24) *is uniquely solvable in the space $\widetilde{C}^{2,2}(]a,b[)$ for every $q \in \widetilde{L}^2_{0,0}(]a,b[)$.*

2 Auxiliary Propositions

2.1 Lemmas on integral inequalities

Now we formulate two lemmas which are proved in [3].

Lemma 2.1. *Let* $\in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ *and*

$$u^{(j-1)}(t_0)=0 \quad (j=1,\dots,m), \qquad \int\limits_{t_0}^{t_1} |u^{(m)}(s)|^2 ds < +\infty. \tag{2.1}$$

Then

$$\int\limits_{t_0}^{t} \frac{(u^{(j-1)}(s))^2}{(s-t_0)^{2m-2j+2}} ds \le \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!}\Big)^2 \int\limits_{t_0}^{t} |u^{(m)}(s)|^2 ds \tag{2.2}$$

for $t_0 \le t \le t_1$.

Lemma 2.2. *Let* $u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$, *and*

$$u^{(j-1)}(t_1)=0 \quad (j=1,\dots,m), \qquad \int\limits_{t_0}^{t_1} |u^{(m)}(s)|^2 ds < +\infty. \tag{2.3}$$

Then

$$\int\limits_{t}^{t_1} \frac{(u^{(j-1)}(s))^2}{(t_1-s)^{2m-2j+2}} ds \le \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!}\Big)^2 \int\limits_{t}^{t_1} |u^{(m)}(s)|^2 ds \tag{2.4}$$

for $t_0 \le t \le t_1$.

Let $t_0, t_1 \in]a,b[,\ u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ and $\tau_j \in M(]a,b[)\ (j=0,\dots,m)$. Then we define the functions $\mu_j : [a,\ (a+b)/2] \times [(a+b)/2,\ b] \times [a,b] \to [a,b]$, $\rho_k :\ [t_0,\, t_1] \to R_+\ (k = 0,1)$, $\lambda_j : [a,b]\times]a,\ (a+b)/2] \times [(a+b)/2,\ b[\times]a,b[\to R_+$, and for any $t_0, t_1 \in [a,\ b]$ the operator $\chi_{t_0,t_1} : C([t_0,\, t_1]) \to C([a,\, b])$, by the equalities

$$\mu_j(t_0,t_1,t) = \begin{cases} \tau_j(t) & \text{for} \quad \tau_j(t) \in [t_0,\; t_1] \\ t_0 & \text{for} \quad \tau_j(t) < t_0 \\ t_1 & \text{for} \quad \tau_j(t) > t_1 \end{cases},$$

$$\rho_k(t) = \Big| \int_t^{t_k} |u^{(m)}(s)|^2 ds \Big|, \quad \lambda_j(c,t_0,t_1,t) = \Big| \int_t^{\mu_j(t_0,t_1,t)} (s-c)^{2(m-j)} ds \Big|^{\frac{1}{2}}, \tag{2.5}$$

$$\chi_{t_0,t_1}(x)(t) = \begin{cases} x(t_0) & \text{for} \quad a \le t < t_0 \\ x(t) & \text{for} \quad t_0 \le t \le t_1 \\ x(t_1) & \text{for} \quad t_1 < t \le b \end{cases}.$$

Let also $\alpha_0 : R_+^2 \times [0,1[\to R_+$, $\alpha_j : R_+^3 \times [0,1[\to R_+$ and $\beta_j \in R_+ \times [0,1[\to R_+$ $(j = 0,\dots,m)$ be the functions defined by the equalities

$$\alpha_0(x,y,\gamma) = \frac{2^{m-1}(b-a)^{m-1/2} x y^\gamma}{(2m-3)!!(2m-1)^{1/2}} \int_a^b \frac{|\varphi(\xi)-\varphi(a)| + |\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi$$

$$\beta_0(x,\gamma) = \Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{x^\gamma}{\sqrt{2\gamma}} \int_a^b \frac{|\varphi(\xi)-\varphi(a)| + |\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi, \tag{2.6}$$

$$\alpha_j(x,y,z,\gamma) = x + \frac{2^{m-j}\, y\, z^\gamma}{(2m-2j-1)!!},$$

$$\beta_j(y,\gamma) = \frac{2^{2m-j-1}}{(2m-2j-1)!!(2m-3)!!} \frac{y^\gamma}{\sqrt{2\gamma}},$$

and

$$G(t,s) = \frac{1}{\varphi(b)-\varphi(a)} \times \begin{cases} \varphi(s)-\varphi(b) & \text{for} \quad s \ge t \\ \varphi(s)-\varphi(a) & \text{for} \quad s < t \end{cases} \tag{2.7}$$

is the Green function of the problem:

$$w'(t) = 0, \qquad \int_a^b w(s) d\varphi(s) = 0, \tag{2.8}$$

where $\varphi : [a,b] \to R$ is a function of bounded variation and $\varphi(b) - \varphi(a) \neq 0$.

Lemma 2.3. *Let $a_0 \in]a, b[$, $t_0 \in]a, a_0[$, $t_1 \in]a_0, b[$, and the function $u \in \widetilde{C}^{m-1}_{loc}(]t_0, t_1[)$ be such that conditions (2.1), (2.3) hold. Moreover, let constants $l_{0\,j} > 0$, $\overline{l}_{0\,0} \geq 0$, $\overline{l}_{0\,j} \geq 0$, $\gamma_{0j} > 0$, and functions $\overline{p}_j \in L_{loc}(]t_0, t_1[)$, $\tau_j \in M(]a, b[)$ be such that the inequalities*

$$(t - t_0)^{2m-1} \int_t^{a_0} [\overline{p}_1(s)]_+ ds \leq l_{0\,1}, \tag{2.9}$$

$$(t - t_0)^{2m-j} \Big| \int_t^{a_0} \overline{p}_j(s) ds \Big| \leq l_{0\,j} \quad (j = 2, \ldots, m), \tag{2.10}$$

$$\begin{gathered} (t - t_0)^{m-1/2-\gamma_{00}} \int_t^{a_0} |\overline{p}_0(s)| ds \leq \overline{l}_{00}, \\ (t - t_0)^{m-\frac{1}{2}-\gamma_{0j}} \int_t^{a_0} |\overline{p}_j(s)| \lambda_j(t_0, t_0, t_1, s) ds \leq \overline{l}_{0\,j} \quad (j = 1, \ldots, m) \end{gathered} \tag{2.11}$$

hold for $t_0 < t \leq a_0$. Then

$$\begin{aligned} \int_t^{a_0} \overline{p}_j(s) u(s) u^{(j-1)}(\mu_j(t_0, t_1, s)) ds \leq \\ \leq \alpha_j(l_{0j}, \overline{l}_{0j}, a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(t) + \overline{l}_{0j} \beta_j(a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(a_0) + \\ + l_{0j} \frac{(2m - j) 2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_0(a_0) \quad (j = 1, \ldots, m) \end{aligned} \tag{2.12}$$

for $t_0 < t \leq a_0$ and

$$\begin{aligned} \int_t^{a_0} \overline{p}_0(s) u(s) \Big(\int_a^b G(\mu_0(t_0, t_1, s), \xi) \chi_{t_0,t_1}(u)(\xi) d\xi \Big) ds \leq \\ \leq \alpha_0(\overline{l}_{00}, a_0 - a, \gamma_{00}) \rho_0^{1/2}(t_1) \rho_0^{1/2}(t) \\ + \overline{l}_{00} \beta_0(a_0 - a, \gamma_{00}) \rho_0^{1/2}(t_1) \rho_0^{1/2}(a_0) \end{aligned} \tag{2.13}$$

for $t_0 < t \leq a_0$, where $\tau^* = \sup\{\mu_j(t_0, t_1, t) : t_0 \leq t \leq a_0, j = 1, \ldots, m\} \leq t_1$.

Proof. In view of the formula of integration by parts, for $t \in [t_0, a_0]$ we have

$$\int_t^{a_0} \overline{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds = \int_t^{a_0} \overline{p}_j(s)u(s)u^{(j-1)}(s)ds+$$

$$+\int_t^{a_0} \overline{p}_j(s)u(s)\Big(\int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi)d\xi\Big)ds = u(t)u^{(j-1)}(t)\int_t^{a_0} \overline{p}_j(s)ds+$$

$$+\sum_{k=0}^{1}\int_t^{a_0}\Big(\int_s^{a_0}\overline{p}_j(\xi)d\xi\Big)u^{(k)}(s)u^{(j-k)}(s)ds+\int_t^{a_0} \overline{p}_j(s)u(s)\Big(\int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi)d\xi\Big)ds \quad (2.14)$$

$(j = 2, \dots, m)$, and

$$\int_t^{a_0} \overline{p}_1(s)u(s)u(\mu_1(t_0,t_1,s))ds \le \int_t^{a_0} [\overline{p}_1(s)]_+u^2(s)ds+$$

$$+\int_t^{a_0} |\overline{p}_1(s)u(s)|\Big|\int_s^{\mu_1(t_0,t_1,s)} u^{'}(\xi)d\xi\Big|ds \le u^2(t)\int_t^{a_0}[\overline{p}_1(s)]_+ds+$$

$$+\,2\int_t^{a_0}\Big(\int_s^{a_0}[\overline{p}_1(\xi)]_+d\xi\Big)|u(s)u'(s)|ds+\int_t^{a_0} |\overline{p}_1(s)u(s)|\Big|\int_s^{\mu_1(t_0,t_1,s)} u^{'}(\xi)d\xi\Big|ds. \quad (2.15)$$

On the other hand, by virtue of conditions (2.1), the Schwartz inequality and Lemma 2.1, we deduce that

$$|u^{(j-1)}(t)| = \frac{1}{(m-j)!}\Big|\int_{t_0}^{t}(t-s)^{m-j}u^{(m)}(s)ds\Big| \le (t-t_0)^{m-j+1/2}\,\rho_0^{1/2}(t) \quad (2.16)$$

for $t_0 \le t \le a_0$ $(j = 1, \dots, m)$. If along with this, in the case where $j > 1$, we take inequality (2.10) and Lemma 2.1 into account, for $t \in [t_0, a_0]$, we obtain the estimates

$$\Big|u(t)u^{(j-1)}(t)\int_t^{a_0}\overline{p}_j(s)ds\Big| \le (t-t_0)^{2m-j}\Big|\int_t^{a_0}\overline{p}_j(s)ds\Big|\rho_0(t) \le l_{0j}\rho_0(t) \quad (2.17)$$

and

$$\sum_{k=0}^{1}\int_t^{a_0}\Big(\int_s^{a_0}\overline{p}_j(\xi)d\xi\Big)u^{(k)}(s)u^{(j-k)}(s)ds \le l_{0j}\sum_{k=0}^{1}\int_t^{a_0}\frac{|u^{(k)}(s)u^{(j-k)}(s)|}{(s-t_0)^{2m-j}}ds \le$$

$$\le l_{0j}\sum_{k=0}^{1}\Big(\int_t^{a_0}\frac{|u^{(k)}(s)|^2ds}{(s-t_0)^{2m-2k}}\Big)^{1/2}\Big(\int_t^{a_0}\frac{|u^{(j-k)}(s)|^2ds}{(s-t_0)^{2m+2k-2j}}\Big)^{1/2} \le$$

$$\le l_{0j}\rho_0(a_0)\sum_{k=0}^{1}\frac{2^{2m-j}}{(2m-2k-1)!!(2m+2k-2j-1)!!}. \tag{2.18}$$

Analogously, if $j=1$, by (2.9) we obtain

$$\begin{gathered}u^2(t)\int_t^{a_0}[\overline{p}_1(s)]_+ds \le l_{01}\rho_0(t),\\ 2\int_t^{a_0}\Big(\int_s^{a_0}[\overline{p}_1(\xi)]_+d\xi\Big)|u(s)u'(s)|ds \le l_{01}\rho_0(a_0)\frac{(2m-1)2^{2m}}{[(2m-1)!!]^2}\end{gathered} \tag{2.19}$$

for $t_0 < t \le a_0$.

By the Schwartz inequality, Lemma 2.1, and the fact that ρ_0 is a nondecreasing function, we get

$$\Big|\int_s^{\mu_j(t_0,t_1,s)}u^{(j)}(\xi)d\xi\Big| \le \frac{2^{m-j}}{(2m-2j-1)!!}\,\lambda_j(t_0,t_0,t_1,s)\,\rho_0^{1/2}(\tau^*) \tag{2.20}$$

for $t_0 < s \le a_0$. Also, due to (2.2), (2.11) and (2.16), we have

$$|u(t)|\int_t^{a_0}|\overline{p}_j(s)|\lambda_j(t_0,t_0,t_1,s)ds = (t-t_0)^{m-1/2}\rho_0^{1/2}(t)\int_t^{a_0}|\overline{p}_j(s)|\lambda_j(t_0,t_0,t_1,s)ds \le$$

$$\le \overline{l}_{0j}\,(t-t_0)^{\gamma_{0j}}\,\rho_0^{1/2}(t)$$

and

$$\int_t^{a_0} |u'(s)|\Big(\int_s^{a_0} |\overline{p}_j(\xi)|\lambda_j(t_0,t_0,t_1,\xi)d\xi\Big)ds \le \overline{l}_{0j}\int_t^{a_0} \frac{|u'(s)|}{(s-t_0)^{m-\frac{1}{2}-\gamma_{0j}}}ds \le$$

$$\le \overline{l}_{0j}\frac{2^{m-1}(a_0-a)^{\gamma_{0j}}}{(2m-3)!!\sqrt{2\gamma_{0j}}}\rho_0^{1/2}(a_0)$$

for $t_0 < t \le a_0$. It is clear from the last three inequalities that

$$\left|\frac{(2m-2j-1)!!}{2^{m-j}\rho_0^{1/2}(\tau^*)}\int_t^{a_0} \overline{p}_j(s)u(s)\Bigg(\int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi)d\xi\Bigg)ds\right| \le$$

$$\le \int_t^{a_0} |\overline{p}_j(s)u(s)|\lambda_j(t_0,t_0,t_1,s)ds \le$$

$$\le |u(t)|\int_t^{a_0} |\overline{p}_j(s)|\lambda_j(t_0,t_0,t_1,s)ds + \int_t^{a_0} |u'(s)|\Big(\int_s^{a_0} |\overline{p}_j(\xi)|\lambda_j(t_0,t_0,t_1,\xi)d\xi\Bigg)ds \le$$

$$\le \overline{l}_{0j}\,(t-t_0)^{\gamma_{0j}}\rho_0^{1/2}(t) + \overline{l}_{0j}\frac{2^{m-1}(a_0-a)^{\gamma_{0j}}}{(2m-3)!!\sqrt{2\gamma_{0j}}}\rho_0^{1/2}(a_0) \quad (2.21)$$

for $t_0 < t \le a_0$. Now we note that, by (2.17)-(2.19) and (2.21), inequality (2.12) follows immediately from from (2.14) and (2.15).

In view of the definition of the function G, the operator $\chi_{t_0\,t_1}$ and condition (2.1), we have

$$\int_t^{a_0} \overline{p}_0(s)u(s)\Bigg(\int_a^b G(\mu_0(t_0,t_1,s),\xi)\chi_{t_0,t_1}(u)(\xi)d\xi\Bigg)ds =$$

$$= \int_t^{a_0} \overline{p}_0(s)u(s)\Bigg(\int_{t_0}^{\mu_0(t_0,t_1,s)} \frac{\varphi(\xi)-\varphi(a)}{\varphi(b)-\varphi(a)}u(\xi)d\xi\Bigg)ds+$$

$$+ \int_t^{a_0} \overline{p}_0(s)u(s)\Bigg(\int_{\mu_0(t_0,t_1,s)}^{t_1} \frac{\varphi(\xi)-\varphi(b)}{\varphi(b)-\varphi(a)}u(\xi)d\xi\Bigg)ds. \quad (2.22)$$

On the other hand, by the carrying out integration by parts and using the Schwartz inequality, we get the inequality

$$\int\limits_{t_0}^{\mu_0(t_0,t_1,s)} \frac{\varphi(\xi)-\varphi(a)}{\varphi(b)-\varphi(a)}u(\xi)d\xi \le \int\limits_{t_0}^{t_1} \Big|\frac{\varphi(\xi)-\varphi(a)}{\varphi(b)-\varphi(a)}\Big|d\xi \times$$

$$\times \bigg(\int\limits_{t_0}^{t_1}(\xi-t_0)^{2(m-1)}d\xi\bigg)^{1/2}\bigg(\int\limits_{t_0}^{t_1}\frac{u'^2(\xi)}{(\xi-t_0)^{2(m-1)}}d\xi\bigg)^{1/2} \tag{2.23}$$

from which, by Lemma 2.1 and the definition of the function μ_0, it follows that

$$\int\limits_{t_0}^{t_1} \frac{\varphi(\xi)-\varphi(a)}{\varphi(b)-\varphi(a)}u(\xi)d\xi \le \frac{2^{m-1}(b-a)^{m-1/2}}{(2m-3)!!(2m-1)^{1/2}}\rho_0^{1/2}(t_1)\int\limits_a^b \Big|\frac{\varphi(\xi)-\varphi(a)}{\varphi(b)-\varphi(a)}\Big|d\xi \tag{2.24}$$

Analogously, by Lemma 2.2, in view of the fact that $\rho_0(t_1)=\rho_1(t_0)$, we get

$$\int\limits_{\mu_0(t_0,t_1,s)}^{t_1} \frac{\varphi(\xi)-\varphi(b)}{\varphi(b)-\varphi(a)}u(\xi)d\xi \le \frac{2^{m-1}(b-a)^{m-1/2}}{(2m-3)!!(2m-1)^{1/2}}\rho_0^{1/2}(t_1)\int\limits_a^b \Big|\frac{\varphi(\xi)-\varphi(a)}{\varphi(b)-\varphi(a)}\Big|d\xi. \tag{2.25}$$

On the other hand by the integration by parts, inequality (2.16), and condition (2.11) we get

$$\int\limits_t^{a_0}|\overline{p}_0(s)u(s)|ds \le |u(s)|\int\limits_t^{a_0}|\overline{p}_0(s)|ds + \int\limits_t^{a_0}|u'(s)|\int\limits_s^{a_0}|\overline{p}_0(\xi)|d\xi ds$$

$$\le (t-t_0)^{\gamma_{00}}\rho_0^{1/2}(t)\overline{l}_{00} + \overline{l}_{00}\int\limits_t^{a_0}\frac{|u'(s)|}{(s-t_0)^{m-1/2-\gamma_{00}}}ds,$$

from which, by the Schwartz inequality and Lemma 2.1, we get

$$\int\limits_t^{a_0}|\overline{p}_0(s)u(s)|ds \le (t-t_0)^{\gamma_{00}}\rho_0^{1/2}(t)\overline{l}_{00} + \frac{2^{m-1}(a_0-a)^{\gamma_{00}}}{(2m-3)!!\sqrt{2\gamma_{00}}}\rho_0^{1/2}(a_0)\overline{l}_{00}. \tag{2.26}$$

From (2.22) by (2.24)-(2.26) and notation (2.6), inequality (2.13) follows immediately. $\square$

The following lemma can be proved similarly to Lemma 2.3.

Lemma 2.4. *Let $b_0 \in]a,\, b[,\ t_1 \in]b_0,\, b[,\ t_0 \in]a,\, b_0[$, and the function $u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ be such that conditions (2.1), (2.3) hold. Moreover, let constants $l_{1\,j} > 0,\ \bar{l}_{1\,0} \geq 0,\ \bar{l}_{1\,j} \geq 0,\ \gamma_{1j} > 0$, and functions $\overline{p}_j \in L_{loc}(]t_0,\, t_1[),\ \tau_j \in M(]a, b[)$ be such that the inequalities*

$$(t_1 - t)^{2m-1} \int_{b_0}^{t} [\overline{p}_1(s)]_+ ds \leq l_{1\,1}, \tag{2.27}$$

$$(t_1 - t)^{2m-j} \Big| \int_{b_0}^{t} \overline{p}_j(s) ds \Big| \leq l_{1\,j} \quad (j = 2, \ldots, m), \tag{2.28}$$

$$\begin{gathered} (t_1 - t)^{m-1/2-\gamma_{10}} \int_{b_0}^{t} |\overline{p}_0(s)| ds \leq \bar{l}_{10}, \\ (t_1 - t)^{m-\frac{1}{2}-\gamma_{1j}} \Big| \int_{b_0}^{t} \overline{p}_j(s) \lambda_j(t_1, t_0, t_1, s) ds \Big| \leq \bar{l}_{1\,j} \quad (j = 1, \ldots, m) \end{gathered} \tag{2.29}$$

hold for $b_0 < t \leq t_1$. Then

$$\begin{aligned} &\int_{b_0}^{t} \overline{p}_j(s) u(s) u^{(j-1)}(\mu_j(t_0, t_1, s)) ds \leq \\ &\qquad \leq \alpha_j(l_{1j}, \bar{l}_{1j}, b - b_0, \gamma_{1j}) \rho_1^{1/2}(\tau_*) \rho_1^{1/2}(t) + \bar{l}_{1j} \beta_j(b - b_0, \gamma_{1j}) \rho_1^{1/2}(\tau_*) \rho_1^{1/2}(b_0) + \\ &\qquad\qquad + l_{1j} \frac{(2m-j) 2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_1(b_0) \end{aligned} \tag{2.30}$$

for $b_0 \leq t < t_1$ and

$$\begin{aligned} &\int_{b_0}^{t} \overline{p}_0(s) u(s) \Big(\int_a^b G(\mu_0(t_0, t_1, s), \xi) \chi_{t_0, t_1}(u)(\xi) d\xi \Big) ds \leq \\ &\qquad \leq \alpha_0(\bar{l}_{10}, b - b_0, \gamma_{10}) \rho_1^{1/2}(t_0) \rho_1^{1/2}(t) + \bar{l}_{10} \beta_0(b - b_0, \gamma_{10}) \rho_1^{1/2}(t_0) \rho_1^{1/2}(b_0), \end{aligned} \tag{2.31}$$

for $b_0 \leq t < t_1$, where $\tau_* = \inf\{\mu_j(t_0, t_1, t) : b_0 \leq t \leq t_1, j = 1, \ldots, m\} \geq t_0$.

2.2 Lemma on a property of functions from $\widetilde{C}^{2m,m-1}(]a,\, b[)$

Lemma 2.5. *Let*

$$w(t)=\sum_{i=1}^{m}\sum_{k=i}^{m}c_{ik}(t)u^{(2m-k)}(t)u^{(i-1)}(t),$$

where $u\in\widetilde{C}^{2m-1,m}(]a,\, b[)$*, and each* $c_{ik}:[a,b]\to R$ *is an* $2m-k-i+1$ *times continuously differentiable function. Moreover, if*

$$u^{(i-1)}(a)=0,\quad u^{(i-1)}(b)=0,\qquad \limsup_{t\to a}|c_{ii}(t)|<+\infty\ \ (i=1,\dots,m),$$

then

$$\liminf_{t\to a}|w(t)|=0,\qquad \liminf_{t\to b}|w(t)|=0.$$

The proof of this Lemma is given in [9].

2.3 Lemmas on the sequences of solutions of auxiliary problems

Remark 2.1. It is easy to verify that the function $\widetilde{u}$ is a solution of problem

$$\widetilde{u}^{(2m)}(t)=\sum_{j=1}^{m}p_j(t)\widetilde{u}^{(j-1)}(\tau_j(t))+p_0(t)\int_a^b G(\tau_0(t),s)\widetilde{u}(s)ds+q(t)\quad\text{for}\quad a<t<b,\ (2.32)$$

$$\widetilde{u}^{(i-1)}(a)=0,\quad \widetilde{u}^{(i-1)}(b)=0\qquad (i=1,\dots,m),\tag{2.33}$$

if and only if the function $u(t)=\int_a^b G(t,s)\widetilde{u}(s)ds$ is a solution of the problem (1.1), (1.2), and analogously $\widetilde{v}$ is a solution of problem

$$\widetilde{v}^{(2m)}(t)=\sum_{j=1}^{m}p_j(t)\widetilde{v}^{(j-1)}(\tau_j(t))+p_0(t)\int_a^b G(\tau_0(t),s)\widetilde{v}(s)ds\quad\text{for}\quad a<t<b,\tag{2.32_0}$$

$$\widetilde{v}^{(i-1)}(a)=0,\quad \widetilde{v}^{(i-1)}(b)=0\qquad (i=1,\dots,m).\tag{2.33_0}$$

if and only if the function $v(t)=\int_a^b G(t,s)\widetilde{v}(s)ds$ is a solution of the problem (1.1_0), (1.2).

Now for every natural k we consider the auxiliary equation

$$\widetilde{u}^{(2m)}(t) = \sum_{j=1}^{m} p_j(t)\widetilde{u}^{(j-1)}(\mu_j(t_{0k}, t_{1k}, t))+$$
$$+ p_0(t) \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s)\chi_{t_{0k}t_{1k}}(\widetilde{u})(s)ds + q_k(t) \quad (2.34)$$

for $t_{0k} \leq t \leq t_{1k}$, with the corresponding homogenous equation

$$\widetilde{u}^{(2m)}(t) = \sum_{j=1}^{m} p_j(t)\widetilde{u}^{(j-1)}(\mu_j(t_{0k}, t_{1k}, t)) + p_0(t) \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s)\chi_{t_{0k}t_{1k}}(\widetilde{u})(s)ds \quad (2.34_0)$$

for $t_{0k} \leq t \leq t_{1k}$, under the boundary conditions

$$\widetilde{u}^{(i-1)}(t_{0k}) = 0, \quad \widetilde{u}^{(j-1)}(t_{1k}) = 0 \qquad (i = 1, \dots, m), \quad (2.35)$$

where

$$a < t_{0k} < t_{1k} < b \quad (k \in N), \qquad \lim_{k\to+\infty} t_{0k} = a, \ \lim_{k\to+\infty} t_{1k} = b. \quad (2.36)$$

Throughout this section, when problems (2.32), (2.33) and (2.34), (2.35) are discussed we assume that

$$p_j \in L_{loc}(]a, b[) \ \ (j = 0, ..., m), \quad q, q_k \in \widetilde{L}^2_{2m-2,\, 2m-2}(]a,\, b[), \quad (2.37)$$

and for an arbitrary $m-1$-times continuously differentiable function $x :]a, b[\to R$, we set

$$\Lambda_k(x)(t) = \sum_{j=1}^{m} p_j(t)x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, t))$$
$$+ p_0(t) \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s)\chi_{t_{0k}t_{1k}}(x)(s)ds, \quad (2.38)$$
$$\Lambda(x)(t) = \sum_{j=1}^{m} p_j(t)x^{(j-1)}(\tau_j(t)) + p_0(t) \int_a^b G(\tau_0(t), s)x(s)ds.$$

Remark 2.2. From the definition of the functions μ_j $(j = 0, \dots, m)$, the estimate

$$|\mu_j(t_{0k}, t_{1k}, t) - \tau_j(t)| \leq \begin{cases} 0 & \text{for } \ \tau_j(t) \in]t_{0k}, \ t_{1k}[\\ \max\{b - t_{1k}, \ \ t_{0k} - a\} & \text{for } \ \tau_j(t) \notin]t_{0k}, \ t_{1k}[\end{cases}$$

follows and thus, if conditions (2.36) hold, then

$$\lim_{k \to +\infty} \mu_j(t_{0k}, t_{1k}, t) = \tau_j(t) \quad (j = 0, \dots, m) \quad \text{uniformly in} \quad]a, b[. \tag{2.39}$$

Let now the sequence of the $m - 1$ times continuously differentiable functions $x_k :]t_{0k}, t_{1k}[\to R$, and functions $x^{(j-1)} \in C([a, b])$ $(j = 1, \dots, m)$ be such that

$$\lim_{k \to +\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \dots, m) \quad \text{uniformly in} \quad]a, b[. \tag{2.40}$$

Remark 2.3. Let the functions $x_k :]t_{0k}, t_{1k}[\to R$, and $x \in C([a, b])$ be such that (2.40) with $j = 1$ holds. Then from the definition of the operators $\chi_{t_{0k}t_{1k}}$ and (2.40) it is clear that

$$\lim_{k \to +\infty} \chi_{t_{0k}t_{1k}}(x_k)(t) = \chi_{t_{0k}t_{1k}}(x)(t), \quad \lim_{k \to +\infty} \chi_{t_{0k}t_{1k}}(x)(t) = x(t) \tag{2.41}$$

uniformly in $]a, b[$.

Lemma 2.6. *Let conditions* (2.36) *hold and the sequence of the $m - 1$-times continuously differentiable functions $x_k :]t_{0k}, t_{1k}[\to R$, and functions $x^{(j-1)} \in C([a, b])$ $(j = 1, \dots, m)$ be such that* (2.40) *holds. Then for any nonnegative function $w \in C([a, b])$ and $t^* \in]a, b[$,*

$$\lim_{k \to +\infty} \int_{t^*}^{t} w(s)\Lambda_k(x_k)(s)ds = \int_{t^*}^{t} w(s)\Lambda(x)(s)ds \tag{2.42}$$

uniformly in $]a, b[$, where Λ_k and Λ are defined by equalities (2.38).

Proof. We have to prove that for any $\delta \in]0, \ \min\{b - t^*, \ t^* - a\}[$, and $\varepsilon > 0$, there exists a constant $n_0 \in N$ such that

$$\Big| \int_{t^*}^{t} w(s)(\Lambda_k(x_k)(s) - \Lambda(x)(s))ds \Big| \leq \varepsilon \quad \text{for} \quad t \in [a + \delta, \, b - \delta], \ k > n_0. \tag{2.43}$$

Let now $w(t_*) = \max\limits_{a \leq t \leq b} w(t)$ and $\varepsilon_1 = \varepsilon\Big(2w(t_*) \sum\limits_{j=0}^{m} \int_{a+\delta}^{b-\delta} |p_j(s)|ds\Big)^{-1}$. Then from the inclusions $x_k^{(j-1)} \in C([a + \delta, \, b - \delta])$, $x^{(j-1)} \in C([a, b])$ $(j = 1, \dots, m)$, conditions (2.39) and

(2.40), it follows the existence of such constant $n_{01} \in N$ that

$$\begin{gathered} |x_k^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s)) - x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s))| \le \varepsilon_1, \\ |x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s)) - x^{(j-1)}(\tau_j(s))| \le \varepsilon_1 \end{gathered} \tag{2.44}$$

for $t \in [a+\delta,\, b-\delta]$, $k > n_{01}$, $j = 1, \dots, m$. Furthermore, (2.39)-(2.41) imply the existence of such constant $n_{02} \in N$ that

$$\begin{aligned} \Big| \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s)\chi_{t_{0k}t_{1k}}(x_k)(s)ds - \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s)\chi_{t_{0k}t_{1k}}(x)(s)ds \Big| \le \\ \le \alpha \int_a^b |\chi_{t_{0k}t_{1k}}(x_k)(s) - \chi_{t_{0k}t_{1k}}(x)(s)|ds \le \varepsilon_1, \end{aligned} \tag{2.45}$$

if $k > n_{02}$, and

$$\begin{aligned} &\Big| \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s)\chi_{t_{0k}t_{1k}}(x)(s)ds - \int_a^b G(\tau_0(t), s)x(s)ds \Big| = \\ &= \Big| \int_a^{\mu_0(t_{0k},t_{1k},t)} \frac{\varphi(s)-\varphi(a)}{\varphi(b)-\varphi(a)}\chi_{t_{0k}t_{1k}}(x)(s)ds - \int_a^{\tau_0(t)} \frac{\varphi(s)-\varphi(a)}{\varphi(b)-\varphi(a)}x(s)ds \Big| + \\ &+ \Big| \int_{\mu_0(t_{0k},t_{1k},t)}^b \frac{\varphi(s)-\varphi(b)}{\varphi(b)-\varphi(a)}\chi_{t_{0k}t_{1k}}(x)(s)ds - \int_{\tau_0(t)}^b \frac{\varphi(s)-\varphi(b)}{\varphi(b)-\varphi(a)}x(s)ds \Big| \le \\ &\le \alpha \int_a^b |\chi_{t_{0k}t_{1k}}(x)(s) - x(s)|ds + 2\alpha \Big| \int_{\tau_0(t)}^{\mu_0(t_{0k},t_{1k},t)} x(s)ds \Big| \le \varepsilon_1, \end{aligned} \tag{2.46}$$

if $k > n_{02}$, where $\alpha = \max\limits_{a \le s \le t \le b} \left\{ \frac{|\varphi(s)-\varphi(t)|}{|\varphi(b)-\varphi(a)|} \right\}$. Thus from (2.43)-(2.46) it is clear that

$$|\Lambda_k(x_k)(s) - \Lambda(x)(s)| \le |\Lambda_k(x_k)(s) - \Lambda_k(x)(s)| + |\Lambda_k(x)(s) - \Lambda(x)(s)| \le 2\varepsilon_1 \sum_{j=0}^m |p_j(t)|,$$

if $k > n_0$, with $n_0 = \max\{n_{01}, n_{02}\}$, and (2.43) follows immediately from the last inequality. $\square$

Lemma 2.7. *Let condition* (2.36) *hold, and for every natural k, problem* (2.34), (2.35) *have a solution $\widetilde{u}_k \in \widetilde{C}^{2m-1}_{loc}(]a,b[)$, and there exist a constant $r_0 > 0$ such that*

$$\int\limits_{t_{0k}}^{t_{1k}} |\widetilde{u}_k^{(m)}(s)|^2 ds \le r_0^2 \quad (k \in N) \tag{2.47}$$

holds. Moreover, let

$$\lim_{k\to+\infty} ||q_k - q||_{\widetilde{L}^2_{2m-2,\,2m-2}} = 0, \tag{2.48}$$

and the homogeneous problem (2.32_0), (2.33_0) *have only the trivial solution in the space* $\widetilde{C}^{2m-1,m}(]a,b[)$. *Then the inhomogeneous problem* (2.32), (2.33) *has a unique solution $\widetilde{u}$ such that*

$$||\widetilde{u}^{(m)}||_{L^2} \le r_0, \tag{2.49}$$

and

$$\lim_{k\to+\infty} \widetilde{u}_k^{(j-1)}(t) = \widetilde{u}^{(j-1)}(t) \quad (j = 1,\dots,2m) \quad \textit{uniformly in} \quad]a,b[\tag{2.50}$$

(that is, uniformly on $[a+\delta, b-\delta]$ for an arbitrarily small $\delta > 0$).

Proof. Suppose that $t_1,\dots,t_{2m}$ are the numbers such that

$$\frac{a+b}{2} = t_1 < \cdots < t_{2m} < b, \tag{2.51}$$

and $g_i(t)$ are the polynomials of $(2m-1)$th degree satisfying the conditions

$$g_j(t_j) = 1, \quad g_j(t_i) = 0 \;\; (i \ne j; \;\; i,j = 1,\dots,2m). \tag{2.52}$$

Then, for every natural k, the solution $\widetilde{u}_k$ of problem (2.34), (2.35) admits the representation

$$\widetilde{u}_k(t) = \sum_{j=1}^{2m} \Big(\widetilde{u}_k(t_j) - \frac{1}{(2m-1)!} \int\limits_{t_1}^{t_j} (t_j - s)^{2m-1} (\Lambda_k(\widetilde{u}_k)(s) + q_k(s)) ds \Big) g_j(t) +$$

$$+ \frac{1}{(2m-1)!} \int\limits_{t_1}^{t} (t-s)^{2m-1} (\Lambda_k(\widetilde{u}_k)(s) + q_k(s)) ds. \tag{2.53}$$

For an arbitrary $\delta \in]0, \frac{a+b}{2}[$, we have

$$\Big|\int\limits_t^{t_1}(s-t)^{2m-j}(q_k(s)-q(s))ds\Big| = (2m-j)\Big|\int\limits_t^{t_1}(s-t)^{2m-j-1}\Big(\int\limits_s^{t_1}(q_k(\xi)-q(\xi))d\xi\Big)ds\Big| \leq$$

$$\leq 2m\Big(\int\limits_t^{t_1}(s-a)^{2m-2j}ds\Big)^{1/2}\Big(\int\limits_t^{t_1}(s-a)^{2m-2}\Big(\int\limits_s^{t_1}(q_k(\xi)-q(\xi))d\xi\Big)^2 ds\Big)^{1/2} \leq$$

$$\leq n\Big|(t_1-a)^{2m-2j+1}-\delta^{2m-2j+1}\Big|^{1/2}||q_k-q||_{\widetilde{L}^2_{2m-2,\,2m-2}} \quad \text{for } \ a+\delta \leq t \leq t_1,$$

$$\Big|\int\limits_{t_1}^{t}(t-s)^{2m-j}(q_k(s)-q(s))ds\Big| \leq 2m\Big|(b-t_1)^{2m-2j+1}-\delta^{2m-2j+1}\Big|^{1/2}\times \tag{2.54}$$

$$\times||q_k-q||_{\widetilde{L}^2_{2m-2,\,2m-2}} \qquad \text{for } \ t_1 \leq t \leq b-\delta \ \ (j=1,\dots,2m-1).$$

Hence, by condition (2.48), we find

$$\lim_{k\to+\infty}\int\limits_t^{t_1}(s-t)^{2m-j}(q_k(s)-q(s))ds = 0 \quad \text{ uniformly in } \ ]a,b[, \tag{2.55}$$

for $(j=1,\dots,2m-1)$. Analogously, one can show that if $t_0 \in]a,b[$, then

$$\lim_{k\to+\infty}\int\limits_{t_0}^{t}(s-t_0)(q_k(s)-q(s))ds = 0 \quad \text{ uniformly on } \ I(t_0), \tag{2.56}$$

where $I(t_0)=[t_0,(a+b)/2]$ for $t_0<(a+b)/2$ and $I(t_0)=[(a+b)/2,t_0]$ for $t_0>(a+b)/2$.

In view of inequalities (2.47), the identities

$$\widetilde{u}_k^{(j-1)}(t) = \frac{1}{(m-j)!}\int\limits_{t_{ik}}^{t}(t-s)^{m-j}\widetilde{u}_k^{(m)}(s)ds \tag{2.57}$$

for $i=0,1;\ j=1,\dots,m;\ k\in N$, yield

$$|\widetilde{u}_k^{(j-1)}(t)| \leq r_j[(t-a)(b-t)]^{m-j+1/2} \tag{2.58}$$

for $t_{0k} \le t \le t_{1k}$ $j = 1, \dots, m;$ $k \in N$, where

$$r_j = \frac{r_0}{(m-j)!}(2m-2j+1)^{-1/2}\Big(\frac{2}{b-a}\Big)^{m-j+1/2}. \tag{2.59}$$

By virtue of the Arzela-Ascoli Lemma and conditions (2.47) and (2.58), the sequence $\{\widetilde{u}_k\}_{k=1}^{+\infty}$ contains a subsequence $\{\widetilde{u}_{k_l}\}_{l=1}^{+\infty}$ such that $\{\widetilde{u}_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ $(j = 1, \dots, m)$ are uniformly convergent in $]a, b[$. Suppose that

$$\lim_{l\to+\infty} \widetilde{u}_{k_l}(t) = \widetilde{u}(t). \tag{2.60}$$

Then, in view of (2.58), $\widetilde{u}^{(j-1)} \in C([a, b])$ $(j = 1, \dots, m)$, and

$$\lim_{l\to+\infty} \widetilde{u}_{k_l}^{(j-1)}(t) = \widetilde{u}^{(j-1)}(t) \quad (j = 1, \dots, m) \quad \text{uniformly in} \quad]a, b[. \tag{2.61}$$

If, along with this, we take conditions (2.36) and (2.55) into account, from (2.53) by Lemma 2.6 we find

$$\begin{aligned}\widetilde{u}(t) = \sum_{j=1}^{2m}\Big(\widetilde{u}(t_j) - \frac{1}{(2m-1)!}\int_{t_1}^{t_j}(t_j - s)^{2m-1}(\Lambda(\widetilde{u})(s) + q(s))ds\Big)g_j(t)+ \\ +\frac{1}{(2m-1)!}\int_{t_1}^{t}(t-s)^{2m-1}(\Lambda(\widetilde{u})(s) + q(s))ds \quad \text{for} \quad a < t < b,\end{aligned} \tag{2.62}$$

$$|\widetilde{u}^{(j-1)}(t)| \le r_j[(t-a)(b-t)]^{m-j+1/2} \quad \text{for} \quad a < t < b \ (j = 1, \dots, m), \tag{2.63}$$

$\widetilde{u} \in \widetilde{C}^{2m-1}_{loc}(]a, b[)$, and

$$\lim_{l\to+\infty} \widetilde{u}_{k_l}^{(j-1)}(t) = \widetilde{u}^{(j-1)}(t) \quad (j = 1, \dots, 2m-1) \quad \text{uniformly in} \quad]a, b[. \tag{2.64}$$

On the other hand, for any $t_0 \in]a, b[$ and natural l, we have

$$(t-t_0)\widetilde{u}_{k_l}^{(2m-1)}(t) = \widetilde{u}_{k_l}^{(2m-2)}(t) - \widetilde{u}_{k_l}^{(2m-2)}(t_0) + \int_{t_0}^{t}(s-t_0)(\Lambda_k(\widetilde{u}_{k_l})(s) + q_{k_l}(s))ds. \tag{2.65}$$

Hence, due to (2.36), (2.56), (2.64), and Lemma 2.6 we get

$$\lim_{l\to+\infty} \widetilde{u}_{k_l}^{(2m-1)}(t) = \widetilde{u}^{(2m-1)}(t) \quad \text{uniformly in} \quad]a, b[. \tag{2.66}$$

Now it is clear that relations (2.64), (2.66), and (2.47) result in (2.49). Consequently, $\widetilde{u} \in \widetilde{C}^{2m-1,\, m}(]a,b[)$. On the other hand, from (2.62) it is obvious that $\widetilde{u}$ is a solution of (2.32), and from (2.63) equalities (2.33) follow, that is, $\widetilde{u}$ is a solution of problem (2.32), (2.33).

To complete the proof of the Lemma, it remains to show that equality (2.50) is satisfied. First note that in the space $\widetilde{C}^{2m-1,m}(]a,b[)$ problem (2.32), (2.33) does not have another solution since in that space the homogeneous problem (2.32_0), (2.33_0) has only the trivial solution. Now let assume the contrary. Then there exist $\delta \in]0, \frac{b-a}{2}[$, $\varepsilon > 0$, and an increasing sequence of natural numbers $\{k_l\}_{l=1}^{+\infty}$ such that

$$\max\Big\{ \sum_{j=1}^{2m} |\widetilde{u}_{k_l}^{(j-1)}(t) - \widetilde{u}^{(j-1)}(t)| : a+\delta \le t \le b-\delta \Big\} > \varepsilon \quad (l \in N). \tag{2.67}$$

By virtue of the Arzela-Ascoli Lemma and condition (2.47), the sequence $\{\widetilde{u}_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ $(j = 1,\dots,m)$, without loss of generality, can be assumed to be uniformly converging in $]a,b[$. Then, in view of what we have shown above, conditions (2.64) and (2.66) hold. However, this contradicts condition (2.67). The obtained contradiction proves the validity of the lemma. $\square$

Lemma 2.8. *Let $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, the functions h_j and the operators f_j be given by equalities* (1.7) *and* (1.8). *Let, moreover, $\tau_j \in M(]a,b[)$, and the constants $l_{k,j} > 0$, $\gamma_{kj} > 0$ $(k = 0,\, 1;\ j = 1,\dots,m)$ be such that conditions* (1.9)-(1.11) *are fulfilled. Then there exists positive constants δ and r_1 such that if $a_0 \in]a,\ a+\delta[$, $b_0 \in]b-\delta, b[$, $t_0 \in]a,\ a_0[$, $t_1 \in]b_0,\ b[$, and $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$, an arbitrary solution $\widetilde{u} \in C^{2m-1}_{loc}(]a,b[)$ of the problem*

$$\begin{gathered}\widetilde{u}^{(2m)}(t) = \sum_{j=1}^{m} p_j(t)\widetilde{u}^{(j-1)}(\mu_j(t_0,t_1,t)) + \\ +p_0(t)\int_a^b G(\mu_j(t_0,t_1,t),s)\chi_{t_0t_1}(\widetilde{u})(s)ds + q(t) \quad for \quad t_0 \le t \le t_1,\end{gathered} \tag{2.68}$$

$$\widetilde{u}^{(j-1)}(t_0) = 0, \quad \widetilde{u}^{(j-1)}(t_1) = 0 \ \ (j = 1,\dots,m) \tag{2.69}$$

satisfies the inequality

$$\int_{t_0}^{t_1} |\widetilde{u}^{(m)}(s)|^2 ds \le r_1 \Big(\Big| \sum_{j=1}^{m} \int_{a_0}^{b_0} p_j(s)\widetilde{u}(s)\widetilde{u}^{(j-1)}(\mu_j(t_0,t_1,s))ds \Big| +$$

$$+ \Big| \int_{a_0}^{b_0} p_0(s)\widetilde{u}(s) \int_{a}^{b} G(\mu_j(t_0,t_1,s),\xi)\chi_{t_0t_1}(\widetilde{u})(\xi)d\xi ds \Big| + ||q||^2_{\widetilde{L}^2_{2m-2,\,2m-2}} \Big). \quad (2.70)$$

Proof. Conditions (1.9) and (1.10) imply the existence of constants $\overline{l}_{kj} \ge 0$ $(k = 0, 1)$ such that

$$(t-a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(a,\tau_j)(t,s) \le \overline{l}_{0j} \quad \text{for} \quad a < t \le s \le a_0,$$

$$(b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(b,\tau_j)(t,s) \le \overline{l}_{1j} \quad \text{for} \quad b_0 \le s \le t < b.$$

Consequently, all the requirements of Lemma 2.3 with $\overline{p}_j(t) = (-1)^m p_j(t)$, $a < t_0 < a_0$, and Lemma 2.4 with $\overline{p}_j(t) = (-1)^m p_j(t)$, $b_0 < t_1 < b$, are fulfilled. Condition (1.11) also guarantees the existence of a $\nu \in]0,1[$ such that

$$\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{kj} < 1 - 2\nu \ \ (k = 0, 1). \quad (2.71)$$

On the other hand, without loss of generality we can assume that $a_0 \in]a, a+\delta[$ and $b_0 \in]b-\delta, b[$, where δ is a constant such that

$$\sum_{j=0}^{m} (\overline{l}_{0j}\beta_j(\delta,\gamma_{0j}) + \overline{l}_{1j}\beta_j(\delta,\gamma_{1j})) < \nu, \quad (2.72)$$

where the functions β_j are defined by (2.6). Let now $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$, u be a solution of problem (2.68), (2.69), and

$$r_1 = \frac{2^{2m}}{(\nu(2m-3)!!)^2}. \quad (2.73)$$

Multiplying both sides of (2.68) by $(-1)^m\widetilde{u}(t)$ and then integrating by parts from t_0 to

t_1, in view of conditions (2.69), we obtain

$$\int\limits_{t_0}^{t_1} |\widetilde{u}^{(m)}(s)|^2 ds = (-1)^m \sum_{j=1}^{m} \int\limits_{t_0}^{t_1} p_j(s)\widetilde{u}(s)\widetilde{u}^{(j-1)}(\mu_j(t_0,t_1,s))ds +$$

$$+ (-1)^m \int\limits_{t_0}^{t_1} p_0(s)\widetilde{u}(s) \int\limits_{a}^{b} G(\mu_j(t_0,t_1,s),\xi)\chi_{t_0t_1}(\widetilde{u})(\xi)d\xi ds +$$

$$+ (-1)^m \int\limits_{t_0}^{t_1} q(s)\widetilde{u}(s)ds. \quad (2.74)$$

Applying Lemmas 2.3 and 2.4 with $\overline{p}_j(t) = (-1)^m p_j(t)$, and using equalities $\rho_0(t_0) = \rho_1(t_1) = 0$, by virtue of (2.71), we get

$$(-1)^m \sum_{j=1}^{m} \int\limits_{t_0}^{a_0} p_j(s)\widetilde{u}(s)\widetilde{u}^{(j-1)}(\mu_j(t_0,t_1,s))ds +$$

$$+ (-1)^m \int\limits_{t_0}^{a_0} p_0(s)\widetilde{u}(s) \int\limits_{a}^{b} G(\mu_j(t_0,t_1,s),\xi)\chi_{t_0t_1}(\widetilde{u})(\xi)d\xi ds \le$$

$$\le \sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{0j}\rho_0(a_0) + \sum_{j=0}^{m} \overline{l}_{0j}\beta_j(a-a_0,\gamma_{0j})\rho_0(t_1) \le$$

$$\le (1-2\nu)\rho_0(a_0) + \sum_{j=0}^{m} \overline{l}_{0j}\beta_j(\delta,\gamma_{0j}) \int\limits_{t_0}^{t_1} |\widetilde{u}^{(m)}(s)|^2 ds, \quad (2.75)$$

and

$$(-1)^m \sum_{j=1}^{m} \int\limits_{b_0}^{t_1} p_j(s)\widetilde{u}(s)\widetilde{u}^{(j-1)}(\mu_j(t_0,t_1,s))ds +$$

$$+ (-1)^m \int\limits_{b_0}^{t_1} p_0(s)\widetilde{u}(s) \int\limits_{a}^{b} G(\mu_j(t_0,t_1,s),\xi)\chi_{t_0t_1}(\widetilde{u})(\xi)d\xi ds \le$$

$$\leq \sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{1j}\rho_1(b_0) + \sum_{j=0}^{m} \bar{l}_{1j}\beta_j(b_0-b,\gamma_{1j})\rho_1(t_0) \leq$$

$$\leq (1-2\nu)\rho_1(b_0) + \sum_{j=0}^{m} \bar{l}_{1j}\beta_j(\delta,\gamma_{1j}) \int_{t_0}^{t_1} |\widetilde{u}^{(m)}(s)|^2 ds. \quad (2.76)$$

If along with this we take into account inequalities (2.72) and $a_0 \leq b_0$, we find

$$(-1)^m \sum_{j=1}^{m} \int_{t_0}^{t_1} p_j(s)\widetilde{u}(s)\widetilde{u}^{(j-1)}(\mu_j(t_0,t_1,s))ds +$$

$$+ (-1)^m \int_{t_0}^{t_1} p_0(s)\widetilde{u}(s) \int_{a}^{b} G(\mu_j(t_0,t_1,s),\xi)\chi_{t_0t_1}(\widetilde{u})(\xi)d\xi ds \leq$$

$$\leq \Big| \sum_{j=1}^{m} \int_{a_0}^{b_0} p_j(s)\widetilde{u}(s)\widetilde{u}^{(j-1)}(\mu_j(t_0,t_1,s))ds \Big| +$$

$$+ \Big| \int_{a_0}^{b_0} p_0(s)\widetilde{u}(s) \int_{a}^{b} G(\mu_j(t_0,t_1,s),\xi)\chi_{t_0t_1}(\widetilde{u})(\xi)d\xi ds \Big| +$$

$$+ (1-2\nu)\Big(\rho_0(a_0)+\rho_1(b_0)\Big) + \nu \int_{t_0}^{t_1} |\widetilde{u}^{(m)}(s)|^2 ds \leq (1-\nu) \int_{t_0}^{t_1} |\widetilde{u}^{(m)}(s)|^2 ds +$$

$$+ \Big| \sum_{j=1}^{m} \int_{a_0}^{b_0} p_j(s)\widetilde{u}(s)\widetilde{u}^{(j-1)}(\mu_j(t_0,t_1,s))ds \Big| +$$

$$+ \Big| \int_{a_0}^{b_0} p_0(s)\widetilde{u}(s) \int_{a}^{b} G(\mu_j(t_0,t_1,s),\xi)\chi_{t_0t_1}(\widetilde{u})(\xi)d\xi ds \Big|. \quad (2.77)$$

On the other hand, if we put $c = (a+b)/2$, then, again on the basis of Lemmas 2.1, 2.2, and the Young inequality, we get

$$\Big| \int_{t_0}^{t_1} q(s)\widetilde{u}(s)ds \Big| \leq \Big| \int_{t_0}^{c} \widetilde{u}'(s)\Big(\int_{s}^{c} q(\xi)d\xi\Big)ds \Big| + \Big| \int_{c}^{t_1} \widetilde{u}'(s)\Big(\int_{c}^{s} q(\xi)d\xi\Big)ds \Big| \leq$$

$$\leq \Big(\int\limits_{t_0}^{c} \frac{\widetilde{u}'^2(s)}{(s-a)^{2m-2}}ds\Big)^{1/2}\Big(\int\limits_{t_0}^{c}(s-a)^{2m-2}\Big(\int\limits_{s}^{c} q(\xi)d\xi\Big)^2 ds\Big)^{1/2}+$$

$$+\Big(\int\limits_{c}^{t_1} \frac{\widetilde{u}'^2(s)}{(b-s)^{2m-2}}ds\Big)^{1/2}\Big(\int\limits_{c}^{t_1}(b-s)^{2m-2}\Big(\int\limits_{c}^{s} q(\xi)d\xi\Big)^2 ds\Big)^{1/2}\leq$$

$$\leq \frac{2^m}{(2m-3)!!}\Big(\int\limits_{t_0}^{t_1}|\widetilde{u}^{(m)}(s)|^2 ds\Big)^{1/2}||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}\leq$$

$$\leq \frac{\nu}{2}\int\limits_{t_0}^{t_1}|\widetilde{u}^{(m)}(s)|^2 ds+\frac{2^{2m}}{\nu((2m-3)!!)^2}||q||^2_{\widetilde{L}^2_{2m-2,\,2m-2}} \qquad (2.78)$$

and without loss of generality we can assume that $\frac{2^{2m}}{\nu((2m-3)!!)^2}\geq 1$. In view of inequalities (2.77), (2.78) and notation (2.73), equality (2.74) results in estimate (2.70). □

Lemma 2.9. *Let* $\tau_j\in M(]a,b[)$, $a_0\in]a,b[$, $b_0\in]a_0,b[$, *conditions* (1.6), (1.9)- (1.11), *hold, where the functions* h_j, β_j *and the operators* f_j *are given by equalities* (1.7), (1.8), *and* l_{kj}, $\overline{l}_{kj}$, γ_{kj} $(k=0,1;\ j=1,\dots,m)$ *are nonnegative numbers. Moreover, let the homogeneous problem* (2.32_0), (2.33_0) *have only the trivial solution in the space* $\widetilde{C}^{2m-1,m}(]a,b[)$. *Then there exist* $\delta\in]0,\frac{b-a}{2}[$ *and* $r>0$ *such that for any* $t_0\in]a,a+\delta]$, $t_1\in]b+\delta,b]$, *and* $q\in\widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$ *problem* (2.68), (2.69) *is uniquely solvable in the space* $\widetilde{C}^{2m-1}(]a,b[)$, *and its solution admits the estimate*

$$\Big(\int\limits_{t_0}^{t_1}|\widetilde{u}^{(m)}(s)|^2 ds\Big)^{1/2}\leq r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}. \qquad (2.79)$$

Proof. We first note that all the requirements of Lemmas 2.7 and 2.8 are fulfilled.

Let now $\delta\in]0,\min\{b-b_0,a_0-a\}]$ be such as in Lemma 2.8 and assume that estimate (2.79) is invalid. Then, for an arbitrary natural k, there exist

$$t_{0k}\in]a,\,a+\delta/k[,\qquad t_{1k}\in]b-\delta/k,\,b[, \qquad (2.80)$$

and a function $q_k\in\widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$ such that problem (2.34), (2.35) has a solution $\widetilde{u}_k\in\widetilde{C}^{2m-1}(]a,b[)$ satisfying the inequality

$$\Big(\int\limits_{t_{0k}}^{t_{1k}}|\widetilde{u}_k^{(m)}(s)|^2 ds\Big)^{1/2}>k||q_k||_{\widetilde{L}^2_{2m-2,\,2m-2}}. \qquad (2.81)$$

In the case when the homogeneous problem (2.34_0), (2.35) has a nontrivial solution, in (2.34) we put that $q_k(t) \equiv 0$ and assume that $\widetilde{u}_k$ is that nontrivial solution of problem (2.34_0), (2.35).

Let now

$$\widetilde{v}_k(t) = \Big(\int\limits_{t_{0k}}^{t_{1k}} |\widetilde{u}_k^{(m)}(s)|^2 ds \Big)^{-1/2} \widetilde{u}_k(t), \quad q_{0k}(t) = \Big(\int\limits_{t_{0k}}^{t_{1k}} |\widetilde{u}_k^{(m)}(s)|^2 ds \Big)^{-1/2} q_k(t). \tag{2.82}$$

Then $\widetilde{v}_k$ is a solution of the problem

$$\begin{gathered} \widetilde{v}^{(2m)}(t) = \sum_{j=1}^{m} p_j(t) \widetilde{v}^{(j-1)}(\tau_j(t)) + \\ + p_0(t) \int\limits_a^b G(\mu_0(t_{0k}, t_{1k}, t), s) \chi_{t_{0k} t_{1k}}(\widetilde{v})(s) ds + q_{0k}(t) \quad \text{for} \quad t_{0k} \leq t \leq t_{1k}, \\ \widetilde{v}^{(i-1)}(t_{0k}) = 0, \qquad \widetilde{v}^{(i-1)}(t_{1k}) = 0 \ \ (i = 1, \dots, m). \end{gathered} \tag{2.83}$$

Moreover, in view of (2.81), it is clear that

$$\int\limits_{t_{0k}}^{t_{1k}} |\widetilde{v}_k^{(m)}(s)|^2 ds = 1, \quad ||q_{0k}||_{\widetilde{L}^2_{2m-2,\,2m-2}} < \frac{1}{k} \ \ (k \in N). \tag{2.84}$$

On the other hand, in view of the fact that problem (2.32_0), (2.33_0) has only the trivial solution in the space $\widetilde{C}^{2m-1,m}(]a,b[)$, by Lemmas 2.7, 2.8, and (2.84) we have

$$\begin{gathered} \lim_{t \to +\infty} \widetilde{v}_k^{(j-1)}(t) = 0 \quad \text{uniformly in} \quad]a,b[\ \ (j = 1, \dots n), \\ 1 < r_0 \Big(\Big| \int\limits_{a_0}^{b_0} \Lambda_k(\widetilde{v}_k)(s) ds \Big| + k^{-2} \Big) \ \ (k \in N), \end{gathered} \tag{2.85}$$

where r_0 is a positive constant independent of k. Now, if we pass to the limit in (2.85) as $k \to +\infty$, by Lemma 2.6 we obtain the contradiction $1 < 0$. Consequently, for any solution of problem (2.68), (2.69), with arbitrary $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$, estimate (2.79) holds. Thus, under conditions (2.69), the homogeneous equation

$$\widetilde{u}^{(2m)}(t) = \sum_{j=1}^{m} p_j(t) \widetilde{u}^{(j-1)}(\mu_j(t_0, t_1, t)) + p_0(t) \int\limits_a^b G(\mu_j(t_0, t_1, t), s) \chi_{t_0 t_1}(\widetilde{u})(s) ds \tag{2.82_0}$$

has only the trivial solution. However, for arbitrarily fixed $t_0 \in]a, a+\delta[$, $t_1 \in]b-\delta, b[$, and $q \in L([t_0, t_1])$ problem (2.68), (2.69) is regular and has the Fredholm property in the space $\widetilde{C}^{2m-1}(]t_0, t_1[)$. Thus, problem (2.68), (2.69) is uniquely solvable. $\square$

Lemma 2.10. *Let $\tau \in M(]a, b[)$, $\alpha \geq 0$, $\beta \geq 0$, and let there exist $\delta \in]0, b-a[$ such that*

$$|\tau(t) - t| \leq k_1 (t-a)^\beta \quad \textit{for} \quad a < t \leq a+\delta. \tag{2.86}$$

Then

$$\Big| \int_t^{\tau(t)} (s-a)^\alpha ds \Big| \leq \begin{cases} k_1 [1 + k_1 \delta^{\beta-1}]^\alpha (t-a)^{\alpha+\beta} & \textit{for} \ \ \beta \geq 1 \\ k_1 [\delta^{1-\beta} + k_1]^\alpha (t-a)^{\alpha\beta+\beta} & \textit{for} \ \ 0 \leq \beta < 1 \end{cases},$$

for $a < t \leq a+\delta$.

Proof. We first note that

$$\Big| \int_t^{\tau(t)} (s-a)^\alpha ds \Big| \leq (\max\{\tau(t), t\} - a)^\alpha |\tau(t) - t| \quad \text{for} \quad a \leq t \leq a+\delta,$$

and $\max\{\tau(t), t\} \leq t + |\tau(t) - t|$ for $a \leq t \leq a+\delta$. Then, in view of condition (2.86), we get

$$\Big| \int_t^{\tau(t)} (s-a)^\alpha ds \Big| \leq k_1 [(t-a) + k_1 (t-a)^\beta]^\alpha (t-a)^\beta \quad \text{for} \quad a \leq t \leq a+\delta.$$

This inequality proves the validity of the lemma. $\square$

Analogously, one can prove

Lemma 2.11. *Let $\tau_j \in M(]a, b[)$, $\alpha \geq 0$, $\beta \geq 0$ and let there exist $\delta \in]0, b-a[$ such that*

$$|\tau_j(t) - t| \leq k_1 (b-t)^\beta \quad \textit{for} \quad b-\delta \leq t < b. \tag{2.87}$$

Then

$$\Big| \int_t^{\tau(t)} (b-t)^\alpha ds \Big| \leq \begin{cases} k_1 [1 + k_1 \delta^{\beta-1}]^\alpha (b-t)^{\alpha+\beta} & \textit{for} \ \ \beta \geq 1 \\ k_1 [\delta^{1-\beta} + k_1]^\alpha (b-t)^{\alpha\beta+\beta} & \textit{for} \ \ 0 \leq \beta < 1 \end{cases},$$

for $b-\delta \leq t < b$.

3 Proofs

Proof of Theorem 1.1. Suppose that problem (1.1_0), (1.2) has only the trivial solution. Then, in view of Remark 2.1, it follows that problem (2.32_0), (2.33_0) also has only the trivial solution. Let now r and δ be the numbers appearing in Lemma 2.9 and

$$t_{0k} = a + \delta/k \qquad t_{1k} = b - \delta/k \quad (k \in N). \tag{3.1}$$

By Lemma 2.9, for every natural k, problem (2.34), (2.35) with $q_k = q$, has a unique solution $\widetilde{u}_k$ in the space $\widetilde{C}^{2m-1}_{loc}(]a,\, b[)$ and

$$\Big(\int\limits_{t_{0k}}^{t_{1k}} |\widetilde{u}_k^{(m)}(s)|^2 ds\Big)^{1/2} \leq r||q||_{\widetilde{L}^2_{2m-2,2m-2}}, \tag{3.2}$$

where the constant r does not depend on q. by Lemma 2.7 with $r_0 = r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}$, it follows from (3.2) that problem (2.32), (2.33) has a unique solution $\widetilde{u} \in \widetilde{C}^{2m-1}_{loc}(]a,\, b[)$ for an arbitrary $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\, b[)$, where

$$\lim_{k\to+\infty} \widetilde{u}_k^{(j-1)}(t) = \widetilde{u}^{(j-1)}(t) \quad (j = 1, \dots, 2m) \quad \text{uniformly in} \quad]a, b[, \tag{3.3}$$

and

$$||\widetilde{u}^{(m)}||_{L^2} \leq r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}.$$

Thus problem (2.32), (2.33) has the Fredholm property and $\widetilde{u} \in \widetilde{C}^{2m-1,m}(]a,\, b[)$ for any $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\, b[)$.

Consequently, it follows from Remark 2.1 that problem (1.1), (1.2) has the Fredholm property in the space $\widetilde{C}^{2m,m+1}(]a,\, b[)$, and its solution u, where $u(t) = \int\limits_a^b G(t,s)\widetilde{u}(s)ds$, i.e. $u'(t) = \widetilde{u}(t)$, admits estimate (1.12). □

Proof of Corollary 1.1. In view of conditions (1.15), there exists a number $\varepsilon > 0$ such that

$$\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\Big(\frac{\kappa_{kj}}{2m-j} + \varepsilon\Big) < 1 \ (k = 0, 1). \tag{3.4}$$

On the other hand, in view of conditions (1.16) and (1.17), we have

$$
\begin{aligned}
&(t-a)^{2m-j}h_j(t,s) \le \frac{\kappa_{0j}}{2m-j} + \kappa_{1j}\int\limits_a^{a_0}\frac{(\xi-a)^{2m-j}}{(b-\xi)^{2m+1-j}}d\xi + \int\limits_a^{a_0}(\xi-a)^{2m-j}p_{0j}(\xi)d\xi \\
&\qquad\qquad \text{for} \quad a < t \le s \le a_0, \\
&(b-t)^{2m-j}h_j(t,s) \le \frac{\kappa_{1j}}{2m-j} + \kappa_{0j}\int\limits_{b_0}^{b}\frac{(b-\xi)^{2m-j}}{(\xi-a)^{2m-j+1}}d\xi + \int\limits_{b_0}^{b}(b-\xi)^{2m-j}p_{0j}(\xi)d\xi \\
&\qquad\qquad \text{for} \quad b_0 \le s \le t < b.
\end{aligned}
\tag{3.5}
$$

Let δ be the constant defined in Lemmas 2.10 and 2.11. Relation (1.16) implies the existence of $a_0 \in]a, a+\delta[$ and $b_0 \in]b-\delta, b[$ such that

$$
|p_1(t)| \le \frac{\kappa}{[(t-a)(b-t)]^{4m}} + p_{01}(t) \quad \text{for} \quad t \in [a, a_0] \cup [b_0, b]. \tag{3.6}
$$

On the other hand, by condition (1.14), it follows from Lemmas 2.10 and 2.11 that there exists a constant k_0 such that

$$
\begin{aligned}
&\Big|\int\limits_t^{\tau_j(t)}(s-a)^{2(m-j)}ds\Big|^{1/2} \le k_0^{1/2}(s-a)^{m-j+\nu_{0j}/2} \quad \text{for} \quad a \le t \le a_0, \\
&\Big|\int\limits_t^{\tau_j(t)}(b-s)^{2(m-j)}ds\Big|^{1/2} \le k_0^{1/2}(b-s)^{m-j+\nu_{1j}/2} \quad \text{for} \quad b_0 \le t \le b.
\end{aligned}
\tag{3.7}
$$

Consequently, if $p_{01} \in L_{n-j,\,2m-j}(]a,b[)$, then, by (1.13) and (3.7), relations (1.16) and (1.17) imply the existence of a nonnegative constant k_2 such that

$$
\begin{aligned}
&(t-a)^{m-1}f_0(a,\tau_0)(t,s) \le \int\limits_a^{a_0}(\xi-a)^{m-1}|p_{00}(\xi)|d\xi + \\
&\quad + \frac{1}{m-1} + \frac{(a_0-a)^m}{(b_0-a_0)^m} \quad \text{for} \quad a \le t < s \le a_0 \\
&(b-t)^{m-1}f_0(b,\tau_0)(t,s) \le \int\limits_{b_0}^{b}(b-\xi)^{m-1}|p_{00}(\xi)|d\xi + \\
&\quad + \frac{1}{m-1} + \frac{(b-b_0)^m}{(b_0-a_0)^m} \quad \text{for} \quad b_0 \le s < t \le b
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
(t-a)^{m-1} f_j(a,\tau_1)(t,s) &\le k_2(a_0-a)^{\varepsilon_0} \quad \text{for} \quad a\le t<s\le a_0,\\
(b-t)^{m-1} f_j(b,\tau_1)(t,s) &\le k_2(b-b_0)^{\varepsilon_0} \quad \text{for} \quad b_0\le s<t\le b,
\end{aligned}
\tag{3.9}
$$

where $0<\varepsilon_0=\min\{\nu_{k1}-4m-2,\ \nu_{kj}-2 : k=0,1;\ j=1,\dots,m\}$. Now, from (3.5), (3.8) and (3.9) it is clear that we can choose $\delta_1\le\delta$ so that if $\max\{b-b_0,\ a_0-a\}\le\delta_1$, then

$$
(t-a)^{2m-j} h_j(t,s) \le \frac{\kappa_{0j}}{2m-j}+\varepsilon \quad \text{for} \quad a<t\le s\le a_0,
$$

$$
(b-t)^{2m-j} h_j(t,s) \le \frac{\kappa_{1j}}{2m-j}+\varepsilon \quad \text{for} \quad b_0\le s\le t<b,
$$

$j\in\{1,\dots,m\}$. From (3.8), (3.9), the last inequalities and (3.4), it is clear that all the assumptions of Theorem 1.1, with $l_{kj}=\frac{\kappa_{kj}}{2m-j}+\varepsilon$, $\gamma_{k0}=\gamma_{kj}=1/2$, $(k=0,1,\quad j=1,\dots,m)$ and $\max\{b-b_0,\ a_0-a\}\le\delta_1$, are fulfilled, and thus the corollary is valid. $\square$

Proof of Theorem 1.2. From Theorem 1.1 by conditions (1.18)-(1.21) it is obvious that problem (1.1), (1.2) has the Fredholm property. Thus, to prove Theorem 1.2, it will suffice to show that the homogeneous problem (1.1_0), (1.2) has only the trivial solution in the space $\widetilde{C}^{2m,\,m+1}(]a,\,b[)$. Suppose that $u\in\widetilde{C}^{2m,\,m+1}(]a,\,b[)$ is a nonzero solution of problem (1.1_0), (1.2) and $\widetilde{u}=u'$. Then, in view of the condition $\varphi(b)-\varphi(a)\neq 0$, it is clear that $u\not\equiv Const$, and it follows from Remark 2.1 that the function $\widetilde{u}$ is a nonzero solution of problem (2.32), (2.33) such that

$$
\rho=\int_a^b |\widetilde{u}^{(m)}(s)|^2 ds<+\infty. \tag{3.10}
$$

Multiplying both sides of (1.1_0) by $(-1)^m\widetilde{u}(t)$ and integrating by parts from s to t, we obtain

$$
\begin{aligned}
w_{2m}(t)-w_{2m}(s)+\int_s^t |\widetilde{u}^{(m)}(\xi)|^2 d\xi &= (-1)^m\sum_{j=1}^m \int_s^t p_j(\xi)\widetilde{u}^{(j-1)}(\tau_j(\xi))\widetilde{u}(\xi)d\xi+\\
&+(-1)^m\int_s^t p_0(s)\widetilde{u}(s)\int_a^b G(s,\xi)\widetilde{u}(\xi)d\xi ds,
\end{aligned}
\tag{3.11}
$$

with $w_{2m}(t)=\sum_{j=1}^m(-1)^{m+j-1}\widetilde{u}^{(2m-j)}(t)\widetilde{u}(t)$, where, due Lemma 2.5, it is obvious that

$$
\liminf_{s\to a}|w_{2m}(s)|=0,\quad \liminf_{t\to b}|w_{2m}(t)|=0. \tag{3.12}
$$

According to (1.20), (1.21) and (3.10), all the conditions of Lemmas 2.3 and 2.4 with $\overline{p}_j(t) = (-1)^m p_j(t)$, $a_0 = b_0 = t^*$, $t_0 = a$, $t_1 = b$ and $\mu_j(t_0, t_1, t) = \tau_j(t)$ hold. Consequently, due to the equalities $\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(t^*) \le \rho$, $\rho_0^{1/2}(b)\rho_0^{1/2}(t^*) \le \rho$, $\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t^*) \le \rho$, $\rho_1^{1/2}(a)\rho_1^{1/2}(t^*) \le \rho$, we have

$$
\begin{aligned}
(-1)^m \int_s^t p_0(s)\widetilde{u}(s) \int_a^b G(s,\xi)\widetilde{u}(\xi)d\xi ds \le \\
\le \overline{l}_{00}\beta_0(t^*-a,\gamma_{00})\rho + \overline{l}_{10}\beta_0(b-t^*,\gamma_{10})\rho + \\
+ \alpha_0(\overline{l}_{00}, a_0-a, \gamma_{00})\rho_0^{1/2}(b)\rho_0^{1/2}(s) + \alpha_0(\overline{l}_{10}, b-b_0, \gamma_{10})\rho_0^{1/2}(a)\rho_1^{1/2}(t)
\end{aligned}
\tag{3.13}
$$

for $a < s < t^* < t < b$ and

$$
\begin{aligned}
(-1)^m \int_s^t p_j(\xi)\widetilde{u}^{(j-1)}(\tau_j(\xi))\widetilde{u}(\xi)d\xi \le \\
\le \overline{l}_{0j}\beta_j(t^*-a,\gamma_{0j})\rho + l_{0j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\rho_0(t^*) + \\
+ \overline{l}_{1j}\beta_j(b-t^*,\gamma_{1j})\rho + l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\rho_1(t^*) + \\
+ \alpha_j(l_{0j}, \overline{l}_{0j}, a_0-a, \gamma_{0j})\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(s) + \alpha_j(l_{1j}, \overline{l}_{1j}, b-b_0, \gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t)
\end{aligned}
\tag{3.14}
$$

for $a < s < t^* < t < b$. On the other hand, due to conditions (1.18) and (1.19), the number $\nu \in]0,1[$ can be chosen such that inequalities

$$
\begin{aligned}
B_0 \equiv \overline{l}_{00}\beta_0(t^*-a,\gamma_{00}) + \\
+ \sum_{j=1}^m \Big(l_{0j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} + \overline{l}_{0j}\beta_j(t^*-a,\gamma_{0j}) \Big) < \frac{1-\nu}{2}, \\
B_1 \equiv \overline{l}_{10}\beta_0(b-t^*,\gamma_{10}) + \\
+ \sum_{j=1}^m \Big(l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} + \overline{l}_{1j}\beta_j(b-t^*,\gamma_{1j}) \Big) < \frac{1-\nu}{2},
\end{aligned}
\tag{3.15}
$$

are satisfied. Thus if we pass to limit with $s \to s$, $t \to b$, in (3.11), according to (3.12)-(3.15), and the fact that $\rho_0(a) = \rho_1(b) = 0$, we get the inequality $\rho \le (1-\nu)\rho$, and consequently, $\rho = 0$. Hence, by

$$|\widetilde{u}(t)| = \frac{1}{(k-1)!}\Big|\int\limits_a^t (t-s)^{m-1}\widetilde{u}^{(m)}(s)ds\Big| \le (t-a)^{m-1/2}\rho \quad \text{for} \quad a<t<b,$$

we have the contradiction with the fact that $\widetilde{u}(t) \equiv 0$. Therefore, our assumption is wrong and, thus, problem (1.1), (1.2) has only the trivial solution in the space $\widetilde{C}^{2m,\,m+1}(]a,\,b[)$.
□

Proof of Remark 1.1. Let u be a solution of problem (1.1), (1.2). Then, by Remark 2.1, the function $\widetilde{u}$, where $u(t) = \int\limits_a^b G(t,s)\widetilde{u}(s)ds$, is a solution of problem (2.32), (2.33) and, in view of Theorem 1.1, the inclusion $u \in \widetilde{C}^{2m,\,m+1}(]a,\,b[)$ holds, i.e.

$$\rho \equiv \int\limits_a^b |u^{(m+1)}(s)|^2 ds\rho = \int\limits_a^b |\widetilde{u}^{(m)}(s)|^2 ds < +\infty. \tag{3.16}$$

Furthermore, if t_{0k}, t_{1k} are defined by equalities (3.1), it is clear from the proof of Theorem 1.1 that for any $k \in N$ problem (2.34), (2.35) has a unique solution $\widetilde{u}_k \in \widetilde{C}^{2m,m-1}(]a,\,b[)$ such that (3.2) and (3.3) hold.

Multiplying equation (2.34) by $(-1)^m\widetilde{u}_k$ and then integrating by parts from t_{0k} to t_{1k}, we obtain

$$\begin{aligned} w_{2m,k}(t) - w_{2m,k}(s) + \int\limits_s^t |\widetilde{u}_k^{(m)}(\xi)|^2 d\xi = (-1)^m \int\limits_s^t q(s)\widetilde{u}_k(s)ds+ \\ +(-1)^m \sum_{j=1}^m \int\limits_s^t p_j(\xi)\widetilde{u}_k^{(j-1)}(\tau_j(\xi))\widetilde{u}_k(\xi)d\xi+ \\ +(-1)^m \int\limits_s^t p_0(s)\widetilde{u}_k(s)\int\limits_a^b G(s,\xi)\chi_{t_{0k}t_{1k}}(\widetilde{u}_k)(\xi)d\xi ds, \end{aligned} \tag{3.17}$$

for $a < s \le t < b$, with $w_{2m,k}(t) = \sum\limits_{j=1}^m (-1)^{m+j-1}\widetilde{u}_k^{(2m-j)}(t)\widetilde{u}_k(t)$, where, due to (3.3), we have

$$\liminf_{k\to+\infty} |w_{2m,k}(t)| = |w_{2m}(t)|, \quad \liminf_{k\to+\infty} |w_{2m,k}(t)| = |w_{2m}(t)|, \tag{3.18}$$

and, therefore, it is obvious from Lemma 2.5 that equalities (3.12) hold. Furthermore, due to conditions (1.18) and (1.19), the number $\nu \in]0,1[$ can be chosen so that inequalities (3.15) hold, and then

$$0 < \nu < 1 - 2\max\{B_0,\, B_1\}. \tag{3.19}$$

It is obvious that the maximum of ν depend only on the numbers l_{kj}, $\bar{l}_{k0}$, $\bar{l}_{kj}$, γ_{k0}, γ_{kj} $(k = 0,1;\ j = 1,\dots,m)$, and a, b, t^*. If we now put $c = (a+b)/2$, then, by using Lemmas 2.1, 2.2, conditions (2.35), and the Young inequality, we get

$$\Big|\int\limits_{t_{0k}}^{t_{1k}} q(\psi)\widetilde{u}_k(\psi)d\psi\Big| \le \Big|\int\limits_{t_{0k}}^{c} q(\psi)\widetilde{u}_k(\psi)d\psi\Big| + \Big|\int\limits_{c}^{t_{1k}} q(\psi)\widetilde{u}_k(\psi)d\psi\Big| =$$

$$= \Big|\int\limits_{t_{0k}}^{c} \widetilde{u}_k'(\psi)\Big(\int\limits_{\psi}^{c} q(\xi)d\xi\Big)d\psi\Big| + \Big|\int\limits_{c}^{t_{1k}} \widetilde{u}_k'(\psi)\Big(\int\limits_{c}^{\psi} q(\xi)d\xi\Big)d\psi\Big| \le$$

$$\le \Big(\int\limits_{t_{0k}}^{c} \frac{\widetilde{u}_k'^2(\psi)}{(\psi-a)^{2m-2}}d\psi\Big)^{1/2} \times \Big(\int\limits_{t_{0k}}^{c} (\psi-a)^{2m-2}\Big(\int\limits_{\psi}^{c} q(\xi)d\xi\Big)^2 d\psi\Big)^{1/2} +$$

$$+\Big(\int\limits_{c}^{t_{1k}} \frac{\widetilde{u}_k'^2(\psi)}{(b-\psi)^{2m-2}}d\psi\Big)^{1/2} \times \Big(\int\limits_{c}^{t_{1k}} (b-\psi)^{2m-2}\Big(\int\limits_{c}^{\psi} q(\xi)d\xi\Big)^2 d\psi\Big)^{1/2} \le$$

$$\le \frac{2^m}{(2m-3)!!}||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}\Big(\int\limits_a^b |\widetilde{u}_k^{(m)}(s)|^2 ds\Big)^{1/2} \le$$

$$\le \frac{\nu}{2}\int\limits_a^b |\widetilde{u}_k^{(m)}(s)|^2 ds + \frac{1}{2\nu}\Big(\frac{2^m}{(2m-1)!!}\Big)^2 ||q||^2_{\widetilde{L}^2_{2m-2,\,2m-2}}. \tag{3.20}$$

Using Lemmas 2.3 and 2.4 and conditions (1.20), (1.21), we get the inequalities (3.13) and (3.14) with $s = t_{0k},\ t = t_{1k}$.

Now if we pass to the limit as $k \to +\infty$ in (3.17), according to (3.3), (3.12), (3.13), (3.14), (3.18), (3.20), and equalities $\rho_0(a) = \rho_1(a) = 0$ we get

$$\rho \le (1-\nu)\rho + \frac{\nu}{2}\rho + \frac{1}{2\nu}\Big(\frac{2^m}{(2m-1)!!}\Big)^2 ||q||^2_{\widetilde{L}^2_{2m-2,\,2m-2}}. \tag{3.21}$$

From (3.19) and (3.21) immediately follows that

$$||u^{(m)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}, \tag{3.22}$$

with

$$r = \frac{2^m}{(1 - 2\max\{B_0,\, B_1\})(2m-1)!!},$$

where it is clear from definition of the numbers B_0, B_1 that r depend only on the numbers l_{kj}, $\overline{l}_{k0}$, $\overline{l}_{kj}$, γ_{k0}, γ_{kj} $(k = 0, 1;\ j = 0, \ldots, m)$, and a, b, t^*. by By virtue of (3.16), the last inequality implies estimate (1.22). □

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