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MULTIPLE SOLUTIONS FOR A CLASS OF $p(x)$-LAPLACIAN PROBLEMS INVOLVING CONCAVE-CONVEX NONLINEARITIES

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Abstract. Using variational methods, we prove a multiplicity result for a class of $p(x)$-Laplacian problems of the form

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda|u|^{r(x)-2}u + f(x,u) & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $p, r \in C(\overline{\Omega})$, $1 < r^- \leq r^+ < p^- \leq p^+ < \min\left\{N, \frac{Np^-}{N-p^-}\right\}$, λ is a positive parameter, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and p^+-superlinear at infinity but does not satisfy the (A-R) type condition.

1. Introduction

In this paper, we are interested in the existence of solutions for a class of $p(x)$-Laplacian problems of the form

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = g(x,u) & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $p \in C(\overline{\Omega})$, $1 < p^- \leq p^+ < N$, and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying subcritical growth condition.

In the case when $p(x) = p$ is a constant, problem (1.1) becomes the p-Laplacian problem of the form

$$\begin{cases} -\Delta_p u = g(x,u) & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Since A. Ambrosetti and P.H. Rabinowitz proposed the mountain pass theorem in 1973 (see [1]), critical point theory has become one of the main tools for finding solutions to elliptic problems of variational type. Especially, elliptic problem (1.2) has been intensively studied for many years. One of the very important hypotheses usually imposed on the nonlinearities is the following Ambrosetti-Rabinowitz type condition ((A-R) type condition for short): There exists $\mu > p$ such that

$$0 < \mu G(x,t) := \mu \int_0^t g(x,s)ds \leq g(x,t)t \tag{1.3}$$

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for all $x \in \Omega$ and $t \in \mathbb{R}\backslash\{0\}$. This condition ensures that the energy functional associated to the problem satisfies the Palais-Smale condition ((PS) condition for short). Clearly, if the condition (A-R) is satisfied then there exist two positive constants d_1, d_2 such that

$$G(x,t) \geq d_1|t|^\mu - d_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

This means that g is p-superlinear at infinity in the sense that

$$\lim_{|t|\to+\infty} \frac{G(x,t)}{|t|^p} = +\infty.$$

In recent years, there have been many authors considering elliptic problem (1.2) without the (A-R) type condition, we refer to some interesting papers on this topic [11, 13, 18, 19, 20, 22, 23, 24, 27, 28, 30, 31, 32] and the references cited there. In [28], O.H. Miyagaki et al. studied problem (1.2) in the semilinear case $p = 2$ by proposing the following non-global condition on the superlinear term $g(x,t)$: There exists $t_0 > 0$ such that

$$\frac{g(x,t)}{t} \text{ is increasing in } t \geq t_0 \text{ and decreasing in } t \leq -t_0, \quad \forall x \in \Omega.$$

Using the mountain pass theorem with the (PS) condition in [1], the authors obtained the existence of a non-trivial weak solution. This result was extended to the p-Laplace operator $-\Delta_p u$ by G. Li et al [23] and to the $p(x)$-Laplace operator $\Delta_{p(x)} u = -\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u\big)$ by C. Ji [19]. Especially, in [23], the authors gave a simpler proof for the existence result by using the mountain pass theorem in [13] with the Cerami condition (see Definition 2.3).

In [2, 3, 4, 33, 34], the authors studied the existence and multiplicity of solutions for problem (1.2) involving concave-convex nonlinearities of the form $g(x,t) = \lambda|t|^{q-2}t + \mu|t|^{r-2}t$, where $1 < q < p < r < p^*$. We also refer the readers to some similar results for the $p(x)$-Laplace operator in recent papers by M. Mihăilescu [26] and R.A. Mashiyev et al. [25].

Motivated by the papers mentioned above, in this work, we will study the existence of multiple solutions for problem (1.1) in a more general case when $g(x,t)$ is defined by

$$g(x,t) = \lambda|t|^{r(x)-2}t + f(x,t), \quad (x,t) \in \overline{\Omega} \times \mathbb{R},$$

where

$$1 < r^- \leq r^+ < p^- \leq p^+ < \min\left\{N, \frac{Np^-}{N-p^-}\right\}, \tag{1.4}$$

and λ is a positive parameter, the function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and p^+-superlinear at infinity but does not satisfy the (A-R) condition (1.3). More precisely, we consider the following $p(x)$-Laplacian problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda |u|^{r(x)-2}u + f(x,u) \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Using the mountain pass theorem with the Cerami condition in [13] combined with the Ekeland variational principle in [15] we show the existence of at least two non-trivial weak solutions for (1.5) provided that $\lambda \in (0, \lambda^*)$, $\lambda^* > 0$ is small enough. In the case when $\lambda = 0$, our result is exactly the one introduced in [19] but our arguments in this present work are clearly different from those presented in [19]. Regarding some estimates of the constant λ^*, we refer the readers to some recent papers [5, 6, 7, 8, 9, 10, 12] in which the authors have studied the existence and multiplicity of weak solutions for elliptic problems involving the $p(x)$-Laplacian. We emphasize that the extension from the p-Laplace operator $\Delta_p u$ to the $p(x)$-Laplace operator involved in (1.5) is interesting and not trivial, since the new operators have a more complicated structure than the p-Laplace operator, for example they are non-homogeneous. Finally, it should be noticed that our result is new even in the case when $p(x) = p$ is a constant, see [2, 3, 4, 23, 28, 33, 34].

Our paper is organized as follows. In Section 2, we will recall some useful results on Sobolev spaces with variable exponents and the mountain pass theorem with the Cerami condition. In section 3, we will state and prove the main result of this paper.

2. Preliminaries

In this section, we recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the book of Musielak [29] and the papers of Kováčik and Rákosník [21], Fan et al. [16, 17] and the lecture notes by L. Diening et al. [14]. Set

$$C_+(\overline{\Omega}) := \{h : \quad h \in C(\overline{\Omega}), \quad h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \text{ and } h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < +\infty$ and continuous functions are dense if $p^+ < +\infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < +\infty$ and p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$ then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

Proposition 2.1 (see [17]). *If $u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$ then the following relations hold*

$$|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{2.1}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{2.2}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \;\Leftrightarrow\; \rho_{p(x)}(u_n - u) \to 0. \tag{2.3}$$

In this paper, we assume that $p \in C_+^{\log}(\overline{\Omega})$, where $C_+^{\log}(\overline{\Omega})$ is the space of all the functions of $C_+(\overline{\Omega})$ which are *logarithmic Hölder continuous*, that is, there exists $R > 0$ such that for all $x, y \in \Omega$ with $0 < |x - y| \le \frac{1}{2}$, $|p(x) - p(y)| \le -\frac{R}{\log|\mathrm{x}-\mathrm{y}|}$, see [14]. We define the space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

Proposition 2.2 (see [17]). *The space $\left(W_0^{1,p(x)}(\Omega), \|.\|\right)$ is a separable and Banach space when $1 < p^- \le p^+ < +\infty$. Moreover, if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then* *the embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is continuous and compact, where* $p^*(x) = \frac{Np(x)}{N-p(x)}$ *if* $p(x) < N$ *or* $p^*(x) = +\infty$ *if* $p(x) \geq N$.

In our proof of the main result, we will use the mountain pass theorem with the Cerami condition in [13]. For the reader's convenience, we recall it below.

Definition 2.3. Let $(X, \|.\|)$ be a real Banach space, $J \in C^1(X, \mathbb{R})$. We say that J satisfies the Cerami condition (write (C_c) condition for short) if any sequence $\{u_m\} \subset X$ such that $J(u_m) \to c$ and $\|J'(u_m)\|_*(1 + \|u_m\|) \to 0$ as $m \to \infty$ has a convergent subsequence.

Proposition 2.4 (see [13])**.** *Let* $(X, \|.\|)$ *be a real Banach space,* $J \in C^1(X, \mathbb{R})$ *satisfies the* (C_c) *condition for any* $c > 0$*,* $J(0) = 0$ *and the following conditions hold:*

(*i*) *There exists a function* $\phi \in X$ *such that* $\|\phi\| > \rho$ *and* $J(\phi) < 0$*;*

(*ii*) *There exist two positive constants* ρ *and* R *such that* $J(u) \geq R$ *for any* $u \in X$ *with* $\|u\| = \rho$.

Then the functional J *has a critical value* $c \geq R$*, i.e. there exists* $u \in X$ *such that* $J'(u) = 0$ *and* $J(u) = c$.

3. Multiple solutions

In this section, we state and prove the main result of this paper. We will use the letter C_i to denote a positive constant whose value may change from line to line. Let us introduce the following hypotheses:

(F_0) There exists $C > 0$ such that

$$|f(x,t)| \leq C(1 + |t|^{q(x)-1})$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, where $q \in C(\overline{\Omega})$ and $p(x) \leq p^+ < q^- \leq q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$;

(F_1) There exists a positive constant $\overline{t} > 0$ such that $F(x,t) \geq 0$ a.e. $x \in \Omega$ and all $t \in [0, \overline{t}]$, where $F(x,t) := \int_0^t f(x,s)\,ds$;

(F_2) $f(x,t) = o(|t|^{p^+-1})$, $t \to 0$, uniformly in $x \in \Omega$;

(F_3) $\lim_{|t|\to+\infty} \frac{F(x,t)}{t^{p^+}} = +\infty$ uniformly in $x \in \Omega$, i.e., f is p^+-superlinear at infinity;

(F_4) There exists a constant $C_* > 0$ such that

$$\mathcal{F}(x,t) \leq \mathcal{F}(x,s) + C_*$$

for any $x \in \Omega$ and $0 < t < s$ or $s < t < 0$, where $\mathcal{F}(x,t) := tf(x,t) - p^+F(x,t)$.

It should be noticed that the condition (F_4) is a consequence of the following condition, which was firstly introduced by O.H. Miyagaki et al. [28] for problem (1.2) in the case $p = 2$ and developed by G. Li et al. [23] and C. Ji [19]:

(F_4') There exists $t_0 > 0$ such that $\frac{f(x,t)}{|t|^{p^+-2}t}$ is increasing in $t \geq t_0$ and decreasing in $t \leq -t_0$ for any $x \in \Omega$.

The readers may consult the proof and comments on this assertion in the papers by G. Li et al. [23] or by O.H. Miyagaki et al. [28] and the references cited there.

Definition 3.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1.5) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx - \lambda \int_\Omega |u|^{r(x)-2}u\varphi\, dx - \int_\Omega f(x,u)\varphi\, dx = 0$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

Our main result of this paper is given by the following theorem.

Theorem 3.2. *Assume that the conditions (1.4), and* (F_0)-(F_4) *are satisfied. Then there exists a positive constant* λ^* *such that for all* $\lambda \in (0, \lambda^*)$, *problem (1.5) has at least two non-trivial weak solutions.*

In the rest of this paper we will use the letter X to denote the Sobolev space $W_0^{1,p(x)}(\Omega)$. Let us introduce the functional $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{r(x)}|u|^{r(x)}\, dx - \int_\Omega F(x,u)\, dx, \quad u \in W_0^{1,p(x)}(\Omega).$$

where $F(x,t) = \int_0^t f(x,s)\, ds$.

By the continuous embeddings obtained from the hypotheses (F_0) and (1.4), some standard arguments assure that the functional J is well defined on X and $J \in C^1(X)$ with the derivative given by

$$J'(u)(\varphi) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx - \lambda \int_\Omega |u|^{r(x)-2}u\varphi\, dx - \int_\Omega f(x,u)\varphi\, dx$$

for all $u, \varphi \in X$. Thus, non-trivial weak solutions of problem (1.5) are exactly the non-trivial critical points of the functional J.

Lemma 3.3. *The functional* J *satisfies the* (C_c) *condition for any* $c > 0$.

Proof. Let $\{u_m\} \subset X$ be a (C_c) sequence of the functional J, that is,

$$J(u_m) \to c, \quad \|J'(u_m)\|_*(1 + \|u_m\|) \to 0 \text{ as } m \to \infty,$$

which shows that

$$c = J(u_m) + o(1), \quad J'(u_m)(u_m) = o(1), \tag{3.1}$$

where $o(1) \to 0$ as $m \to \infty$.

We will prove that the sequence $\{u_m\}$ is bounded in X. Indeed, if $\{u_m\}$ is unbounded in X, we may assume that $\|u_m\| \to +\infty$ as $m \to \infty$. We define the sequence $\{w_m\}$ by $w_m = \frac{u_m}{\|u_m\|}$, $m = 1, 2, \ldots$ It is clear that $\{w_m\} \subset X$ and $\|w_m\| = 1$ for any m. Therefore, up to a subsequence, still denoted by $\{w_m\}$, we have that $\{w_m\}$ converges weakly to some function $w \in X$ and

$$w_m(x) \to w(x), \quad \text{a.e. in } \Omega, \quad m \to \infty, \tag{3.2}$$

$$w_m \to w \quad \text{strongly in } L^{q(x)}(\Omega), \quad m \to \infty, \tag{3.3}$$

$$w_m \to w \quad \text{strongly in } L^{r(x)}(\Omega), \quad m \to \infty, \tag{3.4}$$

$$w_m \to w \quad \text{strongly in } L^{p^+}(\Omega), \quad m \to \infty. \tag{3.5}$$

Let $\Omega_{\neq} := \{x \in \Omega : \ w(x) \neq 0\}$. If $x \in \Omega_{\neq}$ then it follows from (3.2) that $|u_m(x)| = |w_m(x)| \|u_m\| \to +\infty$ as $m \to \infty$. Moreover, from (F_3), we have

$$\lim_{m\to\infty} \frac{F(x, u_m(x))}{|u_m(x)|^{p^+}} |w_m(x)|^{p^+} = +\infty, \quad x \in \Omega_{\neq}. \tag{3.6}$$

Using the condition (F_3), there exists $t_0 > 0$ such that

$$\frac{F(x,t)}{|t|^{p^+}} > 1 \tag{3.7}$$

for all $x \in \Omega$ and $|t| > t_0 > 0$. Since $F(x,t)$ is continuous on $\overline{\Omega} \times [-t_0, t_0]$, there exists a positive constant C_1 such that

$$|F(x,t)| \leq C_1 \tag{3.8}$$

for all $(x,t) \in \overline{\Omega} \times [-t_0, t_0]$. From (3.7) and (3.8) there exists $C_2 \in \mathbb{R}$ such that

$$F(x,t) \geq C_2 \tag{3.9}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. From (3.9), for all $x \in \Omega$ and m, we have

$$\frac{F(x, u_m(x)) - C_2}{\|u_m\|^{p^+}} \geq 0$$

or

$$\frac{F(x, u_m(x))}{|u_m(x)|^{p^+}} |w_m(x)|^{p^+} - \frac{C_2}{\|u_m\|^{p^+}} \geq 0, \quad \forall x \in \Omega, \quad \forall m. \tag{3.10}$$

By (3.1) and the Sobolev embedding, there exists $C_3 > 0$ such that, for m large enough so that $\|u_m\| > 1$, we have

$$
\begin{aligned}
c &= J(u_m) + o(1) \\
&= \int_\Omega \frac{1}{p(x)} |\nabla u_m|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{r(x)} |u_m|^{r(x)}\, dx - \int_\Omega F(x, u_m)\, dx + o(1) \\
&\geq \frac{1}{p^+} \|u_m\|^{p^-} - \frac{\lambda C_3}{r^-} \|u_m\|^{r^+} - \int_\Omega F(x, u_m)\, dx + o(1),
\end{aligned}
$$

which implies since $1 < r^- \leq r^+ < p^-$ that

$$
\int_\Omega F(x, u_m)\, dx \geq \frac{1}{p^+} \|u_m\|^{p^-} - \frac{\lambda C_3}{r^-} \|u_m\|^{r^+} - c + o(1) \to +\infty \text{ as } m \to \infty. \tag{3.11}
$$

We also have

$$
\begin{aligned}
c &= J(u_m) + o(1) \\
&= \int_\Omega \frac{1}{p(x)} |\nabla u_m|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{r(x)} |u_m|^{r(x)}\, dx - \int_\Omega F(x, u_m)\, dx + o(1) \\
&\leq \frac{1}{p^-} \|u_m\|^{p^+} - \lambda \int_\Omega \frac{1}{r(x)} |u_m|^{r(x)}\, dx - \int_\Omega F(x, u_m)\, dx + o(1)
\end{aligned}
$$

and by (3.11),

$$
\begin{aligned}
\|u_m\|^{p^+} &\geq p^- \int_\Omega F(x, u_m)\, dx + \lambda p^- \int_\Omega \frac{1}{r(x)} |u_m|^{r(x)}\, dx + p^- c - o(1) \\
&\geq p^- \int_\Omega F(x, u_m)\, dx + p^- c - o(1) > 0 \text{ for } m \text{ large enough.}
\end{aligned} \tag{3.12}
$$

Next, we will claim that $|\Omega_{\neq}| = 0$. In fact, if $|\Omega_{\neq}| \neq 0$, then by relations (3.6), (3.10), (3.12) and the Fatou lemma, we have

$$
\begin{aligned}
+ &= (+\infty)|\Omega_{\neq}| \\
&= \int_{\Omega_{\neq}} \liminf_{m\to\infty} \frac{F(x,u_m(x))}{|u_m(x)|^{p^+}} |w_m(x)|^{p^+}\,dx - \int_{\Omega_{\neq}} \limsup_{m\to\infty} \frac{C_2}{\|u_m\|^{p^+}}\,dx \\
&= \int_{\Omega_{\neq}} \liminf_{m\to\infty} \left(\frac{F(x,u_m(x))}{|u_m(x)|^{p^+}} |w_m(x)|^{p^+} - \frac{C_2}{\|u_m\|^{p^+}} \right) dx \\
&\leq \liminf_{m\to\infty} \int_{\Omega_{\neq}} \left(\frac{F(x,u_m(x))}{|u_m(x)|^{p^+}} |w_m(x)|^{p^+} - \frac{C_2}{\|u_m\|^{p^+}} \right) dx \\
&\leq \liminf_{m\to\infty} \int_{\Omega} \left(\frac{F(x,u_m(x))}{|u_m(x)|^{p^+}} |w_m(x)|^{p^+} - \frac{C_2}{\|u_m\|^{p^+}} \right) dx \\
&= \liminf_{m\to\infty} \int_{\Omega} \frac{F(x,u_m(x))}{\|u_m\|^{p^+}}\,dx - \limsup_{m\to\infty} \int_{\Omega} \frac{C_2}{\|u_m\|^{p^+}}\,dx \\
&= \liminf_{m\to\infty} \int_{\Omega} \frac{F(x,u_m(x))}{\|u_m\|^{p^+}}\,dx \\
&\leq \liminf_{m\to\infty} \frac{\int_\Omega F(x,u_m(x))\,dx}{p^- \int_\Omega F(x,u_m)\,dx + p^- c - o(1)}.
\end{aligned}
\tag{3.13}
$$

From (3.11) and (3.13), we obtain

$$+\infty \leq \frac{1}{p^-},$$

which is a contradiction. This shows that $|\Omega_{\neq}| = 0$ and thus $w(x) = 0$ a.e. in Ω.

Since the function $t \mapsto J(tu_m)$ is continuous in $t \in [0,1]$, for each m there exists $t_m \in [0,1]$ such that

$$J(t_m u_m) := \max_{t\in[0,1]} J(tu_m), \quad m = 1, 2, ... \tag{3.14}$$

It is clear that $t_m > 0$ and $J(t_m u_m) \geq c > 0 = J(0) = J(0.u_m)$. If $t_m < 1$ then $\frac{d}{dt} J(tu_m)|_{t=t_m} = 0$ which gives $J'(t_m u_m)(t_m u_m) = 0$. If $t_m = 1$, then $J'(u_m)(u_m) = o(1)$. So we always have

$$J'(t_m u_m)(t_m u_m) = o(1). \tag{3.15}$$

Now, we fix a big integer $k \geq 1$ so that $\|u_k\| > 1$ and define the sequence $\{v_m\}$ by

$$v_m = \left(2p^+ \|u_k\|^{p^-}\right)^{\frac{1}{p^-}} w_m, \quad m = 1, 2, ... \tag{3.16}$$

From (F_0) and (F_2), for any $\epsilon > 0$, there exists a positive constant $C(\epsilon)$ such that

$$|F(x,t)| \leq \epsilon |t|^{p^+} + C(\epsilon)|t|^{q(x)}, \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.17}$$

Fix k, since $w_m \to 0$ strongly in the spaces $L^{q(x)}(\Omega)$, $L^{r(x)}(\Omega)$ and $L^{p^+}(\Omega)$ as $m \to \infty$, using (3.17), we deduce that there exists a constant $C_4 > 0$ such that

$$\left| \int_\Omega F(x, v_m)\, dx \right| \leq C_4 \int_\Omega |v_m|^{p^+}\, dx + C_4 \int_\Omega |v_m|^{q(x)}\, dx \to 0 \text{ as } m \to \infty. \tag{3.18}$$

We also have

$$\lim_{m \to \infty} \int_\Omega |v_m|^{r(x)}\, dx = 0. \tag{3.19}$$

Since $\|u_m\| \to +\infty$ as $m \to \infty$, we can find a constant $m_k > k$ depending on k such that

$$0 < \frac{\left(2p^+ \|u_k\|^{p^-}\right)^{\frac{1}{p^-}}}{\|u_m\|} < 1 \text{ for all } m > m_k. \tag{3.20}$$

Hence, using relations (3.14), (3.18)-(3.20), it follows that

$$\begin{aligned}
& J(t_m u_m) \\
& \geq J\left(\frac{\left(2p^+ \|u_k\|^{p^-}\right)^{\frac{1}{p^-}}}{\|u_m\|} u_m \right) \\
& = J(v_m) \\
& = \int_\Omega \frac{1}{p(x)} |\nabla v_m|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{r(x)} |v_m|^{r(x)}\, dx - \int_\Omega F(x, v_m)\, dx \\
& \geq \frac{1}{p^+} \int_\Omega \left(\|u_k\|^{p(x)}.(2p^+)^{\frac{p(x)}{p^-}}.|\nabla w_m|^{p(x)} \right) dx - \frac{\lambda}{r^-} \int_\Omega |v_m|^{r(x)}\, dx - \int_\Omega F(x, v_m)\, dx \\
& \geq 2\|u_k\|^{p^-} - \frac{\lambda}{r^-} \int_\Omega |v_m|^{r(x)}\, dx - \int_\Omega F(x, v_m)\, dx \\
& \geq \|u_k\|^{p^-}
\end{aligned} \tag{3.21}$$

for any $m > m_k > k$ large enough.

On the other hand, using the conditions (F_4) and relation (3.15), for all $m > m_k > k$ large enough, we have

$$
\begin{aligned}
&J(t_m u_m)\\
&= J(t_m u_m) - \frac{1}{p^+} J'(t_m u_m)(t_m u_m) + o(1)\\
&= \int_\Omega \frac{1}{p(x)} |\nabla t_m u_m|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{r(x)} |t_m u_m|^{r(x)}\, dx - \int_\Omega F(x, t_m u_m)\, dx\\
&\quad - \frac{1}{p^+} \int_\Omega |\nabla t_m u_m|^{p(x)}\, dx + \frac{\lambda}{p^+} \int_\Omega |t_m u_m|^{r(x)}\, dx + \frac{1}{p^+} \int_\Omega f(x, t_m u_m) t_m u_m\, dx + o(1)\\
&= \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) |\nabla t_m u_m|^{p(x)}\, dx - \lambda \int_\Omega \left(\frac{1}{r(x)} - \frac{1}{p^+} \right) |t_m u_m|^{r(x)}\, dx\\
&\quad + \frac{1}{p^+} \int_\Omega \mathcal{F}(x, t_m u_m)\, dx\\
&\le \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) |\nabla u_m|^{p(x)}\, dx + \frac{1}{p^+} \int_\Omega \big(\mathcal{F}(x, u_m) + C_* \big)\, dx + o(1)\\
&= \int_\Omega \frac{1}{p(x)} |\nabla u_m|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{r(x)} |u_m|^{r(x)}\, dx - \int_\Omega F(x, u_m)\, dx\\
&\quad - \frac{1}{p^+} \left(\int_\Omega |\nabla u_m|^{p(x)}\, dx - \lambda \int_\Omega |u_m|^{r(x)}\, dx - \int_\Omega f(x, u_m) u_m\, dx \right)\\
&\quad + \lambda \int_\Omega \left(\frac{1}{r(x)} - \frac{1}{p^+} \right) |u_m|^{r(x)}\, dx + \frac{C_* |\Omega|}{p^+} + o(1)\\
&= J(u_m) - \frac{1}{p^+} J'(u_m)(u_m) + \lambda \int_\Omega \left(\frac{1}{r(x)} - \frac{1}{p^+} \right) |u_m|^{r(x)}\, dx + \frac{C_* |\Omega|}{p^+} + o(1)\\
&\le J(u_m) - \frac{1}{p^+} J'(u_m)(u_m) + \lambda C_3 \left(\frac{1}{r^-} - \frac{1}{p^+} \right) \|u_m\|^{r^+} + \frac{C_* |\Omega|}{p^+} + o(1),
\end{aligned}
\tag{3.22}
$$

where C_3 is given by (3.11).

From (3.21) and (3.22), we deduce that for all $m > m_k > k$ large enough,

$$
\|u_k\|^{p^-} \le J(u_m) - \frac{1}{p^+} J'(u_m)(u_m) + \lambda C_3 \left(\frac{1}{r^-} - \frac{1}{p^+} \right) \|u_m\|^{r^+} + \frac{C_* |\Omega|}{p^+} + o(1)
$$

or

$$
\|u_k\|^{p^-} - \lambda C_3 \left(\frac{1}{r^-} - \frac{1}{p^+} \right) \|u_m\|^{r^+} \le J(u_m) - \frac{1}{p^+} J'(u_m)(u_m) + \frac{C_* |\Omega|}{p^+} + o(1) \tag{3.23}
$$

Recall that $k \ge 1$ is an arbitrarily big integer and $m > m_k > k$. In (3.23), let $k \to \infty$ we have $m \to \infty$ and the left hand side of (3.23) tends to $+\infty$ since $r^+ < p^-$. In the right hand side of (3.23), $J(u_m) \to c$ and $\frac{1}{p^+} J'(u_m)(u_m) \to 0$ as $m \to \infty$. Thus, we have a contradiction. This proves that the sequence $\{u_m\}$ is bounded in X.

Now, since the Banach space X is reflexive, there exists $u \in X$ such that passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u in X and converges strongly to u in the spaces $L^{q(x)}(\Omega)$ and $L^{r(x)}(\Omega)$. Using the condition (F_0) and the Hölder inequality, we deduce that

$$\begin{aligned}
\left|\int_\Omega f(x,u_m)(u_m-u)\,dx\right| &\le \int_\Omega |f(x,u_m)||u_m-u|\,dx \\
&\le C\int_\Omega (1+|u_m|^{q(x)-1})|u_m-u|\,dx \\
&\le C_5\left(|1|_{L^{\frac{q(x)}{q(x)-1}}} + \left||u_m|^{q(x)-1}\right|_{L^{\frac{q(x)}{q(x)-1}}(\Omega)}\right)\|u_m-u\|_{L^{q(x)}(\Omega)} \\
&\to 0 \text{ as } m\to\infty,
\end{aligned}$$

which yields

$$\lim_{m\to\infty}\int_\Omega f(x,u_m)(u_m-u)\,dx = 0. \tag{3.24}$$

We also have

$$\begin{aligned}
\left|\int_\Omega |u_m|^{r(x)-2}u_m(u_m-u)\,dx\right| &\le \int_\Omega |u_m|^{r(x)-1}|u_m-u|\,dx \\
&\le C_6\left||u_m|^{r(x)-1}\right|_{L^{\frac{r(x)}{r(x)-1}}(\Omega)}\|u_m-u\|_{L^{r(x)}(\Omega)} \\
&\to 0 \text{ as } m\to\infty.
\end{aligned} \tag{3.25}$$

From (3.24) and (3.25) and the fact that

$$\lim_{m\to\infty} J'(u_m)(u_m-u) = 0$$

we get

$$\lim_{m\to\infty}\int_\Omega |\nabla u_m|^{p(x)-2}\nabla u_m\cdot(\nabla u_m-\nabla u)\,dx = 0. \tag{3.26}$$

Now, using standard arguments we can show that the sequence $\{u_m\}$ converges strongly to u in X and the functional J satisfies the (C_c) condition for any $c>0$. The proof of Lemma 3.3 is complete. □

Lemma 3.4.

(*i*) *There exists $\lambda^*>0$ such that for any $\lambda\in(0,\lambda^*)$, we can choose $R>0$ and $\rho>0$ so that $J(u)\ge R>0$ for all $u\in X$ with $\|u\|=\rho$;*

(*ii*) *There exists $\phi\in X$, $\phi>0$ such that $J(t\phi)\to-\infty$ as $t\to+\infty$;*

(*iii*) *There exists $\psi\in X$, $\psi>0$ such that $J(t\psi)<0$ for all $t>0$ small enough.*

Proof. (i) Since the embeddings $X \hookrightarrow L^{p^+}(\Omega)$ and $X \hookrightarrow L^{q(x)}(\Omega)$ are continuous and compact, there exist constants $C_7, C_8 > 0$ such that

$$\|u\|_{L^{p^+}(\Omega)} \le C_7\|u\|, \quad \|u\|_{L^{q(x)}(\Omega)} \le C_8\|u\|. \tag{3.27}$$

Let $0 < \epsilon < \frac{1}{2p^+ C_7^{p^+}}$, where C_7 is given by (3.27). From (3.17) and (3.27), for all $u \in X$ with $\|u\| < 1$, we have

$$\begin{aligned}
J(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx - \lambda\int_\Omega \frac{1}{r(x)}|u|^{r(x)}\,dx - \int_\Omega F(x,u)\,dx \\
&\ge \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda C_3}{r^-}\|u\|^{r^-} - \epsilon\int_\Omega |u|^{p^+}\,dx - C(\epsilon)\int_\Omega |u|^{q(x)}\,dx \\
&\ge \frac{1}{p^+}\|u\|^{p^+} - \epsilon C_7^{p^+}\|u\|^{p^+} - \frac{\lambda C_3}{r^-}\|u\|^{r^-} - C(\epsilon)C_8^{q^-}\|u\|^{q^-} \\
&\ge \left(\frac{1}{2p^+} - \frac{\lambda C_3}{r^-}\|u\|^{r^- - p^+} - C(\epsilon)C_8^{q^-}\|u\|^{q^- - p^+}\right)\|u\|^{p^+},
\end{aligned} \tag{3.28}$$

where $C_3 > 0$ is given by (3.11).

For each $\lambda > 0$, we consider the function $\gamma_\lambda : (0, +\infty) \to \mathbb{R}$ defined by

$$\gamma_\lambda(t) = \frac{\lambda C_3}{r^-}t^{r^- - p^+} + C(\epsilon)C_8^{q^-}t^{q^- - p^+}. \tag{3.29}$$

It is clear that $\gamma_\lambda(t)$ is a continuous function on $(0, +\infty)$. Since $q^- > p^+ \ge p^- > r^+ \ge r^- > 1$, it follows that

$$\lim_{t\to 0^+}\gamma_\lambda(t) = \lim_{t\to+\infty}\gamma_\lambda(t) = +\infty. \tag{3.30}$$

Hence, we can find $t_* > 0$ such that $0 < \gamma_\lambda(t_*) = \min_{t\in(0,+\infty)}\gamma_\lambda(t)$, in which t_* is defined by the equation

$$0 = \gamma_\lambda'(t_*) = \frac{\lambda C_3}{r^-}(r^- - p^+)t_*^{r^- - p^+ - 1} + C(\epsilon)C_8^{q^-}(q^- - p^+)t_*^{q^- - p^+ - 1}$$

or

$$t_* = \left(\frac{\lambda C_3(p^+ - r^-)}{r^- C(\epsilon)C_8^{q^-}(q^- - p^+)}\right)^{\frac{1}{q^- - r^-}}.$$

Some simple computations imply that

$$\gamma_\lambda(t_*) = C_9.\lambda^{\frac{q^- - p^+}{q^- - r^-}} \to 0 \text{ as } \lambda \to 0^+. \tag{3.31}$$

From relations (3.28), (3.29) and (3.31), there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, we can choose $R > 0$ and $\rho > 0$ so that $J(u) \ge R > 0$ for all $u \in X$ with $\|u\| = \rho$.

(ii) From (F_3), it follows that for any $M > 0$ there exists a constant $C_M = C(M) > 0$ depending on M, such that

$$F(x,t) \ge M|t|^{p^+} - C_M, \quad \text{for a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}. \tag{3.32}$$

Take $\phi \in C_0^\infty(\Omega)$ with $\phi > 0$, from (3.32) and the definition of J, we get

$$\begin{aligned} J(t\phi) &= \int_\Omega \frac{1}{p(x)} |\nabla t\phi|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{r(x)} |t\phi|^{r(x)}\, dx - \int_\Omega F(x, t\phi)\, dx \\ &\le \frac{1}{p^-} \|t\phi\|^{p^+} - M \int_\Omega |t\phi|^{p^+}\, dx - \frac{\lambda}{r^+} \int_\Omega |t\phi|^{r(x)}\, dx + C_M |\Omega| \\ &\le t^{p^+} \left(\frac{1}{p^-} \|\phi\|^{p^+} - M \int_\Omega |\phi|^{p^+}\, dx \right) - \frac{\lambda t^{r^-}}{r^+} \int_\Omega |\phi|^{r(x)}\, dx + C_M |\Omega|, \end{aligned} \tag{3.33}$$

where $t > 1$ is large enough to ensure that $\|t\phi\| > 1$, and $|\Omega|$ denotes the Lebesgue measure of Ω. From (3.33) and the fact that $1 < r^- \le r^+ < p^- \le p^+$, if M is large enough such that

$$\frac{1}{p^-} \|\phi\|^{p^+} - M \int_\Omega |\phi|^{p^+}\, dx < 0,$$

then we have

$$\lim_{t \to +\infty} J(t\phi) = -\infty,$$

which ends the proof of (ii).

(iii) Take $\psi \in C_0^\infty(\Omega)$ with $\psi > 0$, from the definition of J and the condition (F_1) we get for all $t \in \left(0, \min\left\{\frac{1}{\|\psi\|}, \frac{\bar{t}}{\|\psi\|_{L^\infty(\Omega)}}\right\}\right)$ small enough,

$$\begin{aligned} J(t\psi) &= \int_\Omega \frac{1}{p(x)} |\nabla t\psi|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{r(x)} |t\psi|^{r(x)}\, dx - \int_\Omega F(x, t\psi)\, dx \\ &\le \frac{1}{p^-} \|t\psi\|^{p^-} - \frac{\lambda}{r^+} \int_\Omega |t\psi|^{r(x)}\, dx \\ &= \frac{t^{p^-}}{p^-} \|\psi\|^{p^-} - \frac{\lambda t^{r^+}}{r^+} \int_\Omega |\psi|^{r(x)}\, dx. \end{aligned} \tag{3.34}$$

From (3.34), taking

$$0 < \delta < \frac{\lambda p^- \int_\Omega |\psi|^{r(x)}\, dx}{r^+ \|\psi\|^{p^-}}$$

we conclude that $J(t\psi) < 0$ for all $0 < t < \min\left\{\frac{1}{\|\psi\|}, \delta^{\frac{1}{p^- - r^+}}, \frac{\bar{t}}{\|\psi\|_{L^\infty(\Omega)}}\right\}$. The proof of Lemma 3.4 is complete. □

Proof Theorem 3.2. By Lemmas 3.3 and 3.4, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the functional J satisfies all the assumptions of the mountain pass theorem, see Proposition 2.4. Then we deduce u_1 as a non-trivial critical point of the functional J with $J(u_1) = \bar{c} > 0$ and thus a non-trivial weak solution of problem (1.5).

We now prove that there exists a second weak solution $u_2 \in X$ such that $u_2 \ne u_1$. Indeed, by (3.28), the functional J is bounded from below on the ball $\overline{B}_\rho(0)$.

Applying the Ekeland variational principle in [15] to the functional $J : \overline{B}_\rho(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B}_\rho(0)$ such that

$$J(u_\epsilon) < \inf_{u \in \overline{B}_\rho(0)} J(u) + \epsilon,$$

$$J(u_\epsilon) < J(u) + \epsilon \|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

By Lemma 3.4, we have

$$\inf_{u \in \partial B_\rho(0)} J(u) \geq R > 0 \text{ and } \inf_{u \in \overline{B}_\rho(0)} J(u) < 0.$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{u \in \partial B_\rho(0)} J(u) - \inf_{u \in \overline{B}_\rho(0)} J(u).$$

Then, $J(u_\epsilon) < \inf_{u \in \partial B_\rho(0)} J(u)$ and thus, $u_\epsilon \in B_\rho(0)$.

Now, we define the functional $I : \overline{B}_\rho(0) \to \mathbb{R}$ by $I(u) = J(u) + \epsilon\|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I and thus

$$\frac{I(u_\epsilon + tv) - I(u_\epsilon)}{t} \geq 0$$

for all $t > 0$ small enough and all $v \in B_\rho(0)$. The above information shows that

$$\frac{J(u_\epsilon + tv) - J(u_\epsilon)}{t} + \epsilon\|v\| \geq 0.$$

Letting $t \to 0^+$, we deduce that

$$\left\langle J'(u_\epsilon), v \right\rangle \geq -\epsilon\|v\|.$$

It should be noticed that $-v$ also belongs to $B_\rho(0)$, so replacing v by $-v$, we get

$$\left\langle J'(u_\epsilon), -v \right\rangle \geq -\epsilon\| - v\|$$

or

$$\left\langle J'(u_\epsilon), v \right\rangle \leq \epsilon\|v\|,$$

which helps us to deduce that $\|J'(u_\epsilon)\|_{X^*} \leq \epsilon$.

Therefore, there exists a sequence $\{u_m\} \subset B_\rho(0)$ such that

$$J(u_m) \to \underline{c} = \inf_{u \in \overline{B}_\rho(0)} J(u) < 0 \text{ and } J'(u_m) \to 0 \text{ in } X^* \text{ as } m \to \infty. \tag{3.35}$$

From Lemma 3.3, the sequence $\{u_m\}$ converges strongly to some $u_2 \in X$ as $m \to \infty$. Moreover, since $J \in C^1(X, \mathbb{R})$, by (3.35) it follows that $J(u_2) = \underline{c}$ and $J'(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.5).

Finally, we point out the fact that $u_1 \neq u_2$ since $J(u_1) = \overline{c} > 0 > \underline{c} = J(u_2)$. The proof of Theorem 3.2 is complete. $\square$

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References

[1] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical points theory and applications, *J. Funct. Anal.*, **04** (1973), 349-381.

[2] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.*, **122** (1994), 519-543.

[3] A. Ambrosetti, J.G. Azorero and I. Peral, Multiplicity results for some nonlinear elliptic equations, *J. Funct. Anal.*, **137** (1996), 219-242.

[4] T. Bartsch and M. Willem, On an elliptic equation with concave and convex nonlinearities, *Proc. Amer. Math. Soc.*, **123** (1995), 3555-3561.

[5] G. Bonanno and A. Chinnì, Discontinuous elliptic problems involving the $p(x)$-Laplacian, *Math. Nachr.*, **284**(5-6) (2011), 639-652.

[6] G. Bonanno and A. Chinnì, Multiple solutions for elliptic problems involving the $p(x)$-Laplacian, *Le Matematiche*, **66** (2011), 105-113.

[7] G. Bonanno and A. Chinnì, Existence results of infinitely many solutions for $p(x)$-Laplacian elliptic Dirichlet problems, *Complex Variables and Elliptic Equations*, **57**(11) (2012), 1233-1246.

[8] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Analysis*, **75** (2012), 2992-3007.

[9] F. Cammaroto, A. Chinnì and B. Di Bella, Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian, *Nonlinear Analysis*, **71** (2009), 4486-4492.

[10] F. Cammaroto and L. Vilasi, Multiplicity results for a Neumann boundary value problem involving the $p(x)$-Laplacian, *Taiwanese J. Math.*, **16**(2) (2012), 621-634.

[11] Z.H. Chen, Y.T. Shen and Y.X. Yao, Some existence results of solutions for p-Laplacian, *Acta Math. Sci.*, **23B**(4) (2003), 487-496.

[12] A. Chinnì and R. Livrea, Multiple solutions for a Neumann-type differential inclusion problem involving the $p(.)$-Laplacian, *Discrete and Continuous Dynamical Systems (Series S)*, **5**(4) (2012), 753-764.

[13] D.G. Costa and C.A. Magalhaes, Existence results for perturbations of the p-Laplacian, *Nonlinear Analysis (TMA)*, **24** (1995), 409-418.

[14] L. Diening, P. Harjulehto, P. Hasto and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes, vol. 2017, Springer-Verlag, Berlin, 2011.

[15] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324-353.

[16] X.L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.

[17] X.L. Fan, J. Shen and D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **262** (2001), 749-760.

[18] L. Iturriaga, S. Lorca and P. Ubilla, A quasilinear problem without the Ambrosetti-Rabinowitz-type condition, *Proc. Roy. Soc. Edinburgh Sect. A*, **140** (2010), 391-398.

[19] C. Ji, On the superlinear problem involving the $p(x)$-Lapacian, *E. J. Qualitative Theory of Diff. Equ.*, Vol. 2011, No. 40 (2011), 1-9.

[20] A. Kristály, H. Lisei and C. Varga, Multiple solutions for p-Laplacian type equations, *Nonlinear Analysis (TMA)*, **68** (2008), 1375-1381.

[21] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.*, **41** (1991), 592-618.

[22] G.B. Li and H.S. Zhou, Asymptotically "linear" Dirichlet problem for p-Laplacian, *Nonlinear Analysis*, **43** (2001), 1043-1055.

[23] G. Li and C. Yang, The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition, *Nonlinear Analysis (TMA)*, **72** (2010), 4602-4613.

[24] S. Liu, On superlinear problems without the Ambrosetti and Rabinowitz condition, *Nonlinear Analysis (TMA)*, **73** (2010), 788-795.

[25] R.A. Mashiyev, S. Ogras, Z. Yucedag, and M. Avci, The Nehari manifold approach for Dirichlet problem involving the $p(x)$-Laplacian equation, *J. Korean Math. Soc.*, **47**(4) (2010), 845-860.

[26] M. Mihăilescu, Existence and multiplicity of of solutions for an elliptic equation with $p(x)$-growth conditions, *Glasgow Math. J.*, **48** (2006), 411-418.

[27] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A*, **462** (2006), 2625-2641.

[28] O.H. Miyagaki and M.A.S. Souto, Super-linear problems without Ambrosetti and Rabinowitz growth condition, *J. Differential Equations*, **245** (2008), 3628-3638.

[29] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983.

[30] M. Schechter and W. Zou, Superlinear problems, *Pacific J. Math.*, **214** (2004), 145-160.

[31] M.Z. Sun, Multiple solutions of a superlinear p-Laplacian equation without AR-condition, *Applicable Analysis*, **89** (2010), 325-336.

[32] Z. Tan and F. Fang, On superlinear $p(x)$-Laplacian problems without Ambrosetti and Rabinowitz condition, *Nonlinear Analysis (TMA)*, **75** (2012), 3902-3915.

[33] H.Q. Toan and Q.A. Ngo, Multiplicity of weak solutions for a class of non-uniformly elliptic equations of p-Laplacian type, *Nonlinear Analysis (TMA)*, **70** (2009), 1536-1546.

[34] T.F. Wu, On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function, *J. Math. Anal. Appl.*, **318** (2006), 253-270.

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