Electronic Journal of Qualitative Theory of Differential Equations
2005, No. **19**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

REMARKS ON INHOMOGENEOUS ELLIPTIC PROBLEMS ARISING IN ASTROPHYSICS

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Abstract. We deal with the variational study of some type of nonlinear inhomogeneous elliptic problems arising in models of solar flares on the halfplane $\mathbb{R}^n_+$.

1. Introduction

In this paper we study a boundary value problem of type

$$\left\{\begin{array}{l} -\Delta u + c(x)u = \lambda m(y) f(u) \qquad \mathbb{R}^n_+ \\ \quad u(z,0) = h(z) \qquad \forall z \in \mathbb{R}^{n-1} \end{array}\right. \tag{1.1}$$

where $x = (z, y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+ \equiv \mathbb{R}^n_+$ with $\mathbb{R}_+ = \{y \in \mathbb{R} : y > 0\}$ and $n \geq 2$, $f :]-\infty, +\infty[\to \mathbb{R}$ is a function satisfying:

(f-1) There exists $s_0 > 0$ such that $f(s) > 0$ for all $s \in]0, s_0[$.
(f-2) $f(s) = 0$ for $s \leq 0$ o $s \geq s_0$.
(f-3) $f(s) \leq as^\sigma$, a is a positive constant and $1 < \sigma < \frac{n+2}{n-2}$ if $n > 2$ or $\sigma > 1$ if $n = 2$.
(f-4) There exists $l > 0$ such that $|f(s_1) - f(s_2)| \leq l|s_1 - s_2|$, for all s_1, $s_2 \in \mathbb{R}$.

h is a non-negative bounded smooth function, $h \neq 0$, $\min h < s_0$, $c \geq 0$, $c \in L^\infty(\Omega) \bigcap C(\overline{\Omega})$ and $mes\{x \in \Omega : c(x) = 0\} = 0$.
The problem (1.1) is a generalization of an astrophysical gravity model of solar flares in the half plane $\mathbb{R}^2_+$, given in [1], namely:

$$\left\{\begin{array}{rcl} -\Delta u &=& \lambda e^{-\beta y} f(u) \qquad \mathbb{R}^2_+ \\ u(x,0) &=& h(x) \qquad \forall x \in \mathbb{R} \end{array}\right. \tag{1.2}$$

besides the above mentioned conditions for f, h and $\beta > 0$. See [1], [8] and [6] for a detailed description and related problems.
By this, we study the problem (1.1) with $m : \mathbb{R}_+ \to \mathbb{R}_+$ a C^1 function such that

$$\int_0^{+\infty} ym(y)dy < +\infty$$

more general than $e^{-\beta y}$.

1991 *Mathematics Subject Classification.* 35J65, 85A30.
Key words and phrases. Solar flares, variational methods, inhomogeneous semilinear elliptic problems.

We shall follow the ideas of F. Dobarro and E. Lami Dozo in [8]. The authors prove the existence of solutions of (1.1) in the special case $c(x) = 0$. In fact, the result presented here follows from the one obtained by the authors.

First of all we note that problem (1.1) is equivalent to

$$\text{(1.3)} \qquad \begin{cases} -\Delta\omega + c(x)\omega = \lambda m(y) f(\omega + \tau) & \mathbb{R}^n_+ \\ \omega(z, 0) = 0 & \forall z \in \mathbb{R}^{n-1} \end{cases}$$

where $\omega = u - \tau$ and τ is solution of the problem

$$\text{(1.4)} \qquad \begin{cases} -\Delta\tau + c(x)\tau = 0 & \mathbb{R}^n_+ \\ \tau(z, 0) = h(z) & \forall z \in \mathbb{R}^{n-1} \end{cases}$$

We will study (1.3) instead of (1.1).

The problem (1.1), or equivalently (1.3), is interesting not only on whole $\mathbb{R}^n_+$, but also in an arbitrary big but finite domain in $\mathbb{R}^n_+$, for example for semidisks $D_R = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+ : |x|^2 + y^2 < R^2, y > 0\}$, with R big enough.

Motivated by this observation in section 2, we will study the following approximate problem

$$\text{(1.5)} \qquad \begin{cases} -\Delta\omega + c(x)\omega = \lambda m(y) f(\omega + \tau) & D_R \\ \omega = 0 & \partial D_R \end{cases}$$

whose solutions are related to those of (1.3).

Using variational techniques we will prove the existence of an interval $\Lambda \subset R_+$ such that for all $\lambda \in \Lambda$ there exists at least three positive solutions of (1.5), with R large enough.

Finally in section 3 we prove the existence of solutions of (1.3) as limit of a special family of solutions of (1.5) obtained in theorem 5 and its uniqueness to λ small enough.

2. Problem in D_R

Letting Ω be either D_R or $\mathbb{R}^n_+$, we denote by $L^p_m(\Omega)$ the usual weighted L^p space on Ω for a suitable weight m and $1 \le p < \infty$, and by $V^{1,2}_m(\Omega)$, $V^{1,2}_c(\Omega)$ the completion of $C^\infty_0(\Omega)$ in the norm

$$\|u\|^2_{V^{1,2}_m(\Omega)} = \int_\Omega u^2(z, y) m(y) dz dy + \int_\Omega |\nabla u|^2 dz dy$$

and

$$\|u\|^2_{V_c^{1,2}(\Omega)} = \int_\Omega u^2(x)c(x)dx + \int_\Omega |\nabla u|^2 dx$$

Let $m : R_+ \to R_+$ be such that

$$0 < M \equiv \int_0^{+\infty} ym(y)dy < +\infty \tag{2.1}$$

it is easy to prove for all functions $u \in C_0^\infty(\Omega)$ the following inequality holds, see [8].

$$\int_\Omega u^2(x,y)m(y)dxdy \le M \int_\Omega |\nabla u|^2 dxdy \tag{2.2}$$

then $V_m^{1,2}(D_R) \sim H_0^1(D_R) \sim V_c^{1,2}(D_R)$ and $V_m^{1,2}(\mathbb{R}^n_+) \sim D^{1,2}(\mathbb{R}^n_+)$ where $H_0^1(D_R)$ is the usual Sobolev space with the norm $\|\nabla(.)\|_{L^2(D_R)}$ and $D^{1,2}(\mathbb{R}^n_+)$ is the completion of $C_0^\infty(\mathbb{R}^n_+)$ for the norm $\|\nabla(.)\|_{L^2(\mathbb{R}^n_+)}$.
On the other hand if $R' \ge R$, then

$$V_c^{1,2}(D_R) \subset V_c^{1,2}(D_{R'}) \subset V_c^{1,2}(\mathbb{R}^n_+) \subset V_m^{1,2}(\mathbb{R}^n_+) \tag{2.3}$$

There exists many results about immersion of weighted Sobolev spaces into weighted Lebesgue spaces. Here we will take into account one suitable result for our problem.

Let $m : R_+ \to R_+$ be a bounded C^1 function such that there exists $k > 0$ such that

$$|(\log m)'| \le k \tag{2.4}$$

then the identity map is an immersion from $V_m^{1,2}(\Omega)$ into $L^p_{m^{\frac{p}{2}}}(\Omega)$ for

$$\begin{array}{ll} 1 < p < \frac{2n}{n-2} & \text{if } n \ge 3 \\ 1 < p & \text{if n=2} \end{array}$$

More precisely, there exists a constant $K = K(k, \sup m)$ such that

$$\|u\|_{L^p_{m^{\frac{p}{2}}}(\Omega)} \le C_s K \|u\|_{V_m^{1,2}(\Omega)} \tag{2.5}$$

where C_s is the usual Sobolev immersion constant. The immersion is compact if $\Omega = D_R$.

Now we will begin to study (1.3) by variational methods. For this purpose, for all $\lambda \ge 0$ and for all non negative function τ such that $\|\tau\|_{L_m^{\sigma+1}} < +\infty$ we associate the functional $\Psi_{\lambda,\tau} : V_c^{1,2}(\mathbb{R}^n_+) \to R$

$$\Psi_{\lambda,\tau}(u) = \frac{1}{2}\int_{\mathbb{R}^n_+} \{|\nabla u|^2 + c(x)u^2\} - \lambda \int_{\mathbb{R}^n_+} mF(u+\tau) \tag{2.6}$$

where $F(t)=\int_0^t f(s)ds$, $m\in C^1(R_+)$ and $\widehat{m}\equiv m^{\frac{2}{\sigma+1}}$ satisfying (2.1) and (2.4).

$\Psi_{\lambda,\tau}$ is a C^1 functional, so if $u\in V_c^{1,2}(\mathbb{R}^n_+)$ is a critical point of $\Psi_{\lambda,\tau}$ then u is a weak, and by regularity a classical solution of (1.3).

Remark 1. i. If we consider $\Psi_{\lambda,\tau,R}:V_c^{1,2}(D_R)\to R$,

$$\Psi_{\lambda,\tau,R}(u)=\frac{1}{2}\int_{D_R}\{|\nabla u|^2+c(x)u^2\}-\lambda\int_{D_R} mF(u+\tau)$$

its critical points are weak, and by regularity, strong solutions of (1.5). Furthermore if $R\le R'\le+\infty$, then for all $u\in V_c^{1,2}(D_R)$

$$\Psi_{\lambda,\tau,R'}(u)\le\Psi_{\lambda,\tau,R}(u)\le\Psi_{\lambda,0,R}(u)$$

more precisely

$$\Psi_{\lambda,\tau,R'}(u)=\Psi_{\lambda,\tau,R}(u)-\lambda\int_{D_{R'}-D_R} mF(\tau)\le\Psi_{\lambda,\tau,R}(u)$$

Here $D_{R'}$ with $R'=+\infty$ means $\mathbb{R}^n_+$.
ii. Since f is bounded, $\Psi_{\lambda,\tau,R}$ is coercive, bounded from below and verifies Palais-Smale condition for all λ non negative.

Lemma 2. *For each $R>0$ denote $\theta_R:R^n\to R^n$ the map*

$$\theta_R(z,y)\equiv\left(\frac{z}{R},y\right)$$

and Θ_R the scaling $\eta\to\eta_R\equiv\eta o\theta_R$. Then
i. $\forall r>0$, $\Theta_R(V_c^{1,2}(D_r))\subset V_c^{1,2}(\theta_R^{-1}D_r)$ and if $R\ge1$, $V_c^{1,2}(\theta_R^{-1}D_r)\subset V_c^{1,2}(D_{Rr})$.
ii. If $\eta\in C_0^\infty(\mathbb{R}^n_+)$, is non identically 0, then

$$\|\nabla\eta_R\|_{L^2(\mathbb{R}^n_+)}\to+\infty \qquad as \qquad R\to+\infty \tag{2.7}$$

iii. Let f be defined before and m such that verifies (2.1). Then there exists $0<\underline{\lambda}<\infty$ such that if $\lambda>\underline{\lambda}$, $\eta\in C_0^\infty(\mathbb{R}^n_+)$, $\eta\ge0$, non identically 0 and

$$\underline{\lambda}\le Q(\eta)\equiv\frac{\frac{1}{2}\int_{\mathbb{R}^n_+}\{|\nabla\eta|^2+\|c\|_{L^\infty}\eta^2\}}{\int_{\mathbb{R}^n_+} m(y)F(\eta)}<\lambda \tag{2.8}$$

then there exists $r_n>0$: $\eta_R\in V_c^{1,2}(D_{R'})$, $\forall R'$, R: $R'\ge Rr_n\ge r_n$ and for all non negative function τ.
a. $\Psi_{\lambda,\tau,R'}(\eta_R)<0$, $\forall R'$, R: $R'\ge Rr_n\ge r_n$.
b. $\Psi_{\lambda,\tau,Rr_n}(\eta_R)\to-\infty$ as $R\to+\infty$

Proof.- This proof follows almost directly from lemma 6 in [8]. However, by completeness we present all the proof.
i. It is immediate from the definition of Θ_R.
ii. We observe

$$|\nabla \eta_R|^2(z,y) = \frac{1}{R^2}|\nabla \eta|^2_{\theta_R} + \left(1 - \frac{1}{R^2}\right)|\partial_y \eta|^2_{\theta_R(z,y)}$$

thus, changing variables

$$\|\nabla \eta_R\|^2_{L^2(\mathbb{R}^n_+)} = R^{n-1}\left[\frac{1}{R^2}\int_{\mathbb{R}^n_+}|\nabla \eta|^2 + \left(1 - \frac{1}{R^2}\right)\int_{\mathbb{R}^n_+}|\partial_y \eta|^2\right] \tag{2.9}$$

so, since $\int_{\mathbb{R}^n_+}|\partial_y \eta|^2 > 0$, (2.9) implies (2.7).
iii. Set

$$\underline{\lambda} \equiv inf\{Q(\eta) : \eta \in C_0^\infty(\mathbb{R}^n_+), \eta \geq 0, \eta \neq 0\} \tag{2.10}$$

by (f-3) and since F is bounded

$$\frac{b}{2} \equiv \sup_{s>0}\frac{F(s)}{s^2} < +\infty \tag{2.11}$$

so, by (2.2) and since $c(x) \geq 0$

$$\int_{\mathbb{R}^n_+} m(y)F(\eta) \leq \frac{bM}{2}\int_{\mathbb{R}^n_+}|\nabla \eta|^2 + \|c\|_{L^\infty}\eta^2$$

hence

$$0 < \frac{1}{bM} \leq \underline{\lambda} < \infty$$

Let $\lambda > Q(\eta)$ be, since $\eta \in C_0^\infty(\mathbb{R}^n_+)$, there exists $r_n > 0$ such that *supp* $\eta \subset D_{Rr_n}$, for all $R \geq 1$. Then by i. and (2.3) $\eta_R \in V_c^{1,2}(\theta_R^{-1}D_{r_n}) \subset V_c^{1,2}(D_{Rr_n}) \subset V_c^{1,2}(D_{R'})$ for all $R' \geq Rr_n \geq r_n$.
For simplicity from now on we call $Rr_n \equiv R_n$, where $R \geq 1$.
Then, by remark 1

$$\Psi_{\lambda,\tau,R'}(\eta_R) \leq \Psi_{\lambda,\tau,R_n}(\eta_R) \leq \Psi_{\lambda,0,R_n}(\eta_R) \tag{2.12}$$

On the other hand, if we define the function $\xi : R_+ \to R$

$$\begin{aligned}\xi(R) \equiv \frac{1}{R^{n-1}}\|\nabla \eta_R\|^2_{L^2(\mathbb{R}^n_+)} &= \frac{1}{R^2}\int_{\mathbb{R}^n_+}|\nabla \eta|^2 + \left(1 - \frac{1}{R^2}\right)\int_{\mathbb{R}^n_+}|\partial_y \eta|^2 \\ &= \left[\frac{\int_{\mathbb{R}^n_+}|\nabla_z \eta|^2}{R^2\int_{\mathbb{R}^n_+}|\partial_y \eta|^2} + 1\right]\int_{\mathbb{R}^n_+}|\partial_y \eta|^2\end{aligned}$$

is non increasing. So applying $\xi(R) \leq \xi(1)$ to (2.9)

$$\int_{D_{R_n}}|\nabla \eta_R|^2 = \int_{\mathbb{R}^n_+}|\nabla \eta_R|^2 \leq R^{n-1}\int_{\mathbb{R}^n_+}|\nabla \eta|^2$$

furthermore

$$\int_{D_{R_n}} c(x)\eta_R^2 = \int_{\mathbb{R}^n_+} c(x)\eta_R^2 \leq R^{n-1}\|c\|_{L^\infty}\int_{\mathbb{R}^n_+}\eta^2$$

and

$$\int_{D_{R_n}} m(y)F(\eta_R) = \int_{\mathbb{R}^n_+} m(y)F(\eta_R) = R^{n-1}\int_{\mathbb{R}^n_+} m(y)F(\eta)$$

so

$$\Psi_{\lambda,0,R_n} \leq R^{n-1}\left[\frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla\eta|^2 + \|c\|_{L^\infty}\eta^2 - \lambda\int_{\mathbb{R}^n_+} m(y)F(\eta)\right]$$

then

$$\Psi_{\lambda,0,R_n} \leq \frac{R^{n-1}}{2}\int_{\mathbb{R}^n_+}|\nabla\eta|^2 + \|c\|_{L^\infty}\eta^2\left(1 - \frac{\lambda}{Q(\eta)}\right) \tag{2.13}$$

thus, from (2.12) and (2.13) we obtain immediately a and b.

□

Remark 3. i. Let $m : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded C^1 function and let $\widehat{m} \equiv m^{\frac{2}{\sigma+1}}$. It is easy to prove that m verifies (2.4) if and only if $\widehat{m}$ does it. Furthermore, given a positive constant k, $|(\log m)'| \leq k$ if and only if $|(\log \widehat{m})'| \leq \frac{2}{\sigma+1}k$.
ii. If there exists a non negative value $m_1 \geq 0$ such that $\{m > 1\} \subset [0, m_1]$ and

$$0 < \widehat{M} \equiv \int_0^{+\infty} y\widehat{m}(y)dy < +\infty$$

then

$$0 < M \equiv \int_0^{+\infty} ym(y)dy < +\infty$$

Indeed, since $\widehat{m} > 1$ if and only if $m > 1$ and $0 < \frac{2}{\sigma+1} < 1$

$$M = \int_{\widehat{m}>1} ym(y)dy + \int_{\widehat{m}\leq 1} ym(y)dy \leq \left(\sup m\frac{m_1}{2}\right) + \widehat{M} < +\infty$$

Lemma 4. *There exists a positive constant* $C = C(a, \sigma, k, \sup m, \widehat{M})$ *such that for all* $\lambda < \overline{\lambda}(\|\tau\|_{L_m^{\sigma+1}(\mathbb{R}^n_+)})$ *and for all* u : $\|u\|_{V_c^{1,2}(\mathbb{R}^n_+)} = \|\tau\|_{L_m^{\sigma+1}(\mathbb{R}^n_+)}$, $\Psi_{\lambda,\tau}(u) > 0$ *where* $\overline{\lambda}(\|\tau\|_{L_m^{\sigma+1}(\mathbb{R}^n_+)}) \equiv C\|\tau\|^{1-\sigma}_{L_m^{\sigma+1}(\mathbb{R}^n_+)}$.
Moreover $\overline{\lambda}(\|\tau\|_{L_m^{\sigma+1}(\mathbb{R}^n_+)}) \to +\infty$ *as* $\|\tau\|_{L_m^{\sigma+1}(\mathbb{R}^n_+)} \to 0$

Proof.- Let $u \in V_c^{1,2}(\mathbb{R}^n_+)$ be, using (f-3) and Minkowsky inequality with respect to measure $m(y)dxdy$ and (2.2), (2.5) we obtain

$$\begin{aligned}0 \le \int_{\mathbb{R}^n_+} mF(u+\tau) &= \int_{\mathbb{R}^n_+} m\int_0^{u+\tau} f(t)dt \le \frac{a}{\sigma+1}\int_{\mathbb{R}^n_+} m(u+\tau)^{\sigma+1}\\ &\le \frac{a}{\sigma+1}(\|u\|_{L^{\sigma+1}_{\widehat{m}^{\frac{\sigma+1}{2}}}} + \|\tau\|_{L^{\sigma+1}_m})^{\sigma+1}\\ &\le \frac{a}{\sigma+1}(C_sK(1+\widehat{M})^{\frac{1}{2}}\|\nabla u\|_{L^2(\mathbb{R}^n_+)} + \|\tau\|_{L^{\sigma+1}_m})^{\sigma+1}\\ &\le \frac{a}{\sigma+1}(C_sK(1+\widehat{M})^{\frac{1}{2}}\|u\|_{V_c^{1,2}(\mathbb{R}^n_+)} + \|\tau\|_{L^{\sigma+1}_m})^{\sigma+1}\end{aligned}$$

then

(2.14)

$$\Psi_{\lambda,\tau}(u) \ge \frac{1}{2}\|u\|^2_{V_c^{1,2}(\mathbb{R}^n_+)} - \lambda\frac{a}{\sigma+1}(C_sK(1+\widehat{M})^{\frac{1}{2}}\|u\|_{V_c^{1,2}(\mathbb{R}^n_+)} + \|\tau\|_{L^{\sigma+1}_m})^{\sigma+1}$$

then, if we define

$$C \equiv \frac{\sigma+1}{2a}(C_sK(k,\sup m)(1+\widehat{M})^{\frac{1}{2}}+1)^{-\sigma-1}$$

then $\Psi_{\lambda,\tau}(u) > 0$ for all $\lambda < \overline{\lambda} \equiv C\|\tau\|^{1-\sigma}_{L^{\sigma+1}_m(\mathbb{R}^n_+)}$, and since $\sigma > 1$. The lemma is proved.

□

Theorem 5. *Let us assume (f-1-2-3-4), let m:$\mathbb{R}_+ \to \mathbb{R}_+$ be a C^1 function such that m and $\widehat{m} \equiv m^{\frac{2}{\sigma+1}}$ verify (2.1) and (2.4), and let $\tau : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a C^1 function, non identically* 0. *So there exists positive constants $C = C(a,\sigma,k,\sup m,\widehat{M})$ and $\underline{\lambda}$ such that if*

$$\|\tau\|_{L^{\sigma+1}_m(\mathbb{R}^n_+)} < \left(\frac{c}{\underline{\lambda}}\right)^{\frac{1}{\sigma-1}} \tag{2.15}$$

then

$$\forall\lambda : \underline{\lambda} < \lambda < \overline{\lambda} \equiv C\|\tau\|^{1-\sigma}_{L^{\sigma+1}_m(\mathbb{R}^n_+)}$$

there exists a positive $R_0 = R_0(\lambda)$ such that for all $R \ge R_0$, (1.5) has at least three strictly positive solutions.

Proof.- Let $C = C(a,\sigma,k,\sup m,\widehat{M})$ and $\underline{\lambda}$ be the positive constant defined in lemmas 4 and 2 respectively . Since τ verifies (2.15), by lemma 4 and remark 1, for all $\lambda \in]\underline{\lambda},\overline{\lambda}[$ and for all $R \ge 1$

$$\Psi_{\lambda,\tau,R}(u) > 0 \qquad \forall u \in V_c^{1,2}(D_R) : \|u\|_{V_c^{1,2}(D_R)} = \|\tau\|_{L^{\sigma+1}_m(R^n_+)} \tag{2.16}$$

On the other hand, fixed $\lambda \in]\underline{\lambda},\overline{\lambda}[$, $\eta \in C_0^\infty(\mathbb{R}^n_+)$, and letting $r_n > 0$, the radius of any semidisk D_{r_n} such that *supp* $\eta \subset D_{r_n}$, by lemma 2 there exists $R_1 \geq 1$ such that for all $R \geq R_1 r_n$, we have $\eta_{R_1} \in V_c^{1,2}(D_R)$, furthermore

$$(2.17) \quad \|\tau\|_{L_m^{\sigma+1}(\mathbb{R}_+^n)} < \|\nabla \eta_{R_1}\|_{L^2(D_R)} = \|\nabla \eta_{R_1}\|_{L^2(\mathbb{R}_+^n)} < \|\eta_{R_1}\|_{V_c^{1,2}(\mathbb{R}_+^n)}$$

and

$$(2.18) \qquad \Psi_{\lambda,\tau,R}(\eta_{R_1}) < \mu < 0$$

where $\mu \in R$ defined as

$$\mu \equiv \min_{0 \leq t \leq \|\tau\|_{L_m^{\sigma+1}(\mathbb{R}_+^n)}} \frac{1}{2}t^2 - \lambda \frac{a}{\sigma+1}(C_s K(1+\widehat{M})^{\frac{1}{2}} t + \|\tau\|_{L_m^{\sigma+1}})^{\sigma+1}$$

Let $R \geq R_1$, we divide the proof in three steps.
1. *Local minimum.*- Let

$$\nu_R \equiv \inf_{B_\Gamma} \Psi_{\lambda,\tau,R}(u)$$

where $B_\Gamma = \{u \in V_c^{1,2}(D_R) : \|u\|_{V_c^{1,2}(D_R)} < \Gamma \equiv \|\tau\|_{L_m^{\sigma+1}(\mathbb{R}_+^n)}\}$.
Since $\Psi_{\lambda,\tau,R}(0) < 0$, $\nu_R < 0$. Furthermore $\mu \leq \nu_R < 0$, by (2.14) and remark 1 . Therefore $\inf_{\partial B_\Gamma} \Psi_{\lambda,\tau,R} > \nu_R$.
Now we will prove that ν_R is achieved in B_Γ. Using a modification in the proof of proposition 5 and corollaries 6 and 7 in [3], we can obtain a sequence $(u_n)_n$ in B_Γ such that

$$\Psi_{\lambda,\tau,R}(u_n) \to \nu_R$$
$$\Psi'_{\lambda,\tau,R}(u_n) \to 0$$

since $\Psi_{\lambda,\tau,R}$ verifies Palais-Smale condition, there exists a subsequence $(u_{n_k})_k$ such that $u_{n_k} \to u_{1,R}$ in $V_c^{1,2}(D_R)$ and $u_{1,R} \neq 0$ because 0 it is not a critical point of $\Psi_{\lambda,\tau,R}$.
2. *Absolute minimum.*- Let

$$u_R \equiv \inf_{V_c^{1,2}(D_R)} \Psi_{\lambda,\tau,R}$$

Then $u_R < \mu$, by (2.17). Now using similar arguments to *local minimum*, but without any modification, we have that u_R is achieved in $V_c^{1,2}(D_R)$ at a function $u_{2,R}$.
3.*Mountain pass.*- Let

$$c_R \equiv \inf_{\delta \in \Lambda_R} \sup_{u \in \delta} \Psi_{\lambda,\tau,R}(u)$$

where Λ_R is the set of paths

$$\Lambda_R = \{\gamma : \gamma \in C([0,1], V_c^{1,2}(D_R)), \gamma(0) = 0, \gamma(1) = \eta_{R_1}\}$$

Since $\Psi_{\lambda,\tau,R}(0) < 0$, by (2.15), (2.16) and (2.17), $c_R > 0$.
Then by the mountain pass theorem, see [4], c_R is achieved in $V_c^{1,2}(D_R)$ at a function $u_{3,R}$.

On the other hand it is clear that $u_{1,R}$, $u_{2,R}$ and $u_{3,R}$ are different, indeed

$$\Psi_{\lambda,\tau,R}(u_{2,R}) = u_R < \mu \leq \nu_R = \Psi_{\lambda,\tau,R}(u_{1,R}) < 0 < c_R = \Psi_{\lambda,\tau,R}(u_{3,R})$$

$\square$

Remark 6. When λ is small enough it is easy to prove uniqueness for (1.5), so $u_{1,R} = u_{2,R}$, and the local minimum in B_Γ od $\Psi_{\lambda,\tau,R}$ is the absolute in $V_c^{1,2}(D_R)$.

3. Problem in R^n_+

$\Psi_{\lambda,\tau}$ does not verifies Palais-Samale condition, furthermore by lemma 2 and remark 1 $\Psi_{\lambda,\tau}$ is not coercive and not bounded from below. However for λ small enough:

Proposition 7. *Let f be as above, let b be given by (2.11) and suppose m verifies (2.1). Then*
i. For all $\lambda < \frac{1}{bM}$, $\Psi_{\lambda,\tau}$ is coercive and bounded from below.
ii. For all $\lambda < \frac{1}{lM}$, (1.3) has at most one solution in $V_c^{1,2}(\mathbb{R}^n_+)$.
$\lambda < \underline{\lambda}$ holds in both cases.

Proof.- i. By (2.11), (2.2) and Cauchy-Schwartz for the measure $mdxdy$

$$\begin{aligned}
\Psi_{\lambda,\tau}(u) &\geq \frac{1}{2}\|u\|^2_{V_c^{1,2}(\mathbb{R}^n_+)} - \frac{\lambda b}{2}\int_{\mathbb{R}^n_+} m(u+\tau)^2 \\
&\geq \frac{1}{2}\|u\|^2_{V_c^{1,2}(\mathbb{R}^n_+)} - \frac{\lambda b}{2}(M^{\frac{1}{2}}\|\nabla u\|_{L^2(\mathbb{R}^n_+)} + \|\tau\|_{L^2_m(\mathbb{R}^n_+)})^2 \\
&\geq \frac{1}{2}(1-\lambda bM)\|u\|^2_{V_c^{1,2}(\mathbb{R}^n_+)} - (\lambda b M^{\frac{1}{2}}\|\tau\|_{L^2_m(\mathbb{R}^n_+)})\|u\|_{V_c^{1,2}(\mathbb{R}^n_+)} - \\
&\quad - \left(\frac{\lambda b}{2}\|\tau\|^2_{L^2_m(\mathbb{R}^n_+)}\right)
\end{aligned}$$

so, i. is proved.

$\square$

ii. Uniqueness is proved as in [1] using the inequality (2.2) and (f-4). Indeed: if u_1 and u_2 are two solutions of (1.3) then

$$\int_{\mathbb{R}^n_+}(u_1-u_2)^2 m \leq M\int_{\mathbb{R}^n_+}|\nabla(u_1-u_2)|^2 + c(x)(u_1-u_2)^2 \leq Ml\lambda\int_{\mathbb{R}^n_+}(u_1-u_2)^2 m$$

$\square$

Now we will prove a sufficient condition to approximate solutions of (1.3) with solutions of (1.5) with R large enough.

Lemma 8. *Let f and τ be as above and $\lambda \in \mathbb{R}_+$. Suppose $(R_n)_n$ is a sequence $\mathbb{R}_+$ such that $R_n \to +\infty$ and $(u_n)_n$ is a sequence of positive solutions of (1.5) with R_n instead of R, such that for all n, $u_n \in V_c^{1,2}(D_{Rn})$ and $(u_n)_n$ is bounded in $V_c^{1,2}(\mathbb{R}_+^n)$, i.e. there exists $\Gamma' > 0$ such that for all n, $\|u_n\|_{V_c^{1,2}(D_{Rn})} < \Gamma'$. Then, there exists a subsequence (called again $(u_n)_n$)) and a function $u \in V_c^{1,2}(\mathbb{R}_+^n)$ such that $u_n \to u$ weakly in $V_c^{1,2}(\mathbb{R}_+^n)$ and u is a classical solution (1.3).*

Proof.- By the Calderón-Zygmund[1] inequality for all n, $u_n \in H_0^1(D_{R_n}) \bigcap H^{2,p}(D_{R_n})$ and fixed $R' > 0$, for any $\Omega' \subset\subset D_{R'}$

$$\|u_n\|_{H^{2,p}(\Omega')} \leq C(\|u_n\|_{L^p(D_{R'})} + \|\lambda m(y) f(u_n+\tau)\|_{L^p(D_{R'})}) \tag{3.1}$$

for all n such that $R_n > R'$. The constant C depends on $D_{R'}$, n, p and Ω'. Since m is decreasing and strictly positive, and $(u_n)_n$ is bounded in $V_c^{1,2}(\mathbb{R}_+^n)$, by (2.2), (2.5), (3.1) and the hypothesis of f and m, we obtain

$$\|u_n\|_{H^{2,p}(\Omega')} \leq C(m(R')^{-\frac{1}{2}} C_s K (1+M)^{\frac{1}{2}} \Gamma' + \lambda \sup m \sup f |D_{R'}|^{\frac{1}{p}})$$

for p such that

$$\begin{array}{ll} 1 < p < \frac{2n}{n-2} & \text{if } n \geq 3 \\ 1 < p & \text{if n=2} \end{array}$$

and for all n such that $R_n > R'$.
For this and the Sobolev embedding theorem for Ω', there exists a subsequence $(u_n)_n$ such that if n=2,3 $u_n \to u$ in $C^{1,\alpha}(\overline{\Omega'})$ and if $n \geq 4$ and $1 < p < min\left(\frac{n}{2}, \frac{2n}{n-2}\right)$ is fixed, $u_n \to u$ in $L^q(\Omega')$, $1 \leq q < \frac{np}{n-2p}$.
Since Ω' is an arbitrary and relatively compact such that $\Omega' \subset\subset D_{R_n}$ and $R_n \to +\infty$, we obtain that the above convergences are in $C_{loc}^{1,\alpha}(\mathbb{R}_+^n)$ and $L_{loc}^q(\mathbb{R}_+^n)$ respectively. In particular

$$u_n \to u \qquad en \qquad L_{loc}^1(\mathbb{R}_+^n) \tag{3.2}$$

On the other hand, since $(u_n)_n$ is bounded in $V_c^{1,2}(\mathbb{R}_+^n)$, by (2.3), (2.5) and reflexivity

$$u_n \to u \qquad weakly \qquad in \qquad V_c^{1,2}(\mathbb{R}_+^n) \tag{3.3}$$

$$u_n \to u \qquad weakly \qquad in \qquad L^p_{m^{\frac{p}{2}}}(\mathbb{R}_+^n) \tag{3.4}$$

where

$$\begin{array}{ll} 1 < p < \frac{2n}{n-2} & \text{if } n \geq 3 \\ 1 < p & \text{if n=2} \end{array}$$

[1]see theorems 9.9 y 9.11 in [9]

So, if we prove that for all $v \in C_0^\infty(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^n_+} mf(u_n+\tau)v \to \int_{\mathbb{R}^n_+} mf(u+\tau)v$$

our lemma will follow. Based on this and for a fixed $v \in C_0^\infty(R^n_+)$ we consider the function

$$w = v\frac{f(u+\tau)}{u+\tau}m^{\frac{2-p}{2}}$$

It is easy to prove that $w \in L^{p'}_{m^{\frac{p}{2}}}(\mathbb{R}^n_+)$, where $\frac{1}{p}+\frac{1}{p'}=1$. Now

$$\begin{aligned}\int_{\mathbb{R}^n_+} mf(u_n+\tau)v &= \int_{\mathbb{R}^n_+} m\left[f(u_n+\tau)-(u_n+\tau)\frac{f(u+\tau)}{u+\tau}\right]v+ \\ &+\int_{\mathbb{R}^n_+} m^{\frac{p}{2}}(u_n+\tau)w\end{aligned} \tag{3.5}$$

by (3.4), the last term of right hand side of (3.5) tends to $\int_{\mathbb{R}^n_+} mf(u+\tau)v$. On the other hand, by (f-4)

$$\left|\int_{\mathbb{R}^n_+} m\left[f(u_n+\tau)-(u_n+\tau)\frac{f(u+\tau)}{u+\tau}\right]v\right| \le 2l\int_{supp(v)} m|u-u_n||v| \tag{3.6}$$

So, by (3.2) the last term of the right hand side of (3.5) tends to 0.

□

Theorem 9. *Let f, m, and τ as in lemma 8 and let $\Gamma \equiv \|\tau\|_{L^{\sigma+1}_m(\mathbb{R}^n_+)}$. Then for all λ, $0<\lambda<\overline{\lambda}$ the local minima $u_{1,R}$ of $\Psi_{\lambda,\tau,R}$, approximate the local minima of $\Psi_{\lambda,\tau}$ on the ball B_Γ of center 0 and radius Γ in $V^{1,2}_c(\mathbb{R}^n_+)$.*
As a consequence $\nu_\infty \equiv \inf_{B_\Gamma}\Psi_{\lambda,\tau}$, is a minimum and by proposition 7 it is the unique critical point of $\Psi_{\lambda,\tau}$, if λ small enough(i.e. $0<\lambda<\frac{1}{lM}$).

Proof.- We only need to prove that $\nu_R \to \nu_\infty$ as $R\to\infty$. With this aim we consider $(u_R)_R$ in $C_0^\infty(\mathbb{R}^n_+)$ such that $u_R \in V^{1,2}_c(D_R)$ and $\Psi_{\lambda,\tau,R}(u_R) \to \nu_\infty$ as $R\to\infty$. By remark 1 $\lambda\int_{\mathbb{R}^n_+-D_R} mF(\tau)dx \to 0$ as $R\to\infty$, because $\Gamma<+\infty$.
Since

$$\nu_\infty \le \nu_R = \Psi_{\lambda,\tau,R}(u_{1,R}) \le \Psi_{\lambda,\tau,R}(u_R) = \Psi_{\lambda,\tau}(u_R) - \lambda\int_{\mathbb{R}^n_+-D_R} mF(\tau)$$

then $\nu_R \to \nu_\infty$ as $R\to\infty$.

□

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(Received October 23, 2003)

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