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GLOBAL EXISTENCE OF $\varepsilon-$REGULAR SOLUTIONS FOR THE STRONGLY DAMPED WAVE EQUATION

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Abstract. In this paper, we deal with the semilinear wave equation with strong damping. By choosing a suitable state space, we characterize the interpolation and extrapolation spaces of the operator matrix $\mathbf{A}_\theta$, analysis the criticality of the ε-regular nonlinearity with critical growth. Finally, we investigate the global existence of the ε-regular solutions which have bounded $X^{1/2}\times X$ norms on their existence intervals.

1. Introduction and preliminaries

We consider the strongly damped wave equation

$$\begin{cases} u_{tt}+\eta(-\Delta)^\theta u_t+(-\Delta)u=f(u),\ t>0,\ x\in\Omega, \\ u(0,x)=u_0(x), \quad u_t(0,x)=v_0(x),\ x\in\Omega, \\ u(t,x)=0,\ t\geq 0,\ x\in\partial\Omega. \end{cases} \tag{1.1}$$

Here, $\Omega\subseteq\mathbb{R}^n$ $(n\geq 3)$ is a bounded domain with C^2 boundary, $\eta>0$ is the damping coefficient, and $(-\Delta)^\theta$ denotes the fractional power of the negative Laplacian, $\theta\in[1/2,1]$. If we set $X=L^2(\Omega)$ and $A=-\Delta$, then under the homogeneous Dirichlet boundary condition, A is a sectorial operator defined in X for which $\mathcal{D}(A)=H^2(\Omega)\cap H_0^1(\Omega)$, $\mathrm{Re}(\sigma(A))\geq 0$ and A^{-1} is compact. For each $\alpha\in\mathbb{R}$, denote by $X^\alpha=\mathcal{D}(A^\alpha)$ the fractional power space of A endowed with the graph norm. Recall that in this setting, $X^0=X$, $X^{1/2}=H_0^1(\Omega)$, $X^{-1/2}=H^{-1}(\Omega)$ and $X^{-\alpha}=(X^\alpha)^*$ for all $\alpha>0$.

Recently, Pr. (1.1) has been received much more attention by many authors, many works appeared with different themes and methods involved. Using the theory of operator semigroups and interpolation, extrapolation spaces, Carvalho–Cholewa in [1, 2], and lately Carvalho–Cholewa–Dlotko in [3] studied the local existence and regularity of the ε-regular solutions. By employing Galerkin's approximation and the

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theory of potential well, Gazzola–Squassina in [4] focused on the global solutions and finite time blow up. Furthermore, in [5, 6, 7, 8, 9, 10], the authors paid attention to the asymptotic behaviors, including existence and regularity of the global solutions and universal attractors, using the theory of dynamical systems.

In this paper, we restrict ourselves in global existence of the ε-regular solutions, and the tools we used here are sectorial operators and analytic semigroups. Recall that (cf. [1, 3]), if we select $Y = X^{1/2} \times X$ as the work space, and set

$$\mathcal{D}(\mathbf{A}_\theta) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in Y : X^{3/2-\theta} \times X^{1/2}, A^{3/2-\theta}u + \eta v \in X^\theta \right\},$$

$$\mathbf{A}_\theta \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -v \\ A^\theta(A^{1-\theta}u + \eta v) \end{bmatrix}, \quad \forall \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{D}(\mathbf{A}_\theta),$$

then Pr.(1.1) can be abstracted as an abstract parabolic problem, i.e.

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \mathbf{A}_\theta \begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right), \qquad t > 0, \tag{1.2}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \tag{1.3}$$

Here

$$\mathcal{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 0 \\ f(u) \end{bmatrix}.$$

In this setting, $\mathbf{A}_\theta$ is a sectorial operator whose interpolation and extrapolation spaces derived from the space pair (Y, Y_θ^1) ($Y_\theta^1 = \mathcal{D}(\mathbf{A}_\theta)$ endowed with the graph norm) can be characterized as follows :

$$Y_\theta^\alpha = X^{1/2+\alpha(1-\theta)} \times X^{\alpha\theta},$$

and

$$(Y_\theta^{-1})^\alpha \hookleftarrow X^{-1/2-(1-\alpha)(1-\theta)} \times X^{-1/2+\alpha(1-\theta)}$$

for all $\alpha \in [0, 1/2]$. Based on these characterizations, if the nonlinearity f is locally Lipschitz and satisfies a nonlinear growth:

$$|f(u) - f(u')| \le C|u - u'|(1 + |u|^{\rho-1} + |u'|^{\rho-1}), \quad \forall\, u, u' \in \mathbb{R} \qquad (1 \le \rho \le (n+2)/(n-2)), \tag{1.4}$$

then for each initial value $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y$, there is a unique mild solution $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} \in C([0, T], Y) \cap C((0, T], Y_\theta^\varepsilon)$ for some $T > 0$. This solution, called the ε-regular solution, carries a higher regularity than ε. Furthermore, if the growth exponent ρ is subcritical, i.e. $\rho < (n+2)/(n-2)$, then boundedness of $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ in the work space Y on its existence interval infers the global existence. All the results mentioned above can be found in [1, 3, 7], with the references therein.

Here, we focus on the critical case: $\rho = (n+2)/(n-2)$. We will show that, if we select another state space weaker than Y, then the ε-regular operator $\mathcal{F}$ induced by f is not critical anymore. Firstly, we give briefly some notions and results related to Pr. (1.1), and provide some references from the literature.

Let X be a Banach space, on which there is a closed linear operator A with a dense domain. We say that A is a sectorial operator of type $\mathscr{U}(\theta, M, X)$, if $\rho(A) \supseteq \Sigma_\vartheta$, and

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{1+|\lambda|}, \quad \forall\, \lambda \in \Sigma_\vartheta,$$

where, $M > 0$, $\Sigma_\vartheta = \{\lambda \in \mathbb{C} : |\arg\lambda| < \vartheta\} \cup \{0\}$ and $0 < \vartheta < \pi/2$ (see §3.3, [5]). In this case, $-A$ generates an analytic semigroup denoted by $\exp\{-tA\}$. Let $X^1 = \mathcal{D}(A)$ endowed with the graph norm, A^γ ($\gamma \in \mathbb{R}$) be the fractional power of A (§2.6, [11]), and X^α ($0 < \alpha < 1$) be the domain of A^α with $\|\cdot\|_{X^\alpha} = \|A^\alpha \cdot\|_X$. Recall that (refer to §1.3, [12]), if the pure imaginary powers A^{it} are uniformly bounded on $[-\epsilon, \epsilon]$ for any $\epsilon > 0$, then $X^\alpha = [X, X^1]_\alpha$ and $[X^\alpha, X^\beta]_s = X^{(1-s)\alpha + s\beta}$ for $\alpha, \beta > 0$, and $0 < s < 1$ (here $[X, X^1]_\alpha$ denotes the complex interpolation of the space couple (X, X^1)). If X is a Hilbert space, then the above properties are automatically satisfied. For the negative indicator $-\alpha$ ($0 < \alpha < 1$), we can also define the extrapolation space $X^{-\alpha}$ as the completion of $(X, \|A^{-\alpha}\cdot\|_X)$. There are some properties about the fractional powers, interpolation and extrapolation spaces (cf. [13, 14] and Ch. V, [15]):

Proposition 1.1.

(1) For each $\alpha \in (0, \pi/\vartheta)$, $A^\alpha \in \mathscr{U}(\alpha\vartheta, M_\alpha, X^\alpha)$; and
(2) $\widetilde{A^\alpha}$, the extension of A^α from X to $X^{-\alpha}$, also lies in $\mathscr{U}(\alpha\theta, M_{-\alpha}, X^{-\alpha})$, with $\mathcal{D}(\widetilde{A^\alpha}) = X$, and the restriction $\widetilde{A^\alpha}|_X : X \to X^{-\alpha}$ being an isomorphism;
(3) $(X^{-\alpha})^\gamma = X^{-\alpha+\gamma}$ for all $\gamma \in [0, 1]$.

Consider the abstract parabolic problem

$$\begin{cases} x' + Ax = F(x),\, t > 0, \\ x(0) = x_0 \in X, \end{cases} \tag{1.5}$$

where $F \in C((X^{-1})^{1+\varepsilon}, X^{-1})$ for some $0 < \varepsilon < 1$. A mild solution $x \in C([0,T], X)$ is called an ε-regular one of (1.5) on the interval $[0,T]$, if $x \in C((0,T], (X^{-1})^{1+\varepsilon}) \cap C^1((0,T], X^{-1})$ and satisfies the equation in (1.5) in the space X^{-1}. Related to a triple $(\rho, \varepsilon, \gamma)$ with $\rho \geq 1$, $\varepsilon \in [0, 1/\rho)$ and $\rho\varepsilon \leq \gamma < 1$, the nonlinear map F is called

ε-regular, if it takes values in $(X^{-1})^\gamma$ satisfying:

$$\|F(x)-F(y)\|_{(X^{-1})^\gamma} \leq C\|x-y\|_{(X^{-1})^{1+\varepsilon}}(1+\|x\|^{\rho-1}_{(X^{-1})^{1+\varepsilon}}+\|y\|^{\rho-1}_{(X^{-1})^{1+\varepsilon}}),$$
$$\forall\, x,y\in (X^{-1})^{1+\varepsilon}. \tag{1.6}$$

Proposition 1.2. *For any ε-regular map F and $\tilde{x}_0 \in X$, there is a ball $\mathcal{B}_X(\tilde{x}_0, r)$ and a constant $T > 0$ depending on A, $\tilde{x}_0$, ρ, ε and γ, s.t. corresponding to each initial value $x_0 \in \mathcal{B}_X(\tilde{x}_0, r)$, there is a unique ε-regular solution x of (1.5) defined on the interval $[0,T]$ satisfying $x \in C((0,T],(X^{-1})^{1+\gamma}) \cap C^1((0,T],(X^{-1})^{1+\gamma^-})$. Moreover, if $\gamma > \rho\varepsilon$ (correspondingly, F is said to be subcritical), then the radius r can be taken arbitrary. Under this situation, an ε-regular solution x exists globally, whenever the norm $\|x(t)\|_X$ is bounded on its existence interval.*

For other properties and detailed discussions about the ε-regular solutions of (1.5), please refer to [16, 17].

Remark 1.3. There is an open problem about the global existence of x in Prop. 1.2 in case that $\gamma = \rho\varepsilon$ (i.e. F is critical) and the boundedness of $x(t)$ could not be obtained in X^{1+s} for any $s > 0$ but $s = 0$. This problem seems to be much sophisticated, since the domain of F is not X but its subspace $X^{1+\varepsilon}$. In [8], the authors gave a sufficient and necessary condition for this problem. As to the strongly damped wave equations (1.1), the authors in [1, 3, 7] showed that, if the growth indicator ρ of the nonlinearity f is critical, i.e. $\rho = (N+2)/(N-2)$, then the corresponding map $\mathcal{F}$ is also critical associated with the pair (Y, Y_θ^{-1}) provided $1/2 \leq \theta < 1$. We will say that this is not absolutely. By choosing another state space W_θ (see the next section) little weaker than the energy space Y in this paper, we find that despite that the index ρ is critical, the nonlinear perturbation $\mathcal{F}$ is not critical any more, consequently, boundedness of $\begin{bmatrix} u(t) \\ v(t)\end{bmatrix}$ in Y also leads to its global existence.

2. Main results and proofs

Before our discussion, let us introduce a lemma for later use (refer to §5.2, [11] and [3]).

Lemma 2.1. *Suppose X and Y are two Banach spaces on which there are respectively two densely defined linear operators, saying A and B. If $A \in \mathscr{U}(\vartheta, M, X)$ and there exists an isomorphism Q between X and Y s.t. $QA = BQ$, then B also lies in the family $\mathscr{U}(\vartheta, \|Q\|\|Q^{-1}\|M, Y)$ satisfying*

(1) $\sigma(A)=\sigma(B)$,
(2) $e^{-tB}=Qe^{-tA}Q^{-1}$, *and*
(3) $QA^{\alpha}=B^{\alpha}Q$ *for all* $\alpha\in[0,1]$.

Let $Z_\theta=X^\theta\times X^0$, $\mathcal{D}(\mathbf{C}_\theta)=X^1\times X^\theta$, and

$$\mathbf{C}_\theta=\begin{bmatrix}0 & -I\\ A & \eta A^\theta\end{bmatrix}.$$

Remark 2.2. The sectorial property of $\mathbf{C}_\theta$ was treated in [1, 3] for $\theta=1/2$, and in [5] for $\theta=1$. It is worth to point out that, in [3], the basic space X is only an abstract Banach space and $A\in\mathscr{U}(\vartheta,M,X)$ with the extra restriction $\pi/2>\vartheta/2+\arg(\eta/2+\sqrt{\eta^2/4-1})$. For other cases (including $\theta=1/2,1$) in Hilbert spaces, please refer to [18], where A^θ is replaced by another operator B satisfying $A^{1/2}\le B\le A$. Motivated by [3], here we give a proof for the sectorial property of $\mathbf{C}_\theta$ in general Banach spaces for $1/2<\theta\le 1$.

Let X be a Banach space, and $A\in\mathscr{U}(\beta,M,X)$ for some $M>0$ and $\beta\in(0,\pi/2)$. Obviously, the operator matrix $\mathbf{C}_\theta$ defined above is closed in Z_θ. Fixing $\theta\in(1/2,1]$, and introduce two other closed operators associated with $\mathbf{C}_\theta$:

$$\mathbf{B}_\theta=\begin{bmatrix}0 & -I\\ A & \eta A^\theta+\eta^{-1}A^{1-\theta}\end{bmatrix},\quad \mathcal{D}(\mathbf{B}_\theta)=X^1\times X^\theta,$$

$$\mathbf{D}_\theta=\begin{bmatrix}\eta^{-1}A^{1-\theta} & -I\\ 0 & \eta A^\theta\end{bmatrix},\quad \mathcal{D}(\mathbf{D}_\theta)=X^1\times X^\theta.$$

It is easy to check that $\mathbf{P}_\theta\mathbf{B}_\theta=\mathbf{D}_\theta\mathbf{P}_\theta$, where

$$\mathbf{P}_\theta=\begin{bmatrix}I & 0\\ \eta^{-1}A^{1-\theta} & I\end{bmatrix}\in\mathscr{L}(Z_\theta)$$

is an isomorphism with $\mathbf{P}_\theta(X^1\times X^\theta)=X^1\times X^\theta$.

Lemma 2.3. *The operator* $\mathbf{D}_\theta$ *is sectorial in* Z_θ *with* $Re(\sigma(\mathbf{D}_\theta))>0$.

Proof. For each $\lambda\in\mathbb{C}$ with $\mathrm{Re}\lambda\le 0$, consider $\lambda\mathbf{I}-\mathbf{D}_\theta$. For each $\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\in Z_\theta$, by the sectorial properties of A^θ and $A^{1-\theta}$, we can find a pair $\begin{bmatrix}u\\ v\end{bmatrix}\in X^1\times X^\theta$ satisfying $(\lambda I-\eta A^\theta)v=\psi$ in X and $(\lambda I-\eta^{-1}A^{1-\theta})u=\varphi-v$ in X^θ. Thus $\begin{bmatrix}u\\ v\end{bmatrix}\in\mathcal{D}(\mathbf{D}_\theta)$ and $(\lambda\mathbf{I}-\mathbf{D}_\theta)\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}\varphi\\ \psi\end{bmatrix}$. Furthermore,

$$\|v\|_X=\|(\lambda I-\eta A^\theta)^{-1}\psi\|_X\le\frac{M}{\eta+|\lambda|}\|\psi\|_X,$$

and

$$\|u\|_{X^\theta} = \|(\lambda I - \eta^{-1}A^{1-\theta})^{-1}(\varphi - v)\|_{X^\theta} \leq \frac{M}{\eta^{-1} + |\lambda|}(\|\varphi\|_{X^\theta} + \|v\|_{X^\theta}).$$

Since $\|A^\theta(\lambda I - A^\theta)^{-1}\|_{\mathscr{L}(X,X)} \leq 1 + M$, we have $\|v\|_{X^\theta} \leq \eta^{-1}(1+M)\|\psi\|_X$. Consequently

$$\|u\|_{X^\theta} \leq \frac{M_\eta}{1+|\lambda|}(\|\varphi\|_{X^\theta} + \|\psi\|_X)$$

with the constant

$$M_\eta = M \max\left\{1, \frac{1+M}{\eta}\right\} \cdot \sup\left\{\frac{1+|\lambda|}{\eta^{-1}+|\lambda|} : \mathrm{Re}\,\lambda \leq 0\right\}.$$

Therefore, $\lambda \in \rho(\mathbf{D}_\theta)$ and $\|(\lambda\mathbf{I} - \mathbf{D}_\theta)^{-1}\|_{\mathscr{L}(Z_\theta} \leq M_\eta/(1+|\lambda|)$, which leads to the desired results. □

Taking the same procedure as in [3], using 2.1, we can prove

Lemma 2.4. *$\mathbf{B}_\theta$ is a sectorial operator in Z_θ, it has the same spectrum as $\mathbf{D}_\theta$ has, and for all $\alpha \in [0,1]$, the equality $\mathbf{P}_\theta\mathbf{B}_\theta^\alpha = \mathbf{D}_\theta^\alpha\mathbf{P}_\theta$ holds.*

Since the perturbation $\begin{bmatrix} 0 & 0 \\ 0 & -\eta^{-1}A^{1-\theta}\end{bmatrix}$ of $\mathbf{B}_\theta$ satisfies

$$\begin{aligned}
\left\|\begin{bmatrix} 0 & 0 \\ 0 & -\eta^{-1}A^{1-\theta}\end{bmatrix}\begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\|_{Z_\theta} &= \eta^{-1}\|A^{1-\theta}\psi\|_X \\
&= \eta^{-1}\|(A^\theta)^{\theta^{-1}-1}\psi\|_X \\
&\leq C\|A^\theta\psi\|_X^{\theta^{-1}-1}\|\psi\|_X^{2-\theta^{-1}} \\
&\leq C\left\|\mathbf{B}_\theta\begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\|_{Z_\theta}^{\theta^{-1}-1}\left\|\begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\|_{Z_\theta}^{2-\theta^{-1}}
\end{aligned}$$

for all $\begin{bmatrix}\varphi \\ \psi\end{bmatrix} \in \mathcal{D}(\mathbf{B}_\theta)$, then according to p.80, [11], we have

Lemma 2.5. *$\mathbf{C}_\theta$ is a sectorial operator in Z_θ with $Re(\sigma(\mathbf{C}_\theta)) > 0$.*

Let us return to Pr. (1.1). Take $W_\theta = X^{1/2} \times X^{1/2-\theta}$ as a new work space, and define

$$\mathbf{A}_\theta = \begin{bmatrix} 0 & -I \\ \widetilde{A^{\theta-1/2}}A^{3/2-\theta} & \eta\widetilde{A^{\theta-1/2}}A^{1/2}\end{bmatrix}, \quad \mathcal{D}(\mathbf{A}_\theta) = X^{3/2-\theta} \times X^{1/2}.$$

Evidently, $\mathbf{A}_\theta$ is closed in W_θ with a dense domain. As to its sectorial property, let us introduce the operator $\mathbf{Q}_\theta : Z_\theta \to W_\theta$ by

$$\mathbf{Q}_\theta = \begin{bmatrix} \widetilde{A^{\theta-1/2}} & 0 \\ 0 & \widetilde{A^{\theta-1/2}}\end{bmatrix}.$$

This is an isomorphic operator with

$$\mathbf{Q}_\theta^{-1} = \begin{bmatrix} A^{1/2-\theta} & 0 \\ 0 & A^{1/2-\theta} \end{bmatrix}$$

and $\mathbf{Q}_\theta(X^1 \times X^\theta) = X^{3/2-\theta} \times X^{1/2}$. Furthermore, for all $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in X^1 \times X^\theta$, we have

$$\mathbf{Q}_\theta \mathbf{C}_\theta \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -A^{\theta-1/2}v \\ \widetilde{A^{\theta-1/2}}Au + \eta\widetilde{A^{\theta-1/2}}A^\theta v \end{bmatrix} = \mathbf{A}_\theta \mathbf{Q}_\theta \begin{bmatrix} u \\ v \end{bmatrix}.$$

Thus, from the sectorial property of $\mathbf{C}_\theta$ in Z_θ, and Lemma 2.1, we can conclude that

Theorem 2.6. $\mathbf{A}_\theta$ *is a sectorial operator in* W_θ *having the same spectrum as* $\mathbf{C}_\theta$ *has, and* $\mathbf{A}_\theta^\alpha \mathbf{Q}_\theta = \mathbf{Q}_\theta \mathbf{C}_\theta^\alpha$.

Remark 2.7. From this theorem, we can also derive the interpolation and extrapolation spaces of $\mathbf{A}_\theta$ using the corresponding spaces of $\mathbf{C}_\theta$, but here we do not do so, since the domain $\mathcal{D}(\mathbf{A}_\theta)$ itself is a Cartesian product of two spaces.

Let $W_\theta^1 = \mathcal{D}(\mathbf{A}_\theta)$ be endowed with the graph norm. A simple calculation shows that, $W_\theta^1 = X^{3/2-\theta} \times X^{1/2}$ in the sense of isomorphism. Then for any $\varepsilon, \gamma \in (0,1)$, the interpolation and extrapolation spaces of $\mathbf{A}_\theta$ can be characterized as follows:

$$\begin{aligned} W_\theta^\varepsilon &= X^{1/2+\varepsilon(1-\theta)} \times X^{1/2-\theta+\varepsilon\theta}, \\ (W_\theta^{-1})^\gamma &= W_\theta^{-1+\gamma} = (W_\theta^{1-\gamma})^*, \\ W_\theta^{1-\gamma} &= X^{1/2+(1-\gamma)(1-\theta)} \times X^{1/2-\theta+(1-\gamma)\theta} = (X^{1/2})^{(1-\gamma)(1-\theta)} \times (X^{1/2-\theta})^{(1-\gamma)\theta}, \end{aligned}$$

$$\begin{aligned} (W_\theta^{-1})^\gamma &= (X^{1/2})^{-(1-\gamma)(1-\theta)} \times (X^{1/2-\theta})^{-(1-\gamma)\theta} \\ &= X^{1/2-(1-\gamma)(1-\theta)} \times X^{1/2-\theta-(1-\gamma)\theta} \\ &= X^{1/2-(1-\gamma)(1-\theta)} \times X^{1/2-(2-\gamma)\theta}. \end{aligned} \tag{2.1}$$

Remark 2.8. In the new phase space, $\mathbf{A}_\theta$ can be represented as an operator matrix, and consequently all the interpolation and extrapolation spaces can be described as the Cartesian products.

Theorem 2.9. *Under condition (1.4) upon* f *with* $\rho \le (N+2)/(N-2)$*, the* ε*-regular map* $\mathcal{F}$ *is subcritical associated with the space pair* $(W_\theta^{-1}, W_\theta)$ *in the case* $1/2 < \theta \le 1$*. That is, we can find constants* $\varepsilon \in [0,1)$*,* $\gamma \in (0,1]$*, with* $\rho\varepsilon < \gamma$ *s.t. hypothesis (1.6) holds for* $\mathcal{F}$*,* $(W_\theta^{-1})^\gamma$ *and* W_θ^ε *(i.e.* $(W_\theta^{-1})^{1+\varepsilon}$*) as well.*

Proof. We consider the embedding properties for space $X^{1/2-(2-\gamma)}$ appearing in the decomposition (2.1) firstly. Observe that $X^{1/2-(2-\gamma)\theta} = (X^{-1/2+(2-\gamma)\theta})^*$, and $X^{-1/2+(2-\gamma)\theta} \hookrightarrow H^{-1+2(2-\gamma)\theta}(\Omega)$, from which we can deduce that

(i) if $(N+2)/4 \le \theta \le 1$ $(N \ge 4)$, then for any $\gamma \in (0,1)$, $X^{-1/2+(2-\gamma)\theta} \hookrightarrow C(\bar{\Omega})$, and then $X^{1/2-(2-\gamma)\theta} \hookleftarrow L^r(\Omega)$, $\forall\, r \ge 1$;

(ii) if $(N+2)/8 < \theta < (N+2)/4$, then $0 < 2-(N+2)/4\theta < 1$, thus for each $\gamma \in (2-(N+2)/4\theta, 1]$, we have $X^{-1/2+(2-\gamma)\theta} \hookrightarrow L^{r'}(\Omega)$ for all $1 \le r' \le 2N/(N+2-4(2-\gamma)\theta)$, hence $X^{1/2-(2-\gamma)\theta} \hookleftarrow L^r(\Omega)$ for all $r \ge 2N/(N-2+4(2-\gamma)\theta)$.

(iii) if $1/2 < \theta \le (N+2)/8$ then for any $\gamma \in (0,1]$, we have the same imbedding for $X^{1/2-(2-\gamma)\theta}$ as in (ii).

In addition, for the number $\varepsilon \in [0, \varepsilon_0)$ with

$$\varepsilon_0 = \begin{cases} 1, & \text{if } \theta = 1, \\ \min\{1, (N-2)/4(1-\theta)\}, & \text{if } 1/2 \le \theta < 1, \end{cases}$$

we have $X^{1/2+\varepsilon(1-\theta)} \hookrightarrow L^s(\Omega)$ for all $1 \le s \le 2N/(N-2-4\varepsilon(1-\theta))$.

Consequently, in case (i), we can take $\gamma \in (0,1)$ and $\varepsilon \in [0,\varepsilon_0)$, s.t. $\rho\varepsilon < \gamma$, and

$$\begin{aligned} (2.2)\qquad & \|f(u_1) - f(u_2)\|_{X^{1/2-(2-\gamma)}} \\ \le\ & C\|u_1 - u_2\|_{X^{1/2+\varepsilon(1-\theta)}}(1 + \|u_1\|^{\rho-1}_{X^{1/2+\varepsilon(1-\theta)}} + \|u_2\|^{\rho-1}_{X^{1/2+\varepsilon(1-\theta)}}), \\ & \forall\, u_i \in X^{1/2+\varepsilon(1-\theta)},\ i = 1,2, \end{aligned}$$

which leads to (1.6) for $\mathcal{F}$, $(W_\theta^{-1})^\gamma$ and W_θ^ε immediately.

And in case (ii), for each $\gamma \in (2-(N+2)/4\theta, 1]$ and $\varepsilon \in [0,\varepsilon_0)$, we have

$$(2.3)\qquad \|f(u_1) - f(u_2)\|_{L^r(\Omega)} \le C\|u_1 - u_2\|_{L^{\rho r}}(1 + \|u_1\|^{\rho-1}_{L^{\rho r}} + \|u_2\|^{\rho-1}_{L^{\rho r}}),$$

with $\rho r \le 2N/(N-2-4\varepsilon(1-\theta))$ and $r \ge 2N/(N-2+4(2-\gamma)\theta)$, and then

$$(2.4)\qquad \rho \le (N-2+4(2-\gamma)\theta)/(N-2-4\varepsilon(1-\theta)).$$

Consider the difference

$$\begin{aligned} d &= \frac{N-2+4(2-\gamma)\theta}{N-2-4\varepsilon(1-\theta)} - \frac{N+2}{N-2} \\ &= \frac{4}{N-2-4\varepsilon(1-\theta)} \cdot (2\theta - 1 - \gamma\theta + \varepsilon(1-\theta)\frac{N+2}{N-2}) \end{aligned}$$

It is easy to show that, if $\theta = 1$ $(N \ge 6)$, then $d = 4(1-\gamma)/(N-2) > 0$ for all $2-(N+2)/4\theta < \gamma < 1$. If $1/2 < \theta < 1$, then the sufficient and necessary condition for $d > 0$ is that $\rho\varepsilon > (\theta\gamma - (2\theta-1))/(1-\theta)$. Notice also the number on the right hand of the above inequality is smaller than γ for all $0 < \gamma < 1$, therefore we can select $\varepsilon \in (\varepsilon_1, \min\{\varepsilon_0, \rho^{-1}\gamma\})$ in (2.4) with $\varepsilon_1 = \rho^{-1}(\theta\gamma - (2\theta-1))/(1-\theta)$ to make $d > 0$ for all $\rho \le (N+2)/(N-2)$. Putting all the numbers ε, γ and r obtained above into (2.3), using the corresponding embeddings, we obtain (2.2), and hence (1.6) for $\mathcal{F}$, $(W_\theta^{-1})^\gamma$ and W_θ^ε simultaneously.

In case (iii), we have the same results as in case (ii) only with $2-(N+2)/4\theta<\gamma<1$ instead of $\gamma\in(0,1)$.

Summing up all the facts, we conclude that, for each $\theta\in(1/2,\le 1]$, $\mathcal{F}$ is always an ε-regular map of subcritical type. $\square$

Remark 2.10. By replacing the work space Y by W_θ, for the same critical number $\rho=(N+2)/(N-2)$, the criticality of $\mathcal{F}$ has been changed (please compare to [1, 3]).

The following theorem is a natural consequence of Prop. 1.2 and Thm. 2.9.

Theorem 2.11. *Under hypothesis (1.4) upon f, for each pair of initial values $\begin{bmatrix} u_0\\ v_0\end{bmatrix}\in W_\theta$, there is a time interval $[0,T]$ on which, the wave equation with strong damping 1.1 (or equivalently Pr. (1.2)+(1.3)) has a unique ε-regular solution $\begin{bmatrix} u\\ v\end{bmatrix}$ satisfying $\begin{bmatrix} u\\ v\end{bmatrix}\in C((0,T],W_\theta^\gamma)\cap C^1((0,T],W_\theta^{\gamma^-})$. Moreover, if $\begin{bmatrix} u(t)\\ v(t)\end{bmatrix}$ is bounded in W_θ on its existence interval, then $\begin{bmatrix} u\\ v\end{bmatrix}$ survives for ever.*

Remark 2.12. By reviewing the proof of Thm. 2.9, we can find that the indicator γ can be can taken much close to 1. So, every ε-regular solution $\begin{bmatrix} u\\ v\end{bmatrix}$ arising in W_θ drops in $C((0,T],W_\theta^{1^-})\cap C^1((0,T],W_\theta^{1^-})$ definitely. Notice also W_θ^γ contains $X^{1/2}\times X$ if γ is sufficiently close to 1, then the inclusion $\begin{bmatrix} u\\ v\end{bmatrix}\in C((0,\infty),W_\theta^\gamma)$ assures another one $\begin{bmatrix} u\\ v\end{bmatrix}\in C((0,\infty),Y)$. As a direct result, we can conclude that any ε-regular solution $\begin{bmatrix} u\\ v\end{bmatrix}$ starting in Y exists globally, whenever the norm $\left\|\begin{bmatrix} u(t)\\ v(t)\end{bmatrix}\right\|$ is bounded all the time. Moreover, if we lay the dissipative condition on f (see [7, 9]), namely

$$\limsup_{|s|\to\infty}\frac{f(s)}{s}\le 0, \tag{2.5}$$

we then obtain the uniform bounds of $\left\|\begin{bmatrix} u(t)\\ v(t)\end{bmatrix}\right\|_Y$ on the existence interval of $\begin{bmatrix} u\\ v\end{bmatrix}$ by the energy estimate. Under this situation, $\begin{bmatrix} u\\ v\end{bmatrix}$ is a global solution for Pr. (1.2)+(1.3).

Remark 2.13. We must admit that, if $\theta=1/2$, then the ε-regular map $\mathcal{F}$ remains critical in our setting. But this does not change the fact that boundedness of $\begin{bmatrix} u\\ v\end{bmatrix}$ in Y leads to its universal existence. In fact, in this case, $\mathcal{F}$ can also be defined on Y and is weakly continuous from Y to Y_θ^{-1}, which is a sufficient condition for the global existence of $\begin{bmatrix} u\\ v\end{bmatrix}$. For the detailed discussions, please refer to [19].

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