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# Homoclinic orbits for a class of $p$-Laplacian systems with periodic assumption

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**Abstract:** In this paper, by using a linking theorem, some new existence criteria of homoclinic orbits are obtained for the $p$-Laplacian system $d(|\dot{u}(t)|^{p-2}\dot{u}(t))/dt + \nabla V(t,u(t)) = f(t)$, where $p > 1$, $V(t,x) = -K(t,x) + W(t,x)$.

**Keywords:** $p$-Laplacian system; homoclinic orbit; critical point; linking theorem.

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## 1. Introduction and main results

In this paper, we consider the $p$-Laplacian system

$$\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) + \nabla V(t,u(t)) = f(t) \tag{1.1}$$

where $p > 1$, $V(t,x) = -K(t,x) + W(t,x)$, $K, W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}^N$ is a continuous and bounded function. A solution $u(t)$ is nontrivial homoclinic (to 0) if $u(t) \not\equiv 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$. Let $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

When $p = 2$, system (1.1) reduces to the second order Hamiltonian system

$$\ddot{u}(t) + \nabla V(t,u(t)) = f(t) \tag{1.2}$$

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Since 1978, lots of contributions on the existence and multiplicity of homoclinic solutions for system (1.2) have been presented (for example, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 18] and references therein). Most of them considered the following system:

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \tag{1.3}$$

where $L(t)$ is a symmetric matrix value function and $W$ satisfies the following AR-condition:

(W1) there exists $\mu > 2$ such that

$$0 < \mu W(t, x) \leq (\nabla W(t, x), x), \quad \forall\ (t, x) \in \mathbb{R} \times \left(\mathbb{R}^N/\{0\}\right). \tag{1.4}$$

In 2005, Izydorek and Janczewska [14] considered system (1.2), more general than system(1.3), and obtained the following result:

**Theorem A** *Assume that $V$ and $f$ satisfy (W1) and the following conditions:*

(V) $V(t, x) = -K(t, x) + W(t, x)$, *where $K, W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are $C^1$-maps, $T$-periodic with respect to $t$, $T > 0$;*

(K1) *there are constants $b_1, b_2 > 0$ such that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,*

$$b_1|x|^2 \leq K(t, x) \leq b_2|x|^2;$$

(K2) *for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, $K(t, x) \leq (x, \nabla K(t, x)) \leq 2K(t, x)$;*

(W2) $\nabla W(t, x) = o(|x|)$, *as $|x| \to 0$ uniformly with respect to $t$;*

(f) $\bar{b}_1 := \min\{1, 2b_1\} > 2M$ *and* $\|f\|_{L^2(\mathbb{R},\mathbb{R})} < \frac{\bar{b}_1 - 2M}{2C^*}$, *where*

$$M = \sup_{t \in [0,T], |x|=1} W(t, x) \tag{1.5}$$

*and $C^*$ is a positive constant that depends on $T$. When $T \geq 1/2$, $C^* = 1/2$. Then system (1.2) possesses a nontrivial homoclinic solution.*

Since then, several results for system (1.2) in this direction have been obtained (see [11] and [18]). When $p > 1$, the following result can be seen in [17]:

**Theorem B** *Assume that $V$ and $f$ satisfy assumptions (V) and the following conditions:*

*(I1) there exist constants $b > 0$ and $\gamma \in (1, p]$ such that*

$$K(t, 0) = 0, \quad K(t, x) \geq b|x|^\gamma, \quad \textit{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N;$$

*(I2) there is a constant $\theta \geq p$ such that*

$$K(t,x) \leq (\nabla K(t,x), x) \leq \theta K(t,x), \quad \textit{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

*(I3) $W(t,0) \equiv 0$ and $\nabla W(t,x) = o(|x|^{p-1})$, as $|x| \to 0$ uniformly with respect to $t$;*

*(I4) there are two constants $\mu > \theta$ and $\nu \in [0, \mu - \theta)$ such that*

$$0 < \mu W(t,x) \leq (\nabla W(t,x), x) + \nu b|x|^{\gamma}, \quad \textit{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N/\{0\};$$

*(I5)*

$$\liminf_{|x|\to\infty} \frac{W(t,x)}{|x|^{\theta}} > \frac{\pi^p}{pT^p} + m_1 \quad \textit{uniformly with respect to } t,$$

*where*

$$m_1 = \sup\{K(t,x) | t \in [0,T], x \in \mathbb{R}^N, |x| = 1\};$$

*(I6)*

$$\int_{\mathbb{R}} |f(t)|^q dt < \left( \frac{1}{C^{p-1}} \min\left\{ \frac{\delta^{p-1}}{p}, \left(1 - \frac{\nu}{\mu - \gamma}\right) b\delta^{\gamma-1} - M\delta^{\mu-1} \right\} \right)^q,$$

*where $M$ is determined by (1.5), $\frac{1}{p} + \frac{1}{q} = 1$, $C = 2^{\frac{p-1}{p}}(1 + [\frac{1}{2T}])^{1/p}$ and $\delta \in (0,1]$ such that*

$$\left(1 - \frac{\nu}{\mu - \gamma}\right) b\delta^{\gamma-1} - M\delta^{\mu-1} = \max_{x\in[0,1]} \left( \left(1 - \frac{\nu}{\mu - \gamma}\right) bx^{\gamma-1} - Mx^{\mu-1} \right).$$

*Then system (1.1) possesses a nontrivial homoclinic solution.*

For the $p$-Laplacian system (1.1) with $f(t) \equiv 0$ and $K(t,x) \equiv 0$ (or $K(t,x) = (L(t)|x|^{p-2}x, x)$, where $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix), recently, under different assumptions, some results on the existence and multiplicity of periodic solutions, subharmonic solutions and homoclinic solutions have been obtained (for example, see [21, 22, 23, 24, 25, 26]). In [21], the authors considered the existence of subharmonic solutions for system (1.1) with $f(t) \equiv 0$ and $K(t,x) = (L(t)|x|^{p-2}x, x)$, where $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix. Under some reasonable assumptions, they obtained that the system has a sequence of distinct periodic solutions with period $k_jT$ satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$. In [22], the authors considered the existence of homoclinic solutions for system (1.1) with $f(t) \equiv 0$. They assumed that $W$ is asymptotically $p$-linear at infinity, $K$ satisfies (K1) and $W$ and $K$ are not periodic in $t$. In [23]–[26], the authors considered the existence and multiplicity of periodic solutions for system (1.1) with $f(t) \equiv 0$ and $K(t,x) \equiv 0$. Motivated by [11, 14, 17, 18], in this paper, we consider the existence of homoclinic orbits for system (1.1) and present some new existence criteria. Next, we state our main results.

**Theorem 1.1.** *Assume that $f \neq 0$, $W$ and $K$ satisfy (V) and the following conditions:*
*(H1) there exist $\gamma \in (1,p)$ and $a > 0$ such that*

$$K(t,x) \geq a|x|^\gamma, \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^N;$$

*(H2) $K(t,0) \equiv 0, \quad (x, \nabla K(t,x)) \leq pK(t,x), \quad$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$;*
*(H3) (i) there exist $r \in (0,1]$ and $0 < b < a$ such that*

$$W(t,x) \leq b|x|^p, \quad \forall \ |x| \leq r; \tag{1.6}$$

*or (ii) there exist $r > 1$ and $0 < b < ar^{\gamma-p}$ such that (1.6) holds;*
*(H4)*

$$\lim_{|x|\to+\infty} \frac{W(t,x)}{|x|^p} > \frac{\pi^p}{pT^p} + A_0 \quad \textit{uniformly for all } t \in [0,T],$$

*where*

$$A_0 = \max_{|x|=1, t\in[0,T]} K(t,x);$$

*(H5) there exist positive constants $\xi, \eta$ and $\nu \in [0, \gamma - 1)$ such that*

$$0 \leq \left(p + \frac{1}{\xi + \eta|x|^\nu}\right) W(t,x) \leq (\nabla W(t,x), x) \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^N;$$

*(H6) $f \in L^q(\mathbb{R}, \mathbb{R}^N) \cap f \in L^{\frac{p-\nu}{p-\nu-1}}(\mathbb{R}, \mathbb{R}^N)$ and*

$$(i) \quad \|f\|_{L^q(\mathbb{R},\mathbb{R}^N)} < \frac{r^{p-1}}{C_0^{p-1}} \min\left\{\frac{1}{p}, a - b\right\}, \quad \textit{when} \ \ r \in (0,1],$$

$$(ii) \quad \|f\|_{L^q(\mathbb{R},\mathbb{R}^N)} < \frac{r^{p-1}}{C_0^{p-1}} \min\left\{\frac{1}{p}, \frac{a}{r^{p-\gamma}} - b\right\}, \quad \textit{when} \ \ r \in (1,+\infty),$$

*where*

$$C_0 = \left[\max\left\{\frac{1}{2T} + \frac{p}{2q}, \frac{1}{2}\right\}\right]^{1/p}, \quad \textit{when} \ \ p \neq 2,$$

*and*

$$C_0 = \sqrt{\frac{1 + \sqrt{1+4T^2}}{4T}}, \ \textit{when} \ \ p = 2.$$

*Then system (1.1) possesses a nontrivial homoclinic solution.*

Next, we present an example of $K$ and $W$, which satisfies (H1)–(H5) but does not satisfy those conditions in [11, 14, 17, 18].

**Example 1.1.** Let $p = 5$,

$$K(t,x) = \ln(\frac{1}{2^5}+2)|x|^4 + |x|^5, \quad W(t,x) = |x|^5 \ln(|x|^5+1).$$

Choose $\gamma = 4$ and $a = \ln(\frac{1}{2^5}+2)$. Then it is easy to verify that (H1) and (H2) hold. If one chooses $r = \frac{1}{2}$, then

$$W(t,x) \le \ln(\frac{1}{2^5}+1)|x|^5, \quad \forall |x| \le r.$$

Choose $b = \ln(\frac{1}{2^5}+1)$. Then (H3)(i) holds. Obviously,

$$\lim_{|x|\to+\infty} \frac{W(t,x)}{|x|^5} = +\infty \text{ uniformly for all } t \in [0,T].$$

(H4) holds. Moreover, note that

$$5\xi|x|^5 \ge \ln(|x|^5+1) \text{ and } 5\eta|x|^2 \ge \ln(|x|^5+1), \quad \text{for all } x \in \mathbb{R}^N,$$

when we choose sufficiently large $\xi$ and $\eta$. Hence

$$\begin{aligned} & 5\xi|x|^5 + 5\eta|x|^7 \ge \ln(|x|^5+1) + \ln(|x|^5+1)|x|^5 \\ \Longleftrightarrow\ & 5(\xi+\eta|x|^2)|x|^5 \ge \ln(|x|^5+1)(|x|^5+1) \\ \Longleftrightarrow\ & 5(\xi+\eta|x|^2)|x|^{10} \ge |x|^5\ln(|x|^5+1)(|x|^5+1) \\ \Longleftrightarrow\ & \frac{5|x|^{10}}{|x|^5+1} \ge \frac{|x|^5\ln(|x|^5+1)}{\xi+\eta|x|^2} \\ \Longleftrightarrow\ & (\nabla W(t,x),x) - 5W(t,x) \ge \frac{W(t,x)}{\xi+\eta|x|^2}, \quad \text{for all } x \in \mathbb{R}^N, \end{aligned}$$

which implies that (H5) holds.

**Theorem 1.2.** *Assume that $f \ne 0$, $W$ and $K$ satisfy (V), (H1)–(H5) and the following conditions:*

*(H6)′ $f \in L^1(\mathbb{R},\mathbb{R}^N)$ and*

$$(i) \quad \|f\|_{L^1(\mathbb{R},\mathbb{R}^N)} < \frac{r^{p-1}}{C_0^p}\min\left\{\frac{1}{p}, a-b\right\}, \quad \textit{when } r \in (0,1],$$

$$(ii) \quad \|f\|_{L^1(\mathbb{R},\mathbb{R}^N)} < \frac{r^{p-1}}{C_0^p}\min\left\{\frac{1}{p}, \frac{a}{r^{p-\gamma}}-b\right\}, \quad \textit{when } r \in (1,+\infty).$$

*Then system (1.1) possesses a nontrivial homoclinic solution.*

**Theorem 1.3.** *Assume that $f \neq 0$, $W$ and $K$ satisfy (V), (H2), (H4), (H5) and the following conditions:*

*(H1)′ there exists $a > 0$ such that*

$$K(t,x) \geq a|x|^p \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^N;$$

*(H3)′ there exist $r > 0$ and $0 < b < a$ such that*

$$W(t,x) \leq b|x|^p, \quad \forall\ |x| \leq r;$$

*(H6)″ $f \in L^q(\mathbb{R},\mathbb{R}^N) \cap f \in L^{\frac{p-\nu}{p-\nu-1}}(\mathbb{R},\mathbb{R}^N)$ and*

$$\|f\|_{L^q(\mathbb{R},\mathbb{R}^N)} < \frac{r^{p-1}}{C_0^{p-1}} \min\left\{\frac{1}{p}, a-b\right\}.$$

*Then system (1.1) possesses a nontrivial homoclinic solution.*

**Theorem 1.4.** *Assume that $f \neq 0$, $W$ and $K$ satisfy (V), (H1)′, (H2), (H3)′, (H4), (H5) and the following condition:*

*(H6)‴ $f \in L^1(\mathbb{R},\mathbb{R}^N)$ and*

$$\|f\|_{L^1(\mathbb{R},\mathbb{R}^N)} < \frac{r^{p-1}}{C_0^{p}} \min\left\{\frac{1}{p}, a-b\right\}.$$

*Then system (1.1) possesses a nontrivial homoclinic solution.*

**Remark 1.1.** Theorem 1.3 and Theorem 1.4 show that $f$ can be large when $r$ is large, which is different from Theorem A and Theorem B. Moreover, in Theorem 1.1 and Theorem 1.2, if $r \in (1,+\infty)$, it is also possible that $f$ can be large.

**Theorem 1.5.** *Assume that $f \equiv 0$, $W$ and $K$ satisfy (H1), (H4) and the following conditions:*

*(H2)′ $K(t,0) \equiv 0$, $\quad K(t,x) \leq (x, \nabla K(t,x)) \leq pK(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$;*

*(H3)″ there exist $r > 0$ and $0 < b < ar^{\gamma-p}$ such that*

$$W(t,x) \leq b|x|^p, \quad \forall\ |x| \leq r;$$

*(H5)′ there exist positive constants $\xi, \eta$ and $\nu \in [0,\gamma)$ such that*

$$0 \leq \left(p + \frac{1}{\xi + \eta|x|^\nu}\right) W(t,x) \leq (\nabla W(t,x), x), \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^N;$$

*(H7)* $Y(0) < \min\{1, a\}$, *where the function* $Y : [0, +\infty) \to [0, +\infty)$ *is defined by*

$$Y(s) = \max_{\substack{t\in[0,T] \\ 0<|x|\le s}} \frac{(\nabla W(t,x), x)}{|x|^p}$$

*for* $s > 0$ *and*

$$Y(0) = \lim_{s\to 0^+} Y(s) = \lim_{s\to 0^+} \max_{\substack{t\in[0,T] \\ 0<|x|\le s}} \frac{(\nabla W(t,x), x)}{|x|^p}.$$

*Then system (1.1) possesses a nontrivial homoclinic solution.*

**Theorem 1.6.** *Assume that* $f \equiv 0$, $W$ *and* $K$ *satisfy (H1)′, (H2)′, (H3)′, (H4), (H7) and the following conditions:*

*(H5)″ there exist positive constants* $\xi, \eta$ *and* $\nu \in [0, p)$ *such that*

$$0 \le \left(p + \frac{1}{\xi + \eta|x|^\nu}\right) W(t,x) \le (\nabla W(t,x), x) \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^N.$$

*Then system (1.1) possesses a nontrivial homoclinic solution.*

## 2. Preliminaries

Similar to [11, 14, 17, 18], we will obtain the homoclinic orbit of system (1.1) as a limit of solutions of a sequence of differential systems:

$$\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) + \nabla V(t, u(t)) = f_k(t), \tag{2.1}$$

where $f_k : \mathbb{R} \to \mathbb{R}^N$ is a $2kT$-periodic extension of restriction of $f$ to the interval $[-kT, kT), k \in \mathbb{N}$.

For $p > 1$, let $L^p_{2kT}(\mathbb{R}, \mathbb{R}^N)$ denote the Banach space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ and the norm defined by

$$\|u\|_{L^p_{2kT}} = \left(\int_{-kT}^{kT} |u(t)|^p dt\right)^{1/p}.$$

Let $L^\infty_{2kT}(\mathbb{R}, \mathbb{R}^N)$ denote a space of $2kT$-periodic essential bounded (measurable) functions from $\mathbb{R}$ to $\mathbb{R}^N$ equipped with the norm

$$\|u\|_{L^\infty_{2kT}} = \operatorname{ess\,sup}\{|u(t)|, t \in [-kT, kT]\}.$$

For each $k \in \mathbb{N}$, define $E_k = W^{1,p}_{2kT}$ by

$$W^{1,p}_{2kT} = \{u : \mathbb{R} \to \mathbb{R}^N | u(t) \text{ is absolutely continuous on } [-kT, kT], u(t + 2kT) = u(t) \\ \text{and } \dot{u} \in L^p([-kT, kT]; \mathbb{R}^N)\}.$$

On $W^{1,p}_{2kT}$, we define the norm as follows:

$$\|u\|_{E_k} = \Big[\int_{-kT}^{kT} |u(t)|^p dt + \int_{-kT}^{kT} |\dot{u}(t)|^p dt\Big]^{1/p}, \quad u \in W^{1,p}_{2kT}.$$

Then $\left(W^{1,p}_{2kT}, \|\cdot\|_{E_k}\right)$ is a reflexive and uniformly convex Banach space (see [19], Theorem 3.3 and Theorem 3.6).

**Lemma 2.1.** *Let $c > 0$ and $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$. Then for every $t \in \mathbb{R}$, the following inequalities hold:*

$$|u(t)| \le (2c)^{-1/p} \left(\int_{t-c}^{t+c} |u(s)|^p ds\right)^{1/p} + \frac{c^{1/q}}{2^{1/p}(q+1)^{1/q}} \left(\int_{t-c}^{t+c} |\dot{u}(s)|^p ds\right)^{1/p}, \tag{2.2}$$

$$|u(t)| \le 2^{-1/p} \left(\int_{t-1}^{t+1} |u(s)|^p ds + \int_{t-1}^{t+1} |\dot{u}(s)|^p ds\right)^{1/p} \tag{2.3}$$

*and*

$$|u(t)| \le \left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |u(s)|^p ds + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{u}(s)|^p ds\right)^{1/p} \tag{2.4}$$

**Proof.** Fix $t \in \mathbb{R}$. Then for every $\tau \in \mathbb{R}$,

$$u(t) = u(\tau) + \int_\tau^t \dot{u}(s) ds. \tag{2.5}$$

Set

$$\phi(s) = \begin{cases} s - t + c, & t - c \le s \le t, \\ t + c - s, & t \le s \le t + c. \end{cases}$$

Integrating (2.5) on $[t - c, t + c]$ and using the Hölder's inequality, we have

$$\begin{aligned} 2c|u(t)| &\le \int_{t-c}^{t+c} |u(\tau)| d\tau + \int_{t-c}^{t+c} \int_\tau^t |\dot{u}(s)| ds d\tau \\ &\le \int_{t-c}^{t+c} |u(\tau)| d\tau + \int_{t-c}^{t} \int_\tau^t |\dot{u}(s)| ds d\tau + \int_t^{t+c} \int_t^\tau |\dot{u}(s)| ds d\tau \\ &\le \int_{t-c}^{t+c} |u(\tau)| d\tau + \int_{t-c}^{t} \Big(s - t + c\Big) |\dot{u}(s)| ds + \int_t^{t+c} \Big(t + c - s\Big) |\dot{u}(s)| ds \end{aligned}$$

$$
\begin{aligned}
&= \int_{t-c}^{t+c} |u(\tau)|d\tau + \int_{t-c}^{t+c} \phi(s)|\dot{u}(s)|ds \\
&\le (2c)^{1/q}\left(\int_{t-c}^{t+c} |u(\tau)|^p d\tau\right)^{1/p} + \left(\int_{t-c}^{t+c} [\phi(s)]^q ds\right)^{1/q}\left(\int_{t-c}^{t+c} |\dot{u}(s)|^p ds\right)^{1/p} \\
&= (2c)^{1/q}\left(\int_{t-c}^{t+c} |u(\tau)|^p d\tau\right)^{1/p} + \frac{2^{1/q}c^{(q+1)/q}}{(q+1)^{1/q}}\left(\int_{t-c}^{t+c} |\dot{u}(s)|^p ds\right)^{1/p}. \qquad (2.6)
\end{aligned}
$$

So (2.2) holds. Let $c = 1$ and $c = 1/2$, respectively. Then (2.3) and (2.4) hold.

**Remark 2.1.** When $p = 2$, Lemma 2.1 reduces to Lemma 2.2 in [12] and (2.4) improved Lemma 2.2 in [17].

The following (2.8) and its proof have been given in [11] (see [11], Lemma 2.2). Here, for readers' convenience, we also present it. In our Lemma 2.2, our main aim is to present the following (2.7) which generalizes Lemma 2.2 in [11] in some sense.

**Lemma 2.2.** *For every $k \in \mathbb{N}$, if $p > 1$ and $u \in E_k$, then*

$$
\|u\|_{L^\infty_{2kT}} \le \left[\max\left\{\frac{1}{2kT} + \frac{p-1}{2}, \frac{1}{2}\right\}\right]^{1/p}\left(\int_{-kT}^{kT} |u(s)|^p ds + \int_{-kT}^{kT} |\dot{u}(s)|^p ds\right)^{1/p}; \qquad (2.7)
$$

If $p = 2$ and $u \in E_k$, then the following better result holds:

$$
\|u\|_{L^\infty_{2kT}} \le \sqrt{\frac{1+\sqrt{1+4(kT)^2}}{4kT}}\left(\int_{-kT}^{kT} |u(s)|^2 ds + \int_{-kT}^{kT} |\dot{u}(s)|^2 ds\right)^{1/2}. \qquad (2.8)
$$

**Proof.** Let $\bar{t} \in [-kT, kT]$ and $t^* \in [\bar{t}, \bar{t} + 2kT]$ such that

$$
|u(\bar{t})|^p = \frac{1}{2kT}\int_{-kT}^{kT} |u(s)|^p ds \quad \text{and} \quad |u(t^*)| = \max_{t\in[-kT,kT]} |u(t)|.
$$

Then

$$
|u(t^*)|^p = |u(\bar{t})|^p + p\int_{\bar{t}}^{t^*} (|u(s)|^{p-2}u(s), \dot{u}(s))ds \qquad (2.9)
$$

and

$$
|u(t^* - 2kT)|^p = |u(\bar{t})|^p - p\int_{t^*-2kT}^{\bar{t}} (|u(s)|^{p-2}u(s), \dot{u}(s))ds \qquad (2.10)
$$

It follows from (2.9), (2.10) and Young's inequality that

$$
\begin{aligned}
|u(t^*)|^p &= \frac{1}{2}\left[|u(t^*)|^p + |u(t^* - 2kT)|^p\right] \\
&= \frac{1}{2}|u(\bar{t})|^p + \frac{1}{2}|u(\bar{t})|^p + \frac{p}{2}\int_{\bar{t}}^{t^*} (|u(s)|^{p-2}u(s), \dot{u}(s))ds \\
&\quad -\frac{p}{2}\int_{t^*-2kT}^{\bar{t}} (|u(s)|^{p-2}u(s), \dot{u}(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq |u(\bar{t})|^p + \frac{p}{2}\int_{\bar{t}}^{t^*} |u(s)|^{p-1}|\dot{u}(s)|ds + \frac{p}{2}\int_{t^*-2kT}^{\bar{t}} |u(s)|^{p-1}|\dot{u}(s)|ds \\
&= |u(\bar{t})|^p + \frac{p}{2}\int_{t^*-2kT}^{t^*} |u(s)|^{p-1}|\dot{u}(s)|ds \\
&= \frac{1}{2kT}\int_{-kT}^{kT} |u(s)|^p ds + \frac{p}{2}\int_{-kT}^{kT} |u(s)|^{p-1}|\dot{u}(s)|ds \qquad (2.11) \\
&\leq \frac{1}{2kT}\int_{-kT}^{kT} |u(s)|^p ds + \frac{p}{2}\int_{-kT}^{kT} \left[\frac{|u(s)|^p}{q} + \frac{|\dot{u}(s)|^p}{p}\right] ds \\
&\leq \max\left\{\frac{1}{2kT} + \frac{p}{2q}, \frac{1}{2}\right\}\left[\int_{-kT}^{kT} |u(s)|^p ds + \int_{-kT}^{kT} |\dot{u}(s)|^p ds\right] \\
&= \max\left\{\frac{1}{2kT} + \frac{p-1}{2}, \frac{1}{2}\right\}\left[\int_{-kT}^{kT} |u(s)|^p ds + \int_{-kT}^{kT} |\dot{u}(s)|^p ds\right]
\end{aligned}
$$

When $p = 2$, it follows from (2.11) and Young's inequality that

$$
\begin{aligned}
|u(t^*)|^2 &\leq \frac{1}{2kT}\int_{-kT}^{kT} |u(s)|^2 ds + \int_{-kT}^{kT} |u(s)||\dot{u}(s)|ds \\
&\leq \frac{1}{2kT}\int_{-kT}^{kT} |u(s)|^2 ds + \frac{kT}{1+\sqrt{1+4(kT)^2}}\int_{-kT}^{kT} |u(s)|^2 ds \\
&\quad + \frac{1+\sqrt{1+4(kT)^2}}{4kT}\int_{-kT}^{kT} |\dot{u}(s)|^2 ds \\
&= \frac{1+\sqrt{1+4(kT)^2}}{4kT}\left[\int_{-kT}^{kT} |u(s)|^2 ds + \int_{-kT}^{kT} |\dot{u}(s)|^2 ds\right].
\end{aligned}
$$

**Corollary 2.1.** *For every $k \in \mathbb{N}$, if $p > 1$ and $u \in E_k$, then*

$$
\|u\|_{L^\infty_{2kT}} \leq \left[\max\left\{\frac{1}{2T} + \frac{p-1}{2}, \frac{1}{2}\right\}\right]^{1/p}\left(\int_{-kT}^{kT} |u(s)|^p ds + \int_{-kT}^{kT} |\dot{u}(s)|^p ds\right)^{1/p}; \qquad (2.12)
$$

*If $p = 2$ and $u \in E_k$, then the following better result holds:*

$$
\|u\|_{L^\infty_{2kT}} \leq \sqrt{\frac{1+\sqrt{1+4T^2}}{4T}}\left(\int_{-kT}^{kT} |u(s)|^2 ds + \int_{-kT}^{kT} |\dot{u}(s)|^2 ds\right)^{1/2}. \qquad (2.13)
$$

**Remark 2.2.** It is easy to verify that Corollary 2.1 improves Corollary 2.1 in [17].

**Corollary 2.2.** *If $p > 1$ and $u \in E_k$, then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,*

$$
\|u\|_{L^\infty_{2kT}} \leq C^*\left(\int_{-kT}^{kT} |u(s)|^p ds + \int_{-kT}^{kT} |\dot{u}(s)|^p ds\right)^{1/p} \qquad (2.14)
$$

where $C^* > \left[\max\left\{\frac{p-1}{2}, \frac{1}{2}\right\}\right]^{1/p}$.

**Proof.** It follows from sequences $\left\{\left[\max\left\{\frac{1}{2kT}+\frac{p-1}{2},\frac{1}{2}\right\}\right]^{1/p}\right\}$ and $\left\{\sqrt{\frac{1+\sqrt{1+4k^2T^2}}{4kT}}\right\}$ are decreasing and

$$\left[\max\left\{\frac{1}{2kT}+\frac{p-1}{2},\frac{1}{2}\right\}\right]^{1/p}\to\left[\max\left\{\frac{p-1}{2},\frac{1}{2}\right\}\right]^{1/p}, \quad \text{as } k\to\infty$$

and

$$\sqrt{\frac{1+\sqrt{1+4k^2T^2}}{4kT}}\to\frac{\sqrt{2}}{2}, \quad \text{as } k\to\infty.$$

**Remark 2.3.** Corollary 2.2 generalizes (3.3) in [11].

Define $\eta : E_k \to [0,+\infty)$ by

$$\eta_k(u)=\left(\int_{-kT}^{kT}[|\dot{u}(t)|^p+pK(t,u(t))]dt\right)^{1/p}$$

and $\varphi_k : E_k \to \mathbb{R}$ by

$$\begin{aligned}\varphi_k(u) &= \int_{-kT}^{kT}\left[\frac{1}{p}|\dot{u}(t)|^p - V(t,u(t))\right]dt+\int_{-kT}^{kT}(f_k(t),u(t))dt\\ &= \frac{1}{p}\eta_k^p(u)-\int_{-kT}^{kT}W(t,u(t))dt+\int_{-kT}^{kT}(f_k(t),u(t))dt.\end{aligned}$$

It is easy to obtain that $\varphi\in C^1(E_k,\mathbb{R})$ and for $u,v\in E_k$,

$$\begin{aligned}(\varphi_k'(u),v) &= \int_{-kT}^{kT}\left[(|\dot{u}(t)|^{p-2}\dot{u}(t),\dot{v}(t))-(\nabla V(t,u(t)),v(t))\right]dt+\int_{-kT}^{kT}(f_k(t),v(t))dt\\ &= \int_{-kT}^{kT}\left[(|\dot{u}(t)|^{p-2}\dot{u}(t),\dot{v}(t))+(\nabla K(t,u(t)),v(t))-(\nabla W(t,u(t)),v(t))\right]dt\\ &\quad+\int_{-kT}^{kT}(f_k(t),v(t))dt.\end{aligned}$$

By (H2) or (H2)′, for all $u\in E_k$, we obtain

$$\begin{aligned}(\varphi_k'(u),u) &\le \int_{-kT}^{kT}\left[|\dot{u}(t)|^{p-2}+pK(t,u(t))\right]dt-\int_{-kT}^{kT}(\nabla W(t,u(t)),u(t))dt\\ &\quad+\int_{-kT}^{kT}(f_k(t),u(t))dt.\end{aligned}$$

It is well known that critical points of $\varphi$ correspond to solutions of system (1.1).

Different from [11, 14, 17], we shall use one linking method in [20] to obtain the critical points of $\varphi$ (the details can be seen in [20]). Let $(E,\|\cdot\|)$ be a Banach space. Define the continuous map $\Gamma : [0,1] \times E \to E$ by $\Gamma(t,x) = \Gamma(t)x$, where $\Gamma(t)$ satisfies the following conditions:

1) $\Gamma(0) = I$, the identity map.

2) For each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of $E$ onto $E$ and $\Gamma^{-1}(t) \in C(E \times [0,1), E)$.

3) $\Gamma(1)E$ is a single point in $E$ and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$.

4) For each $t_0 \in [0,1)$ and each bounded set $A \subset E$,

$$\sup_{\substack{0 \le t \le t_0 \\ u \in A}} \{\|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\|\} < \infty.$$

Let $\Phi$ be the set of all continuous maps $\Gamma$ as defined above.

**Definition 2.1.** (see [20], Definition 3.2) *We say that $A$ links $B$[hm] if $A$ and $B$ are subsets of $E$ such that $A \cap B = \emptyset$, and for each $\Gamma \in \Phi$, there is a $t' \in (0,1]$ such that $\Gamma(t')A \cap B \neq \emptyset$.*

**Example 1.** (see [20], page 21) Let $B$ be an open set in $E$, and let $A$ consist of two points $e_1, e_2$ with $e_1 \in B$ and $e_2 \notin \bar{B}$. Then $A$ links $\partial B$[hm].

We use the following theorem to prove our main results.

**Theorem 2.1.** (see [20], Theorem 3.4 and Theorem 2.12) *Let $E$ be a Banach space, $\varphi \in C^1(E, \mathbb{R})$ and $A$ and $B$ two subsets of $E$ such that $A$ links $B$[hm]. Assume that*

$$\sup_A \varphi \le \inf_B \varphi$$

*and*

$$c := \inf_{\Gamma \in \Phi} \sup_{\substack{s \in [0,1] \\ u \in A}} \varphi(\Gamma(s)u) < \infty.$$

*Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0,\infty)$ satisfying $\int_0^\infty \psi(r)dr = \infty$. Then there exists a sequence $\{u_n\} \subset E$ such that $\varphi(u_n) \to c$ and $\varphi'(u_n)/\psi(\|u_n\|) \to 0$, as $n \to \infty$. Moreover, if $c = \sup_A \varphi$, then there is a sequence $\{u_n\} \subset E$ satisfying $\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0,$ and $d(u_n, B) \to 0, \quad$ as $\ n \to \infty$.*

**Remark 2.4.** Since $A$ links $B$, by Definition 2.1, it is easy to know that $c \geq \inf_B \varphi$. By [20], if we let $\psi(r) = \frac{1}{1+r}$, the sequence $\{u_n\}$ is the Cerami sequence, that is $\{u_n\}$ satisfying

$$\varphi(u_n) \to c, \quad (1 + \|u_n\|)\|\varphi'(u_n)\| \to 0, \quad \text{as } n \to \infty.$$

# 3. Proofs of theorems

For convenience, we denote by $C_i,\ i = 1, \ldots$ various positive constants. When $p > 1$ and $p \neq 2$, let

$$C_0 = \left[\max\left\{\frac{1}{2T} + \frac{p-1}{2}, \frac{1}{2}\right\}\right]^{1/p}$$

and when $p = 2$, let

$$C_0 = \sqrt{\frac{1 + \sqrt{1 + 4T^2}}{4T}}.$$

**Lemma 3.1.** *Suppose that (H2) or (H2)′ holds. Then*

$$K(t, x) \leq K\left(t, \frac{x}{|x|}\right)|x|^p \quad \textit{for all } t \in \mathbb{R},\ |x| \geq 1;$$
$$K(t, x) \geq K\left(t, \frac{x}{|x|}\right)|x|^p \quad \textit{for all } t \in \mathbb{R},\ |x| \leq 1.$$

**Proof.** Since the function $\xi \in (0, +\infty) \to K(t, \xi^{-1}x)\xi^p$ is nondecreasing, the proof is easy to be completed.

**Lemma 3.2.** *Suppose that (H1) or (H1)′ holds. Then for any $u \in E_k$,*

$$\eta_k^p(u) \geq \min\{\|u\|_{E_k}^p, paC_0^{\gamma-p}\|u\|_{E_k}^\gamma\}, \quad \forall k \in \mathbb{N}.$$

**Proof.** It follows from (2.7), (H1) or (H1)′ and $\gamma \leq p$ that for any $u \in E_k$,

$$\begin{aligned}
\eta_k^p(u) &= \int_{-kT}^{kT} [|\dot{u}(t)|^p + pK(t, u(t))]\, dt \\
&\geq \int_{-kT}^{kT} [|\dot{u}(t)|^p + pa|u(t)|^\gamma]\, dt \\
&\geq \int_{-kT}^{kT} \left[|\dot{u}(t)|^p + pa\|u\|_{L_{2kT}^\infty}^{\gamma-p}|u(t)|^p\right] dt \\
&\geq \int_{-kT}^{kT} |\dot{u}(t)|^p dt + pa(C_0\|u\|_{E_k})^{\gamma-p} \int_{-kT}^{kT} |u(t)|^p dt
\end{aligned}$$

$$\begin{aligned} &\geq \min\{1, pa(C_0\|u\|_{E_k})^{\gamma-p}\}\|u\|_{E_k}^p \\ &= \min\{\|u\|_{E_k}^p, paC_0^{\gamma-p}\|u\|_{E_k}^\gamma\}. \end{aligned}$$

**Proof of Theorem 1.1.** We divide the proof into the following Lemma 3.3–Lemma 3.5.

**Lemma 3.3.** *Under the assumptions of Theorem 1.1, for every $k \in \mathbb{N}$, system (2.1) has a nontrivial solution $u_k$ in $E_k$.*

**Proof.** We first construct $A$ and $B$ which satisfy assumptions in Theorem 2.1.

(i) when $r \in (0, 1]$, by Corollary 2.1, (H1), (H3)(i), Hölder inequality and $\gamma < p$, for $u \in E_k$ with $\|u\|_{E_k} = r/C_0$, we have

$$\begin{aligned} \varphi_k(u) &\geq \frac{1}{p}\eta_k^p(u) - b\int_{-kT}^{kT} |u(t)|^p dt - \left(\int_{-kT}^{kT} |f(t)|^q dt\right)^{1/q} \left(\int_{-kT}^{kT} |u(t)|^p dt\right)^{1/p} \\ &\geq \frac{1}{p}\int_{-kT}^{kT} [|\dot{u}(t)|^p + pa|u(t)|^\gamma]\, dt - b\int_{-kT}^{kT} |u(t)|^p dt \\ &\quad - \left(\int_{-kT}^{kT} |f(t)|^q dt\right)^{1/q} \left(\int_{-kT}^{kT} |u(t)|^p dt\right)^{1/p} \\ &\geq \frac{1}{p}\int_{-kT}^{kT} |\dot{u}(t)|^p dt + a(C_0\|u\|_{E_k})^{\gamma-p}\int_{-kT}^{kT} |u(t)|^p dt - b\int_{-kT}^{kT} |u(t)|^p dt \\ &\quad -\|f\|_{L^q(\mathbb{R};\mathbb{R}^N)}\|u\|_{E_k} \\ &\geq \min\left\{\frac{1}{p}, ar^{\gamma-p} - b\right\}\|u\|_{E_k}^p - \|f\|_{L^q(\mathbb{R};\mathbb{R}^N)}\|u\|_{E_k} \\ &\geq \min\left\{\frac{1}{p}, a - b\right\}\|u\|_{E_k}^p - \|f\|_{L^q(\mathbb{R};\mathbb{R}^N)}\|u\|_{E_k}. \end{aligned} \tag{3.1}$$

(H6)(i) implies that there exists $\alpha > 0$ such that

$$\varphi_k(u) \geq \alpha > 0, \quad \text{for all } u \in E_k \text{ with } \|u\|_{E_k} = \frac{r}{C_0}, \quad \forall k \in \mathbb{N}.$$

(ii) when $r \in (1, +\infty)$, by Corollary 2.1, (H1), Hölder's inequality and $\gamma < p$, for $u \in E_k$ with $\|u\|_{E_k} = r/C_0$, we have

$$\begin{aligned} \varphi_k(u) &\geq \frac{1}{p}\int_{-kT}^{kT} |\dot{u}(t)|^p dt + a(C_0\|u\|_{E_k})^{\gamma-p}\int_{-kT}^{kT} |u(t)|^p dt - b\int_{-kT}^{kT} |u(t)|^p dt \\ &\quad -\|f\|_{L^q(\mathbb{R};\mathbb{R}^N)}\|u\|_{E_k} \\ &\geq \min\left\{\frac{1}{p}, ar^{\gamma-p} - b\right\}\|u\|_{E_k}^p - \|f\|_{L^q(\mathbb{R};\mathbb{R}^N)}\|u\|_{E_k}. \end{aligned} \tag{3.2}$$

(H6)(ii) implies that there exists $\alpha > 0$ such that

$$\varphi_k(u) \geq \alpha > 0, \quad \text{for all } u \in E_{kT} \text{ with } \|u\|_{E_k} = \frac{r}{C_0}, \quad \forall k \in \mathbb{N}.$$

By Lemma 3.1 and the periodicity of $K$, there exists a constant $B_0 > 0$ such that

$$K(t,x) \le A_0|x|^p + B_0, \quad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{3.3}$$

where

$$A_0 = \max_{|x|=1, t\in[0,T]} K(t,x).$$

By (H4), we know that there exist $\varepsilon_0 > 0$ and $L > 0$ such that

$$W(t,x) \ge \left(\frac{\pi^p}{pT^p} + A_0 + \varepsilon_0\right)|x|^p, \quad \text{for all } t\in\mathbb{R} \text{ and } \quad \forall |x| \ge L. \tag{3.4}$$

By (3.4) and the periodicity of $W$, there exists a constant $B_1 > 0$ such that

$$W(t,x) \ge \left(\frac{\pi^p}{pT^p} + A_0 + \varepsilon_0\right)|x|^p - B_1, \quad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{3.5}$$

Define $w_k \in E_k$ by

$$w_k(t) = \begin{cases} (|\sin\frac{\pi}{T}t|, 0, \ldots, 0) & \text{if } t\in[-T,T] \\ 0 & \text{if } t \in [-kT,kT]/[-T,T]. \end{cases}$$

Since $K(t,0)\equiv 0$ and $W(t,0)\equiv 0$ which is implied by (H5), we have $\varphi_k(\xi w_k) = \varphi_1(\xi w_1)$ for all $\xi\in\mathbb{R}$. Then by (3.5), we have

$$\begin{aligned}
\varphi_k(\xi w_k) &= \varphi_1(\xi w_1) \\
&= \int_{-T}^{T}\left[\frac{1}{p}|\xi\dot{w}_1(t)|^p + K(t,\xi w_1(t)) - W(t,\xi w_1(t))\right]dt + \int_{-T}^{T}(f_1(t),\xi w_1(t))dt \\
&\le \frac{|\xi|^p\pi^p}{pT^p}\int_{-T}^{T}|\cos\frac{\pi}{T}t|^p dt + A_0|\xi|^p\int_{-T}^{T}|\sin\frac{\pi}{T}t|^p dt + 2TB_0 \\
&\quad -\left(\frac{\pi^p}{pT^p} + A_0 + \varepsilon_0\right)|\xi|^p\int_{-T}^{T}|\sin\frac{\pi}{T}t|^p dt + 2TB_1 \\
&\quad +|\xi|\left(\int_{-T}^{T}|f_1(t)|^q dt\right)^{\frac{1}{q}}\left(\int_{-T}^{T}|\sin\frac{\pi}{T}t|^p dt\right)^{\frac{1}{p}} \\
&= -\varepsilon_0|\xi|^p\int_{-T}^{T}|\cos\frac{\pi}{T}t|^p dt + 2TB_0 \\
&\quad +2TB_1 + |\xi|\left(\int_{-T}^{T}|f_1(t)|^q dt\right)^{\frac{1}{q}}\left(\int_{-T}^{T}|\sin\frac{\pi}{T}t|^p dt\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.6}$$

So there exists $\xi_0\in\mathbb{R}$ such that $\|\xi_0 w_k\| > \frac{r}{C_0}$ and $\varphi(\xi_0 w_k) < 0$. Moreover, it is clear that $\varphi_k(0) = 0$. Let $e_1 = \xi_0 w_k$ and

$$A = \{0, e_1\}, \quad B = \{u \in E_k : \|u\| < \frac{r}{C_0}\}.$$

Then $0 \in B$ and $e_1 \notin \bar{B}$. So by Example 1 in Section 2, we know that $A$ links $\partial B$ [hm]. So by Theorem 2.1 and Remark 2.4, we have

$$c_k = \inf_{\Gamma \in \Phi} \sup_{\substack{s \in [0,1] \\ u \in A}} \varphi_k(\Gamma(s)u) \geq \inf_{\partial B} \varphi_k > \alpha > 0, \tag{3.7}$$

and there exists a sequence $\{u_n\} \subset E_k$ such that

$$\varphi_k(u_n) \to c_k, \quad (1 + \|u_n\|)\, \|\varphi_k'(u_n)\| \to 0.$$

Then there exists a constant $C_{1k} > 0$ such that

$$|\varphi_k(u_n)| \leq C_{1k}, \quad (1 + \|u_n\|)\, \|\varphi_k'(u_n)\| \leq C_{1k} \text{ for all } n \in \mathbb{N}. \tag{3.8}$$

It follows from (H5) and the periodicity and continuity of $W$ that

$$[(\nabla W(t,x), x) - pW(t,x)](\zeta + \eta |x|^\nu) \geq W(t,x), \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{3.9}$$

So by (3.5), there exists $C_2 > 0$ such that

$$\begin{aligned} [(\nabla W(t,x), x) - pW(t,x)] &\geq \frac{W(t,x)}{\zeta + \eta |x|^\nu} \\ &\geq \frac{\left(\frac{\pi^p}{pT^p} + A_0 + \varepsilon_0\right)|x|^p - B_1}{\zeta + \eta |x|^\nu} \\ &\geq \frac{\frac{\pi^p}{pT^p} + A_0 + \varepsilon_0}{\eta} |x|^{p-\nu} - C_2, \forall\ x \in \mathbb{R}^N. \end{aligned} \tag{3.10}$$

Hence, it follows from (H2), (3.8) and (3.10) that

$$\begin{aligned} & pC_{1k} + C_{1k} \\ \geq\ & p\varphi_k(u_n) - \langle \varphi_k'(u_n), u_n \rangle \\ \geq\ & \int_{-kT}^{kT} [(\nabla W(t, u_n(t)), u_n(t)) - pW(t, u_n(t))] dt \\ & + (p-1) \int_{-kT}^{kT} (f(t), u_n(t)) dt \qquad (3.11) \\ \geq\ & \left( \frac{\frac{\pi^p}{pT^p} + A_0 + \varepsilon_0}{\eta} \right) \int_{-kT}^{kT} |u_n(t)|^{p-\nu} dt \\ & - (p-1) \int_{-kT}^{kT} |f(t)||u_n(t)| dt - 2kTC_2 \\ \geq\ & \left( \frac{\pi^p}{p\eta T^p} + \frac{A_0}{\eta} + \frac{\varepsilon_0}{\eta} \right) \int_{-kT}^{kT} |u_n(t)|^{p-\nu} dt - 2kTC_2 \\ & - (p-1) \left( \int_{-kT}^{kT} |f(t)|^{\frac{p-\nu}{p-\nu-1}} dt \right)^{\frac{p-\nu-1}{p-\nu}} \left( \int_{-kT}^{kT} |u_n(t)|^{p-\nu} dt \right)^{1/(p-\nu)}. \qquad (3.12) \end{aligned}$$

The fact $p-\nu>1$ and the above inequality show that $\int_{-kT}^{kT}|u_n(t)|^{p-\nu}dt$ is bounded. It follows from (H5) that

$$[(\nabla W(t,x),x)-pW(t,x)](\zeta+\eta|x|^{\nu})\geq W(t,x)\geq 0. \tag{3.13}$$

By (H1), (H6), (3.8), (3.11), (3.13), Hölder's inequality and (2.12), there exist $C_5>0$ and $C_6>0$ such that

$$\begin{aligned}
&\frac{1}{p}\|u_n\|_{E_k}^p\\
= {}& \varphi_k(u_n)-\int_{-kT}^{kT}K(t,u_n(t))dt+\int_{-kT}^{kT}W(t,u_n(t))dt+\frac{1}{p}\int_{-kT}^{kT}|u_n(t)|^p dt\\
&-\int_{-kT}^{kT}(f(t),u_n(t))dt\\
\leq {}& \varphi_k(u_n)+\int_{-kT}^{kT}[(\nabla W(t,u_n(t)),u_n(t))-pW(t,u_n(t))](\zeta+\eta|u_n(t)|^{\nu})dt\\
&+\frac{1}{p}\int_{-kT}^{kT}|u_n(t)|^p dt+\left(\int_{-kT}^{kT}|u_n(t)|^p\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{\frac{1}{q}}\\
\leq {}& C_{1k}+\frac{1}{p}\int_{-kT}^{kT}|u_n(t)|^p dt+\|u_n\|_{E_k}\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{\frac{1}{q}}\\
&+(\zeta+\eta\|u_n\|_{L_{2kT}^\infty}^{\nu})\int_{-kT}^{kT}[(\nabla W(t,u_n(t)),u_n(t))-pW(t,u_n(t))]dt\\
\leq {}& C_{1k}+\frac{1}{p}\|u_n\|_{L_{2kT}^\infty}^{\nu}\int_{-kT}^{kT}|u_n(t)|^{p-\nu}dt+\|u_n\|_{E_k}\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{\frac{1}{q}}\\
&+(\zeta+\eta\|u_n\|_{L_{2kT}^\infty}^{\nu})\left[(p+1)C_{1k}+(p-1)\|u_n\|_{E_k}\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{\frac{1}{q}}\right]\\
\leq {}& C_{1k}+\frac{C_0^{\nu}}{p}\|u_n\|_{E_k}^{\nu}\int_{-kT}^{kT}|u_n(t)|^{p-\nu}dt+\|u_n\|_{E_k}\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{\frac{1}{q}}\\
&+(\zeta+\eta C_0^{\nu}\|u_n\|_{E_k}^{\nu})\left[(p+1)C_{1k}+(p-1)\|u_n\|_{E_k}\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.14}$$

Since $\nu<\gamma-1<p-1$, (3.14) implies that $\|u_n\|_{E_k}$ is bounded. Similar to the argument of Lemma 2 in [10], next we prove that in $E_k$, $\{u_n\}$ has a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n\to u_k$, as $n\to\infty$. Since $W_{2kT}^{1,p}$ is a reflexive Banach space, then there is a renamed subsequence $\{u_n\}$ such that

$$u_n\rightharpoonup u_k \ \text{ weakly in } W_{2kT}^{1,p}. \tag{3.15}$$

Furthermore, by Proposition 1.2 in [4], we have

$$u_n \to u_k \ \text{ strongly in } \ C([-kT, kT], \mathbb{R}^N). \tag{3.16}$$

Note that

$$\begin{aligned}
&\langle {\varphi_k}'(u_n), u_n - u_k\rangle \\
= \ &\int_{-kT}^{kT} (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_k(t))dt + \int_{-kT}^{kT} (\nabla K(t, u_n(t)), u_n(t) - u_k(t))dt \\
&- \int_{-kT}^{kT} (\nabla W(t, u_n(t)), u_n(t) - u_k(t))dt + \int_{-kT}^{kT} (f_k(t), u_n(t) - u_k(t))dt
\end{aligned} \tag{3.17}$$

Since $\{\|u_n\|\}$ is bounded and ${\varphi_k}'(u_n) \to 0$, we have

$$\langle {\varphi_k}'(u_n), u_n - u_k\rangle \to 0 \quad \text{ as } \ n \to \infty. \tag{3.18}$$

By assumption (V) and (3.16), we have

$$\int_{-kT}^{kT} \big(\nabla K(t, u_n(t)), u_n(t) - u_k(t)\big)dt \to 0 \quad \text{ as } n \to \infty \tag{3.19}$$

and

$$\int_{-kT}^{kT} \big(\nabla W(t, u_n(t)), u_n(t) - u_k(t)\big)dt \to 0 \quad \text{ as } n \to \infty. \tag{3.20}$$

Since $f_k(t)$ is bounded, (3.16) also implies that

$$\int_{-kT}^{kT} (f_k(t), u_n(t) - u_k(t))dt \to 0 \quad \text{ as } n \to \infty. \tag{3.21}$$

Hence, it follows from (3.18), (3.19), (3.20) and (3.21) that

$$\int_{-kT}^{kT} (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_k(t))dt \to 0 \quad \text{ as } \quad n \to \infty. \tag{3.22}$$

On the other hand, it is easy to derive from (3.16) and the boundedness of $\{u_n\}$ that

$$\int_{-kT}^{kT} (|u_n(t)|^{p-2}u_n(t), u_n(t) - u_k(t))dt \to 0 \quad \text{ as } \quad n \to \infty. \tag{3.23}$$

Set

$$\psi_k(u_k) = \frac{1}{p}\left(\int_{-kT}^{kT} |u_k(t)|^p dt + \int_{-kT}^{kT} |\dot{u}_k(t)|^p dt\right).$$

Then we have

$$\begin{aligned}
\langle \psi_k'(u_n), u_n - u_k\rangle \ = \ &\int_{-kT}^{kT} (|u_n(t)|^{p-2}u_n(t), u_n(t) - u_k(t))dt \\
&+ \int_{-kT}^{kT} (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_k(t))dt,
\end{aligned} \tag{3.24}$$

and

$$\langle \psi_k'(u_k), u_n - u_k \rangle = \int_{-kT}^{kT} (|u_k(t)|^{p-2} u_k(t), u_n(t) - u_k(t)) dt + \int_{-kT}^{kT} (|\dot{u}_k(t)|^{p-2} \dot{u}_k(t), \dot{u}_n(t) - \dot{u}_k(t)) dt. \tag{3.25}$$

From (3.22) and (3.23), we obtain

$$\langle \psi_k'(u_n), u_n - u_k \rangle \to 0 \qquad \text{as} \quad n \to \infty. \tag{3.26}$$

On the other hand, it follows from (3.15) that

$$\langle \psi_k'(u_k), u_n - u_k \rangle \to 0 \qquad \text{as} \quad n \to \infty. \tag{3.27}$$

By (3.24), (3.25) and the Hölder's inequality, we get

$$\begin{aligned}
& \langle \psi_k'(u_n) - \psi_k'(u_k), u_n - u_k \rangle \\
= & \int_{-kT}^{kT} (|u_n(t)|^{p-2} u_n(t), u_n(t) - u_k(t)) dt + \int_{-kT}^{kT} (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}_n(t) - \dot{u}_k(t)) dt \\
& - \int_{-kT}^{kT} (|u_k(t)|^{p-2} u_k(t), u_n(t) - u_k(t)) dt - \int_{-kT}^{kT} (|\dot{u}_k(t)|^{p-2} \dot{u}_k(t), \dot{u}_n(t) - \dot{u}_k(t)) dt \\
= & \|u_n\|_{E_k}^p + \|u_k\|_{E_k}^p - \int_{-kT}^{kT} (|u_n(t)|^{p-2} u_n(t), u_k(t)) dt - \int_{-kT}^{kT} (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}_k(t)) dt \\
& - \int_{-kT}^{kT} (|u_k(t)|^{p-2} u_k(t), u_n(t)) dt - \int_{-kT}^{kT} (|\dot{u}_k(t)|^{p-2} \dot{u}_k(t), \dot{u}_n(t)) dt \\
\geq & \|u_n\|_{E_k}^p + \|u_k\|_{E_k}^p - \left( \|u_n\|_{L_{2kT}^p}^{p-1} \|u_k\|_{L_{2kT}^p} + \|\dot{u}_n\|_{L_{2kT}^p}^{p-1} \|\dot{u}_k\|_{L_{2kT}^p} \right) \\
& - \left( \|u_k\|_{L_{2kT}^p}^{p-1} \|u_n\|_{L_{2kT}^p} + \|\dot{u}_k\|_{L_{2kT}^p}^{p-1} \|\dot{u}_n\|_{L_{2kT}^p} \right) \\
\geq & \|u_n\|_{E_k}^p + \|u_k\|_{E_k}^p - \left( \|u_k\|_{L_{2kT}^p}^p + \|\dot{u}_k\|_{L_{2kT}^p}^p \right)^{1/p} \left( \|u_n\|_{L_{2kT}^p}^p + \|\dot{u}_n\|_{L_{2kT}^p}^p \right)^{1/q} \\
& - \left( \|u_n\|_{L_{2kT}^p}^p + \|\dot{u}_n\|_{L_{2kT}^p}^p \right)^{1/p} \left( \|u_k\|_{L_{2kT}^p}^p + \|\dot{u}_k\|_{L_{2kT}^p}^p \right)^{1/q} \\
= & \|u_n\|_{E_k}^p + \|u_k\|_{E_k}^p - \|u_k\|_{E_k} \, \|u_n\|_{E_k}^{p-1} - \|u_n\|_{E_k} \, \|u_k\|_{E_k}^{p-1} \\
= & \left( \|u_n\|_{E_k}^{p-1} - \|u_k\|_{E_k}^{p-1} \right) \left( \|u_n\|_{E_k} - \|u_k\|_{E_k} \right).
\end{aligned}$$

It follows that

$$0 \leq \left( \|u_n\|_{E_k}^{p-1} - \|u_k\|_{E_k}^{p-1} \right) \left( \|u_n\|_{E_k} - \|u_k\|_{E_k} \right) \leq \langle \psi'(u_n) - \psi'(u_k), u_n - u_k \rangle,$$

which, together with (3.26) and (3.27) yields $\|u_n\|_{E_k} \to \|u_k\|_{E_k}$ (see [10]). By the uniform convexity of $E_k$ and (3.15), it follows from the Kadec–Klee property (see [27]) that $\|u_n -$

$u_k\|_{E_k} \to 0$. Moreover, by the continuity of $\varphi_k$ and $\varphi_k'$ , we obtain $\varphi_k'(u_k) = 0$ and $\varphi_k(u_k) = c_k > 0$. It is clear that $u_k \neq 0$ and so $u_k$ is a desired nontrivial solution of system (2.1). The proof is complete.

**Lemma 3.4.** *Let $\{u_k\}_{k\in\mathbb{N}}$ be the solution of system* (2.1). *Then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in\mathbb{N}}$ convergent to a certain function $u_0 \in C^1(\mathbb{R}, \mathbb{R}^N)$ in $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$.*

**Proof.** First, we prove that the sequence $\{c_k\}_{k\in\mathbb{N}}$ is bounded and the sequence $\{u_k\}_{k\in\mathbb{N}}$ is uniformly bounded. Second, we prove $\{\dot{u}_k\}_{k\in\mathbb{N}}$ is also uniformly bounded. Finally, we prove both $\{u_k\}$ and $\{\dot{u}_k\}$ are equicontinuous and then by using the Arzelà–Ascoli Theorem, we obtain the conclusion. We only prove the first step. The rest of proof is the same as Lemma 3.2 in [17]. For every $k \in \mathbb{N}$, define $\Gamma_k : [0,1] \times E_k \to E_k$ by

$$\Gamma_k(s)v = (1-s)v, \quad v \in E_k.$$

Then $\Gamma \in \Phi$. Note that set $A = \{0, e_1\}$. So (3.7) implies that

$$\varphi_k(u_k) = c_k \leq \sup_{\substack{s\in[0,1]\\ u\in A}} \varphi_k((1-s)u) = \sup_{s\in[0,1]} \varphi_k((1-s)e_1) = \sup_{s\in[0,1]} \varphi_1((1-s)e_1) := M_0,$$

where $M_0$ is independent of $k \in \mathbb{N}$. Moreover, $\varphi_k'(u_k) = 0$. Then it follows from (H2) and (3.10) that

$$\begin{aligned} pM_0 \geq pc_k &= p\varphi_k(u_k) - \langle \varphi_k'(u_k), u_k\rangle \\ &\geq \int_{-kT}^{kT} [(\nabla W(t, u_k(t)), u_k(t)) - pW(t, u_k(t))]dt \\ &\quad +(p-1)\int_{-kT}^{kT} (f(t), u_k(t))dt \\ &\geq \int_{-kT}^{kT} \frac{W(t,u_k(t))}{\xi + \eta|u_k(t)|^\nu}dt + (p-1)\int_{-kT}^{kT}(f(t), u_k(t))dt. \end{aligned}$$

So

$$\int_{-kT}^{kT} \frac{W(t,u_k(t))}{\xi + \eta|u_k(t)|^\nu}dt \leq pM_0 - (p-1)\int_{-kT}^{kT}(f(t), u_k(t))dt.$$

Then

$$\begin{aligned} \eta_k^p(u_k) &= p\varphi_k(u_k) + p\int_{-kT}^{kT} \frac{W(t,u_k(t))}{\xi + \eta|u_k(t)|^\nu}(\xi + \eta|u_k(t)|^\nu)dt - p\int_{-kT}^{kT}(f(t), u_k(t))dt \\ &\leq p\varphi_k(u_k) + p(\xi + \eta\|u_k\|_\infty^\nu)\int_{-kT}^{kT} \frac{W(t,u_k(t))}{\xi + \eta|u_k(t)|^\nu}dt - p\int_{-kT}^{kT}(f(t), u_k(t))dt \end{aligned}$$

$$
\begin{aligned}
&\leq \; p\varphi_k(u_k) + p(\xi + \eta C_0\|u_k\|_{E_k}^{\nu})\left(pM_0 - (p-1)\int_{-kT}^{kT}(f(t),u_k(t))dt\right) \\
&\quad -p\int_{-kT}^{kT}(f(t),u_k(t))dt \\
&\leq \; pM_0 + p^2\xi M_0 + p^2\eta C_0 M_0\|u_k\|_{E_k}^{\nu} - p(p-1)\xi\int_{-kT}^{kT}(f(t),u_k(t))dt \\
&\quad -p(p-1)\eta C_0\|u_k\|_{E_k}^{\nu}\int_{-kT}^{kT}(f(t),u_k(t))dt - p\int_{-kT}^{kT}(f(t),u_k(t))dt \\
&\leq \; (p+p^2\xi)M_0 + [p(p-1)\xi + p]\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{1/q}\left(\int_{-kT}^{kT}|u_k(t)|^p dt\right)^{1/p} \\
&\quad +p^2\eta C_0 M_0\|u_k\|_{E_k}^{\nu} + p(p-1)\eta C_0\|u_k\|_{E_k}^{\nu}\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{1/q}\left(\int_{-kT}^{kT}|u_k(t)|^p dt\right)^{1/p} \\
&\leq \; (p+p^2\xi)M_0 + [p(p-1)\xi + p]\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{1/q}\|u_k\|_{E_k} + p^2\eta C_0 M_0\|u_k\|_{E_k}^{\nu} \\
&\quad +p(p-1)\eta C_0\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{1/q}\|u_k\|_{E_k}^{\nu+1}. \qquad (3.28)
\end{aligned}
$$

Thus (3.28) and Lemma 3.2 imply that

$$
\begin{aligned}
&(p+p^2\xi)M_0 + [p(p-1)\xi + 1]\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{1/q}\|u_k\|_{E_k} + p^2\eta C_0 M_0\|u_k\|_{E_k}^{\nu} \\
&+p(p-1)\eta C_0\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{1/q}\|u_k\|_{E_k}^{\nu+1} \\
\geq \; &\min\{\|u_k\|_{E_k}^{p}, paC_0^{\gamma-p}\|u_k\|_{E_k}^{\gamma}\}.
\end{aligned}
$$

Note that $\gamma > \nu + 1$. So (H6) implies there exists $M_1 > 0$ (independent of $k$) such that

$$\|u_k\|_{E_k} \leq M_1 \;\text{ for every } k \in \mathbb{N}.$$

By Corollary 2.1,

$$\|u_k\|_{L_{2kT}^{\infty}} \leq C_0 M_1 := M_2 \;\text{ for every } k \in \mathbb{N}.$$

Thus the proof is complete.

**Lemma 3.5.** *Let $u_0 \in C^1(\mathbb{R}, \mathbb{R}^N)$ be determined by Lemma 3.4. When $f \neq 0$, $u_0$ is a nontrivial solution of system (1.1) such that $u_0(t) \to 0$ and $\dot{u}_0(t) \to 0$ as $t \to \pm\infty$.*

**Proof.** The proof is the same as Step 1–Step 3 in the proof of Lemma 3.3 in [17].

**Proof of Theorem 1.2.** The proof is easy to be completed by replacing

$$\int_{-kT}^{kT}(f(t),u(t))dt \leq \left(\int_{-kT}^{kT}|f(t)|^q dt\right)^{1/q}\left(\int_{-kT}^{kT}|u(t)|^p dt\right)^{1/p} \leq \|u\|_{E_k}\left(\int_{\mathbb{R}}|f(t)|^q dt\right)^{1/q}$$

with

$$\int_{-kT}^{kT}(f(t),u(t))dt \leq \|u\|_{L^\infty_{2kT}}\int_{-kT}^{kT}|f(t)|dt \leq C_0\|u\|_{E_k}\int_{\mathbb{R}}|f(t)|dt,$$

in the proofs of Lemma 3.3 and Lemma 3.4.

**Proofs of Theorem 1.3 and Theorem 1.4.** We only note that in the proof of Lemma 3.3, when $\gamma = p$, we dot not need $r \in (0,1]$ and it is sufficient that $r > 0$. The remaining parts of the proofs are the same as the proofs of Theorem 1.1 and Theorem 1.2, respectively.

**Proof of Theorem 1.5.** Note that $f \equiv 0$. By (H1), (H3)$''$ and $\gamma < p$, for $u \in E_k$ with $\|u\|_{E_k} = r/C_0$, we have

$$\begin{aligned}
\varphi_k(u) &\geq \frac{1}{p}\eta_k^p(u) - b\int_{-kT}^{kT}|u(t)|^p dt \\
&\geq \frac{1}{p}\int_{-kT}^{kT}\left[|\dot{u}(t)|^p + pa|u(t)|^\gamma\right]dt - b\int_{-kT}^{kT}|u(t)|^p dt \\
&\geq \frac{1}{p}\int_{-kT}^{kT}|\dot{u}(t)|^p dt + a(C_0\|u\|_{E_k})^{\gamma-p}\int_{-kT}^{kT}|u(t)|^p dt - b\int_{-kT}^{kT}|u(t)|^p dt \\
&\geq \min\left\{\frac{1}{p}, ar^{\gamma-p} - b\right\}\frac{r^p}{C_0^p}.
\end{aligned}$$

So (H3)$''$ implies that there exists $\alpha > 0$ such that

$$\varphi_k(u) \geq \alpha > 0, \quad \text{for all } u \in E_k \text{ with } \|u\|_{E_k} = \frac{r}{C_0}, \quad \forall k \in \mathbb{N}.$$

(H5)$'$ implies that $W(t,0) \equiv 0$ and (H2)$'$ implies that (H2). So (3.6) holds with $f_1(t) \equiv 0$. Hence there exists $\xi_0 \in \mathbb{R}$ such that $\|\xi_0 w_k\| > \frac{r}{C_0}$ and $\varphi(\xi_0 w_k) < 0$. Moreover, it is clear that $\varphi_k(0) = 0$. Let $e_1 = \xi_0 w_k$ and

$$A = \{0, e_1\}, \quad B = \{u \in E_k : \|u\| < \frac{r}{C_0}\}$$

Then $0 \in B$ and $e_1 \notin \bar{B}$. So by Example 1 in Section 2, we know that $A$ links $\partial B$ [hm]. So by Theorem 2.1 and Remark 2.4,

$$c_k = \inf_{\Gamma\in\Phi}\sup_{\substack{s\in[0,1]\\ u\in A}}\varphi_k(\Gamma(s)u) \geq \inf_{\partial B}\varphi_k > \alpha > 0,$$

and there exists a sequence $\{u_n\} \subset E_k$ such that

$$\varphi_k(u_n) \to c_k, \quad (1 + \|u_n\|)\,\|\varphi_k'(u_n)\| \to 0,$$

Then there exists a constant $C_{1k} > 0$ such that

$$|\varphi_k(u_n)| \le C_{1k}, \quad (1 + \|u_n\|)\,\|\varphi_k'(u_n)\| \le C_{1k} \text{ for all } n \in \mathbb{N}.$$

Similar to the argument in Lemma 3.3 and Lemma 3.4 with $f(t) \equiv 0$, noting that it is sufficient $\nu < \gamma < p$ when $f \equiv 0$, we can obtain that $u_k$ is a desired nontrivial solution of system (2.1). By the Step 1–Step 3 in the proof of Lemma 3.3 in [17], we obtain that $u_0(t) \to 0$ and $\dot{u}_0(t) \to 0$ as $t \to \pm\infty$. Next, we prove, when $f \equiv 0$, $u_0$ is nontrivial. The proof is the similar to that in [18] and same as step 4 in the proof of Lemma 3.3 in [17] (with $\gamma = p$ and $b = a$ there). Here, for readers' convenience, we also present it. It is easy to see that the function $Y$ defined in (H7) is continuous, nondecreasing, $Y(s) \ge Y(0) \ge 0$. By the definition of $Y$, we have

$$(\nabla W(t, u_k(t)), u_k(t)) \le Y(\|u_k\|_{L^\infty_{2kT}})|u_k(t)|^p.$$

Integrating the above inequality on the interval $[-kT, kT]$, we obtain that for every $k \in \mathbb{N}$,

$$\int_{-kT}^{kT} (\nabla W(t, u_k(t)), u_k(t))dt \le Y(\|u_k\|_{L^\infty_{2kT}})\|u_k\|^p_{E_k}. \tag{3.29}$$

Note that $(\varphi_k'(u_k), u_k) = 0$. Hence,

$$\int_{-kT}^{kT} (\nabla W(t, u_k(t)), u_k(t))dt = \int_{-kT}^{kT} |\dot{u}_k(t)|^p dt + \int_{-kT}^{kT} (\nabla K(t, u_k(t)), u_k(t))dt. \tag{3.30}$$

By (3.29), (3.30), (H1)$'$ and (H2)$'$, we obtain that

$$Y(\|u_k\|_{L^\infty_{2kT}})\|u_k\|^p_{E_k} \ge \min\{1, a\}\|u_k\|^p_{E_k}.$$

Then

$$Y(\|u_k\|_{L^\infty_{2kT}}) \ge \min\{1, a\}.$$

The remainder of the proof is the same as in [7, 11, 17, 18]. If $\|u_k\|_{L^\infty_{2kT}} \to 0$ as $k \to \infty$, we would have $Y(0) \ge \min\{1, a\}$, a contradiction to (H7). Thus there is $m > 0$, which is independent of $k$, such that

$$\|u_k\|_{L^\infty_{2kT}} \ge m \tag{3.31}$$

for every $k \in \mathbb{N}$. Now to complete the proof, observe that by the $T$-periodicity of $V$ and $f \equiv 0$, whenever $u_k(t)$ is a $2kT$-periodic solution of system (2.1), so is $u_k(t+jT)$ for every

$j \in \mathbb{Z}$. Hence, by replacing earlier, if necessary, $u_k$ by $u_k(t+jT)$ for some $j \in [-k,k] \cap \mathbb{Z}$, one can assume that the maximum of $u_k$ occurs in $[-T,T]$. Suppose, contrary to our claim, that $u_0 \equiv 0$. Then by Lemma 3.4,

$$\|u_{k_j}\|_{L^\infty_{2k_jT}} = \max_{t\in[-T,T]} |u_{k_j}(t)| \to 0, \quad \text{as} \quad j \to \infty.$$

which contradicts (3.31).

**Proof of Theorem 1.6.** Similar to the argument of Lemma 3.3 and Lemma 3.4, it is easy to obtain that, under the conditions of Theorem 1.6, $u_k$ is a desired nontrivial solution of system (2.1). Then by the proof of Theorem 1.5, we know that $u_0$ is nontrivial.

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