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The limit of vanishing viscosity for doubly nonlinear parabolic equations

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Abstract. We show that solutions of the doubly nonlinear parabolic equation

$$\frac{\partial b(u)}{\partial t} - \epsilon \operatorname{div}(a(\nabla u)) + \operatorname{div}(f(u)) = g$$

converge in the limit $\epsilon \searrow 0$ of vanishing viscosity to an entropy solution of the doubly nonlinear hyperbolic equation

$$\frac{\partial b(u)}{\partial t} + \operatorname{div}(f(u)) = g\,.$$

The difficulty here lies in the fact that the functions a and b specifying the diffusion are nonlinear.

Keywords: vanishing viscosity, doubly nonlinear evolution equations, conservation law, quasilinear, degenerate.

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1 Introduction

An m-dimensional doubly nonlinear hyperbolic system of first order on a domain $\Omega \subset \mathbb{R}^n$ has the form

$$\frac{\partial b(u)}{\partial t} + \operatorname{div}(f(u)) = g\,, \tag{1.1}$$

where $b = d\phi_b \colon \mathbb{R}^m \to \mathbb{R}^m$ is the derivative of a convex C^1-function $\phi_b : \mathbb{R}^m \to \mathbb{R}$, the flux $f \colon \mathbb{R}^m \to \mathbb{R}^m \otimes \mathbb{R}^n$ is a C^1-function and g is an inhomogeneity or nonlinearity. As stated, (1.1) is merely a balance law and should be accompanied with conditions on b and f which guarantee hyperbolicity, but we are also interested in the case where hyperbolicity is violated, especially due to singularity or degeneracy of b. The focus of this article lies on the case $g = 0$, where (1.1) is called a conservation law.

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Equation (1.1) may be viewed as a limit of the doubly nonlinear parabolic system

$$\frac{\partial b(u)}{\partial t} - \epsilon \operatorname{div}(a(\nabla u)) + \operatorname{div}(f(u)) = g \tag{1.2}$$

for $\epsilon \searrow 0$, where the viscosity tensor $\epsilon a\colon \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^m \otimes \mathbb{R}^n$ converges to zero. Therefore, (1.1) is said to be the limit of vanishing viscosity of (1.2). Note that the viscosity tensor ϵa is allowed to be nonlinear, and even degenerate or singular, thus much more general physical models are covered by (1.2) than by parabolic equations with linear viscosity $a(\nabla u) = \nabla u$.

The aim of this article is to prove convergence of the solutions u_ϵ of the parabolic equation (1.2) to an entropy solution u of the hyperbolic equation (1.1) as $\epsilon \searrow 0$. To the best of our knowledge there are no articles which explicitly prove convergence in the case (1.2) of nonlinear diffusion. The scalar case with $b(u) = u$ and nonlinear diffusion is handled in [11], however, the proof of the entropy inequality [11, Proposition 3.2] is omitted. The limit of an equation with nonlinear diffusion and dispersion is studied in [4].

As equation (1.2) is often known to be a good model of a physical system, it is relevant to study the limit of vanishing viscosity for nonlinear diffusions. Therefore, it is interesting to know whether the behaviour of this system is governed by (1.1) when viscosity effects become small or are neglected. There are many articles which prove existence of solutions for (1.1) by different approaches and under more general conditions, e.g. for only continuous f and b with noncontinuous inverse b^{-1} (see [5]), or with general boundary conditions (see [15, 2]). However, note that it is not obvious how to obtain solutions of (1.1) as a limit of solutions of (1.2) in the case of nonlinear diffusion $\epsilon \operatorname{div}(a(\nabla u))$ instead of ordinary viscosity $\epsilon \Delta u$ (as can also be seen from the nontrivial proof of Theorem 1.1).

Let us emphasize that for general systems the weak solution of (1.1) selected by the limit of vanishing viscosity may depend on the nonlinear diffusion in the approximating equations. This phenomenon is taken into account by an explicit dependence of the notion of admissible entropies on the functions a and b which specify the diffusion, see Definition 3.1. A similar phenomenon is observed for equations with a discontinuous flux f, where non-equivalent notions of entropy correspond to different applications, see [3]. However, at least for a scalar equation the admissibility of entropies depends in an obvious way only on b and not on a.

Outline of the paper

The second section reviews those results about the doubly nonlinear parabolic equation (1.2) which are needed latter in the limit process $\epsilon \searrow 0$.

In the third section, we give a definition of admissible entropies and entropy fluxes, which depends on the nonlinear diffusion terms a and b. It is shown that the limit of vanishing viscosity (provided that it exists) satisfies the corresponding entropy inequalities.

Existence of the limit of vanishing viscosity as a measure-valued entropy solution of (1.1) is established in the fourth section by using Young measures. Further, it is shown how compensated compactness can be used to prove the existence of a traditional entropy solution (i.e. a solution which is a function and not a measure) in the doubly nonlinear case. A consequence is the following theorem for a one-dimensional doubly nonlinear scalar conservation law (i.e. $m = 1$, $n = 1$) on the real line $\Omega = \mathbb{R}$.

Theorem 1.1. *Let $1 < p, q < \infty$, let $b \in C(\mathbb{R})$ be strictly monotone, q-coercive, $(q-1)$-growing, and assume that b^{-1} is differentiable. Let $f \in C^1(\mathbb{R})$ be genuinely nonlinear, i.e. f' does not vanish on any open interval, let $u_0 \in L^\infty(\mathbb{R})$, and denote by u_ϵ the weak solution of* (1.2) *to the initial value u_0 in the* *case $g = 0$ for a monotone, p-coercive, $(p-1)$-growing viscosity $a \in C(\mathbb{R})$. Then for every sequence $\epsilon_k \to 0$ there exists a subsequence u_{ϵ_k} which converges weakly* in $L^\infty((0,T)\times\mathbb{R})$ and strongly in every $L^r_{loc}((0,T)\times\mathbb{R})$, $1 < r < \infty$, to a traditional entropy solution u of (1.1).*

The proof of this theorem is given in section 4.1. Note explicitly that it is allowed to use functions b which are not differentiable, e.g. a signed power $b(u) = |u|^{q-2}u$ with $1 < q < 2$. In this case the parabolic equation (1.2) is not only degenerate resp. singular at points with $\nabla u = 0$, e.g. in the case $a(\nabla u) = |\nabla u|^{p-2}\nabla u$ with $p \neq 2$, but also singular at points where $u = 0$, and the later also holds for the hyperbolic equation (1.1).

2 Doubly nonlinear diffusion equations with transport terms

Doubly nonlinear parabolic equations like (1.2) have been studied by many authors and in many articles, see e.g. [7, 1, 10]. Here we just review some of the results that allow for the limit process $\epsilon \searrow 0$ in (1.2). For simplicity we mainly discuss the case $g = 0$, where no sources or reaction terms are present. In this case (1.2) is called a doubly nonlinear diffusion equation with transport terms. Further, we restrict ourselves to the case where b depends only on u (and not on t, x) and a depends only on ∇u (and not on t, x, u, $b(u)$).

Let us first formulate standard assumptions on the functions a, b and f specifying the doubly nonlinear diffusion. Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain, let $1 < p < \infty$ and suppose that

(A1) $a\colon \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^m \otimes \mathbb{R}^n$ is continuous,

(A2) a satisfies the growth condition $|a(\xi)| \leq C|\xi|^{p-1}$ with a constant $C < \infty$,

(A3) a satisfies the coercivity condition $a(\xi)\cdot\xi \geq c|\xi|^p$ with a constant $c > 0$,

(A4) a is monotone, i.e. $(a(\xi) - a(\tilde{\xi}))\cdot(\xi - \tilde{\xi}) \geq 0$ holds for arbitrary $\xi, \tilde{\xi} \in \mathbb{R}^m \otimes \mathbb{R}^n$.

By (A1–A2) the mapping a induces a superposition operator

$$A\colon W_0^{1,p}(\Omega,\mathbb{R}^m) \to W^{-1,p'}(\Omega,\mathbb{R}^m)\,,\ \langle Au, v\rangle := \int_\Omega a(\nabla u)\cdot\nabla v\,dx\,,$$

which is coercive and monotone due to (A3–A4). Further, let $1 < q < \infty$ and assume that

(B1) $b\colon \mathbb{R}^m \to \mathbb{R}^m$ is continuous,

(B2) b satisfies the growth condition $|b(u)| \leq C|u|^{q-1}$ with a constant $C < \infty$,

(B3) b satisfies the coercivity condition $b(u)\cdot u \geq c|u|^q$ with a constant $c > 0$,

(B4) b is strictly monotone, i.e. $(b(u) - b(\tilde{u}))\cdot(u - \tilde{u}) > 0$ holds for arbitrary $u \neq \tilde{u}$,

(B5) b has a potential ϕ_b, i.e. $b(u) = d\phi_b(u)$.

By (B1–B2) b induces a superposition operator

$$B\colon L^q(\Omega,\mathbb{R}^m) \to L^{q'}(\Omega,\mathbb{R}^m)\,,\ (Bu)(x) := b(u(x))\,,$$

and by coercivity (B3) and strict monotonicity (B4) the operator B is continuously invertible. Note that the potential ϕ_b of b is strictly convex by strict monotonicity, and that (B5) is trivially satisfied for scalar equations by setting $\phi_b(u) := \int_0^u b(u)\,du$. Finally, let us assume that

(F1) $f\colon \mathbb{R}^m \to \mathbb{R}^m \otimes \mathbb{R}^n$ is continuous,

(F2) f satisfies the growth condition $|f(u)| \le C|u|^{q/p'}$ with a constant $C < \infty$.

Then f induces by (F1–F2) a superposition operator

$$F\colon L^q(\Omega, \mathbb{R}^m) \to W^{-1,p'}(\Omega, \mathbb{R}^m)\,,\ \langle Fu, v\rangle := -\int_\Omega f(u)\cdot \nabla v\, dx\,,$$

and by Hölder's and Young's inequality

$$\left|\int_\Omega f(u)\cdot \nabla u\, dx\right| \le \|u\|_q^{q/p'}\|\nabla u\|_p \le \frac{\epsilon c}{p}\|\nabla u\|_p^p + C_\epsilon \|u\|_q^q \le \frac{\epsilon c}{p}\|\nabla u\|_p^p + C_\epsilon \hat{\Phi}_B(u)$$

is valid with the constant c from (A3), a constant $C_\epsilon < \infty$ and the Legendre transform $\hat{\Phi}_B(u) = \int_\Omega \phi_b^*(b(u(x)))\, dx$ of the potential of B in dependence on u, see e.g. [16, (8.218)]. Thus the operator $\epsilon A + F$ is B-pseudomonotone (see [10]) and semicoercive, as

$$\langle(\epsilon A + F)u, u\rangle \ge \frac{\epsilon c}{p'}\|\nabla u\|_p^p - C_\epsilon \hat{\Phi}_B(u)$$

holds. Hence, under the condition $q < p^*$, which guarantees compactness of $L^q(\Omega') \cap W_0^{1,p}(\Omega')$ in $L^q(\Omega')$ for bounded subdomains $\Omega' \subset \Omega$, the abstract theory for doubly nonlinear parabolic equations with transport term

$$\frac{\partial Bu}{\partial t} + \epsilon Au + Fu = 0$$

guarantees existence of weak solutions to initial values in $L^q(\Omega)$.

Theorem 2.1. *Let $1 < p, q < \infty$, $T > 0$, $\epsilon > 0$, and assume $q < p^*$, (A1–A4), (B1–B5), (F1–F2), then to every $u_0 \in L^q(\Omega)$ there exists a weak solution u of the equation (1.2) without sources (i.e. $g = 0$). More precisely, there exists $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^q(\Omega))$ such that $Bu \in L^\infty(0, T; L^{q'}(\Omega))$ has the initial value Bu_0 and a weak derivative $\frac{\partial Bu}{\partial t} \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ satisfying $\frac{\partial Bu}{\partial t} + \epsilon Au + Fu = 0$ as an equation in $L^{p'}(0, T; W^{-1,p'}(\Omega))$.*

The basic a priori estimates to prove this theorem are derived by testing the equation with u, which yields the energy inequality

$$\frac{d}{dt}\hat{\Phi}_B(u) + \frac{\epsilon c}{p'}\|\nabla u\|_p^p \le C_\epsilon \hat{\Phi}_B(u) \tag{2.1}$$

for almost every $t \in (0, T)$ and via Gronwall's inequality bounds of u in $L^\infty(0, T; L^q(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$. Note that without the transport term the right hand side of (2.1) would vanish and especially

$$\epsilon \int_0^T \|\nabla u\|_p^p\, dt \le C \tag{2.2}$$

would hold with a constant $C < \infty$ independent of ϵ.

For our purposes the abstract setting has to be modified in two aspects: C^1-transport terms have to be handled which do not satisfy any growth condition, and solvability of (1.2) for initial values $u_0 \in L^\infty(\Omega)$ has to be assured. To do this, note that because of $\operatorname{div}(f(u)) = \operatorname{div}(f(u) - f(0))$ without loss of generality it can be assumed that the C^1-function $f\colon \mathbb{R}^m \to \mathbb{R}^m \otimes \mathbb{R}^n$ satisfies $f(0) = 0$. Further, in the hope that an L^∞-estimate can be established, let us consider a C^1-cut-off of f with a compact support such that $f(u)$ coincides with the original values for $|u| \le \|u_0\|_{L^\infty(\Omega)}$.

Then in the case $q \le p'$ there is a constant C such that the growth estimate $|f(u)| \le C|u|^{q/p'}$ of (F2) is valid for the cut-off of the original function (in the case $p' < q$ a further approximation f_ϵ of f is needed, which converges uniformly on compact subsets of Ω and satisfies the growth condition). Thus to an initial value $u_0 \in L^q(\Omega) \cap L^\infty(\Omega)$ there exists a weak solution u of (1.2) with the original transport term replaced by its cut-off.

Now at least in the case $m = 1$ of a single scalar equation, logarithmic Gagliardo–Nirenberg inequalities allow to prove an L^∞-estimate even in the presence of a transport term (and more generally also for initial values not in $L^\infty(\Omega)$, see [9, Theorem 2.9] and [13]), and due to $u_0 \in L^\infty(\Omega)$ here this L^∞-estimate reads as

$$\|u(t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \text{ for a.e. } t \in (0,T)\,. \tag{2.3}$$

By this L^∞-estimate the cut-off coincides with the original values $f(u)$ of the transport term, thus on the one hand u eventually solves the original equation. On the other hand due to $u \in L^\infty((0,T) \times \Omega)$ also $df(u) \in L^\infty((0,T) \times \Omega)$ holds uniformly w.r.t. ϵ, and a test of (1.2) by u gives

$$\frac{d}{dt}\hat{\Phi}_B(u) + \epsilon c\|\nabla u\|_p^p \le -\int_{\partial\Omega}\left(\int_0^u \langle df(\tilde{u}) \cdot \tilde{u}, d\tilde{u}\rangle\right) dS$$

where the right hand side is uniformly bounded w.r.t. ϵ. Thus (2.2) even holds in presence of a transport term.

Finally, every initial value $u_0 \in L^\infty(\Omega)$ can be approximated by smooth $u_{0k} \in L^q(\Omega) \cap L^\infty(\Omega)$, and via a limit process a function $u \in L^\infty((0,T) \times \Omega)$ satisfying (2.3) and solving (1.2) in the sense of distributions can be obtained.

Regarding the uniqueness of solutions of the parabolic equation, let us mention the validity of an L^1-contraction principle proved by [14] which guarantees uniqueness of so-called entropy solutions to initial values $b(u_0) \in L^1(\Omega)$, see also [5], and the uniqueness result of [8] for bounded solutions.

Remark 2.2. Recall that the validity of the L^∞-estimate (2.3) is crucial for the former conclusions, but as far as we know an L^∞-estimate has generally been established only in the scalar case $m = 1$. Thus, to use the same methods for systems the validity of an L^∞-estimate has to be assured separately, e.g. like in section 4.2 for the system (4.2) with artificial viscosity on the right hand side.

3 Admissible entropies and entropy solutions

Weak solutions of first order hyperbolic equations obtained by the limit of vanishing viscosity have the special property that they satisfy an entropy condition. In the doubly nonlinear case (1.1), the usual notion of an entropy – entropy flux pair has to be modified. To obtain an approproriate notion we assume from now on that at least one of the functions b or b^{-1} is differentiable. This property is also useful in a discussion of strong solutions of (1.2), see [12].

Definition 3.1. A pair of smooth functions $\Phi\colon \mathbb{R}^m \to \mathbb{R}$ (entropy) and $\Psi\colon \mathbb{R}^m \to \mathbb{R}^n$ (entropy flux) is called an admissible entropy - entropy flux pair for equation (1.1) w.r.t. the nonlinear functions a and b, if

- $d^2\Phi(u)(\xi, a(\xi)) \geq 0$ holds for arbitrary $u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^m \otimes \mathbb{R}^n$, and $d\Phi(u)db^{-1}(b(u)) \cdot df(u) = d\Psi(u)$ in the case that b^{-1} is differentiable[1],

- $d^2\Phi(b(u))(db(u)\xi, a(\xi)) \geq 0$ holds for arbitrary $u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^m \otimes \mathbb{R}^n$, and $d\Phi(b(u)) \cdot df(u) = d\Psi(u)$ in the case that b is differentiable.

Note that the nonlinear functions a and b specifying the nonlinear diffusion generally determine which type of entropies are admissible. However, in the scalar case $m = 1$ the requirements on $d^2\Phi$ are equivalent to convexity of Φ. In fact, for $m = 1$ the first condition reads as $d^2\Phi(u)(\xi, a(\xi)) = \Phi''(u)\langle \xi, a(\xi)\rangle \geq 0$, and $\langle \xi, a(\xi)\rangle \geq 0$ by monotonicity of a. Similarly, the second condition reads as $d^2\Phi(b(u))(db(u)\xi, a(\xi)) = \Phi''(b(u))b'(u)\langle \xi, a(\xi)\rangle \geq 0$, thus convexity of Φ follows from monotonicity of a and b.

The aim of this section is to show that if there is a limit of the solutions u_ϵ of (1.2), then this limit is a (measure-valued or traditional) entropy solution. In other words, the weak solution of the hyperbolic equation singled out by the nonlinear diffusion via the limit of vanishing viscosity is an entropy solution.

Definition 3.2. A function u is called a (traditional) entropy solution of (1.1) if u is a weak solution of (1.1), i.e.

$$-\int_0^T \int_\Omega (b(u(t)) - b(u_0))\frac{\partial v}{\partial t}(t) + f(u(t)) \cdot \nabla v(t)\, dx\, dt = \int_0^T \int_\Omega g(u(t))v(t)\, dx\, dt$$

holds for every $v \in C^1(0, T; C_c^1(\Omega))$ with $v(T) = 0$, and satisfies for every admissible entropy - entropy flux pair (Φ, Ψ) w.r.t. a and b the entropy condition

$$\int_0^\infty \int_\Omega \Phi(u)\frac{\partial v}{\partial t} + \Psi(u) \cdot \nabla v\, dx\, dt \geq 0$$

(in the case that b^{-1} is differentiable) resp.

$$\int_0^\infty \int_\Omega \Phi(b(u))\frac{\partial v}{\partial t} + \Psi(u) \cdot \nabla v\, dx\, dt \geq 0$$

(in the case that b is differentiable) whenever $v \in C_c^1((0, T) \times \Omega)$ is a nonnegative function.

Sometimes the requirement that u is a weak solutions is too strong, but still we want to speak about entropy solutions. Therefore, a measure-valued solution of (1.1) (see Section 4) is called an entropy solution, if it satisfies the measure-valued analogon of the entropy condition.

Let us now prove that in the case of convergence we obtain in the limit of vanishing viscosity an entropy solution of (1.1). Therefore, let u_ϵ be a solution of (1.2) to an initial value

[1] In coordinates $d^2\Phi(u)(\xi, a(\xi)) = \sum_{i,k} \frac{\partial^2\Phi}{\partial x_i \partial x_k} \sum_{j=1}^n \xi_{ij} a_{jk}(\xi)$, i.e. $(\xi, a(\xi))$ is the element of $\mathbb{R}^m \times \mathbb{R}^m$ obtained by contraction of $\xi, a(\xi) \in \mathbb{R}^m \otimes \mathbb{R}^n$ in the index $j = 1, \ldots, n$. Similarly, in the second expression the linear form $d\Phi(u)$ on $\mathbb{R}^m$ maps the $(m \times m)$-matrix $db^{-1}(b(u))$ to a vector in $\mathbb{R}^m$, and by the product this vector is contracted with $\frac{\partial f_{ij}}{\partial x_k}$ in the index $i = 1, \ldots, m$.

$u_0 \in L^\infty(\Omega)$, and recall the L^∞-estimate (2.3) and the a priori estimate (2.2). Then

$$\begin{aligned}
&\frac{\partial \Phi(u_\epsilon)}{\partial t} + \operatorname{div}(\Psi(u_\epsilon)) \\
&\quad = d\Phi(u_\epsilon) db^{-1}(b(u_\epsilon)) \cdot \frac{\partial b(u_\epsilon)}{\partial t} + d\Psi(u_\epsilon) \cdot \nabla u_\epsilon \\
&\quad = d\Phi(u_\epsilon) db^{-1}(b(u_\epsilon)) \cdot \left(\frac{\partial b(u_\epsilon)}{\partial t} + df(u_\epsilon) \cdot \nabla u_\epsilon \right) \\
&\quad = db^{-1}(b(u_\epsilon))^* d\Phi(u_\epsilon) \cdot \left(\frac{\partial b(u_\epsilon)}{\partial t} + \operatorname{div}(f(u_\epsilon)) \right) \\
&\quad = db^{-1}(b(u_\epsilon))^* d\Phi(u_\epsilon) \cdot \operatorname{div}(\epsilon a(\nabla u_\epsilon)) \\
&\quad = db^{-1}(b(u_\epsilon))^* \left(\operatorname{div}(d\Phi(u_\epsilon) \cdot \epsilon a(\nabla u_\epsilon)) - d^2\Phi(u_\epsilon)(\nabla u_\epsilon, \epsilon a(\nabla u_\epsilon)) \right) \\
&\quad \le db^{-1}(b(u_\epsilon))^* \operatorname{div}(d\Phi(u_\epsilon) \cdot \epsilon a(\nabla u_\epsilon))
\end{aligned}$$

in the case that b^{-1} is differentiable (where $db^{-1}(b(u_\epsilon))^*$ denotes the adjoint of $db^{-1}(b(u_\epsilon))$), while

$$\begin{aligned}
&\frac{\partial \Phi(b(u_\epsilon))}{\partial t} + \operatorname{div}(\Psi(u_\epsilon)) \\
&\quad = d\Phi(b(u_\epsilon)) \frac{\partial b(u_\epsilon)}{\partial t} + d\Psi(u_\epsilon) \cdot \nabla u_\epsilon \\
&\quad = d\Phi(b(u_\epsilon)) \left(\frac{\partial b(u_\epsilon)}{\partial t} + df(u_\epsilon) \cdot \nabla u_\epsilon \right) \\
&\quad = d\Phi(b(u_\epsilon)) \left(\frac{\partial b(u_\epsilon)}{\partial t} + \operatorname{div}(f(u_\epsilon)) \right) \\
&\quad = d\Phi(b(u_\epsilon)) \cdot \operatorname{div}(\epsilon a(\nabla u_\epsilon)) \\
&\quad = \operatorname{div}(d\Phi(b(u_\epsilon)) \cdot \epsilon a(\nabla u_\epsilon)) - d^2\Phi(b(u_\epsilon))(db(u_\epsilon)\nabla u_\epsilon, \epsilon a(\nabla u_\epsilon)) \\
&\quad \le \operatorname{div}(d\Phi(b(u_\epsilon)) \cdot \epsilon a(\nabla u_\epsilon))
\end{aligned}$$

in the case that b is differentiable. In both cases the right hand side converges to zero as $\epsilon \searrow 0$. In fact, in the first case $\operatorname{div}(d\Phi(u_\epsilon) \cdot \epsilon a(\nabla u_\epsilon)) \to 0$ holds in $L^{p'}(0,T;(W_0^{1,p}(\Omega))^*)$ because of

$$\begin{aligned}
&\left| \int_0^T \int_\Omega d\Phi(u_\epsilon) \cdot \epsilon a(\nabla u_\epsilon) \cdot \nabla \phi \, dx \, dt \right| \\
&\quad \le \int_0^T \| d\Phi(u_\epsilon) \|_\infty \| \epsilon a(\nabla u_\epsilon) \|_{p'} \| \nabla \phi \|_p \, dt \\
&\quad \le C\epsilon \int_0^T \| \nabla u_\epsilon \|_p^{p-1} \| \nabla \phi \|_p \, dt \\
&\quad \le C\epsilon \left(\int_0^T \| \nabla u_\epsilon \|_p^p \, dt \right)^{1/p'} \left(\int_0^T \| \nabla \phi \|_p^p \, dt \right)^{1/p} \\
&\quad \le C\epsilon^{1/p} \left(\int_0^T \| \nabla \phi \|_p^p \, dt \right)^{1/p},
\end{aligned}$$

where boundedness of $d\Phi$ and the a priori estimate $\int_0^T \|\nabla u_\epsilon\|_p^p \, dt \le \frac{C}{\epsilon}$ (i.e. (2.2)) was used. Thus uniform boundedness of $db^{-1}(b(u_\epsilon))$ in $L^\infty((0,T) \times \Omega)$ w.r.t. ϵ implies the validity of

$db^{-1}(b(u_\epsilon))^* \operatorname{div}(d\Phi(u_\epsilon)\cdot \epsilon a(\nabla u_\epsilon)) \to 0$ in $L^{p'}(0,T;(W_0^{1,p}(\Omega))^*)$. Similary, in the second case

$$\begin{aligned}
&\left|\int_0^T \int_\Omega d\Phi(b(u_\epsilon))\cdot \epsilon a(\nabla u_\epsilon)\cdot \nabla\phi \,dx\,dt\right| \\
&\quad\le \int_0^T \|d\Phi(b(u_\epsilon))\|_\infty \|\epsilon a(\nabla u_\epsilon)\|_{p'} \|\nabla\phi\|_p \,dt \\
&\quad\le C\epsilon \int_0^T \|\nabla u_\epsilon\|_p^{p-1} \|\nabla\phi\|_p \,dt \\
&\quad\le C\epsilon \left(\int_0^T \|\nabla u_\epsilon\|_p^p \,dt\right)^{1/p'} \left(\int_0^T \|\nabla\phi\|_p^p \,dt\right)^{1/p} \\
&\quad\le C\epsilon^{1/p} \left(\int_0^T \|\nabla\phi\|_p^p \,dt\right)^{1/p}
\end{aligned}$$

implies $\operatorname{div}(d\Phi(u_\epsilon)\cdot \epsilon a(\nabla u_\epsilon)) \to 0$ in $L^{p'}(0,T;(W_0^{1,p}(\Omega))^*)$ as $\epsilon \searrow 0$. Hence, in both cases the right-hand side converges to zero in $L^{p'}(0,T;(W_0^{1,p}(\Omega))^*)$ as $\epsilon \searrow 0$. Therefore, in case of convergence we obtain in the limit of vanishing viscosity a (traditional) entropy solution of (1.1).

4 Existence of the limit of vanishing viscosity

In the former section we precisely showed that if $u_\epsilon \overset{*}{\rightharpoonup} u$, $b(u_\epsilon) \overset{*}{\rightharpoonup} b(u)$, $f(u_\epsilon) \overset{*}{\rightharpoonup} f(u)$ and also $\Phi(u_\epsilon) \overset{*}{\rightharpoonup} \Phi(u)$ as well as $\Psi(u_\epsilon) \overset{*}{\rightharpoonup} \Psi(u)$ (in the case that b^{-1} is differentiable, similarly for differentiable b) in $L^\infty((0,T)\times\Omega)$, then u is a (traditional) entropy solution of (1.1).

It remains to prove these convergences. Clearly, as $u_\epsilon \in L^\infty((0,T)\times\Omega)$ is uniformly bounded w.r.t. ϵ by (2.3), also $b(u_\epsilon)$ is uniformly bounded w.r.t. ϵ in $L^\infty((0,T)\times\Omega)$, thus there is a subsequence of $\epsilon \searrow 0$ such that $b(u_\epsilon) \overset{*}{\rightharpoonup} b(u)$ in $L^\infty((0,T)\times\Omega)$.

With this weak*-convergent subsequence $b(u_\epsilon)$ a Young measure $(t,x)\mapsto \mu_{(t,x)}$ can be associated such that $h(b(u_\epsilon)) \overset{*}{\rightharpoonup} \langle \mu, h\rangle$ for any continuous function h, see e.g. [9, Chapter 3]. However, due to existence and continuity of b^{-1} this implies that $\tilde h(u_\epsilon) \overset{*}{\rightharpoonup} \langle \nu, \tilde h\rangle$ for any continuous function $\tilde h$ with the measure $\nu := b^{-1}(\mu)$.

The Young measure ν may already be interpreted as an entropy solution. In fact, $b(u_\epsilon) \overset{*}{\rightharpoonup} \langle \nu, b\rangle = b(u)$ and $f(u_\epsilon) \overset{*}{\rightharpoonup} \langle \nu, f\rangle$ hold by definition of the Young measure, thus

$$\frac{\partial}{\partial t} b(u) + \operatorname{div}(\langle \nu, f\rangle) = 0$$

is valid in the sense of distributions, since the limit of the viscosity tensor is zero in the space $L^{p'}(0,T;(W_0^{1,p}(\Omega))^*)$. Further, by the same arguments as in the former section, ν satisfies the measure-valued analogue of the entropy admissibility condition

$$\frac{\partial}{\partial t}\langle \nu, \Phi\rangle + \operatorname{div}(\langle \nu, \Psi\rangle) \le 0$$

(in the case that b^{-1} is differentiable) resp.

$$\frac{\partial}{\partial t}\langle \nu, (\Phi\circ b)\rangle + \operatorname{div}(\langle \nu, \Psi\rangle) \le 0$$

(in the case that b is differentiable) in the sense of distributions for entropy – entropy flux pairs (Φ,Ψ). Therefore, ν is called a measure-valued entropy solution.

However, in the following we want to show for special cases that this measure-valued entropy solution is a traditional entropy solution. Thus, we have to show that the Young measure $\nu_{(t,x)}$ is $\delta_{u(t,x)}$, because then $u_\epsilon \overset{*}{\rightharpoonup} u$, $f(u_\epsilon) \overset{*}{\rightharpoonup} f(u)$, $\Phi(u_\epsilon) \overset{*}{\rightharpoonup} \Phi(u)$ and $\Psi(u_\epsilon) \overset{*}{\rightharpoonup} \Psi(u)$ in $L^\infty((0,T)\times\Omega)$, so that u is a traditional entropy solution.

To do this, we want to use compensated compactness in the form of the div-curl-lemma and special entropies, see [19]. In order to apply the div-curl-lemma we want to establish compactness in $W^{-1,2}((0,T)\times\Omega)$, see [9, Lemma III.3.12] or [6, Lemma 16.2.2]. In the case that b^{-1} is differentiable we already concluded

$$\frac{\partial\Phi(u_\epsilon)}{\partial t} + \operatorname{div}(\Psi(u_\epsilon)) = db^{-1}(b(u_\epsilon))^* \left(\epsilon \operatorname{div}(d\Phi(u_\epsilon)\cdot a(\nabla u_\epsilon)) - \epsilon d^2\Phi(u_\epsilon)(\nabla u_\epsilon, a(\nabla u_\epsilon))\right)$$

in the former section. The second term in the bracket is a uniformly bounded measure on $[0,T]\times\Omega$, because the a priori estimate $\epsilon\int_0^T\int_\Omega a(\nabla u_\epsilon)\cdot\nabla u_\epsilon \le C$ holds by (2.3) and $d^2\Phi(u_\epsilon)$ as well as $db^{-1}(b(u_\epsilon))$ are uniformly bounded. The first term in the bracket is compact in $L^{p'}(0,T;(W_0^{1,p}(\Omega))^*) \subset W^{-1,p'}((0,T)\times\Omega)$, because it converges to zero as $\epsilon \searrow 0$. Further, the left hand side is bounded in every $W^{-1,r}((0,T)\times\Omega)$, $1<r<\infty$, because $\Phi(u_\epsilon)$ and $\Psi(u_\epsilon)$ are uniformly bounded. Thus the left hand side is also compact in $W^{-1,2}((0,T)\times\Omega)$.

In the case that b is differentiable,

$$\frac{\partial\Phi(u_\epsilon)}{\partial t} + \operatorname{div}(\Psi(u_\epsilon)) = \epsilon\operatorname{div}(d\Phi(b(u_\epsilon))\cdot a(\nabla u_\epsilon)) - \epsilon d^2\Phi(b(u_\epsilon))(db(u_\epsilon)\nabla u_\epsilon, a(\nabla u_\epsilon))$$

has been shown in the former section. Again the second term is a uniformly bounded measure $[0,T]\times\Omega$ by (2.3) and by uniform boundedness of $d^2\Phi(b(u_\epsilon))$ and $db(u_\epsilon)$, and the first term converges in $L^{p'}(0,T;(W_0^{1,p}(\Omega))^*)$ so that it is compact. Thus due to boundedness of the left hand side in $W^{-1,r}((0,T)\times\Omega)$, $1<r<\infty$, even compactness of the left hand side in $W^{-1,2}((0,T)\times\Omega)$ holds.

Note that in the former proof of compactness it is not important whether Φ resp. $\Phi\circ b$ satisfy the convexity assumptions of Definition 3.1, only the pointwise relations $d\Phi(u)db^{-1}(b(u))\cdot df(u) = d\Psi(u)$ resp. $d\Phi(b(u))\cdot df(u) = d\Psi(u)$ required from an entropy – entropy flux pair are needed.

The following two subsections show, how compactness in $W^{-1,2}((0,T)\times\Omega)$ leads in two special cases to a traditional entropy solution of (1.1).

4.1 One-dimensional doubly nonlinear scalar conservation laws

With the results of the former sections at hand we are now prepared to prove Theorem 1.1. In fact, in the scalar one-dimensional case $m=1$, $n=1$, $\Omega=\mathbb{R}$, we can consider the two entropy – entropy flux pairs (b,f) and (f,g) (even if f is not convex), where the entropy flux g corresponding to f is given by $g(\cdot) := \int_0^\cdot (b^{-1})'(b(v))(f'(v))^2\,dv$. Note that $\operatorname{div}_{(t,x)}(\Phi(u),\Psi(u)) = \frac{\partial\Phi(u)}{\partial t} + \frac{\partial\Psi(u)}{\partial x} = \operatorname{curl}_{(t,x)}(\Psi(u),-\Phi(u))$, and compactness of this expression in $W^{-1,2}((0,T)\times\Omega)$ has just been shown. Thus we can apply the div-curl-lemma to $\operatorname{div}(b(u_\epsilon),f(u_\epsilon))$ and $\operatorname{curl}(g(u_\epsilon),-f(u_\epsilon))$ and obtain the Murat–Tartar relation (see [9, 3.3])

$$\langle\nu, bg - f^2\rangle = \langle\nu,b\rangle\langle\nu,g\rangle - (\langle\nu,f\rangle)^2\,.$$

By the relation between f and g

$$(f(\tilde u)-f(u))^2 = \left(\int_u^{\tilde u} f'(v)\,dv\right)^2 \le \left(\int_u^{\tilde u}\frac{1}{(b^{-1})'(b(v))}\,dv\right)(g(\tilde u)-g(u))\,,$$

where the Cauchy–Schwarz inequality

$$\int_u^{\tilde u} f'(v)\,dv \le \left(\int_u^{\tilde u} \frac{1}{(b^{-1})'(b(v))}\right)^{1/2} \left(\int_u^{\tilde u} (b^{-1})'(b(v))(f'(v))^2\,dv\right)^{1/2} \tag{4.1}$$

was applied. Further, $\frac{1}{(b^{-1})'(b(v))}\,dv = d(b(v))$ as measures and thus

$$(f(\tilde u) - f(u))^2 \le (b(\tilde u) - b(u))(g(\tilde u) - g(u))\,.$$

Hence, if we consider $\tilde u$ as a variable and u as the function defined by $b(u_\epsilon) \overset{*}{\rightharpoonup} b(u)$, then due to $\langle \nu, 1\rangle = 1$

$$\begin{aligned} 0 &\le \langle \nu, (b - b(u))(g - g(u)) - (f - f(u))^2\rangle \\ &= \langle \nu, b\rangle\langle \nu, g\rangle - b(u)\langle \nu, g\rangle - g(u)\langle \nu, b\rangle + b(u)g(u) - (\langle \nu, f\rangle - f(u))^2 \\ &= (\langle \nu, b\rangle - b(u))(\langle \nu, g\rangle - g(u)) - (\langle \nu, f\rangle - f(u))^2 = -(\langle \nu, f\rangle - f(u))^2\,. \end{aligned}$$

In the last step we used that we know already $\langle \nu, b\rangle = b(u)$ due to $b(u_\epsilon) \overset{*}{\rightharpoonup} b(u)$. Therefore, $\langle \nu, f\rangle = f(u)$ or in other words $f(u_\epsilon) \overset{*}{\rightharpoonup} f(u)$.

As a consequence, $\langle \nu, (b - b(u))(g - g(u)) - (f - f(u))^2\rangle = 0$ holds, i.e. the inequality $(f(\tilde u) - f(u))^2 \le (b(\tilde u) - b(u))(g(\tilde u) - g(u))$ has to hold as an equality for $\tilde u$ in the support of ν. However, the Cauchy–Schwarz inequality (4.1) is an equality only if f' is constant on the interval from $\tilde u$ to $u(t,x)$. Thus, if there is no interval on which f' is constant, i.e. if f is genuinely nonlinear, then the support of $\nu_{(t,x)}$ has to be the single point $\{u(t,x)\}$, so that $\nu_{(t,x)} = \delta_{\{u(t,x)\}}$.

Finally, if the Young measure associated with a weak* convergent sequence $b(u_\epsilon)$ reduces to a Dirac measure, then $b(u_\epsilon)$ converges even strongly in every $L^r_{loc}((0,T)\times\Omega)$, see e.g. [9, Theorem III.2.31], thus also u_ϵ converges strongly, and the proof of Theorem 1.1 is finished.

4.2 One-dimensional doubly nonlinear systems of two conservation laws

For general systems of conservation laws it still seems to be an open problem to establish boundedness of u_ϵ in $L^\infty((0,T)\times\Omega)$. However, once boundedness has been shown the former methods apply at least to genuinely nonlinear systems, for which a rich set of entropy pairs exists. Instead of discussing general doubly nonlinear systems let us discuss as a concrete example a doubly nonlinear generalisation of the system of isentropic elasticity (see [6, Section 16.7]).

Consider the doubly nonlinear wave equation of second order

$$\frac{\partial}{\partial t} b\left(\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\sigma\left(\frac{\partial u}{\partial x}\right) = 0$$

for the deformation u of a nonlinear one-dimensional elastic body with nonlinear momentum b and nonlinear stress tensor σ. This wave equation can equivalently be written as a first order system

$$\begin{aligned} \frac{\partial b(v)}{\partial t} - \frac{\partial \sigma(w)}{\partial x} &= 0\,, \\ \frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} &= 0\,, \end{aligned} \tag{4.2}$$

where $v := \frac{\partial u}{\partial t}$, $w := \frac{\partial u}{\partial x}$, and the second equation is a compatibility condition. In the case that b^{-1} is a C^2-function and $\sigma''(w) \cdot w > 0$ for $w \neq 0$ (note that in the case $\sigma''(0) = 0$ the system is not genuinely nonlinear along the line $u = 0$) it is possible to establish L^∞-boundedness of solutions (v_ϵ, w_ϵ) to the corresponding doubly nonlinear parabolic equation with artificial viscosity, where the right hand side of (4.2) is replaced by $\epsilon(\frac{\partial}{\partial x} a(\frac{\partial v}{\partial x}), \frac{\partial}{\partial x} a(\frac{\partial w}{\partial x}))$.

In fact, in analogy to the proof of [6, Theorem 15.7.2] let us consider the Riemann invariants $R_\pm(z, w) := \int_0^z \sqrt{(b^{-1})'(\tilde{z})}\, d\tilde{z} \pm \int_0^w \sqrt{\sigma'(\tilde{w})}\, d\tilde{w}$ of the system

$$\begin{aligned} \frac{\partial z}{\partial t} - \frac{\partial \sigma(w)}{\partial x} &= 0\,, \\ \frac{\partial w}{\partial t} - \frac{\partial b^{-1}(z)}{\partial x} &= 0\,, \end{aligned} \tag{4.3}$$

which is equivalent to (4.2) by substituting $z := b(v)$, and let us prove positive invariance of $B_M := \{\, (v, w) \,|\, |R_\pm(b(v), w)| \leq M \,\}$ under the flow generated by the doubly nonlinear parabolic equation with artificial viscosity corresponding to (4.2) for arbitrary $M > 0$. Note that for bounded initial data $(v_\epsilon(0), w_\epsilon(0))$ there exists a M such that $(v_\epsilon(0), w_\epsilon(0)) \in B_M$, thus positive invariance of B_M implies $(v_\epsilon(t), w_\epsilon(t)) \in B_M$ for all $t \geq 0$ and hence provides an L^∞-bound.

To prove that B_M is positively invariant, let us construct special entropies – entropy fluxes (Φ_μ, Ψ_μ) by separation of variables. Definition 3.1 requires

$$\begin{aligned} \frac{\partial \Psi_\mu}{\partial w} &= -\sigma'(w) \frac{\partial \Phi_\mu}{\partial z}\,, \\ \frac{\partial \Psi_\mu}{\partial z} &= -(b^{-1})'(z) \frac{\partial \Phi_\mu}{\partial w}\,, \end{aligned}$$

and thus $(b^{-1})'(z) \frac{\partial^2 \Phi_\mu}{\partial w^2} = \sigma'(w) \frac{\partial^2 \Phi_\mu}{\partial z^2}$. The ansatz $\Phi_\mu(z, w) = Z(z)W(w) - 1$ for the solution of this PDE leads to the ODEs

$$\begin{aligned} Z''(z) &= \mu^2 (b^{-1})'(z) Z(z)\,, \\ W''(w) &= \mu^2 \sigma'(w) W(w)\,, \end{aligned}$$

with a constant $\mu \geq 0$, which we solve under the initial conditions $Z(0) = 1$, $Z'(0) = 0$, $W(0) = 1$, $W'(0) = 0$. The corresponding entropy flux to Φ_μ is given by $\Psi_\mu(z, w) = \frac{1}{\mu^2} Z'(z) W'(w))$.

To obtain some information about the solutions Z and W, multiply the ODEs with Z' resp. W' to conclude

$$\begin{aligned} (\mu^2 (b^{-1})' Z^2 - (Z')^2)' &= \mu^2 (b^{-1})'' Z^2\,, \\ (\mu^2 \sigma' W^2 - (W')^2)' &= \mu^2 \sigma'' W^2\,. \end{aligned}$$

Because $(b^{-1})''$ and σ'' change sign at 0, the functions $\mu^2 (b^{-1})' Z^2 - (Z')^2$ and $\mu^2 \sigma' W^2 - (W')^2$ attain their minima at $z = 0$ resp. $w = 0$. Thus due to the choice of the initial conditions we have $\mu^2 (b^{-1})' Z^2 - (Z')^2 \geq \mu^2 (b^{-1})'(0) \geq 0$ and $\mu^2 \sigma' W^2 - (W')^2 \geq \mu^2 \sigma'(0) \geq 0$, so that

$$\begin{aligned} \frac{\partial^2 \Phi_\mu}{\partial z^2} \frac{\partial^2 \Phi_\mu}{\partial w^2} - \left(\frac{\partial^2 \Phi_\mu}{\partial z \partial w} \right)^2 &= Z'' W'' Z W - (Z' W')^2 \\ &= \mu^2 (b^{-1})' Z^2 (\mu^2 \sigma' W^2 - \frac{(Z')^2}{\mu^2 (b^{-1})' Z^2} (W')^2) \\ &\geq \mu^2 (b^{-1})' Z^2 (\mu^2 \sigma' W^2 - (W')^2) \geq 0 \end{aligned}$$

due to $\mu^2(b^{-1})'Z^2 \geq 0$ and $\frac{(Z')^2}{\mu^2(b^{-1})'Z^2} \leq 1$. Therefore, the entropy Φ_μ is convex and attains its minimum 0 at $(0,0)$. Further, Z resp. W behave like

$$\left(1+O\left(\frac{1}{\mu}\right)\right)\exp\left(\mu\left|\int_0^z \sqrt{(b^{-1})'(\tilde{z})}\,d\tilde{z}\right|\right) \text{ resp.}$$
$$\left(1+O\left(\frac{1}{\mu}\right)\right)\exp\left(\mu\left|\int_0^w \sqrt{\sigma'(\tilde{w})}\,d\tilde{w}\right|\right)$$

as $\mu \to \infty$[2]. Hence $\Phi_\mu^{\frac{1}{\mu}}$ behaves like

$$\left(1+O\left(\frac{1}{\mu}\right)\right)\exp\left(R_+(z,w)\right) \text{ for } z>0\,,\ w>0\,,$$
$$\left(1+O\left(\frac{1}{\mu}\right)\right)\exp\left(R_-(z,w)\right) \text{ for } z>0\,,\ w<0\,,$$
$$\left(1+O\left(\frac{1}{\mu}\right)\right)\exp\left(-R_-(z,w)\right) \text{ for } z<0\,,\ w>0\,,$$
$$\left(1+O\left(\frac{1}{\mu}\right)\right)\exp\left(-R_+(z,w)\right) \text{ for } z<0\,,\ w<0\,,$$

as $\mu \to \infty$.

Now if (v_ϵ, w_ϵ) is the solution of the doubly nonlinear parabolic equation with artificial viscosity corresponding to (4.2) to initial data $(v_\epsilon(0), w_\epsilon(0)) \in B_M$, then

$$\frac{\partial \Phi_\mu(b(v_\epsilon), w_\epsilon)}{\partial t} + \frac{\partial \Psi_\mu(b(v_\epsilon), w_\epsilon)}{\partial x} \leq 0$$

holds with the entropy Φ_μ and the entropy flux Ψ_μ. Integration of this inequality over $[0,t]\times\mathbb{R}$ gives

$$\int_{\mathbb{R}} \Phi_\mu(b(v_\epsilon(t)), w_\epsilon(t))\,dx \leq \int_{\mathbb{R}} \Phi_\mu(b(v_\epsilon(0)), w_\epsilon(0))\,dx\,. \tag{4.4}$$

Because $\Phi_\mu(b(v_\epsilon(t)), w_\epsilon(t))$ behaves like $((1+O(1/\mu))\exp(\pm R_\pm(z,w)))^\mu$ and the convergence $(\int_{\mathbb{R}} |f(x)|^\mu\,dx)^{1/\mu} \to \|f\|_\infty$ holds as $\mu \to \infty$, we can conclude from (4.4) that the validity of $|R_\pm(b(v_\epsilon(0,x)), w_\epsilon(0,x))| \leq M$ for a.e. $x \in \mathbb{R}$ implies $|R_\pm(b(v_\epsilon(t,x)), w_\epsilon(t,x))| \leq M$ for a.e. $x \in \mathbb{R}$. Thus, if $(v_\epsilon(0), w_\epsilon(0))$ attains a.e. value in B_M, then also $(v_\epsilon(t), w_\epsilon(t))$ does for $t \geq 0$, i.e. B_M is positively invariant and an L^∞-bound has been established.

Using this L^∞-bound the machinery of section 4 can be applied to obtain a measure-valued entropy solution ν of (4.2). Finally, we want to show that this solution is a traditional entropy solution. Similarly to [6, Section 15.6] we have to show that the smallest rectangle $[z_-, z_+] \times [w_-, w_+]$ which contains the support of the measure ν degenerates to a point. To do this, we only have to modify the proof of [6, Theorem 15.7.1] so that instead of the Riemann invariants there now $R_\pm(b(v), w)$ are used, but these are minor changes, as also in the doubly nonlinear case $R_\pm(z,w)$ is the sum of a term depending on w and a term depending on $z = b(v)$. Hence, a traditional entropy solution of the doubly nonlinear system (4.2) exists. Let us formulate the result as a theorem.

[2]In fact, $H(z) := \exp\left(\mu\int_0^z \sqrt{(b^{-1})'(\tilde{z})}\,d\tilde{z}\right)$ satisfies $H'' = \mu^2(b^{-1})'H + \mu\frac{(b^{-1})''}{2\sqrt{(b^{-1})'}}$, but the last term is of order μ and not of order μ^2, i.e. for the asymptotics it is irrelevant.

Theorem 4.1. *Let $1 < p, q < \infty$, let $b \in C(\mathbb{R})$ be strictly monotone, q-coercive, $(q-1)$-growing, and assume that $b^{-1} \in C^2(\mathbb{R})$ satisfies $(b^{-1})''(z) \cdot z > 0$ for $z \neq 0$. Let $\sigma \in C^2(\mathbb{R})$ be strictly monotone such that $\sigma''(w) \cdot w > 0$ holds for $w \neq 0$, let $v_0, w_0 \in L^\infty(\mathbb{R})$, and denote for $\epsilon > 0$ by (v_ϵ, w_ϵ) weak solutions of the parabolic perturbation of (4.2), where the right hand side of (4.2) is replaced by the artificial viscosity terms $\epsilon(\frac{\partial}{\partial x} a(\frac{\partial v}{\partial x}), \frac{\partial}{\partial x} a(\frac{\partial w}{\partial x}))$ for a monotone, p-coercive, $(p-1)$-growing $a \in C(\mathbb{R})$. Then for every sequence $\epsilon_k \to 0$ there exists a subsequence $(v_{\epsilon_k}, w_{\epsilon_k})$ which converges weakly* in $L^\infty((0,T) \times \mathbb{R})$ and strongly in every $L^r_{loc}((0,T) \times \mathbb{R})$, $1 < r < \infty$, to a traditional entropy solution (v, w) of (4.2).*

Let us remark that this result would be much nicer if the artificial viscosity could have been replaced by the physical viscosity $\epsilon(\frac{\partial}{\partial x} a(\frac{\partial v}{\partial x}), 0)$ on the right hand side of (4.2). For the standard case $b(v) = v$ and $a(\nabla u) = \nabla u$ this seems to be shown by [17] via compensated compactness in the L^p-framework, see also [18].

5 Conclusion

We considered the limit of vanishing viscosity for a doubly nonlinear diffusion equation with transport terms and were able to prove – at least in the one-dimensional scalar case and for the one-dimensional doubly nonlinear system of isentropic elasticity – that this limit exists and is a traditional entropy solution of the corresponding doubly nonlinear conservation law. Hereby we used a definition of admissible entropy pairs which in general depends on the functions a and b specifying the doubly nonlinear diffusion. In the same way generally doubly nonlinear systems may be handled if L^∞-boundedness can be established and sufficiently many entropies are available. As a generalization of the setting discussed in this paper it seems possible to discuss non-continuous multivalued b as long as b^{-1} exists and is continuously differentiable.

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