New existence theorems of positive solutions for singular boundary value problems*

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Abstract: *In this paper, some nonexistence, existence and multiplicity of positive solutions are established for a class of singular boundary value problem. The authors also obtain the relation between the existence of the solutions and the parameter λ. The arguments are based upon the fixed point index theory and the upper and lower solutions method.*

1 Introduction

Consider the following second-order singular boundary value problem (BVP)

$$\begin{cases} \frac{1}{p(t)}(p(t)x'(t))' + \lambda g(t)f(x(t)) = 0, & 0 < t < 1, \\ ax(0) - b\lim\limits_{t\to 0^+} p(t)x'(t) = 0, \\ cx(1) + d\lim\limits_{t\to 1^-} p(t)x'(t) = 0, \end{cases} \tag{1.1$_\lambda$}$$

where $a \geq 0, b \geq 0, c \geq 0, d \geq 0, ac + bc + ad > 0; \lambda > 0$.

Differential equations with singularity arise in the fields of gas dynamics, nuclear physics, theory of boundary layer, nonlinear optics and so on. Nonlinear singular boundary value problems has become an important area of investigation in previous years; see[1-5, 7-15] and references therein.

When $p(t) = 1, \beta = \delta = 0, \alpha = \gamma = 1$, the BVP $(1.1)_\lambda$ reduces to

$$\begin{cases} x''(t) + \lambda g(t)f(x(t)) = 0, & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases} \tag{1.2}$$

which is a special case of the BVP $(1.2)_\lambda$.

In the special cases *i)* $f(t,x) = q(t)x^{-\lambda_1}, \lambda_1 > 0$, and *ii)* $f(t,x) = q(t)x^{\lambda_1}, 0 < \lambda_1 < 1$, where $q > 0$ for $t \in (0,1)$, the existence and uniqueness of positive solutions for the BVP (1.2) as $\lambda = 1$ have been studied completely by Taliaferro in [1] with the shooting method and by Zhang [2] with the sub-super solutions method, respectively. In the special case iii)$f(t,x) = q(t)g(x)$, $q(t)$ is singular only at $t = 0$ and $g(x) \geq e^x$, the existence of multiple positive solutions for the BVP (1.2) have been studied by Ha and Lee in [3] with the sub-super solutions method. In the special case *iv)* $f(t,x) = q(t)g(x)$, $q(t)$ is singular only at $t = 0$ and $g(x) \in C(-\infty,+\infty), [0. +\infty)$,

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the existence of multiple positive solutions for the BVP (1.2) have been studied by Wong in [4] with the shooting method.

Motivated by the results mentioned above, in this paper we study the existence, multiplicity, and nonexistence of positive solutions for the BVP (1.1) by new technique(different from [3,5,11,12,13,14]) to overcome difficulties arising from the appearances of $p(t)$ and $p(t)$ is singular at $t = 0$ and $t = 1$. On the other hand, to the best of our knowledge, there are very few literatures considering the existence, multiplicity, and nonexistence of positive solutions for the case when $p(t)$ is singular at $t = 0$ and $t = 1$. The arguments are based upon the fixed point index theory and the upper and lower solutions method.

Fixed point index theorems have been applied to various boundary value problems to show the existence of multiple positive solutions. An overview of such results can be found in Guo and Lakshmikantham V., [16] and in Guo and Lakshmikantham V., Liu X.Z.,[17] and in Guo, [18] and in K. Deimling, [19] and in M. Krasnoselskii, [20].

Lemma 1.1.[16,17,18,19,20] Let P be a cone of real Banach space E, Ω be a bounded open subset of E and $\theta \in \Omega$. Suppose $A : P \cap \bar{\Omega} \to P$ is a completely continuous operator, and satisfies

$$Ax = \mu x, x \in P \cap \partial\Omega \Longrightarrow \mu < 1.$$

Then $i(A, P \cap \Omega, P) = 1$.

Lemma 1.2.[16,17,18,19,20] Suppose $A : P \cap \bar{\Omega} \to P$ is a completely continuous operator, and satisfies

(1) $\inf\limits_{x \in P \cap \partial\Omega} ||Ax|| > 0$;

(2)$Ax = \mu x, x \in P \cap \partial\Omega \Longrightarrow \mu \notin (0, 1]$.

Then $i(A, P \cap \Omega, P) = 0$.

The paper is organized in the following fashion. In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with the BVP $(1.1)_\lambda$. In Section 3, the main results will be stated and proved. Finally some examples are worked out to demonstrate our main results in this section.

2 Preliminaries

Let $J = [0, 1]$. The basic space used in this paper is $E = C[0, 1]$. It is well known that E be a Banach space with the norm $|| \cdot ||$ defined by $||x|| = \max\limits_{0 \le t \le 1} |x(t)|$. Let $S = \{\lambda > 0 | (1.1)_\lambda \text{ has at least one solution}\}$ and $P = \{x \in E | x(t) \ge 0, t \in [0, 1]\}$. It is clear that P is a cone of E.

The following assumptions will stand throughout this paper:

(H_1) $p \in C((0, 1), (0, +\infty))$ and $0 < \int_0^1 \frac{dt}{p(t)} < +\infty$;

(H_2) $g \in C((0, 1), (0, +\infty))$ and $0 < \int_0^1 G(s, s)p(s)g(s)ds < +\infty$;

(H_3) $f \in C([0, +\infty), (0, +\infty))$ is nondecreasing and there exist $\bar{\delta} > 0, m \ge 2$ such that $f(x) > \bar{\delta} x^m, x \in (0, +\infty)$.

Let $G(t, s)$ be *Green's* function of the following BVP

$$\begin{cases} \frac{1}{p(t)}(p(t)x'(t))' = 0, \qquad 0 < t < 1, \\ ax(0) - b \lim\limits_{t \to 0^+} p(t)x'(t) = 0, \\ cx(1) + d \lim\limits_{t \to 1^-} p(t)x'(t) = 0. \end{cases}$$

Then $G(t,s)$ is defined by

$$G(t,s)=\frac{1}{\Delta}\begin{cases}(b+a\int_0^s\frac{dr}{p(r)})(d+c\int_t^1\frac{dr}{p(r)}), & if\ \ 0\le s\le t\le 1,\\ (b+a\int_0^t\frac{dr}{p(r)})(d+c\int_s^1\frac{dr}{p(r)}), & if\ \ 0\le t\le s\le 1,\end{cases} \tag{2.1}$$

where $\Delta = ad + ac\int_0^1\frac{dr}{p(r)} + bc$. It is easy to prove that $G(t,s)$ has the following properties.

Property 2.1. For all $t,s\in[0,1]$ we have

$$G(t,s)\le G(s,s)\le\frac{1}{\Delta}(b+a\int_0^1\frac{dr}{p(r)})(d+c\int_0^1\frac{dr}{p(r)})<+\infty. \tag{2.2}$$

Property 2.2. For all $t,s\in J_\theta=[\theta,1-\theta],\theta\in(0,\frac{1}{2})$ we have

$$G(t,s)\ge\frac{1}{\Delta}(b+a\int_0^\theta\frac{dr}{p(r)})(d+c\int_{1-\theta}^1\frac{dr}{p(r)})>0. \tag{2.3}$$

Property 2.3. For all $t\in J_\theta, s\in[0,1]$ we have

$$G(t,s)\ge\sigma_0 G(s,s), \tag{2.4}$$

where

$$\sigma_0=\ \min\left\{\frac{b+a\int_0^\theta\frac{dr}{p(r)}}{b+a\int_0^1\frac{dr}{p(r)}},\frac{d+c\int_{1-\theta}^1\frac{dr}{p(r)}}{d+c\int_0^1\frac{dr}{p(r)}}\right\}. \tag{2.5}$$

It is easy to see that $0<\sigma_0<1$.

Definition 2.1. Letting $x(t)\in C[0,1]\cap C^1(0,1)$, we say $x(t)$ is a lower solution for the BVP $(1.1)_\lambda$ if $x(t)$ satisfies :

$$\begin{cases}-\frac{1}{p(t)}(p(t)x'(t))'\le\lambda g(t)f(x(t)), & 0<t<1,\\ ax(0)-b\lim\limits_{t\to0^+}p(t)x'(t)\le0,\\ cx(1)+d\lim\limits_{t\to1^-}p(t)x'(t)\le0.\end{cases}$$

Definition 2.2. Letting $y(t)\in C[0,1]\cap C^1(0,1)$, we say $y(t)$ is an upper solution for the BVP $(1.1)_\lambda$ if $y(t)$ satisfies :

$$\begin{cases}-\frac{1}{p(t)}(p(t)y'(t))'\ge\lambda g(t)f(y(t)), & 0<t<1,\\ ay(0)-b\lim\limits_{t\to0^+}p(t)y'(t)\ge0,\\ cy(1)+d\lim\limits_{t\to1^-}p(t)y'(t)\ge0.\end{cases}$$

Firstly, we consider the following BVP:

$$\begin{cases}\frac{1}{p(t)}(p(t)x'(t))'+\lambda g(t)f(x(t))=0, & 0<t<1,\\ ax(0)-b\lim\limits_{t\to0^+}p(t)x'(t)=0,\\ cx(1)+d\lim\limits_{t\to1^-}p(t)x'(t)=\rho\ge0.\end{cases} \tag*{$(2.6)_\lambda^\rho$}$$

Define $T_\lambda^\rho: E\to E$ by

$$T_\lambda^\rho x(t)=\int_0^1 G(t,s)\lambda p(s)g(s)f(x(s))ds+\rho h(t), \tag{2.7}$$

where $h(t)=\frac{1}{\triangle}(b+a\int_0^t\frac{dr}{p(r)})$. It is not difficult to see that $0<h(t)\le 1$ for $c\ge 1$.

It is easy to obtain the following Lemma 2.1 by (2.7).

Lemma 2.1. Let $(H_1)-(H_3)$ be satisfied. Then the BVP $(1.1)_\lambda$ has a solution x if and only if x is a fixed point of T_λ^0.

Proof. It is easy to prove Lemma 2.1 by calculation.

In order to prove the following results we define a cone by

$$Q=\{x\in C[0,1]|x(t)\ge 0,\min_{t\in J_\theta}x(t)\ge\sigma_0||x||\} \tag{2.8}$$

Where σ_0 is given by (2.5), $\theta\in(0,\frac{1}{2})$. It is easy to see that Q is a closed convex cone of E and $Q\subset P$.

Lemma 2.2. Let $(H_1)-(H_3)$ be satisfied. Then $T_\lambda^0(Q)\subset Q$ and $T_\lambda^0:Q\to Q$ is completely continuous and nondecreasing.

Proof. For any $x\in P$, we have by (2.2) and (2.7)

$$\begin{aligned}T_\lambda^0x(t)&=\int_0^1G(t,s)\lambda p(s)g(s)f(x(s))ds\\&\le\int_0^1\lambda G(s,s)p(s)g(s)f(x(s))ds.\end{aligned}$$

therefore $$||T_\lambda^0x||\le\int_0^1\lambda G(s,s)p(s)g(s)f(x(s))ds.$$

On the other hand, for any $t\in J_\theta$, we have by (2.4) and (2.7)

$$\begin{aligned}\min_{t\in J_\theta}T_\lambda^0x(t)&=\min_{t\in J_\theta}\int_0^1G(t,s)\lambda p(s)g(s)f(x(s))ds\\&\ge\lambda\sigma_0\int_0^1G(s,s)p(s)g(s)f(x(s))ds\\&\ge\sigma_0||T_\lambda^0x||.\end{aligned}$$

Hence $T_\lambda^0x\in Q$. Therefore $T_\lambda^0P\subset Q$ and therefore $T_\lambda^0Q\subset Q$ by $Q\subset P$. By similar arguments in [1,3], $T_\lambda^0:Q\to Q$ is completely continuous. It is clear that T_λ^0 is nondecreasing on $[0,+\infty)$ by (H_3).

Remark 2.1. Similar to proving Lemma 2.1-Lemma 2.2, we have $T_\lambda^\rho:Q\to Q$ is completely continuous; $x(t)$ is a solution of $(2.6)_\lambda^\rho$ if and only if $x(t)$ is a fixed point of T_λ^ρ.

Lemma 2.3. Suppose $\lambda\in S,S_1=(\lambda,+\infty)\cap S\not\equiv\emptyset$. Then there exists $R(\lambda)>0$, such that $||x_{\lambda'}||\le R(\lambda)$, where $\lambda'\in S_1,x_{\lambda'}\in Q$ is a solution of $(1.1)_{\lambda'}$.

Proof. For any $\lambda'\in S$, let $x_{\lambda'}$ is a solution of the BVP $(1.1)_{\lambda'}$. Then we have

$$\begin{aligned}x_{\lambda'}(t)&=T_{\lambda'}^0x_{\lambda'}(t)\\&=\int_0^1G(t,s)\lambda'p(s)g(s)f(x_{\lambda'}(s))ds.\end{aligned}$$

Let $R(\lambda)=\max\{[\lambda'\sigma_0^{m+1}\bar{\delta}\int_\theta^{1-\theta}G(s,s)p(s)g(s)ds]^{-1},1\}$, next we prove $||x_{\lambda'}||\le R(\lambda)$. Indeed, if $||x_{\lambda'}||<1$, the result is easily obtained. On the other hand, if $||x_{\lambda'}||\ge 1$, then we have by (H_3)

$$\begin{aligned}\frac{1}{||x_{\lambda'}||}&\ge\frac{\min\limits_{t\in J_\theta}x_{\lambda'}(t)}{||x_{\lambda'}||^2}\\&=\frac{1}{||x_{\lambda'}||^2}\min_{t\in J_\theta}\int_0^1G(t,s)\lambda'p(s)g(s)f(x_{\lambda'}(s))ds\\&\ge\frac{1}{||x_{\lambda'}||^2}\sigma_0\int_\theta^{1-\theta}G(s,s)\lambda'p(s)g(s)\bar{\delta}(x_{\lambda'}(s))^mds\\&\ge\frac{1}{||x_{\lambda'}||^2}\sigma_0^{m+1}\int_\theta^{1-\theta}G(s,s)\lambda'p(s)g(s)\bar{\delta}||x_{\lambda'}||^mds\\&\ge\lambda'\sigma_0^{m+1}\bar{\delta}\int_\theta^{1-\theta}G(s,s)p(s)g(s)ds.\end{aligned}$$

Hence $||x_{\lambda'}|| \le R(\lambda)$. It follows the result of Lemma 2.3.

Lemma 2.4. (see [3]) Suppose $f : [0, +\infty) \to (0, +\infty)$ is continuous and increasing. For given s, s_0 and M such that $0 < s < s_0, M > 0$, then there exist $\bar{s} \in (s, s_0), \rho_0 \in (0, 1)$ such that

$$sf(x + \rho) < \bar{s}f(x), x \in [0, M], \rho \in (0, \rho_0). \tag{2.9}$$

3 Main results

In this section, we give our main results and proofs. Our approach depends on the upper and lower solutions method and the fixed point index theory. In addition, we let $c \ge 1$ when we prove the conclusion (3) of the Theorem 3.1.

Theorem 3.1. Let $(H_1) - (H_4)$ be satisfied. Then there exists $0 < \lambda^* < +\infty$ such that:

(1) the BVP $(1.1)_\lambda$ has no solution for $\lambda > \lambda^*$;

(2) the BVP $(1.1)_\lambda$ has at least one positive solution for $\lambda = \lambda^*$;

(3) the BVP $(1.1)_\lambda$ has at least two positive solutions for $0 < \lambda < \lambda^*$.

Proof. Firstly, we prove the conclusion (1) of Theorem 3.1 is held. Let $\beta(t)$ is a solution of the following BVP

$$\begin{cases} \frac{1}{p(t)}(p(t)x^{'}(t))^{'} + g(t) = 0, & 0 < t < 1, \\ ax(0) - b \lim\limits_{t \to 0^+} p(t)x^{'}(t) = 0, \\ cx(1) + d \lim\limits_{t \to 1^-} p(t)x^{'}(t) = 0. \end{cases} \tag{3.1}$$

therefore we have by Lemma 2.1 $\beta(t) = \int_0^1 G(t, s)p(s)g(s)ds$.

Let $\beta_0 = \max\limits_{t \in [0,1]} \beta(t)$. Therefore by (H_3) and (2.4) we have

$$T_\lambda^0 \beta(t) \le T_\lambda^0 \beta_0 = \int_0^1 G(t, s)\lambda p(s)g(s)f(\beta_0)ds < \beta(t), \quad \forall 0 < \lambda < \frac{1}{f(\beta_0)}.$$

This implies that $\beta(t)$ is an upper solution of T_λ^0. On the other hand, let $\alpha(t) \equiv 0, t \in [0, 1]$. Then it is clear that $\alpha(t)$ is a lower solution of T_λ^0, and $\alpha(t) < \beta(t), t \in [0, 1]$. By Lemma 2.2, T_λ^0 is completely continuous on $[\alpha, \beta]$. Therefore, T_λ^0 has a fixed point $x_\lambda \in [\alpha, \beta]$, and x_λ is a solution of the BVP $(1.1)_\lambda$ by Lemma 2.1. Hence, for any $0 < \lambda < \frac{1}{f(\beta_0)}$, we have $\lambda \in S$, which implies $S \ne \emptyset$.

On the other hand, if $\lambda_1 \in S$, then we must have $(0, \lambda_1) \subset S$. In fact, let x_{λ_1} be a solution of the BVP $(1.1)_{\lambda_1}$. Then we have by Lemma 2.1

$$x_{\lambda_1}(t) = T_{\lambda_1}^0 x_{\lambda_1}(t), t \in [0, 1].$$

Therefore, for any $\lambda \in (0, \lambda_1)$, we have by (2.7)

$$\begin{aligned} T_\lambda^0 x_{\lambda_1}(t) &= \int_0^1 G(t, s)\lambda p(s)g(s)f(x_{\lambda_1}(s)ds \\ &\le \int_0^1 G(t, s)\lambda_1 p(s)g(s)f(x_{\lambda_1}(s)ds \\ &= T_{\lambda_1}^0 x_{\lambda_1}(t) \\ &= x_{\lambda_1}(t), \end{aligned}$$

which implies that x_{λ_1} is an upper solution of T_λ^0. Combining this with the fact that $\alpha(t) \equiv 0 \ \ (t \in [0, 1])$ is a lower solution of T_λ^0, then by Lemma 2.1, the BVP $(1.1)_\lambda$ has a solution. Thus $\lambda \in S$ and we have $(0, \lambda_1) \subset S$.

Let $\lambda^* = supS$. Now we prove $\lambda^* < +\infty$. If not, then we must have $N \subset S$, where N denotes natural number set. Therefore, for any $n \in N$, by Lemma 2.1, there exists $x_n \in Q$ satisfying

$$\begin{aligned} x_n &= T_n^0 x_n \\ &= \int_0^1 G(t,s)np(s)g(s)f(x_n(s)ds. \end{aligned}$$

Let $K = [\bar{\delta}\sigma_0^{m+1}\int_\theta^{1-\theta} G(s,s)p(s)g(s)ds]^{-1}$ and $\|x_n\| \geq 1$. Then, by Lemma 2.1 and (H_3), we have

$$\begin{aligned} 1 &\geq \tfrac{1}{||x_n||} \\ &\geq \tfrac{\min\limits_{\theta\leq t\leq 1-\theta} x_n(t)}{||x_n||^2} \\ &= \tfrac{1}{||x_n||^2} \min_{\theta\leq t\leq 1-\theta} \int_0^1 G(t,s)np(s)g(s)f(x_n(s)ds \\ &\geq \tfrac{1}{||x_n||^2}\sigma_0 \int_\theta^{1-\theta} G(s,s)np(s)g(s)\bar{\delta}(x_n(s))^m ds \\ &\geq \tfrac{1}{||x_n||^2}\sigma_0^{m+1} \int_\theta^{1-\theta} G(s,s)np(s)g(s)\bar{\delta}||x_n||^m ds \\ &\geq n\sigma_0^{m+1}\bar{\delta}\int_\theta^{1-\theta} G(s,s)p(s)g(s)ds. \end{aligned}$$

If $\|x_n\| \leq 1$, then we have

$$1 \geq \|x_n\| \geq \min_{\theta\leq t\leq 1-\theta}\int_0^1 G(t,s)np(s)g(s)f(x_n(s)ds \geq \sigma_0\int_\theta^{1-\theta} G(s,s)np(s)g(s)f(0)ds.$$

Hence $n \leq \{K, [\sigma_0\int_\theta^{1-\theta} G(s,s)p(s)g(s)f(0)ds]^{-1}\}$, this contradicts the fact that N is unbounded, therefore $\lambda^* < +\infty$, and therefore the proof of the conclusion (1) is complete.

Secondly, we verify the conclusion (2) of Theorem 3.1. Let$\{\lambda_n\} \subset [\frac{\lambda^*}{2}, \lambda^*), \lambda_n \to \lambda^*(n \to \infty)$, $\{\lambda_n\}$ be an increasing sequence. Suppose x_n is solution of $(1.1)_{\lambda_n}$, by Lemma 2.3, there exists $R(\frac{\lambda^*}{2}) > 0$ such that $||x_n|| \leq R(\frac{\lambda^*}{2}), n = 1, 2, \cdots$. Hence x_n is a bounded set. It is clear that $\{x_n\}$ is equicontinuous set of C[0,1]. Therefore we have by Ascoli-Arzela theorem $\{x_n\}$ is compact set, and therefore $\{x_n\}$ has convergent subsequence. No loss of generality, we suppose x_n is convergent: $x_n \to x^*(n \to +\infty)$. Since $x_n = T_{\lambda_n}^0 x_n$, by control convergence theorem, we have $x^* = T_{\lambda^*}^0 x^*$. Therefore, by Lemma 2.1, x^* is a solution of the BVP $(1.1)_{\lambda^*}$. Hence the conclusion (2) of Theorem 3.1 is held.

Finally, we prove the conclusion (3) of Theorem 3.1.

Let $\alpha(t) \equiv 0(t \in [0,1])$. Then for any $\lambda \in (0, \lambda^*)$, $\alpha(t)$ is a lower solution of the BVP $(2.6)_\lambda^\rho$.

On the other hand, By Lemma 2.3, there exists $R(\lambda) > 0$ such that $||x_{\lambda'}|| \leq R(\lambda), \lambda^{'} \in [\lambda, \lambda^*]$, where $x_{\lambda'}$ is a solution of the BVP $(1.1)_{\lambda'}$. And by Lemma 2.4, there exist $\bar{\lambda} \in [\lambda, \lambda^*], \rho_0 \in (0,1)$ satisfying

$$\lambda f(x+\rho) < \bar{\lambda} f(x), x \in [0, R(\lambda)], \rho \in (0, \rho_0).$$

Let $x_{\bar{\lambda}}$ be a solution of the BVP $(1.1)_{\bar{\lambda}}$. Suppose $\bar{x}_\lambda(t) = x_{\bar{\lambda}} + \rho, \rho \in (0, \rho_0)$. Then

$$\begin{aligned} \bar{x}_\lambda(t) &= x_{\bar{\lambda}} + \rho \\ &= \int_0^1 G(t,s)\bar{\lambda}g(s)f(x_{\bar{\lambda}}(s))ds + \rho \\ &\geq \rho + \int_0^1 G(t,s)\lambda g(s)f(x_{\bar{\lambda}}(s)+\rho)ds \\ &\geq \rho h(t) + \int_0^1 G(t,s)\lambda g(s)f((x_{\bar{\lambda}}(s)+\rho))ds \\ &= T_\lambda^\rho \bar{x}_\lambda(t) \end{aligned}$$

Combining this with $a\bar{x}(0) - b\lim_{t\to 0^+} p(t)\bar{x}'(t) \geq 0$, $c\bar{x}(1) + d\lim_{t\to 1^-} p(t)\bar{x}'(t) \geq \rho$, we have that $\bar{x}_\lambda(t)$ is an upper solution of the BVP $(2.6)_\lambda^\rho$. Therefore the BVP $(2.6)_\lambda^\rho$ has solution and let $v_\lambda(t)$ be a solution of the BVP $(2.6)_\lambda^\rho$. Let $\Omega = \{y \in Q | y(t) < v_\lambda(t), t \in [0,1]\}$. It is clear that $\Omega \subset Q$ is a bounded open set. If $y \in \partial\Omega$, then there exists $t_0 \in [0,1]$, such that $y(t_0) = v_\lambda(t_0)$. Therefore, for any $\mu \geq 1, \rho \in (0, \rho_0), y \in \partial\Omega$, we have

$$\begin{aligned} T_\lambda^0 y(t_0) &< \rho h(t) + T_\lambda^0 y(t_0) \\ &= \rho h(t) + T_\lambda^0 v_\lambda(t_0) \\ &= T_\lambda^\rho v_\lambda(t_0) \\ &= v_\lambda(t_0) \\ &= y(t_0) \\ &\leq \mu y(t_0). \end{aligned}$$

Hence for any $\mu \geq 1$, we have $T_\lambda^0 y \neq \mu y, y \in \partial\Omega$. Therefore by Lemma 1.1 we have

$$i(T_\lambda^0, \Omega, Q) = 1. \tag{3.2}$$

It remains to prove that the conditions of Lemma 1.2 are held.

Firstly, we check the condition (1) of Lemma 1.2 is satisfied. In fact, for any $x \in Q$, by (H_4) and (2.5) we have

$$\begin{aligned} T_\lambda^0 x(\tfrac{1}{2}) &= \int_0^1 G(\tfrac{1}{2}, s)\lambda p(s)g(s)f(x(s))ds \\ &\geq \int_\theta^{1-\theta} G(\tfrac{1}{2}, s)\lambda p(s)g(s)\bar{\delta}\sigma_0^m ||x||^m ds \\ &= ||x||^m \int_\theta^{1-\theta} G(\tfrac{1}{2}, s)\lambda p(s)g(s)\bar{\delta}\sigma_0^m ds \\ &= ||x||^{m-1} \int_\theta^{1-\theta} G(\tfrac{1}{2}, s)\lambda p(s)g(s)\bar{\delta}\sigma_0^m ds ||x|| \end{aligned} \tag{3.3}$$

Taking $\bar{R} > 0$, such that $\bar{R}^{m-1}\int_\theta^{1-\theta} G(\frac{1}{2}, s)\lambda p(s)g(s)\bar{\delta}\sigma_0^m ds > 1$. Therefore, for any $R > \bar{R}$ and $B_R \subset Q$, we have by (3.3)

$$||T_\lambda^0 x|| > ||x|| > 0, x \in \partial B_R, \tag{3.4}$$

where $B_R = \{x \in Q | ||x|| < R\}$. Hence the condition (1) of Lemma 1.2 is held.

Now we prove the condition (2) of Lemma 1.2 is satisfied. In fact, if the condition (2) of Lemma 1.2 is not held, then there exist $x_1 \in Q \bigcap \partial B_R, 0 < \mu_1 \leq 1$, such that $T_\lambda^0 x_1 = \mu_1 x_1$. Therefore $||T_\lambda^0 x_1|| \leq ||x_1||$. This conflicts with (3.4). Hence the condition (2) of Lemma 1.2 is satisfied. By Lemma 1.2 we have

$$i(T_\lambda^0, B_R, Q) = 0. \tag{3.5}$$

Consequently, by the additivity of the fixed point index,

$$0 = i(T_\lambda^0, B_R, Q) = i(T_\lambda^0, \Omega, Q) + i(T_\lambda^0, B_R \setminus \bar{\Omega}, Q).$$

Since $i(T_\lambda^0, \Omega, Q) = 1, i(T_\lambda^0, B_R\bar{\Omega}, Q) = -1$. Therefore, by the solution property of the fixed point index, there is a fixed point of T_λ^0 in Ω and a fixed point of T_λ^0 in $B_R \setminus \bar{\Omega}$, respectively. Therefore the BVP $(1.1)_\lambda$ by Lemma 2.1 has at least two solutions. Furthermore, the BVP $(1.1)_\lambda$ has at least two positive solutions by $(H_1) - (H_3)$. The proof of Theorem 3.1 is complete.

Example. consider the following BVP

$$\begin{cases} \sqrt{t}(1-t)(\frac{1}{\sqrt{t(1-t)}}x'(t))' + \lambda\frac{1-t}{\sqrt[3]{t}}2^{2x} = 0, \quad 0 < t < 1, \\ x(0) = x(1) = 0. \end{cases} \tag*{$(3.6)_\lambda$}$$

where $\lambda > 0$. It is clear that the BVP $(3.6)_\lambda$ is not resolved by the results of [1-15].

Let $p(t) = \frac{1}{\sqrt{t(1-t)}}$, $g(t) = \frac{1-t}{\sqrt[3]{t}}$, $f(x) = 2^{2x}$, $0 < t < 1$, $a = c = 1$, $b = d = 0$. It is clearly that $p(t), g(t)$ are singular at $t = 0$ and/or at $t = 1$ respectively. It is easy to prove that (H_1) and (H_2) hold. By calculation, we obtain the *Green's* function of the BVP $(3.6)_\lambda$:

$$G(t,s) = \begin{cases} \frac{1}{15}s^{\frac{3}{2}}(5-3s)(2-5t^{\frac{3}{2}}+3t^{\frac{5}{2}}), & 0 \le s \le t \le 1, \\ \frac{1}{15}t^{\frac{3}{2}}(5-3t)(2-5t^{\frac{3}{2}}+3s^{\frac{5}{2}}), & 0 \le t \le s \le 1. \end{cases}$$

It is easy to see that $0 \le G(s,s) \le 1$. In addition, for $\bar{\delta} = 1 > 0, m = 2$, $f(x) = 2^{2x} = \bar{\delta}2^{2x} > x^2 = x^m > 0$. Hence (H_3) and (H_4) are held.

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