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Energy decay of solutions for a wave equation with a constant weak delay and a weak internal feedback

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Abstract. In this paper, we consider the wave equation with a weak internal constant delay term:

$$u''(x,t) - \Delta_x u(x,t) + \mu_1(t)\, u'(x,t) + \mu_2(t)\, u'(x,t-\tau) = 0$$

in a bounded domain. Under appropriate conditions on μ_1 and μ_2, we prove global existence of solutions by the Faedo–Galerkin method and establish a decay rate estimate for the energy using the multiplier method.

Keywords: nonlinear wave equation, delay term, decay rate, multiplier method.

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1 Introduction

In this paper we investigate the decay properties of solutions for the initial boundary value problem for the linear wave equation of the form

$$\begin{cases} u''(x,t) - \Delta_x u(x,t) + \mu_1(t)\, u'(x,t) + \mu_2(t)\, u'(x,t-\tau) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) & \text{on } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Omega\times]0,\tau[, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau > 0$ is a time delay and the initial data (u_0, u_1, f_0) belong to a suitable function space.

In absence of delay ($\mu_2 = 0$), the energy of problem (P) is exponentially decaying to zero provided that μ_1 is constant, see, for instance, [3, 4, 7, 8] and [12]. On the contrary, if $\mu_1 = 0$ and $\mu_2 > 0$ (a constant weight), that is, there exists only the internal delay, the system (P)

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becomes unstable (see, for instance, [5]). In recent years, the PDEs with time delay effects have become an active area of research since they arise in many practical problems (see, for example, [1, 19]). In [5], it was shown that a small delay in a boundary control could turn a well-behaved hyperbolic system into a wild one and, therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [13, 15, 20]). For instance, the authors of [13] studied the wave equation with a linear internal damping term with constant delay ($\tau = \text{const}$ in the problem (P) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be unstable if $\mu_2 \geq \mu_1$. The main approach used in [13] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting on the boundary. We also recall the result by Xu, Yung and Li [20], where the authors proved a result similar to the one in [13] for the one-space dimension by adopting the spectral analysis approach.

In [17], Nicaise, Pignotti and Valein extended the above result to higher space dimensions and established an exponential decay.

Our purpose in this paper is to give an energy decay estimate of the solution of problem (P) in the presence of a delay term with a weight depending on time. We use the Galerkin approximation scheme and the multiplier technique to prove our results.

2 Preliminaries and main results

First assume the following hypotheses:

(H1) $\mu_1 : \mathbb{R}_+ \to]0, +\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_1(t)}\right| \leq M, \tag{2.1}$$

(H2) $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta \mu_1(t), \tag{2.2}$$
$$|\mu_2'(t)| \leq \tilde{M} \mu_1(t), \tag{2.3}$$

for some $0 < \beta < 1$ and $\tilde{M} > 0$.

We now state a Lemma needed later.

Lemma 2.1 (Martinez [10]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Assume that there exist $\sigma > -1$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,dt \leq \frac{1}{\omega} E^{\sigma}(0)E(S), \quad 0 \leq S < +\infty. \tag{2.4}$$

Then

$$E(t) = 0 \quad \forall t \geq \frac{E(0)^{\sigma}}{\omega|\sigma|}, \quad \text{if} \quad -1 < \sigma < 0, \tag{2.5}$$

$$E(t) \leq E(0)\left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{\frac{1}{\sigma}} \forall t \geq 0, \quad \text{if} \quad \sigma > 0, \tag{2.6}$$

$$E(t) \leq E(0)e^{1-\omega\phi(t)} \forall t \geq 0, \quad \text{if} \quad \sigma = 0. \tag{2.7}$$

We introduce, as in [13], the new variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1), \quad t > 0. \tag{2.8}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ \text{ in } \Omega \times (0,1) \times (0,+\infty). \tag{2.9}$$

Therefore, problem (P) takes the form:

$$\begin{cases} u''(x,t) - \Delta_x u(x,t) + \mu_1(t)u'(x,t) + \mu_2(t)z(x,1,t) = 0, & x \in \Omega, t > 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \Omega, \rho \in (0,1), t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ z(x,0,t) = u'(x,t), & x \in \Omega, t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ z(x,\rho,0) = f_0(x,-\tau\rho), & x \in \Omega, \rho \in (0,1). \end{cases} \tag{2.10}$$

Let $\overline{\xi}$ be a positive constant such that

$$\tau\beta < \overline{\xi} < \tau(2-\beta). \tag{2.11}$$

We define the energy of the solution by:

$$E(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\|\nabla_x u(t)\|_2^2 + \frac{\xi(t)}{2}\int_\Omega \int_0^1 z^2(x,\rho,t)\,d\rho\,dx, \tag{2.12}$$

where

$$\xi(t) = \overline{\xi}\mu_1(t).$$

We have the following theorem.

Theorem 2.2. *Let $(u_0,u_1,f_0) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega) \times H_0^1(\Omega;H^1(0,1))$ satisfy the compatibility condition*

$$f_0(\cdot,0) = u_1.$$

Assume that $(H1)$ and $(H2)$ hold. Then problem (P) admits a unique global weak solution

$$u \in L^\infty_{\text{loc}}((-\tau,\infty);\ H^2(\Omega)\cap H_0^1(\Omega)),\ u' \in L^\infty_{\text{loc}}((-\tau,\infty);\ H_0^1(\Omega)),\ u'' \in L^\infty_{\text{loc}}((-\tau,\infty);\ L^2(\Omega)).$$

Moreover, for some positive constants c, ω, we obtain the following decay property:

$$E(t) \leq cE(0)e^{-\omega\int_0^t \mu_1(s)\,ds}, \qquad \forall\, t \geq 0. \tag{2.13}$$

Lemma 2.3. *Let (u,z) be a solution to the problem (2.10). Then, the energy functional defined by (2.12) satisfies*

$$\begin{aligned} E'(t) &\leq -\left(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\|u_t(x,t)\|_2^2 - \left(\frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\|z(x,1,t)\|_2^2 \\ &\leq 0. \end{aligned} \tag{2.14}$$

Proof. Multiplying the first equation in (2.10) by $u_t(x,t)$, integrating over Ω and using Green's identity, we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\|u_t(x,t)\|_2^2 + \frac{1}{2}\frac{\partial}{\partial t}\|\nabla u(x,t)\|_2^2 \\ &\quad + \mu_1(t)\|u_t(x,t)\|_2^2 + \mu_2(t)\int_\Omega u_t(x,t-\tau)u_t(x,t)\,dx = 0. \end{aligned} \tag{2.15}$$

We multiply the second equation in (2.10) by $\xi(t)z$ and integrate over $\Omega\times(0,1)$ to obtain:

$$\xi(t)\tau\int_\Omega\int_0^1 z_t(x,\rho,t)z(x,\rho,t)\,d\rho\,dx + \xi(t)\int_\Omega\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t))\,d\rho\,dx = 0. \tag{2.16}$$

This yields

$$\frac{\xi(t)\tau}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx + \frac{\xi(t)}{2}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t))\,d\rho\,dx = 0,$$

which gives

$$\begin{aligned} &\frac{\tau}{2}\left[\frac{d}{dt}\left(\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right) - \xi'(t)\int_0^1\int_\Omega z^2(x,\rho,t)\,d\rho dx\right] \\ &\qquad + \frac{\xi(t)}{2}\int_\Omega z^2(x,1,t)\,dx - \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx = 0. \end{aligned}$$

Consequently,

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\left(\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right) \\ &\quad = \frac{\tau}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx - \frac{\xi(t)}{2}\int_\Omega z^2(x,1,t)\,dx + \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx. \end{aligned} \tag{2.17}$$

Combination of (2.15) and (2.17) leads to

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left[\|u_t(x,t)\|_2^2 + \|\nabla u(x,t)\|_2^2 + \xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right] \\ &\quad = -\mu_1(t)\|u_t(x,t)\|_2^2 - \mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\,dx \\ &\qquad + \frac{1}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx - \frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)\,dx + \frac{\xi(t)}{2\tau}\|u_t(x,t)\|_2^2. \end{aligned}$$

Recalling the definition of $E(t)$ in (2.12), we arrive at

$$\begin{aligned} E'(t) = & -\left(\mu_1(t) - \frac{\xi(t)}{2\tau}\right)\|u_t(x,t)\|_2^2 - \mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\,dx \\ & + \frac{1}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx - \frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)\,dx. \\ \leq & -\left(\mu_1(t) - \frac{\xi(t)}{2\tau}\right)\|u_t(x,t)\|_2^2 - \mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\,dx \\ & - \frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)\,dx. \end{aligned} \tag{2.18}$$

Due to Young's inequality, we have

$$\int_\Omega z(x,1,t)u_t(x,t)\,dx \leq \frac{1}{2}\|u_t(x,t)\|_2^2 + \frac{1}{2}\|z(x,1,t)\|_2^2. \tag{2.19}$$

Inserting (2.19) into (2.18), we obtain

$$\begin{aligned} E'(t) &\leq -\left(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\|u_t(x,t)\|_2^2 - \left(\frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\|z(x,1,t)\|_2^2 \\ &\leq -\mu_1(t)\left(1 - \frac{\overline{\xi}}{2\tau} - \frac{\beta}{2}\right)\|u_t(x,t)\|_2^2 - \mu_1(t)\left(\frac{\overline{\xi}}{2\tau} - \frac{\beta}{2}\right)\|z(x,1,t)\|_2^2 \leq 0. \end{aligned} \tag{2.20}$$

This completes the proof of the lemma. □

3 Global existence

Throughout this section we assume $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, $f_0 \in L^2(\Omega; H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2(\Omega)\cap H_0^1(\Omega)$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2(\Omega\times(0,1))$ such that $(\phi_j)_j$ form a basis of $L^2(\Omega; H^1(0,1))$ and denote by Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(u_k, z_k), k = 1,2,3,\ldots$, in the form

$$u_k(t) = \sum_{j=1}^k g_{jk}(t)w_j, \quad z_k(t) = \sum_{j=1}^k h_{jk}(t)\phi_j,$$

where g_{jk} and h_{jk} $(j = 1,2,\ldots,k)$ are determined by the following system of ordinary differential equations:

$$\begin{cases} (u_k''(t), w_j) + (\nabla_x u_k(t), \nabla_x w_j) + \mu_1(t)(u_k', w_j) + \mu_2(t)(z_k(.,1), w_j) = 0, \\ 1 \leq j \leq k, \\ z_k(x,0,t) = u_k'(x,t), \end{cases} \tag{3.1}$$

associated with the initial conditions

$$u_k(0) = u_{0k} = \sum_{j=1}^{k}(u_0, w_j)w_j \to u_0 \text{ in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \to +\infty, \tag{3.2}$$

$$u_k'(0) = u_{1k} = \sum_{j=1}^{k}(u_1, w_j)w_j \to u_1 \text{ in } H_0^1(\Omega) \text{ as } k \to +\infty, \tag{3.3}$$

and

$$\begin{cases} (\tau z_{kt} + z_{k\rho}, \phi_j) = 0, \\ \qquad 1 \le j \le k, \end{cases} \tag{3.4}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k}(f_0, \phi_j)\phi_j \to f_0 \text{ in } L^2(\Omega; H^1(0,1)) \text{ as } k \to +\infty. \tag{3.5}$$

By virtue of the theory of ordinary differential equations, the system (3.1)–(3.5) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \le +\infty$) by Zorn lemma. Note that $u_k(t)$ is of class C^2.

In the next step, we obtain a priori estimates for the solution of the system (3.1)–(3.5), so that it can be extended beyond $[0, T_k[$ to obtain a solution defined for all $t > 0$. Then, we utilize a standard compactness argument for the limiting procedure.

The first estimate. Since the sequences u_{0k}, u_{1k} and z_{0k} converge, then from (2.14) we can find a positive constant C independent of k such that

$$E_k(t) + \int_0^t a_1(s)\|u_k'(s)\|_2^2\, ds + \int_0^t a_2(s)\|z_k(x,1,s)\|_2^2\, ds \le E_k(0) \le C, \tag{3.6}$$

where

$$E_k(t) = \frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\|\nabla_x u_k(t)\|_2^2 + \frac{\xi(t)}{2}\int_\Omega \int_0^1 z_k^2(x,\rho,t)\, d\rho\, dx,$$

$$a_1(t) = \mu_1(t)\left(1 - \frac{\overline{\xi}}{2\tau} - \frac{\beta}{2}\right) \text{ and } a_2(t) = \mu_1(t)\left(\frac{\overline{\xi}}{2\tau} - \frac{\beta}{2}\right).$$

These estimates imply that the solution (u_k, z_k) exists globally in $[0, +\infty[$.

Estimate (3.6) yields

$$(u_k) \text{ is bounded in } L^\infty_{\text{loc}}(0, \infty; H_0^1(\Omega)), \tag{3.7}$$

$$(u_k') \text{ is bounded in } L^\infty_{\text{loc}}(0, \infty; L^2(\Omega)), \tag{3.8}$$

$$\mu_1(t)(u_k'^2(t)) \text{ is bounded in } L^1(\Omega \times (0,T)), \tag{3.9}$$

$$\mu_1(t)(z_k^2(x,\rho,t)) \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty; L^1(\Omega \times (0,1))), \tag{3.10}$$

$$\mu_1(t)(z_k^2(x,1,t)) \text{ is bounded in } L^1(\Omega \times (0,T)). \tag{3.11}$$

The second estimate. We first estimate $u_k''(0)$. Replacing w_j by $u_k''(t)$ in (3.1) and taking $t = 0$, we obtain:

$$\begin{aligned} \|u_k''(0)\|_2 &\le \|\Delta_x u_{0k}\|_2 + \mu_1(0)\|u_{1k}\|_2 + |\mu_2(0)|\|z_{0k}\|_2 \\ &\le \|\Delta_x u_0\|_2 + \mu_1(0)\|u_1\|_2 + |\mu_2(0)|\|z_0\|_2 \\ &\le C. \end{aligned}$$

Differentiating (3.1) with respect to t, we get

$$(u_k'''(t) + \Delta_x u_k'(t) + \mu_1(t)u_k''(t) + \mu_1'(t)u_k'(t) + \mu_2(t)z_k'(1,t) + \mu_2'(t)z_k(1,t), w_j) = 0.$$

Multiplying by $g_{jk}''(t)$, summing over j from 1 to k, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u_k''(t)\|_2^2 + \|\nabla_x u_k'(t)\|_2^2\right) + \mu_1(t)\int_\Omega u''^2_k(t)\,dx + \mu_1'(t)\int_\Omega u''_k(t)u_k'(t)\,dx \\ + \mu_2(t)\int_\Omega u''_k(t)z_k'(x,1,t)\,dx + \mu_2'(t)\int_\Omega u''_k(t)z_k(x,1,t)\,dx = 0.\end{aligned} \tag{3.12}$$

Differentiating (3.4) with respect to t, we get

$$\left(\tau z_k''(t) + \frac{\partial}{\partial\rho}z_k', \phi_j\right) = 0.$$

Multiplying by $h_{jk}'(t)$, summing over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|z_k'(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z_k'(t)\|_2^2 = 0. \tag{3.13}$$

Taking the sum of (3.12) and (3.13), we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u_k''(t)\|_2^2 + \|\nabla_x u_k'(t)\|_2^2 + \tau\int_0^1 \|z_k'(x,\rho,t)\|^2_{L^2(\Omega)}\,d\rho\right) \\ &\qquad +\mu_1(t)\int_\Omega u''^2_k(t)\,dx + \frac{1}{2}\int_\Omega |z_k'(x,1,t)|^2\,dx \\ &\quad = -\mu_2(t)\int_\Omega u''_k(t)z_k'(x,1,t)\,dx - \mu_1'(t)\int_\Omega u''_k(t)u_k'(t)\,dx \\ &\qquad -\mu_2'(t)\int_\Omega u''_k(t)z_k(x,1,t)\,dx + \frac{1}{2}\|u_k''(t)\|_2^2\end{aligned}$$

Using (**H1**), (**H2**), Cauchy–Schwarz and Young's inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u_k''(t)\|_2^2 + \|\nabla_x u_k'(t)\|_2^2 + \int_0^1 \tau\|z_k'(x,\rho,t)\|^2_{L^2(\Omega)}\,d\rho\right) \\ &\qquad +\mu_1(t)\int_\Omega u''^2_k(t)\,dx + \frac{1}{2}\int_\Omega |z_k'(x,1,t)|^2\,dx \\ &\quad \le |\mu_2(t)|\|u_k''(t)\|_2\|z_k'(x,1,t)\|_2 + |\mu_1'(t)|\|u_k''(t)\|_2\|u_k'(t)\|_2 \\ &\qquad +|\mu_2'(t)|\|u_k''(t)\|_2\|z_k(x,1,t)\|_2 + \frac{1}{2}\|u_k''(t)\|_2^2 \\ &\quad \le \frac{|\mu_2(t)|^2}{2}\|u_k''(t)\|_2^2 + \frac{1}{2}\|z_k'(x,1,t)\|_2^2 + \frac{|\mu_1'(t)|}{4}\|u_k''(t)\|_2^2 + |\mu_1'(t)|\|u_k'(t)\|_2^2 \\ &\qquad +\frac{|\mu_2'(t)|}{4}\|u_k''(t)\|_2^2 + |\mu_2'(t)|\|z_k(x,1,t)\|_2^2 + \frac{1}{2}\|u_k''(t)\|_2^2 \\ &\quad \le c'\|u_k''(t)\|_2^2 + |\mu_1'(t)|\|u_k'\|_2^2 + |\mu_2'(t)|\|z_k(x,1,t)\|_2^2 + \frac{1}{2}\|z_k'(x,1,t)\|_2^2. \\ &\quad \le c'\|u_k''(t)\|_2^2 + M\mu_1(t)\|u_k'\|_2^2 + \tilde{M}\mu_1(t)\|z_k(x,1,t)\|_2^2 + \frac{1}{2}\|z_k'(x,1,t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using (3.6), we get

$$\begin{aligned}
&\left(\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\tau\int_0^1\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\,d\rho\right)\\
&\quad\le\left(\|u_k''(0)\|_2^2+\|\nabla_x u_k'(0)\|_2^2+\tau\int_0^1\|z_k'(x,\rho,0)\|_{L^2(\Omega)}^2\,d\rho\right)+2M\int_0^t\mu_1(s)\|u_k'(s)\|_2^2\,ds\\
&\qquad+2\tilde{M}\int_0^t\mu_1(s)\|z_k(x,1,s)\|_2^2\,ds+2c'\int_0^t\|u_k''(s)\|_2^2\,ds\\
&\quad\le C+C'\int_0^t\left(\|u_k''(s)\|_2^2+\|\nabla_x u_k'(s)\|_2^2+\tau\int_0^1\|z_k'(x,\rho,s)\|_{L^2(\Omega)}^2\,d\rho\right)ds.
\end{aligned}$$

Using Gronwall's lemma, we deduce that

$$\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\tau\int_0^1\|z_k'(x,\rho,t)\|_{L^2(\Omega)}^2\,d\rho\le Ce^{C'T}$$

for all $t\in\mathbb{R}^+$, therefore, we conclude that

$$(u_k'') \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)), \tag{3.14}$$
$$(u_k') \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;H_0^1(\Omega)), \tag{3.15}$$
$$(\tau z_k') \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega\times(0,1))). \tag{3.16}$$

Applying Dunford–Pettis' theorem, we deduce from (3.7), (3.8), (3.9), (3.10), (3.11), (3.14), (3.15) and (3.16), replacing the sequence u_k with a subsequence if necessary, that

$$u_k\to u \text{ weak-star in } L^\infty_{\text{loc}}(0,\infty;H^2(\Omega)\cap H_0^1(\Omega)), \tag{3.17}$$
$$u_k'\to u' \text{ weak-star in } L^\infty_{\text{loc}}(0,\infty;H_0^1(\Omega)),$$
$$u''_k\to u'' \text{ weak-star in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)), \tag{3.18}$$
$$u_k'\to\chi \text{ weak in } L^2(\Omega\times(0,T);\mu_1(t)),$$
$$z_k\to z \text{ weak-star in } L^\infty_{\text{loc}}(0,\infty;H_0^1(\Omega;L^2(0,1))),$$
$$z_k'\to z' \text{ weak-star in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega\times(0,1))), \tag{3.19}$$
$$z_k(x,1,t)\to\psi \text{ weak in } L^2(\Omega\times(0,T),\mu_1(t))$$

for suitable functions

$$u\in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)),\quad z\in L^\infty(0,T;L^2(\Omega\times(0,1))),$$
$$\chi\in L^2(\Omega\times(0,T);\mu_1(t)),\qquad \psi\in L^2(\Omega\times(0,T);\mu_1(t)),$$

for all $T\ge 0$. We have to show that u is a solution of (P).

From (3.15) we have that (u_k') is bounded in $L^\infty(0,T;H_0^1(\Omega))$. Then (u_k') is bounded in $L^2(0,T;H_0^1(\Omega))$. Since (u_k'') is bounded in $L^\infty(0,T;L^2(\Omega))$, then it is bounded in $L^2(0,T;L^2(\Omega))$, too. Consequently, (u_k') is bounded in $H^1(Q)$.

Since the embedding $H^1(Q)\hookrightarrow L^2(Q)$ is compact, using the Aubin–Lions theorem [9], we can extract a subsequence (u_ς) of (u_k) such that

$$u_\varsigma'\to u' \text{ strongly in } L^2(Q). \tag{3.20}$$

Therefore

$$u_\varsigma'\to u' \text{ strongly and a.e. in } Q. \tag{3.21}$$

Similarly we obtain

$$z_\varsigma \to z \text{ strongly in } L^2(\Omega\times(0,1)\times(0,T)) \tag{3.22}$$

and

$$z_\varsigma \to z \text{ strongly and a.e. in } \Omega\times(0,1)\times(0,T). \tag{3.23}$$

It follows at once from (3.17), (3.18), (3.19), (3.20) and (3.22) that for each fixed $v\in L^2(0,T;L^2(\Omega))$ and $w\in L^2(0,T;L^2(\Omega)\times(0,1))$

$$\begin{aligned}\int_0^T\int_\Omega \big(u_\varsigma'' - \Delta_x u_\varsigma + \mu_1(t)u_\varsigma' + \mu_2(t)z_\varsigma\big)v\,dx\,dt \\ \to \int_0^T\int_\Omega \big(u'' - \Delta_x u + \mu_1(t)u' + \mu_2(t)z\big)v\,dx\,dt,\end{aligned} \tag{3.24}$$

$$\int_0^T\int_0^1\int_\Omega \Big(\tau z_\varsigma' + \tfrac{\partial}{\partial\rho}z_\varsigma\Big)w\,dx\,d\rho\,dt \to \int_0^T\int_0^1\int_\Omega \Big(\tau z' + \tfrac{\partial}{\partial\rho}z\Big)w\,dx\,d\rho\,dt \tag{3.25}$$

as $\varsigma\to+\infty$. Thus the problem (P) admits a global weak solution u.

Uniqueness. Let (u_1,z_1) and (u_2,z_2) be two solutions of problem (2.10). Then $(w,\tilde w) = (u_1,z_1)-(u_2,z_2)$ satisfies

$$\begin{cases} w''(x,t) - \Delta_x w(x,t) + \mu_1(t)w'(x,t) + \mu_2(t)\tilde w(x,1,t) = 0, & \text{in } \Omega\times]0,+\infty[,\\ \tau\tilde w'(x,\rho,t) + \tilde w_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+\infty[\\ w(x,t) = 0, & \text{on } \partial\Omega\times]0,+\infty[\\ \tilde w(x,0,t) = w'(x,t), & \text{on } \Omega\times]0,+\infty[\\ w(x,0) = 0,\ w'(x,0) = 0, & \text{in } \Omega\\ \tilde w(x,\rho,0) = 0, & \text{in } \Omega\times]0,1[\end{cases} \tag{3.26}$$

Multiplying the first equation in (3.26) by w', integrating over Ω and using integration by parts, we get

$$\frac12\frac{d}{dt}(\|w'\|_2^2 + \|\nabla_x w\|_2^2) + \mu_1(t)\|w'\|_2^2 + \mu_2(t)(\tilde w(x,1,t),w') = 0. \tag{3.27}$$

Multiplying the second equation in (3.26) by $\tilde w$, integrating over $\Omega\times(0,1)$ and using integration by parts, we get

$$\frac{\tau}{2}\frac{d}{dt}\|\tilde w\|_2^2 + \frac12\frac{d}{d\rho}\|\tilde w\|_2^2 = 0. \tag{3.28}$$

Then

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1\|\tilde w\|_2^2\,d\rho + \frac12(\|\tilde w(x,1,t)\|_2^2 - \|w'\|_2^2) = 0. \tag{3.29}$$

From (3.27), (3.29), using the Cauchy–Schwarz inequality we get

$$\begin{aligned}&\frac12\frac{d}{dt}\left(\|w'\|_2^2 + \|\nabla_x w\|_2^2 + \tau\int_0^1\|\tilde w\|_2^2\,d\rho\right) + \mu_1(t)\|w'\|_2^2 + \frac12\|\tilde w(x,1,t)\|_2^2\\ &\quad= -\mu_2(t)(\tilde w(x,1,t),w') + \frac12\|w'\|_2^2\\ &\quad\le |\mu_2(t)|\|\tilde w(x,1,t)\|_2\|w'\|_2 + \frac12\|w'\|_2^2.\end{aligned}$$

Using Young's inequality, we obtain

$$\frac12\frac{d}{dt}\left(\|w'\|_2^2 + \|\nabla_x w\|_2^2 + \tau\int_0^1\|\tilde w\|_2^2\,d\rho\right) \le c\|w'\|_2^2,$$

where c is a positive constant. Then integrating over $(0,t)$, using Gronwall's lemma, we conclude that

$$\|w'\|_2^2 + \|\nabla_x w\|_2^2 + \tau \int_0^1 \|\tilde{w}\|_2^2 \, d\rho = 0.$$

Hence, uniqueness follows.

4 Asymptotic behavior

From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (2.10) by $\phi' E^q u$, where ϕ is a bounded function satisfying all the hypotheses of Lemma 2.1. We obtain

$$\begin{aligned}
0 = & \int_S^T E^q \phi' \int_\Omega u \Big(u'' - \Delta u + \mu_1(t) u' + \mu_2(t) z(x,1,t) \Big) \, dx \, dt \\
= & \left[E^q \phi' \int_\Omega u u' \, dx \right]_S^T - \int_S^T (q E' E^{q-1} \phi' + E^q \phi'') \int_\Omega u u' \, dx \, dt \\
& - 2 \int_S^T E^q \phi' \int_\Omega u'^2 \, dx \, dt + \int_S^T E^q \phi' \int_\Omega \left(u'^2 + |\nabla u|^2 \right) dx dt \\
& + \int_S^T E^q \phi' \mu_1(t) \int_\Omega u u' \, dx \, dt + \int_S^T E^q \phi' \mu_2(t) \int_\Omega u z(x,1,t) \, dx \, dt.
\end{aligned}$$

Similarly, we multiply the second equation of (2.10) by $E^q \phi' \xi(t) e^{-2\tau\rho} z(x,\rho,t)$ and get

$$\begin{aligned}
0 = & \int_S^T E^q \phi' \int_\Omega \int_0^1 e^{-2\tau\rho} \xi(t) z \Big(\tau z_t + z_\rho \Big) \, dx \, d\rho \, dt \\
= & \left[\frac{1}{2} E^q \phi' \xi(t) \tau \int_\Omega \int_0^1 e^{-2\tau\rho} z^2 \, dx \, d\rho \right]_S^T - \frac{1}{2} \int_S^T \int_\Omega \int_0^1 \left(E^q \phi' \xi(t) \tau e^{-2\tau\rho} \right)' z^2 \, dx \, d\rho \, dt \\
& + \int_S^T E^q \phi' \int_\Omega \int_0^1 \xi(t) \left(\frac{1}{2} \frac{\partial}{\partial \rho} \left(e^{-2\tau\rho} z^2 \right) + \tau e^{-2\tau\rho} z^2 \right) dx \, d\rho \, dt \\
= & \left[\frac{1}{2} E^q \phi' \xi(t) \tau \int_\Omega \int_0^1 e^{-2\tau(t)\rho} z^2 \, dx \, d\rho \right]_S^T - \frac{\tau}{2} \int_S^T \left(E^q \phi' \xi(t) \right)' \int_\Omega \int_0^1 e^{-2\tau\rho} z^2 \, dx \, d\rho \, dt \\
& + \frac{1}{2} \int_S^T E^q \phi' \xi(t) \int_\Omega \Big(e^{-2\tau} z^2(x,1,t) - z^2(x,0,t) \Big) dx \, dt + \int_S^T E^q \phi' \xi(t) \tau \int_0^1 \int_\Omega e^{-2\tau\rho} z^2 \, dx \, d\rho \, dt.
\end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned}
A \int_S^T E^{q+1} \phi' \, dt \leq & - \left[E^q \phi' \int_\Omega u u' \, dx \right]_S^T + \int_S^T (q E' E^{q-1} \phi' + E^q \phi'') \int_\Omega u u' \, dx \, dt \\
& + 2 \int_S^T E^q \phi' \int_\Omega u'^2 \, dx \, dt - \int_S^T \mu_1(t) E^q \phi' \int_\Omega u u' \, dx \, dt \\
& - \int_S^T \mu_2(t) E^q \phi' \int_\Omega u z(x,1,t) \, dx \, dt - \left[\frac{1}{2} E^q \phi' \xi(t) \tau \int_\Omega \int_0^1 e^{-2\tau\rho} z^2 \, dx \, d\rho \right]_S^T \\
& + \frac{1}{2} \int_S^T \left(E^q \phi' \xi(t) \right)' \tau \int_\Omega \int_0^1 e^{-2\tau\rho} z^2 \, dx \, d\rho \, dt \\
& - \frac{1}{2} \int_S^T E^q \phi' \xi(t) \int_\Omega \Big(e^{-2\tau(t)} z^2(x,1,t) - z^2(x,0,t) \Big) \, dx \, dt,
\end{aligned} \tag{4.1}$$

where $A = 2\min\{1, e^{-2\tau_1}\}$. Using the Cauchy–Schwarz and Poincaré's inequalities and the definition of E and assuming that ϕ' is a bounded non-negative function on $\mathbb{R}^+$, we get

$$\left| E^q(t)\phi' \int_\Omega uu'\,dx \right| \leq cE(t)^{q+1}.$$

By recalling (2.14), we have

$$\begin{aligned}\int_S^T \left| qE'E^{q-1}\phi' \int_\Omega uu'\,dx \right| dt &\leq c\int_S^T E^q(t)|E'(t)|\,dt \leq c\int_S^T E^q(t)(-E'(t))\,dt \\ &\leq cE^{q+1}(S),\end{aligned}$$

$$\int_S^T E^q\phi'' \int_\Omega uu'\,dx\,dt \leq c\int_S^T E^{q+1}(t)(-\phi'') \leq cE^{q+1}(S)\int_S^T (-\phi'')\,dt \leq cE^{q+1}(S),$$

and

$$\begin{aligned}\int_S^T E^q\phi' \int_\Omega u'^2\,dxdt &\leq c\int_S^T E^q\phi' \frac{1}{\mu_1(t)} \int_\Omega \mu_1(t)u'^2\,dxdt \\ &\leq \int_S^T E^q \frac{\phi'}{\mu_1(t)} \left(-E'\right)\,dt. \end{aligned} \tag{4.2}$$

Define

$$\phi(t) = \int_0^t \mu_1(s)\,ds. \tag{4.3}$$

It is clear that ϕ is a non-decreasing function of class C^1 on $\mathbb{R}^+$, ϕ is bounded and

$$\phi(t) \to +\infty \text{ as } t \to +\infty. \tag{4.4}$$

So, we deduce, from (4.2), that

$$\int_S^T E^q\phi' \int_\Omega u'^2\,dxdt \leq c\int_S^T E^q(-E')\,dt \leq cE^{q+1}(S), \tag{4.5}$$

By the hypothesis (**H1**), Young's and Poincaré's inequality and (2.14), we have

$$\begin{aligned}\left| \int_S^T E^q\phi' \int_\Omega uu'\,dx\,dt \right| &\leq c\int_S^T E^q\phi' \|u\|_2\|u'\|_2\,dt \\ &\leq c\varepsilon' \int_S^T E^q\phi' \|u\|_2^2\,dt + c(\varepsilon')\int_S^T E^q\phi'\|u'\|_2^2\,dt \\ &\leq \varepsilon' c_* \int_S^T E^q\phi' \|\nabla_x u\|_2^2\,dt + c(\varepsilon')\int_S^T E^q\phi'\|u'\|_2^2\,dt \\ &\leq \varepsilon' c_* \int_S^T E^{q+1}\phi'\,dt + cE^{q+1}(S).\end{aligned}$$

Recalling that $\xi' \leq 0$ and the definition of E, we have

$$\begin{aligned}\int_S^T \left(E^q\xi(t)\right)'\tau \int_\Omega \int_0^1 e^{-2\tau\rho}z^2\,dx\,d\rho\,dt &\leq \int_S^T \left(E^q\right)'\xi(t)\tau \int_\Omega \int_0^1 e^{-2\tau\rho}z^2\,dx\,d\rho\,dt \\ &\leq c\int_S^T E^q|E'|\,dt \\ &\leq c\int_S^T E^q(-E'(t))\,dt \\ &\leq cE^{q+1}(S),\end{aligned}$$

$$\begin{aligned}\int_S^T E^q\xi(t)\int_\Omega e^{-2\tau}z^2(x,1,t)\,dx\,dt &\le c\int_S^T E^q\xi(t)\int_\Omega z^2(x,1,t)\,dx\,dt\\ &\le c\int_S^T E^q\left(-E'\right)\,dt\\ &\le cE^{q+1}(S),\end{aligned}$$

$$\begin{aligned}\int_S^T E^q\xi(t)\int_\Omega z^2(x,0,t)\,dx\,dt &= \int_S^T E^q\xi(t)\int_\Omega u'^2(x,t)\,dx\,dt\\ &\le cE^{q+1}(S),\end{aligned}$$

where we have also used the Cauchy–Schwarz inequality. Combining these estimates and choosing ε' sufficiently small, we conclude from (4.1) that

$$\int_S^T E^{q+1}\phi'\,dt \le CE^{q+1}(S).$$

This ends the proof of Theorem 2.2.

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