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# Existence of solutions for fourth order elliptic equations of Kirchhoff type on $R^N$

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**Abstract.** In this paper, we study the positive solutions to a class of fourth order elliptic equations of Kirchhoff type on $R^N$ by using variational methods and the truncation method.

**Keywords:** Kirchhoff type problems, variational methods, truncation method.

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## 1 Introduction

The purpose of this work is to study the existence of positive solutions for the fourth order elliptic equations:

$$\Delta^2 u - \left(a + b\int_{R^N} |\nabla u|^2\,dx\right)\Delta u + cu = f(u) \tag{1.1}$$

where $N > 4$, $\Delta^2$ is the biharmonic operator, and $\nabla u$ denotes the spatial gradient of $u$, and $a$, $b$, $c$ are positive constants. Usually, the proof is based on either variational approach or topological methods. For example, in [7, 8, 13], T. F. Ma, F. Wang et al. applied the variational methods to study the existence and multiplicity of solutions for a nonlocal fourth order equation of Kirchhoff type:

$$\begin{cases} u'''' - M\left(\int_0^1 |u'|^2\,dx\right)u'' = h(x)f(x,u), \\ u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0, \end{cases}$$

and

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u), \quad \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N$ is a bounded smooth domain, $M\colon R \to R$ is continuous, and satisfies

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(H) For some $m_0 > 0$, $M(t) > m_0$, $\forall\, t > 0$. In addition, there exist $m' > m_0$ and $t_0 > 0$, such that $M(t) = m'$, $\forall\, t > t_0$.

In [10], by using the fixed point theorems in cones of ordered Banach spaces, T. F. Ma studied the existence of positive solutions for

$$u'''' - M\left(\int_0^1 |u'|^2\,dx\right) u'' = h(x)f(x,u,u').$$

In [9, 11] also fourth order problems with nonlinear boundary conditions are studied. In this case the Kirchhoff function is possibly degenerate and multiplies lower order terms rather than the leading fourth order term. More recently, in [14], F. Wang et al. studied the existence of nontrivial solutions for the fourth order elliptic equations:

$$\begin{cases} \Delta^2 u - \lambda \left(a + b\int_\Omega |\nabla u|^2\,dx\right) \Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\lambda$ is a positive parameter, $a$, $b$ are positive constants, $\Omega \subset R^N$ is a bounded smooth domain, $f\colon \Omega \times R \to R$ is locally Lipschitz continuous. The authors show that there exists a $\lambda^*$ such that the fourth order elliptic equation has nontrival solutions for $0 < \lambda < \lambda^*$ by using the mountain pass techniques and the truncation method.

In addition, the problem treated here presents the Kirchhoff function multiplying a lower order term, while in general it multiplies the leading (fourth order) operator. Some results related to problems involving Kirchhoff functions in front of lower order terms are obtained in [2, 3]. Some other Kirchhoff problems are also been studied. For example, in [1, 5, 6, 15], the authors studied the existence of positive solutions of second order non-degenerate Kirchhoff-type problems; in [4], F. Colasuonno and P. Pucci studied a higher order elliptic Kirchhoff equation, under Dirichlet boundary conditions. The novelty there is to take the Kirchhoff function possibly zero at zero, that is to cover also the degenerate case.

The object of this paper is to study the existence of a positive solution to the fourth order elliptic equation (1.1) of Kirchhoff type on $R^N$ by using variational methods. In particular, we use a cut-off functional to obtain bounded (PS)-sequences. The main result can be described as follows.

**Theorem 1.1.** *Assume that the following conditions hold:*

*(H1)* $f\colon R^+ \to R^+$ *is continuous,* $f(t) \equiv 0$, *if* $t \le 0$ *and satisfies*

$$f(t) \le C(1 + t^p) \quad \forall\, t \in R^+,$$

*where* $1 < p < \frac{N+4}{N-4}$ *if* $N > 4$;

*(H1)* $\lim_{t\to 0} \frac{f(t)}{t} = 0$;

*(H1)* $\lim_{t\to +\infty} \frac{f(t)}{t} = +\infty$.

*Then there exists* $b^* > 0$ *such that problem* (1.1) *has at least one positive solution for* $0 \le b < b^*$.

## 2 Preliminaries

In this section, we show examples how theorems, definitions, lists and formulae should be formatted.

Let $\mathbf{H} = \{u \in \mathbf{H}^2(R^N) : u(x) \text{ is radial}\}$, where $\mathbf{H}^2(R^N)$ is the usual Sobolev space. We equip $\mathbf{H}$ with the inner product

$$(u, v) = \int_{R^N} (\Delta u \Delta v + a\nabla u \nabla v + cuv)\, dx,$$

and the deduced norm

$$\|u\|^2 = \int_{R^N} |\Delta u|^2 dx + a \int_{R^N} |\nabla u|^2 + cu^2\, dx.$$

For the Kirchhoff problem (1.1), the associated function is defined on $\mathbf{H}$ as follows

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{R^N} |\nabla u|^2\, dx\right)^2 - \int_{R^N} F(u)\, dx$$

where $F(t) = \int_0^t f(s)\, ds$. By (H1), we know that $J$ is well defined, and is $C^1$. To overcome the difficulty of finding bounded Palais–Smale sequences for the associate functional $J$, we modify the functional $J$ as follows

$$J_\lambda^T(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\psi\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u|^2\, dx\right)^2 - \lambda \int_{R^N} F(u)\, dx \tag{2.1}$$

where $T > 0$, the cut-off functional $\psi(t)$ is defined by

$$\begin{cases} \psi(t) = 1, & t \in [0, 1], \\ 0 \le \psi(t) \le 1, & t \in (1, 2), \\ \psi(t) = 0, & t \in [2, +\infty), \\ \|\psi'(t)\|_\infty \le 2. \end{cases}$$

The following lemma is important to our arguments.

**Lemma 2.1** ([12]). *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset R^+$ an interval. Consider the family of $C^1$ functionals on $X$*

$$J_\lambda = A(u) - \lambda B(u), \quad \lambda \in I,$$

*with $B$ nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $J_\lambda(0) = 0$.*

*For any $\lambda \in I$, we set*

$$\Gamma_\lambda = \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ J_\lambda(\gamma(1)) < 0\}.$$

*If for every $\lambda \in I$, the set $\Gamma_\lambda$ is nonempty and*

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(\gamma(t)) > 0,$$

*then for almost every $\lambda \in I$ there is a sequence $\{u_n\} \subset X$ such that*

*(i) $\{u_n\}$ is bounded;*

*(ii)* $J_\lambda(u_n) \to c_\lambda$;

*(iii)* $J'_\lambda(u_n) \to 0$ *in the dual* $X^{-1}$ *of* $X$.

Throughout this paper, let

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\psi\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u|^2\, dx\right)^2, \quad B(u) = \int_{R^N} F(u)dx.$$

Now, we show that $J_\lambda^T$ satisfies the conditions of Lemma 2.1.

**Lemma 2.2.** *$\Gamma_\lambda \neq \emptyset$ for all $\lambda \in I = [\delta, 1]$, where $\delta \in (0,1)$ is a positive constant.*

*Proof.* We choose $\phi \in C_0^\infty(R^N)$ satisfying the following conditions:

$$\begin{cases} \phi \geq 0, \\ \|\phi\| = 1, \\ \operatorname{supp}\phi \subset B(0,R) \text{ for some } R > 0. \end{cases}$$

By (H3), we have that for any $C_1 > 0$ with $\delta C_1 \int_{B(0,R)} \phi^2\, dx > \frac{1}{2}$, there exists $C_2 > 0$ such that

$$F(t) \geq C_1 t^2 - C_2, \quad t \in R^+.$$

Then for $t^2 > 2T^2$, we have

$$\begin{aligned} J_\lambda^T(t\phi) &= \frac{1}{2}\|t\phi\|^2 + \frac{b}{4}\psi\left(\frac{\|t\phi\|^2}{T^2}\right)\left(\int_{R^N} |\nabla(t\phi)|^2\, dx\right)^2 - \lambda\int_{R^N} F(t\phi)\, dx \\ &= \frac{1}{2}\|t\phi\|^2 - \lambda\int_{R^N} F(t\phi)\, dx \\ &\leq \frac{1}{2}t^2 - \lambda\int_{R^N} C_1 t^2\phi^2\, dx + C_3 \\ &\leq \frac{1}{2}t^2 - \lambda C_1 t^2 \int_{B(0,R)} \phi^2\, dx + C_3 \\ &\leq \frac{1}{2}t^2 - \delta C_1 t^2 \int_{B(0,R)} \phi^2\, dx + C_3. \end{aligned}$$

If $t$ is sufficiently large, we have $J_\lambda^T(t\phi) < 0$. The proof is completed. □

**Lemma 2.3.** *There exists a constant $c > 0$ such that $c_\lambda \geq c > 0$ for any $\lambda \in I$.*

*Proof.* By (H1) and (H2), we see that for any $\varepsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for all $t \in R^+$, one has

$$F(t) \leq \frac{1}{2}\varepsilon t^2 + C_\epsilon t^{p+1}.$$

Furthermore, combining with the Sobolev inequality, we have

$$\begin{aligned} J_\lambda^T(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\psi\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u)|^2\, dx\right)^2 - \lambda\int_{R^N} F(u)\, dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\epsilon}{2}\int_{R^N} |u|^2\, dx - C_\epsilon \int_{R^N} |u|^{p+1}\, dx \\ &\geq \left(\frac{1}{2} - \frac{\epsilon}{2}\right)\|u\|^2 - C_\epsilon \int_{R^N} |u|^{p+1}\, dx. \end{aligned}$$

Then for $\epsilon$ sufficiently small, there exists $\rho > 0$ such that $J_\lambda^T(u) > 0$ for any $\lambda \in I$, $u \in \mathbf{H}$ with $0 < \|u\| \le \rho$. In particular, for $\|u\| = \rho$, we have $J_\lambda^T(u) \ge c > 0$. Fix $\lambda \in I$ and $\gamma \in \Gamma_\lambda$. By the definition of $\Gamma_\lambda$, $\|\gamma(1)\| > \rho$. By continuity, there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\lambda \in I$, we have

$$c_\lambda \ge \inf_{\gamma \in \Gamma_\lambda} J_\lambda^T(\gamma(t_\gamma)) \ge c > 0.$$

□

**Lemma 2.4.** *For any $\lambda \in I$ and $8bT^2 \le 1$, each bounded (PS)-sequence of the functional $J_\lambda^T$ admits a convergent subsequence.*

*Proof.* Let $\lambda \in I$ and $\{u_n\}$ be a bounded (PS)-sequence of $J_\lambda^T$, namely

$$\begin{cases} \{u_n\} \text{ is bounded}, \\ \{J_\lambda^T(u_n)\} \text{ is bounded}, \\ (J_\lambda^T)'(u_n) \to 0 \text{ in } \mathbf{H}'. \end{cases}$$

Up to a subsequence, there exists $u \in \mathbf{H}$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } \mathbf{H}, \\ u_n \to u & \text{in } L^{p+1}(R^N), \\ u_n \to u & \text{a.e. in } R^N. \end{cases}$$

By the definition of $J_\lambda^T$, we get

$$\begin{aligned} \left((J_\lambda^T)'(u_n), u_n - u\right) = &\left[1 + \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u_n|^2\,dx\right)^2\right](u_n, u_n - u) \\ &+ b\psi\left(\frac{\|u_n\|^2}{T^2}\right)\int_{R^N} |\nabla u_n|^2\,dx \int_{R^N} \nabla u_n \cdot \nabla(u_n - u)\,dx \\ &- \lambda \int_{R^N} f(u_n)(u_n - u)\,dx. \end{aligned}$$

Furthermore, from $\left|\psi'\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u|^2\,dx\right)^2\right| \le 8T^4$, $8bT^2 \le 1$, we easily obtain

$$\begin{aligned} &\left[1 + \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u_n|^2\,dx\right)^2\right](u_n - u, u_n - u) \\ &= \left((J_\lambda^T)'(u_n), u_n - u\right) - \left[1 + \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u_n|^2\,dx\right)^2\right](u, u_n - u) \\ &\quad - b\psi\left(\frac{\|u_n\|^2}{T^2}\right)\int_{R^N} |\nabla u_n|^2\,dx \int_{R^N} \nabla(u_n - u) \cdot \nabla(u_n - u)\,dx \\ &\quad - b\psi\left(\frac{\|u_n\|^2}{T^2}\right)\int_{R^N} |\nabla u_n|^2\,dx \int_{R^N} \nabla u \cdot \nabla(u_n - u)\,dx + \lambda \int_{R^N} f(u_n)(u_n - u)\,dx \\ &\le \left((J_\lambda^T)'(u_n), u_n - u\right) - \left[1 + \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u_n|^2\,dx\right)^2\right](u, u_n - u) \\ &\quad - b\psi\left(\frac{\|u_n\|^2}{T^2}\right)\int_{R^N} |\nabla u_n|^2\,dx \int_{R^N} \nabla u \cdot \nabla(u_n - u)\,dx + \lambda \int_{R^N} f(u_n)(u_n - u)\,dx. \end{aligned}$$

Firstly, it is clear to know that $((J_\lambda^T)'(u_n), u_n - u) \to 0$; secondly, since $u_n \rightharpoonup u$ in $\mathbf{H}$, then we have

$$\left[1 + \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u_n|^2\,dx\right)^2\right](u, u_n - u) \to 0;$$

thirdly, by the fact that the imbedding $\mathbf{H} \hookrightarrow \widetilde{\mathbf{H}}$ is continuous (see [6, p. 2287]), where $\widetilde{\mathbf{H}} = \{u \in L^2(R^N) : \nabla u \in [L^2(R^N)]^N\}$ is endowed with the norm $\|u\|_{\widetilde{\mathbf{H}}} = (\int_{R^N} |\nabla u|^2\,dx)^{\frac{1}{2}}$, we have $u_n \rightharpoonup u$ in $\widetilde{\mathbf{H}}$ and this implies

$$b\psi\left(\frac{\|u_n\|^2}{T^2}\right)\int_{R^N} |\nabla u_n|^2\,dx \int_{R^N} \nabla u \cdot \nabla(u_n - u)\,dx \to 0.$$

Finally, from (H1) and (H2), it follows that for any $\varepsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for all $t \in R^+$, one has

$$f(t) \le \varepsilon t + C_\epsilon |t|^p.$$

Then, we obtain

$$\begin{aligned}\left|\int_{R^N} f(u_n)(u_n - u)\,dx\right| &\le \int_{R^N} |f(u_n)||(u_n - u)|\,dx \\ &\le \varepsilon |u_n|_{L^2}|u_n - u|_{L^2} + C_\epsilon \int_{R^N} |u_n|^p|u_n - u|\,dx \\ &\le \epsilon\|u_n\|\|u_n - u\| + C_\epsilon |u_n|^p_{L^{p+1}}|u_n - u|_{L^{p+1}} \\ &\to 0.\end{aligned}$$

Therefore, we can get

$$\left[1 + \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u_n|^2\,dx\right)^2\right](u_n - u, u_n - u) \to 0,$$

which implies that $u_n \to u$ in $\mathbf{H}$. □

From Lemmas 2.1–2.4, we obtain the following result.

**Lemma 2.5.** *Let $8bT^2 \le 1$, then for almost every $\lambda \in I$, there exists $u_\lambda \in \mathbf{H}\backslash\{0\}$ such that $(J_\lambda^T)'(u_\lambda) = 0$ and $J_\lambda^T(u_\lambda) = c_\lambda$.*

**Lemma 2.6.** *Let $N \ge 5$. If $u \in \mathbf{H}$ is a critical point of $J_\lambda^T(u)$, namely, $u$ a weak solution of*

$$\left(1 + \frac{b}{2T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u|^2\,dx\right)^2\right)Lu - b\psi\left(\frac{\|u\|^2}{T^2}\right)\int_{R^N} |\nabla u|^2\,dx\,\Delta u = \lambda f(u),$$

*where $Lu = \Delta^2 u - a\Delta u + cu$, then the following Pohozaev identity holds:*

$$\begin{aligned}\lambda N \int_{R^N} F(u)\,dx = \frac{b(N-2)}{2}\psi\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N} |\nabla u|^2\,dx\right)^2 + \left(\frac{1}{2} + \frac{b}{4T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\right) \times \\ \times \left[(N-4)\int_{R^N} |\Delta u|^2\,dx + a(N-2)\int_{R^N} |\nabla u|^2\,dx + cN\int_{R^N} |u|^2\,dx\right].\end{aligned}$$

*Proof.* Let $T(t)\colon \mathbf{H} \to \mathbf{H}$ be a family of transformations such that

$$T(t)u(x) = u\left(\frac{x}{t}\right), \quad t > 0,$$

and consequently

$$T(1) = id.$$

If $u \in \mathbf{H}$ is a critical point of $J_\lambda^T$, then we have

$$\begin{aligned} J_\lambda^T(T(t)u) = {} & \frac{t^{N-4}}{2}\int_{R^N}|\Delta u|^2\,dx + \frac{at^{N-2}}{2}\int_{R^N}|\nabla u|^2\,dx + \frac{ct^N}{2}\int_{R^N}|u|^2\,dx \\ & + \frac{bt^{2N-4}}{4}\psi\left(\frac{\int_{R^N} t^{N-4}|\Delta u|^2 + at^{N-2}|\nabla u|^2 + ct^N|u|^2\,dx}{T^2}\right)\left(\int_{R^N}|\nabla u|^2\,dx\right)^2 \\ & - \lambda t^N\int_{R^N}F(u)\,dx \end{aligned}$$

and

$$\begin{aligned} 0 & = \frac{\partial}{\partial t}J_\lambda^T(T(t)u)|_{t=1} \\ & = \frac{N-4}{2}\int_{R^N}|\Delta u|^2\,dx + \frac{a(N-2)}{2}\int_{R^N}|\nabla u|^2\,dx + \frac{cN}{2}\int_{R^N}|u|^2\,dx \\ & \quad + \frac{b(2N-4)}{4}\psi\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u|^2\,dx\right)^2 + \frac{b}{4T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u|^2\,dx\right)^2 \times \\ & \quad \times\left((N-4)\int_{R^N}|\Delta u|^2\,dx + a(N-2)\int_{R^N}|\nabla u|^2\,dx + cN\int_{R^N}|u|^2\,dx\right) \\ & \quad - \lambda N\int_{R^N}F(u)\,dx. \end{aligned}$$

Then, if $N \geq 5$, the Pohozaev identity of the fourth order elliptic equation:

$$\begin{aligned} & \left(1 + \frac{b}{2T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u|^2\,dx\right)^2\right)[\Delta^2 u - a\Delta u + cu] \\ & \qquad - b\psi\left(\frac{\|u\|^2}{T^2}\right)\int_{R^N}|\nabla u|^2\,dx\Delta u = \lambda f(u), \end{aligned}$$

takes the form

$$\begin{aligned} \lambda N\int_{R^N}F(u)\,dx = {} & \left(\frac{1}{2} + \frac{b}{4T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\right)\times \\ & \times\left[(N-4)\int_{R^N}|\Delta u|^2\,dx + a(N-2)\int_{R^N}|\nabla u|^2\,dx + cN\int_{R^N}|u|^2\,dx\right] \\ & + \frac{b(2N-4)}{4}\psi\left(\frac{\|u\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u|^2\,dx\right)^2. \end{aligned}$$

□

**Lemma 2.7.** *Let $u_n$ be a critical point of $J_{\lambda_n}^T$ at level $c_{\lambda_n}$. Then for $T > 0$ sufficiently large, there exists $b^* = b^*(T)$ with $8b^*T^2 \leq 1$ such that for any $b \in [0, b^*)$, up to a subsequence, $\|u_n\| < T$.*

*Proof.* According to Lemma 2.5, there exists a sequence $\{\lambda_n\} \subset I$ with $\lambda_n \to 1^-$, and $\{u_n\} \subset \mathbf{H}$ such that

$$J_{\lambda_n}^T(u_n) = c_{\lambda_n}, \quad \left(J_{\lambda_n}^T\right)'(u_n) = 0.$$

Firstly, since $(J_{\lambda_n}^T)'(u_n) = 0$, from Lemma 2.6, it follows that

$$\begin{aligned} &\lambda_n N \int_{R^N} F(u_n)\,dx \\ &= \frac{b(2N-4)}{4}\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2 + \left(\frac{1}{2} + \frac{b}{4T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\right)\times \\ &\quad \times\left[(N-4)\int_{R^N}|\Delta u_n|^2\,dx + a(N-2)\int_{R^N}|\nabla u_n|^2\,dx + cN\int_{R^N}|u_n|^2\,dx\right]. \end{aligned} \tag{2.2}$$

Secondly, using $J_{\lambda_n}^T(u_n) = c_{\lambda_n}$, we have that

$$\frac{N}{2}\|u_n\|^2 + \frac{Nb}{4}\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2 - \lambda_n N\int_{R^N}F(u_n)\,dx = c_{\lambda_n}N \tag{2.3}$$

Finally, from (2.2) and (2.3), we have

$$\begin{aligned} &\left(2\int_{R^N}|\Delta u_n|^2\,dx + a\int_{R^N}|\nabla u_n|^2\,dx\right) \\ &\le \left(2 + \frac{b}{T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\right)\int_{R^N}|\Delta u|^2\,dx + a\left(1 + \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\right)\int_{R^N}|\nabla u|^2\,dx \\ &= \frac{N}{2}\|u_n\|^2 + \frac{Nb}{4T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^2 + \frac{b(2N-4)}{4}\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2 \\ &\quad - \lambda_n N\int_{R^N}F(u_n)\,dx \\ &= \frac{N}{2}\|u_n\|^2 + \frac{Nb}{4T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^2 + \frac{b(2N-4)}{4}\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2 \\ &\quad + c_{\lambda_n}N - \frac{N}{2}\|u_n\|^2 - \frac{Nb}{4}\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2 \\ &= c_{\lambda_n}N + \frac{Nb}{4T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^2 + \frac{b(N-4)}{4}\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2. \end{aligned}$$

Now we show some estimates on the right-hand side.

$$\begin{aligned} c_{\lambda_n} &\le \max_t J_{\lambda_n}^T(t\phi) \\ &\le \max_t \frac{1}{2}\|t\phi\|^2 + \frac{b}{4}\psi\left(\frac{\|t\phi\|^2}{T^2}\right)\left(\int_{R^N}|\nabla t\phi)|^2\,dx\right)^2 - \lambda\int_{R^N}F(t\phi)\,dx \\ &\le \frac{1}{2}t^2 - \lambda\int_{R^N}C_1t^2\phi^2\,dx + C_3 \\ &\le \max_t \frac{1}{2}t^2 - \lambda C_1t^2\int_{B(0,R)}\phi^2\,dx + C_3 + \max_t \frac{b}{4}\psi\left(\frac{t^2}{T^2}\right)\left(\int_{R^N}|\nabla\phi|\,dx\right)^2 t^4 \\ &\le \max_t \frac{1}{2}t^2 - \delta C_1t^2\int_{B(0,R)}\phi^2\,dx + C_3 + \max_t \frac{b}{4}\psi\left(\frac{t^2}{T^2}\right)t^4. \end{aligned}$$

Since $\psi\left(\frac{t^2}{T^2}\right) = 0$ for $t^2 \geq 2T^2$, we can obtain

$$c_{\lambda_n} \leq C_3 + bT^4,$$
$$\frac{Nb}{4T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^2 \leq Nb,$$
$$\frac{b(N-4)}{4}\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2 \leq b(N-4)T^4.$$

Then we have

$$2\int_{R^N}|\Delta u_n|^2\,dx + a\int_{R^N}|\nabla u_n|^2\,dx \leq C_3 + bT^4 + b(N-4)T^4 + Nb.$$

Finally, we show that there exists $T > 0$ such that $\|u_n\| \leq T$. On the contrary, there exists no subsequence of $\{u_n\}$ which is uniformly bounded by $T$, namely, $\|u_n\| > T$. By (H1) and (H2), we have

$$\begin{aligned}
&\|u_n\|^2 + \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2\|u_n\|^2 + b\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2\\
&= \lambda_n\int_{R^N} f(u_n)u_n\,dx\\
&\leq \varepsilon|u_n|_{L^2}^2 + C_\epsilon|u_n|_{L^{2^*}}^{2^*}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
(1-\epsilon)\|u_n\|^2 &\leq C_\epsilon|u_n|_{L^{2^*}}^{2^*} - \frac{b}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2\|u_n\|^2\\
&\leq C_4|\nabla u_n|_{L^2}^{2^*} + 8bT^2\\
&\leq C_4\left(\frac{C_3 + bT^4 + b(N-4)T^4 + Nb}{a}\right)^{\frac{2^*}{2}} + 8bT^2.
\end{aligned}$$

Then we get the following inequality

$$T < \|u_n\| \leq C_5\left(C_3 + bT^4 + b(N-4)T^4 + Nb\right)^{\frac{2^*}{2}} + 8bT^2.$$

However, this inequality in not true for sufficiently large $T$ with $8bT^2 < 1$, and this implies the conclusion. □

*Proof of Theorem 1.1.* Let $T$ and $b^* = \frac{1}{8T^2}$ be defined as in Lemma 2.7, and $u_n$ be a critical point for $J_{\lambda_n}^T$ at level $c_{\lambda_n}$. Then from Lemma 2.7, we know that $\|u_n\| \leq T$. So

$$\begin{aligned}
J_{\lambda_n}^T(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\psi\left(\frac{\|u_n\|^2}{T^2}\right)\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2 - \lambda_n\int_{R^N}F(u_n)\,dx\\
&= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\left(\int_{R^N}|\nabla u_n|^2\,dx\right)^2 - \lambda_n\int_{R^N}F(u_n)\,dx.
\end{aligned}$$

Since $\lambda_n \to 1$, we have

$$(J'(u_n), v) = \left((J_{\lambda_n}^T)'(u_n), v\right) + (\lambda_n - 1)\int_{R^N} f(u_n)v\,dx, \quad v \in \mathbf{H},$$

which implies that $J'(u_n) \to 0$. Then combining with the boundedness of $\{u_n\}$, we show that $\{u_n\}$ also is a (PS)-sequence of $J$. By Lemma 2.4, $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$ with $u_{n_k} \to u$. Consequently, $J'(u) = 0$. According to Lemma 2.3, we have

$$J(u) = \lim_{k\to\infty} J(u_{n_k}) = \lim_{k\to\infty} J^T_{\lambda_{n_k}}(u_{n_k}) \geq c > 0,$$

and $u$ is a positive solution of (1.1) by (H1). The proof is completed. □

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