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Weighted L^p estimates for the elliptic Schrödinger operator

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Abstract. In this paper we study weighted L^p estimates for the elliptic Schrödinger operator $P = -\Delta + V(x)$ with non-negative potentials $V(x)$ on $\mathbb{R}^n$ $(n \geq 3)$ which belongs to certain reverse Hölder class.

Keywords: weighted, regularity, L^p estimates, elliptic, Schrödinger operator.

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1 Introduction

Shen [19] proved the L^p boundedness with $1 < p \leq 2$ of the nontangential maximal function of ∇u for the L^p-Neumann problem of the elliptic Schrödinger operator

$$P = -\Delta + V(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3 \tag{1.1}$$

with $V \in V_\infty$ (see Definition 1.1) in a domain $\Omega \subset \mathbb{R}^n$. Moreover, Shen [20] has obtained the following L^p estimates for (1.1)

$$\int_{\mathbb{R}^n} \left| D^2 \left(-\triangle + V(x) \right)^{-1} f \right|^p dx \leq C \int_{\mathbb{R}^n} |f|^p \, dx$$

for $1 < p \leq q$, assuming that $V \in V_q$ for some $q \geq n/2$. In this paper we consider weighted L^p estimates for the elliptic Schrödinger operator (1.1)

$$P = -\Delta + V(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3$$

with $V \in V_\infty$, where $x = (x^1, \dots, x^n)$ and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Definition 1.1. *The function $V(x)$ is said to belong to the reverse Hölder class V_q for some $1 < q \leq \infty$ if $V \in L^q_{loc}(\mathbb{R}^n)$, $V \geq 0$ almost everywhere and there exists a constant C such that for all balls B_r of $\mathbb{R}^n$,*

$$\left(\fint_{B_r} V^q(x) \, dx \right)^{1/q} \leq C \fint_{B_r} V(x) \, dx,$$

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with

$$\fint_{B_r} V(x)\,dx = \frac{1}{|B_r|}\int_{B_r} V(x)\,dx.$$

If $q=\infty$, then the left-hand side is the essential supremum on B_r, i.e.,

$$\sup_{B_r}|V(x)| \le C\fint_{B_r} V(x)\,dx.$$

In fact, if $V \in V_\infty$, it clearly implies $V \in V_q$ for every $q>1$.

We can refer to [2, 20, 21] regarding the reverse Hölder class. In particular, $V(x)=|x|^\alpha \in V_q$ if $\alpha q > -n$.

We use the Hardy–Littlewood maximal function which controls the local behavior of a function.

Definition 1.2. *Let v be a locally integrable function. The Hardy–Littlewood maximal function $\mathcal{M}v(x)$ is defined as*

$$\mathcal{M}v(x) = \sup \fint_Q |v(y)|dy,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing x.

It is well known that the maximal functions satisfy strong p-p estimate for any $1<p<\infty$ and weak $(1,1)$ estimate (see [21]).

We now introduce the weighted Lebesgue spaces (see [11, 12, 15, 16, 18, 21, 22]).

Definition 1.3. *A_q for some $q>1$ is the class of the Muckenhoupt weights: $w\in A_q$ if $w \in L^1_{loc}(\mathbb{R}^n)$, $w>0$ almost everywhere and there exists a constant C such that for all balls B_r in $\mathbb{R}^n$,*

$$\left(\fint_{B_r} w(x)\,dx\right)\left(\fint_{B_r} w(x)^{\frac{-1}{q-1}}\,dx\right)^{q-1} \le C.$$

Moreover, we denote

$$A_\infty = \bigcup_{1<q<\infty} A_q$$

and

$$w(\Omega) = \int_\Omega w(x)\,dx,$$

where $\Omega \subset \mathbb{R}^n$. Furthermore, the corresponding weighted Lebesgue space $L^q_w(\Omega)$ consists of all functions h which satisfy

$$\|h\|_{L^q_w(\Omega)} := \left(\int_\Omega |h|^q\, w(x)\,dx\right)^{1/q} < \infty.$$

Remark 1.4. We remark that $A_{q_1}\subset A_{q_2}$ for any $1<q_1\le q_2<\infty$ (see [21, p. 195]).

Lemma 1.5. *If $w\in A_q$ with $q>q_1>1$, then we have*

$$L^q_w(B_r)\subset L^{q_1}(B_r).$$

Proof. From Hölder's inequality we have

$$\int_{B_r} |f|^{q_1}\,dx = \int_{B_r} |f|^{q_1}\, w(x)^{\frac{q_1}{q}} w(x)^{-\frac{q_1}{q}}\,dx$$
$$\leq \left(\int_{B_r} |f|^q\, w(x)\,dx\right)^{\frac{q_1}{q}} \left(\int_{B_r} w(x)^{-\frac{q_1}{q-q_1}}\,dx\right)^{1-\frac{q_1}{q}}.$$

Since $w \in A_q$ with $q > q_1 > 1$, from Remark 1.4 we find that $w \in A_{q/q_1}$. Furthermore, we conclude that

$$⨍_{B_r} w(x)^{-\frac{q_1}{q-q_1}}\,dx = ⨍_{B_r} w(x)^{-\frac{1}{q/q_1-1}}\,dx \leq C\left(\frac{|B_r|}{w(B_r)}\right)^{\frac{q_1}{q-q_1}}.$$

Thus, if $f \in L^q_w(B_r)$, then we have

$$\int_{B_r} |f|^{q_1}\,dx \leq C\left(\int_{B_r} |f|^q\, w(x)\,dx\right)^{\frac{q_1}{q}} |B_r|^{1-\frac{q_1}{q}} \left(\frac{|B_r|}{w(B_r)}\right)^{\frac{q_1}{q}} \leq C,$$

since $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ almost everywhere. This finishes our proof. □

Lemma 1.6 (see [5, 6, 12, 15, 16, 21, 22])**.** *Assume that $w(x) \in A_q$ for some $q > 1$ and $g \in L^q_w(\mathbb{R}^n)$. Then we have*

(1)

$$\|\mathcal{M}g\|_{L^q_w(\mathbb{R}^n)} \leq C\|g\|_{L^q_w(\mathbb{R}^n)}.$$

(2)

$$w\left(\{x \in \mathbb{R}^n : \mathcal{M}g(x) > \mu\}\right) \leq \frac{C}{\mu^q}\int_{\mathbb{R}^n} |g|^q\, w(x)\,dx.$$

(3)

$$\int_{\mathbb{R}^n} |g|^q w(x)\,dx = q\int_0^\infty \mu^{q-1} w\left(\{x \in \mathbb{R}^n : |g| > \mu\}\right)d\mu.$$

Next, we shall give some lemmas on the properties of A_q weight.

Lemma 1.7 (see [5, 6, 15, 16, 22])**.** *If $w \in A_q$ for some $q > 1$ and $B_r \subset B_R \subset \mathbb{R}^n$, then there exists a constant $C_1 > 0$ such that*

$$\frac{w(B_R)}{w(B_r)} \leq C_1\left(\frac{|B_R|}{|B_r|}\right)^q.$$

Furthermore, we have the following reverse Hölder inequality.

Lemma 1.8 (see [22, Theorem 3.5 in Chapter 9])**.** *If $w \in A_q$ for some $q > 1$, then there exists a small positive constant $\epsilon_0 < 1$ and a constant $C_2 > 1$ such that*

$$\left(⨍_{B_r} w(x)^{1+\epsilon_0}\,dx\right)^{\frac{1}{1+\epsilon_0}} \leq C_2 ⨍_{B_r} w(x)\,dx$$

for any ball $B_r \subset \mathbb{R}^n$.

Lemma 1.9. *If $w \in A_q$ for some $q > 1$ and $B_r \subset B_R \subset \mathbb{R}^n$, then there exists $\sigma > 0$ such that*

$$\frac{w(B_r)}{w(B_R)} \leq C_2 \left(\frac{|B_r|}{|B_R|} \right)^{\sigma}.$$

Proof. We first conclude that

$$\begin{aligned} w(B_r) &= \int_{B_r} w(x)\, dx \\ &\leq \left(\int_{B_r} w(x)^{1+\epsilon_0}\, dx \right)^{\frac{1}{1+\epsilon_0}} \cdot |B_r|^{\frac{\epsilon_0}{1+\epsilon_0}} \\ &\leq \left(\fint_{B_R} w(x)^{1+\epsilon_0}\, dx \right)^{\frac{1}{1+\epsilon_0}} \cdot |B_R|^{\frac{1}{1+\epsilon_0}} \cdot |B_r|^{\frac{\epsilon_0}{1+\epsilon_0}} \end{aligned}$$

by using Hölder's inequality. Thus, it follows from the lemma above that

$$w(B_r) \leq C_2 \fint_{B_R} w(x)\, dx \cdot |B_R|^{\frac{1}{1+\epsilon_0}} \cdot |B_r|^{\frac{\epsilon_0}{1+\epsilon_0}} = C_2 w\left(B_R\right) \left(\frac{|B_r|}{|B_R|} \right)^{\frac{\epsilon_0}{1+\epsilon_0}},$$

which finishes our proof by selecting $\sigma = \epsilon_0/(1+\epsilon_0)$. □

Now let us state the main results of this work: Theorem 1.10 and Theorem 1.11. We shall give the direct proofs of the main results via the maximal function approach which was employed by [1, 5, 7, 13, 15, 16, 17].

Theorem 1.10. *Assume that $w(x) \in A_p$ for some $p > 1$ and $f \in L^p_w(\mathbb{R}^n)$. If u is the solution of the Poisson equation*

$$-\Delta u = f(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3, \tag{1.2}$$

then we have

$$\int_{\mathbb{R}^n} \left|D^2 u\right|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f|^p w(x)\, dx.$$

Theorem 1.11. *Assume that $w(x) \in A_p$ for some $p > 1$, $V \in V_\infty$ and $f \in L^p_w(\mathbb{R}^n)$. If $u \in C_0^\infty(\mathbb{R}^n)$ is the solution of the following elliptic Schrödinger equation*

$$-\Delta u(x) + V(x)u(x) = f(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3, \tag{1.3}$$

then we have

$$\int_{\mathbb{R}^n} |Vu|^p w(x)\, dx + \int_{\mathbb{R}^n} \left|D^2 u\right|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f|^p w(x)\, dx.$$

Remark 1.12. Assume that $u \in C_0^\infty(\mathbb{R}^n)$ and $V \in V_q$ with $1 < p \leq q$ and $q \geq n/2$. The authors of [4] proved that

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \leq C \left(\|f\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right)$$

for (1.3) and the general case.

2 Proofs of the main results

In this section we shall finish the proofs of the main results: Theorem 1.10 and Theorem 1.11.

2.1 Proof of Theorem 1.10

We first give the following Calderón–Zygmund decomposition, which is much influenced by [14].

Lemma 2.1. *Let D be a cube in $\mathbb{R}^n$ and $A, B \subset D$ be measurable sets. Assume that $0 < w(A) < \mu w(D)$ for $0 < \mu < 1$. Then there exists a sequence of disjoint cubes $\{Q_k\}_{k\in\mathbb{N}}$ satisfying*

(1) $w\left(A \setminus \bigcup_{k\in\mathbb{N}} Q_k\right) = 0$,

(2) $w\left(A \cap Q_k\right) > \mu w\left(Q_k\right)$,

(3) $w\left(A \cap \widetilde{Q_k}\right) \le \mu w\left(\widetilde{Q_k}\right)$ *if $\widetilde{Q_k}$ is the predecessor (father) of Q_k.*

Furthermore, if for any Q_k, its predecessor $\widetilde{Q_k}$ satisfies

$$w\left(B \cap \widetilde{Q_k}\right) > \alpha w\left(\widetilde{Q_k}\right) \quad \text{for } 0 < \alpha < 1, \tag{2.1}$$

then we have

$$w(A) \le \frac{\mu}{\alpha} w(B).$$

Proof. **1.** We first divide D into 2^n $\left(\text{denote by } \{Q_1^{j_1}\}_{j_1=1}^{2^n}\right)$ disjoint cubes (daughters) with the same size. Choose those cubes satisfying $w(A \cap Q_1^{j_1}) > \mu w(Q_1^{j_1})$ and continue to divide every remaining cube $Q_1^{j_1}$ into 2^n $\left(\text{denote by } \{Q_2^{j_1,j_2}\}_{j_2=1}^{2^n}\right)$ disjoint cubes with the same size. Therefore, we obtain a sequence of disjoint cubes $\{Q_k\}_{k\in\mathbb{N}}$ which satisfies (2)–(3) by repeating the process above. If $x \in D \setminus \{Q_k\}_{k\in\mathbb{N}}$, then there is a sequence of cubes P_i containing x with the diameters of P_i converging to 0 and

$$w\left(A \cap P_i\right) \le \mu w\left(P_i\right).$$

From elementary measure theory and the fact that $w(x) > 0$ almost everywhere we can conclude that for almost every $x \in D \setminus \{Q_k\}_{k\in\mathbb{N}}$, $x \in D \setminus A$. That is say, (1) is true.

2. Let $\widetilde{Q_k}$ be the predecessor (father) of Q_k. Now we choose a disjoint predecessor subsequence $\left\{\widetilde{Q_{k_j}}\right\}$ $\left(\text{still denoted by } \left\{\widetilde{Q_k}\right\}\right)$ such that $\bigcup_{k\in\mathbb{N}} Q_k \subset \bigcup_{k\in\mathbb{N}} \widetilde{Q_k}$. Thus, from (1), (3) and the hypothesis (2.1) we deduce that

$$w(A) = \sum_k w\left(A \cap \widetilde{Q_k}\right) \le \mu \sum_k w\left(\widetilde{Q_k}\right) < \frac{\mu}{\alpha} \sum_k w\left(B \cap \widetilde{Q_k}\right) \le \frac{\mu}{\alpha} w\left(B\right),$$

which finishes our proof. □

Next, we shall prove the following important result.

Lemma 2.2. *Assume that $1 < q < p$. For any $\mu, \alpha \in (0,1)$ there exist two constants $M_2 = M_2(n) > 1$ and $\delta = \delta(n, \mu) \in (0,1)$, such that if*

$$\left|\left\{x \in \widetilde{Q} : \mathcal{M}\left(\left|D^2 u\right|^q\right)(x) \le 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}\left(|f|^q\right)(x) \le \delta^q\right\}\right| > \alpha \left|\widetilde{Q}\right|, \tag{2.2}$$

then we have

$$\left|\left\{x \in Q : \mathcal{M}\left(\left|D^2 u\right|^q\right)(x) \ge M_2^q\right\}\right| \le \mu \left|Q\right|.$$

Proof. **1.** From the hypothesis (2.2) there exists $x_0 \in \widetilde{Q}$ such that

$$\mathcal{M}\left(\left|D^2u\right|^q\right)(x_0) \leq 1 \quad \text{and} \quad \mathcal{M}\left(|f|^q\right)(x_0) \leq \delta^q. \tag{2.3}$$

Since $x_0 \in \widetilde{Q} \subset 3Q$, we conclude that

$$\fint_{4Q} \left|D^2u\right|^q dx \leq 1 \quad \text{and} \quad \fint_{4Q} |f|^q dx \leq \delta^q. \tag{2.4}$$

Let v_1 be the solution of

$$-\Delta v_1 = \bar{f} \qquad \text{on } \mathbb{R}^n,$$

where $\bar{f}$ is the zero extention of f from $4Q$ to $\mathbb{R}^n$. Then from the elementary L^p-type estimates we have

$$\int_{\mathbb{R}^n} \left|D^2v_1\right|^q dx \leq \int_{\mathbb{R}^n} |\bar{f}|^q dx,$$

which implies that

$$\int_{4Q} \left|D^2v_1\right|^q dx \leq \int_{\mathbb{R}^n} \left|D^2v_1\right|^q dx \leq \int_{\mathbb{R}^n} |\bar{f}|^q dx = \int_{4Q} |f|^q dx. \tag{2.5}$$

Therefore, from (2.4) we conclude that

$$\fint_{4Q} \left|D^2v_1\right|^q dx \leq \fint_{4Q} |f|^q dx \leq \delta^q. \tag{2.6}$$

Set $h_1 = u - v_1$. From the definition of $\bar{f}$, we find that h_1 satisfies

$$-\Delta h_1 = 0 \qquad \text{in } 4Q. \tag{2.7}$$

Moreover, it follows from $W^{2,\infty}_{loc}$ regularity that

$$\sup_{3Q} \left|D^2h_1\right| \leq M_1,$$

where $M_1 > 1$ only depends on n.

2. The proof is totally similar to the proof of Lemma 2.8. Here we omit the details. □

Corollary 2.3 (cf. Corollary 2.9)**.** *Assume that $1 < q < p$ and $w \in A_p$. For any $\mu \in (0,1)$ there exist two constants $M_3 = M_3(n) > 1$ and $\delta = \delta(n, \sigma, \mu) \in (0,1)$ such that if*

$$w\left(\left\{x \in \widetilde{Q} : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \leq 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}\left(|f|^q\right)(x) \leq \delta^q\right\}\right) > \frac{1}{2} w\left(\widetilde{Q}\right), \tag{2.8}$$

then we have

$$w\left(\left\{x \in Q : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \geq M_3^q\right\}\right) \leq \mu w(Q).$$

Corollary 2.4 (cf. Corollary 2.10)**.** *Let D be a cube in $\mathbb{R}^n$. Assume that q, w, μ, δ, M_3 satisfy the same conditions as those in Corollary 2.3. If*

$$w\left(\left\{x \in D : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \geq M_3^q\right\}\right) \leq \mu w(D),$$

then we have

$$\begin{aligned} & w\left(\left\{x \in D : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \geq M_3^q\right\}\right) \\ & \leq 2\mu\Big[w\left(\left\{x \in D : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) > 1\right\}\right) \\ & \quad + w\left(\left\{x \in D : \mathcal{M}\left(|f|^q\right)(x) > \delta^q\right\}\right)\Big]. \end{aligned} \tag{2.9}$$

Corollary 2.5 (cf. Corollary 2.11)**.** *Assume that $\mu \in (0,1)$ with $C_2\mu^\sigma < 1$ and q, w, δ, M_3 satisfy the same conditions as those in Corollary 2.3. For any $\lambda > 0$ we have*

$$\begin{aligned} &w\left(\left\{x \in \mathbb{R}^n : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \geq \lambda^q M_3^q\right\}\right) \\ &\quad \leq 2C_2\mu^\sigma\Big[w\left(\left\{x \in \mathbb{R}^n : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) > \lambda^q\right\}\right) \\ &\qquad + w\left(\left\{x \in \mathbb{R}^n : \mathcal{M}\left(|f|^q\right)(x) > \lambda^q\delta^q\right\}\right)\Big]. \end{aligned} \tag{2.10}$$

The rest of the proof of Theorem 1.10 is totally similar to that of Theorem 1.11 in §2.2.

2.2 Proof of Theorem 1.11

We first recall the following result (see [21, p. 195]).

Lemma 2.6. *If $V \in V_\infty$, then there exist $t \in [1,\infty)$ and $C > 0$ such that*

$$⨍_Q g\,dx \leq \left(\frac{C}{V(Q)}\int_Q V g^t\,dx\right)^{\frac{1}{t}}$$

holds for any nonnegative function g and all cubes Q, where

$$V(Q) = \int_Q V\,dx.$$

Furthermore, we have the following local boundedness property.

Lemma 2.7. *Assume that $V \in V_\infty$. If $h(x)$ satisfies $-\Delta h(x) + V(x)h(x) = 0$ in $2Q$, then*

$$\sup_Q |h| \leq \frac{C}{V(2Q)}\int_{2Q} V|h|\,dx,$$

where C depends on n.

Proof. Since $V \in V_\infty$ and $u \in C_0^\infty(\mathbb{R}^n)$ satisfies $-\Delta u(x) + V(x)u(x) = f(x)$, we may as well assume that

$$\operatorname{supp} u \subset B_{r_0}, \quad V(x) \equiv 0 \quad \text{in } \mathbb{R}^n \setminus B_{r_0} \quad \text{and} \quad |V(x)| \leq C \quad \text{in } \mathbb{R}^n$$

for some $r_0 > 0$. Recalling the elementary local boundedness property of the second-order elliptic equation (see [9, Theorem 9.20], or [10, Theorem 4.1]), we have

$$\sup_Q |h| \leq C\left(⨍_{2Q} |h|^r\,dx\right)^{\frac{1}{r}}$$

for any $r > 0$. Then using the above inequality and Lemma 2.6 with $r = \frac{1}{t}$, we find that

$$\sup_Q |h| \leq C\left(⨍_{2Q} |h|^{\frac{1}{t}}\,dx\right)^t \leq \frac{C}{V(2Q)}\int_{2Q} V|h|\,dx.$$

This completes our proof. □

Next, we shall prove the following important result.

Lemma 2.8. *For any $\mu, \alpha \in (0,1)$ there exist two constants $N_2 = N_2(n) > 1$ and $\delta = \delta(n,\mu) \in (0,1)$, such that if*

$$\left|\left\{x \in \widetilde{Q} : \mathcal{M}\left(V|u|\right)(x) \le 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}\left(|f|\right)(x) \le \delta\right\}\right| > \alpha\left|\widetilde{Q}\right|, \tag{2.11}$$

then we have

$$\left|\left\{x \in Q : \mathcal{M}\left(V|u|\right)(x) \ge N_2\right\}\right| \le \mu\left|Q\right|.$$

Proof. **1.** From the hypothesis (2.11) there exists $x_0 \in \widetilde{Q}$ such that

$$\mathcal{M}\left(V|u|\right)(x_0) \le 1 \quad \text{and} \quad \mathcal{M}\left(|f|\right)(x_0) \le \delta. \tag{2.12}$$

Since $x_0 \in \widetilde{Q} \subset 3Q$, we conclude that

$$\fint_{4Q} |Vu|\,dx \le 1 \quad \text{and} \quad \fint_{4Q} |f|\,dx \le \delta. \tag{2.13}$$

Let v be the solution of

$$-\Delta v(x) + V(x)v(x) = \bar{f} \quad \text{on } \mathbb{R}^n,$$

where $\bar{f}$ is the zero extention of f from $4Q$ to $\mathbb{R}^n$. Then recalling the well-known L^1 estimate (see [3, 8]), we have

$$\int_{\mathbb{R}^n} V|v|\,dx \le \int_{\mathbb{R}^n} |\bar{f}|\,dx,$$

which implies that

$$\int_{4Q} V|v|\,dx \le \int_{\mathbb{R}^n} V|v|\,dx \le \int_{\mathbb{R}^n} |\bar{f}|\,dx = \int_{4Q} |f|\,dx. \tag{2.14}$$

Therefore, from (2.13) we conclude that

$$\fint_{4Q} V|v|\,dx \le \fint_{4Q} |f|\,dx \le \delta. \tag{2.15}$$

Set $h = u - v$. From the definition of $\bar{f}$, we find that h satisfies

$$-\Delta h(x) + V(x)h(x) = 0 \quad \text{in } 4Q. \tag{2.16}$$

Moreover, it follows from (2.13) and (2.15) that

$$\fint_{4Q} V|h|\,dx \le \fint_{4Q} V|v|\,dx + \fint_{4Q} V|u|\,dx < 2.$$

Then from the above inequality and Lemma 2.7 we find that

$$\sup_{3Q} V|h| \le C \sup_{4Q} V \; \left[V\left(4Q\right)\right]^{-1} \int_{4Q} V|h|\,dx \le C \sup_{4Q} V \left(\fint_{4Q} V\,dx\right)^{-1},$$

which implies that

$$\sup_{3Q} V|h| \le N_1, \tag{2.17}$$

since $V \in V_\infty$, where $N_1 > 1$ depends on n.

2. Next, we shall prove that

$$\{x \in Q : \mathcal{M}(V|u|)(x) > N_2\} \subset \{x \in Q : \mathcal{M}(|Vv|)(x) > N_1\}, \tag{2.18}$$

where $N_2 := \max\{2N_1, 3^n\}$. Actually, from (2.17) we find that

$$V|u| \le V|v| + V|h| \le V|v| + N_1 \quad \text{for any} \quad x \in 3Q.$$

Let x be a point in $\{x \in Q : \mathcal{M}(V|v|)(x) \le N_1\}$. If $x \in Q_1 \subset 3Q$, then we have

$$⨍_{Q_1} V|u|dx \le ⨍_{Q_1} V|v|dx + N_1 \le 2N_1. \tag{2.19}$$

Moreover, if $x \in Q_1 \not\subset 3Q$, then we have $x \in Q \subset Q_1$ and $3Q \subset 3Q_1$. Therefore, from (2.12) we find that

$$⨍_{Q_1} V|u|dy \le 3^n ⨍_{3Q_1} V|u|dy \le 3^n, \tag{2.20}$$

since $x_0 \in \widetilde{Q} \subset 3Q \subset 3Q_1$ and $\mathcal{M}(V|u|)(x_0) \le 1$. Thus, it follows from (2.19) and (2.20) that $\mathcal{M}(V|u|)(x) \le N_2$, which implies that (2.18) is true. Finally, from (2.15), (2.18) and the weak $(1,1)$ estimate of the maximal functions we have

$$\begin{aligned} &|\{x \in Q : \mathcal{M}(V|u|)(x) > N_2\}| \\ &\quad \le |\{x \in Q : \mathcal{M}(|Vv|)(x) > N_1\}| \\ &\quad \le C \int_Q |Vv|\, dx \le C \int_{4Q} |Vv|\, dx \le C\delta\, |4Q| \le C\delta\, |Q| \le \mu\, |Q|, \end{aligned}$$

by choosing δ small enough satisfying the last inequality. Thus we complete the proof. □

Furthermore, we can directly obtain the following result from the lemma above.

Corollary 2.9. *Assume that $w \in A_p$ for $p > 1$. For any $\mu \in (0,1)$ there exist two constants $N_3 = N_3(n) > 1$ and $\delta = \delta(n, \sigma, \mu) \in (0,1)$ such that if*

$$w\left(\left\{x \in \widetilde{Q} : \mathcal{M}(V|u|)(x) \le 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}(|f|)(x) \le \delta\right\}\right) > \frac{1}{2} w\left(\widetilde{Q}\right), \tag{2.21}$$

then we have

$$w(\{x \in Q : \mathcal{M}(V|u|)(x) \ge N_3\}) \le \mu w(Q).$$

Proof. From Lemma 1.9 and (2.21) we have

$$\begin{aligned} &\frac{\left|\left\{x \in \widetilde{Q} : \mathcal{M}(V|u|)(x) \le 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}(|f|)(x) \le \delta\right\}\right|}{\left|\widetilde{Q}\right|} \\ &\quad \ge \left[\frac{w\left(\left\{x \in \widetilde{Q} : \mathcal{M}(V|u|)(x) \le 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}(|f|)(x) \le \delta\right\}\right)}{C_2 w\left(\widetilde{Q}\right)}\right]^{\frac{1}{\sigma}} \\ &\quad \ge (2C_2)^{-\frac{1}{\sigma}} \in (0,1), \end{aligned}$$

since $C_2 > 1$ and $\sigma > 0$. So, for any $\mu_1 \in (0,1)$ with $C_2 \mu_1^{\sigma} < 1$, it follows from Lemma 2.8 that there exist two positive constants $N_3 = N_3(n) > 1$ and $\delta = \delta(n, \sigma, C_2, \mu_1) \in (0,1)$ such that

$$|\{x \in Q : \mathcal{M}(V|u|)(x) \ge N_3\}| \le \mu_1 |Q|.$$

Then Lemma 1.9 implies that

$$w\left(\{x\in Q:\mathcal{M}\left(V|u|\right)(x)\geq N_3\}\right)\leq C_2\mu_1^{\sigma}w\left(Q\right),$$

which completes our proof by selecting $\mu = C_2\mu_1^{\sigma}$. □

Furthermore, we can obtain the following result.

Corollary 2.10. *Let D be a cube in $\mathbb{R}^n$. Assume that w, μ, δ, N_3 satisfy the same conditions as those in Corollary 2.9. If*

$$w\left(\{x\in D:\mathcal{M}\left(V|u|\right)(x)\geq N_3\}\right)\leq \mu w\left(D\right),$$

then we have

$$\begin{aligned} &w\left(\{x\in D:\mathcal{M}\left(V|u|\right)(x)\geq N_3\}\right)\\ &\quad\leq 2\mu\Big[w\left(\{x\in D:\mathcal{M}\left(V|u|\right)(x)>1\}\right)+w\left(\{x\in D:\mathcal{M}\left(|f|\right)(x)>\delta\}\right)\Big].\end{aligned}\tag{2.22}$$

Proof. We denote

$$A = w\left(\{x\in D:\mathcal{M}\left(V|u|\right)(x)\geq N_3\}\right)$$

and

$$B = w\left(\{x\in D:\mathcal{M}\left(V|u|\right)(x)>1\}\cup\{x\in D:\mathcal{M}\left(|f|\right)(x)>\delta\}\right).$$

Then $A, B \subset D$ and $w\left(A\right) \leq \mu w(D)$. Therefore, it follows from Lemma 2.1 that there exists a sequence of disjoint cubes $\{Q_k\}$ satisfying

(1) $w\left(A\setminus \bigcup_{k\in\mathbb{N}} Q_k\right) = 0$,

(2) $w\left(A\cap Q_k\right) > \mu w\left(Q_k\right)$,

(3) $w\left(A\cap \widetilde{Q_k}\right) \leq \mu w\left(\widetilde{Q_k}\right)$ if $\widetilde{Q_k}$ is the predecessor (father) of Q_k.

If $w\big(\widetilde{Q_k}\cap B\big) \leq \frac{1}{2}w\big(\widetilde{Q_k}\big)$, where $\widetilde{Q_k}$ is the predecessor of Q_k, then we obtain (2.21) with $\widetilde{Q}$ repacing by $\widetilde{Q_k}$. Furthermore, it follows from Corollary 2.9 that

$$w\left(A\cap Q_k\right)\leq w\left(\{x\in Q_k:\mathcal{M}\left(V|u|\right)(x)\geq N_3\}\right)\leq \mu w\left(Q_k\right).$$

So, we get a contradiction with (2) and then know that $w\big(\widetilde{Q_k}\cap B\big) > \frac{1}{2}w\big(\widetilde{Q_k}\big)$. Finally, we can use Lemma 2.1 again to get that

$$w\left(A\right)\leq 2\mu w\left(B\right),$$

which implies (2.22) is true. Thus, we finish the proof. □

Corollary 2.11. *Assume that $\mu \in (0,1)$ with $C_2\mu^{\sigma} < 1$ and w, δ, N_3 satisfy the same conditions as those in Corollary 2.9. For any $\lambda > 0$ we have*

$$\begin{aligned} &w\left(\{x\in\mathbb{R}^n:\mathcal{M}\left(V|u|\right)(x)\geq \lambda N_3\}\right)\\ &\quad\leq 2C_2\mu^{\sigma}\big[w\left(\{x\in\mathbb{R}^n:\mathcal{M}\left(V|u|\right)(x)>\lambda\}\right)\\ &\qquad+w\left(\{x\in\mathbb{R}^n:\mathcal{M}\left(|f|\right)(x)>\lambda\delta\}\right)\big].\end{aligned}\tag{2.23}$$

Proof. Without loss of generality, we may as well assume that $\lambda = 1$. Let

$$\mathbb{R}^n = \bigcup_{i=1}^{\infty} \overline{Q_i},$$

where $\{Q_i\}$ is a sequence of disjoint same side-length cubes. Moreover, from the weak 1-1 estimate and L^1 estimate (see [3, 8]) we conclude that

$$|\{x \in \mathbb{R}^n : \mathcal{M}(V|u|)(x) \geq N_3\}| \leq \frac{C}{N_3}\|Vu\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{N_3}\|f\|_{L^1(\mathbb{R}^n)}.$$

We may as well assume that $f \in C_0^\infty(\mathbb{R}^n)$ via an elementary approximation argument. So, we can obtain

$$|\{x \in Q_i : \mathcal{M}(V|u|)(x) \geq N_3\}| \leq \mu |Q_i|$$

by selecting $|Q_i|$ large enough for $i \in \mathbb{N}$. Furthermore, from Lemma 1.9 we have

$$w(\{x \in Q_i : \mathcal{M}(V|u|)(x) \geq N_3\}) \leq C_2\mu^\sigma w(Q_i).$$

Thus, by Corollary 2.10 we obtain

$$\begin{aligned} & w(\{x \in Q_i : \mathcal{M}(V|u|)(x) \geq N_3\}) \\ & \quad \leq 2C_2\mu^\sigma \left[w(\{x \in Q_i : \mathcal{M}(V|u|)(x) > 1\}) + w(\{x \in Q_i : \mathcal{M}(|f|)(x) > \delta\}) \right], \end{aligned}$$

which implies that the desired estimate (2.23) is true. This finishes our proof. □

Now we are ready to prove Theorem 1.11.

Proof. From Lemma 1.6 (3) and Corollary 2.11 we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |\mathcal{M}(V|u|)|^p w(x)\,dx \\ & \quad = p\int_0^\infty (N_3\lambda)^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}(V|u|)(x) > N_3\lambda\})\,d[N_3\lambda] \\ & \quad \leq 2C_2p\mu^\sigma \int_0^\infty (N_3\lambda)^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}(V|u|)(x) > \lambda\})\,d[N_3\lambda] \\ & \quad\quad +2C_2p\mu^\sigma \int_0^\infty (N_3\lambda)^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}(|f|)(x) > \lambda\delta\})\,d[N_3\lambda] \\ & \quad \leq C_3\mu^\sigma \int_{\mathbb{R}^n} |\mathcal{M}(V|u|)|^p w(x)\,dx + C_4\int_{\mathbb{R}^n} |\mathcal{M}(|f|)|^p w(x)\,dx \end{aligned}$$

for any $\mu \in (0,1)$ with $C_2\mu^\sigma < 1$, where $C_3 = C_3(p,n)$ and $C_4 = C_4(p,n,\mu,\sigma)$. Without loss of generality we may as well assume that $f \in C_0^\infty(\mathbb{R}^n)$. Then choosing a suitable μ such that $C_3\mu^\sigma < 1$, we obtain

$$\int_{\mathbb{R}^n} |\mathcal{M}(V|u|)|^p w(x)\,dx \leq C\int_{\mathbb{R}^n} |\mathcal{M}(|f|)|^p w(x)\,dx \leq C\int_{\mathbb{R}^n} |f|^p\, w(x)\,dx$$

in view of Lemma 1.6 (1). Thus, we can obtain

$$\int_{\mathbb{R}^n} |Vu|^p w(x)\,dx \leq C\int_{\mathbb{R}^n} |f|^p w(x)\,dx$$

by using the fact that $V|u|(x) \leq \mathcal{M}(V|u|)(x)$. Thus from Theorem 1.10 we observe that

$$\int_{\mathbb{R}^n} |D^2u|^p w(x)\,dx \leq C\int_{\mathbb{R}^n} |f|^p w(x)\,dx,$$

which completes the proof.

□

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