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PRINCIPAL SOLUTION OF HALF-LINEAR DIFFERENTIAL EQUATION: LIMIT AND INTEGRAL CHARACTERIZATION

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Abstract. We investigate integral and limit characterizations of the principal solution of the nonoscillatory half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2},\ p > 1.$$

In particular, we supplement and extend the results of the previous papers [2, 3, 5, 8, 11].

1. Introduction

In this paper we deal with the second order half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2},\ p > 1, \tag{1}$$

where r, c are continuous functions for $t \geq 0$ and $r(t) > 0$.

The qualitative theory of (1) attracted considerable attention in recent years. It was shown that many properties of solutions of (1) are very similar to those of the linear Sturm-Liouville differential equation (which is the special case $p = 2$ in (1))

$$(r(t)x')' + c(t)x = 0. \tag{2}$$

On the other hand, several phenomena have been indicated, where the behavior of solutions of (1) and (2) is completely different. We refer to the recent book [12] for the comprehensive treatment of the theory of half-linear differential equations. For the reader who is not familiar with the elements of the theory of half-linear equations let us recall at least that the terminology *half-linear* equation comes from the Hungarian mathematicians I. Bihari [1] and Á. Elbert [13] and it is motivated by the fact that the solution space of (1) has just one half of the properties which characterize linearity, namely homogeneity, but generally not additivity.

Our principal concern is to investigate properties of the so-called *principal solution* of (1). The concept of the principal solution of the linear equation (2) was introduced

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by Leighton and Morse [18] and basic properties of this solution were investigated by Hartman (see [16] for a basic survey). It was shown, that *nonoscillatory* equation (2) (i.e., there exists $T \in \mathbb{R}$, such that any nontrivial solution of (2) has at most one zero point on $[T, \infty)$) possesses a unique (up to a nonzero multiplicative factor) solution $\tilde{x}$, called the *principal solution*, with the property that

$$\lim_{t\to\infty} \frac{\tilde{x}(t)}{x(t)} = 0 \tag{3}$$

for any solution x linearly independent of $\tilde{x}$. An equivalent characterization of the principal solution $\tilde{x}$ is

$$\int^{\infty} \frac{dt}{r(t)\tilde{x}^2(t)} = \infty \tag{4}$$

since this integral is convergent for any solution linearly independent of $\tilde{x}$.

A closer examination of the treatment given in [16, Chap. XI] reveals that characterizations (3) and (4) are based on the additivity of the solution space and the Wronskian identity for solutions of (2), respectively, but none of these two properties extends to (1). However, as observed Mirzov [19] and later independently Elbert and Kusano [15], there is another equivalent characterization of the principal solution of (2) (less known than (3) and (4)) that *does extend* to (1). It is based on the fact that the solution $\tilde{w} := r\tilde{x}'/\tilde{x}$ of the Riccati equation

$$w' + c(t) + \frac{w^2}{r(t)} = 0 \tag{5}$$

corresponding to the principal solution $\tilde{x}$ of (1) is smaller than any other solution of (5) for large t .

Mirzov [19] observed that a similar situation holds for the half-linear equation (1) and the *associated Riccati equation*

$$w' + c(t) + (p-1)\frac{|w|^q}{r^{q-1}(t)} = 0, \quad q = \frac{p}{p-1}, \tag{6}$$

which is related to (1) by the Riccati substitution

$$\tilde{w}(t) = r(t)\Phi(\tilde{x}'(t)/\tilde{x}(t)).$$

More precisely, Mirzov showed that among all solutions w of (6) there exists the *minimal* one $\tilde{w}$, minimal in the sense that for any other solution w of (6) we have

$$\tilde{w}(t) < w(t) \quad \text{for large } t. \tag{7}$$

The *principal solution* of (1) is that solution $\tilde{x}$ which "generates" the minimal solution $\tilde{w}$ via the Riccati substitution, i.e., it is given by the formula

$$\tilde{x}(t) = C \exp\left\{ \int^t r^{1-q}(s)\Phi^{-1}(\tilde{w}(s))\, ds \right\},$$

where $\Phi^{-1}(x) = |x|^{q-2}x$ is the inverse function of Φ. Obviously, the principal solution is unique up to a constant factor.

Comparing the above three properties of principal solutions of (2) – the "Riccati property" (7), the limit property (3) and the integral property (4), the last one is specific: it involves just one solution $\tilde{x}$, without comparing it with other solutions. The extension of the integral property to half-linear equations seems to be a difficult problem. As we will show, the universal integral characterization which would contain only one solution likely does not exist in general.

In this paper we continue in the investigation initiated in [2, 3, 4, 5, 6, 8, 11], where various integral characterizations of the principal solution of (1) have been offered and the limit characterization (3) has been investigated (see [2]–[5]). Our main result shows how the limit characterization (3) of the principal solution $\tilde{x}$ is related with two types of integral characterizations. We also present an example showing that one of the results of the paper [8] is not correct.

We will use the following notation:

$$J_r = \int^\infty \frac{dt}{r^{q-1}(t)}, \quad J_c = \int^\infty |c(t)|\, dt.$$

Note that if both integrals J_r and J_c are divergent, then (1) is oscillatory, see [12].

2. Survey of the known results

We recall known results concerning integral and limit characterizations of the principal solution of (1). The first attempt to find an integral characterization of the principal solution of (1) was made in the paper of Mirzov [19]. To introduce it, denote

$$F(t) := \begin{cases} \frac{1-|t|^q}{1-t} + \Phi^{-1}(1-t) & t \neq 1, \\ q & t = 1, \end{cases}$$

and

$$m_* = \min\{F(t) :\ t \in [0,1]\}, \quad m^* = \max\{F(t) :\ t \in [0,1]\}.$$

Mirzov's integral characterization of the pricipal solution of (1) is given in the next proposition.

Proposition 1. *Suppose that* $J_r = \infty$. *If* $\tilde{x}$ *is the principal solution, then*

$$\int^{\infty} \frac{dt}{r^{q-1}(t)|\tilde{x}(t)|^{m_*}} = \infty,$$

and, conversely, if

$$\int^{\infty} \frac{dt}{r^{q-1}(t)|\tilde{x}(t)|^{m^*}} = \infty,$$

then $\tilde{x}$ *is the principal solution of* (1).

However, the exponents m_*, m^* coincide (and $m_* = m^* = 2$) only in the linear case $p = 2$ and generally $m^* > m_*$, so Mirzov's integrals characterize principal solution of (1) *equivalently* only in the linear case.

The second attempt to find an integral characterization of the principal solution of (1) comes from the paper [8]. The main result of that paper reads as follows.

Proposition 2. *Let* $\tilde{x}$ *be a solution of* (1) *such that* $\tilde{x}'(t) \neq 0$ *for large* t. *Then we have the following statements:*

(i) *Let* $p \in (1, 2]$. *If*

$$I(\tilde{x}) := \int^{\infty} \frac{dt}{r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}} = \infty, \tag{8}$$

then $\tilde{x}$ *is the principal solution of* (1).

(ii) *Let* $p \geq 2$. *If* $\tilde{x}$ *is the principal solution of* (1) *then* (8) *holds.*

(iii) *Suppose that* $p \geq 2$, $J_r = \infty$, *the function* $\gamma(t) := \int_t^\infty c(s)\, ds$ *exists, and* $\gamma(t) \geq 0$, *but* $\gamma(t) \not\equiv 0$ *for large* t. *Then* $\tilde{x}$ *is the principal solution of* (1) *if and only if* (8) *holds.*

Note that the statement (iii) of the previous proposition is stated in [8] without the assumption $p \geq 2$. As the next example shows, the implication: "$\tilde{x}$ is the principal solution $\Longrightarrow$ $I(\tilde{x}) = \infty$" may fail to hold for $p \in (1, 2)$.

Example 1. Consider the equation

$$((x')^{1/2})' + \frac{15t^{-3/2}}{(t^9 - 1)^{1/2}} x^{1/2} = 0, \quad t \geq 1. \tag{9}$$

This equation has a solution $\tilde{x}(t) = 1 - 1/t^9$ which satisfies $\tilde{x}(t) \to 1$ and its quasiderivative

$$\tilde{x}^{[1]} := (\tilde{x}')^{1/2} = 3/t^5 \to 0$$

as $t \to \infty$. According to the uniqueness result of [17] (see also [3, Theorem B]), all bounded solutions of (1) are uniquely determined up to a multiplicative constant.

Hence,

$$\lim_{t\to\infty} \frac{\tilde{x}(t)}{x(t)} = 0 \tag{10}$$

for any solution $x \neq \lambda\tilde{x}$. Suppose that $\tilde{x}$ is not a principal solution of (9) and let x_0 be a positive principal solution. By (7) we have

$$x_0'(t)/x_0(t) < \tilde{x}'(t)/\tilde{x}(t) \quad \text{for large } t,$$

that is the function $\tilde{x}/x_0$ is positive and increasing for large t. On the other hand, we have by (10) that $\tilde{x}/x_0 \to 0$, which is a contradiction. Thus $\tilde{x}$ is the principal solution of (9). However

$$I(\tilde{x}) = \int^{\infty} \frac{\tilde{x}'(t)}{\tilde{x}^2(t)\tilde{x}^{[1]}(t)}\,dt \sim \int^{\infty} (\tilde{x}'(t))^{1/2}\,dt \sim 3\int_1^{\infty} \frac{dt}{t^5} < \infty.$$

Hence, Proposition 2, part (iii), is not generally true for $p \in (1, 2)$.

Another integral characterization of the principal solution has been suggested in the papers [2, 3, 4], and the main results of these papers along this line are summarized in the next statement.

Proposition 3. *Suppose that either*
- (i) $c(t) < 0$ *for large* t*, or*
- (ii) *both integrals* J_r*,* J_c *are convergent.*

Then a solution $\tilde{x}$ *of* (1) *is principal if and only if*

$$\int^{\infty} \frac{dt}{r^{q-1}\tilde{x}^2(t)} = \infty. \tag{11}$$

Note that if $J_r = \infty$, $c(t) > 0$, and $p > 2$, then the principal solution of (1) need not satisfy (11), as the next example shows.

Example 2. Consider the half-linear Euler equation

$$(\Phi(x'))' + \left(\frac{\gamma}{t}\right)^p \Phi(x) = 0, \quad \gamma = (p-1)/p$$

The solution $\tilde{x}(t) = t^{\gamma}$ of this equation is principal (see e.g. [12, p. 40]) but does not satisfy (11) for $p > 2$. Note that any linearly independent nonprincipal solution is given by the asymptotic formula

$$x(t) = t^{\gamma}(\log t)^{\frac{2}{p}}\bigl(1 + O(\log^{-1} t)\bigr),$$

i.e., the limit characterization of the principal solution (10) holds in this case.

When $c(t) > 0$ for large t and one of the integrals J_r or J_c is divergent, the integral characterization of the principal solution of (1) has been studied in [3, 5] and is summarized in the following two statements (see [5, Theorem 2 and Theorem 3]):

Proposition 4. *Suppose that $c(t) > 0$ for large t, (1) is nonoscillatory and $J_r = \infty$. In addition, if both integrals*

$$\int^{\infty} \frac{1}{r^{q-1}(t)} \Big(\int_t^{\infty} c(s)\, ds \Big)^{q-1} dt, \quad \int^{\infty} c(t) \left(\int^t \frac{ds}{r^{q-1}(s)} \right)^{p-1} dt \tag{12}$$

are divergent, assume $p > 2$. Then a solution $\tilde{x}$ of (1) is principal if and only if

$$\int^{\infty} \frac{c(t)(\tilde{x}(t))^{p-2}}{\big(r(t)\Phi(\tilde{x}'(t))\big)^2}\, dt = \infty.$$

Proposition 5. *Suppose that $c(t) > 0$ for large t, (1) is nonoscillatory and $J_c = \infty$. In addition, if both integrals*

$$\int^{\infty} \frac{1}{r^{q-1}(t)} \Big(\int^{t} c(s)\, ds \Big)^{q-1} dt, \quad \int^{\infty} c(t) \left(\int_t^{\infty} \frac{ds}{r^{q-1}(s)} \right)^{p-1} dt \tag{13}$$

are divergent, assume $p < 2$. Then a solution $\tilde{x}$ of (1) is principal if and only if (8) holds.

Concerning the limit characterization of the half-linear principal solution, we have following statement proved in [2] (the case $c(t) < 0$), [3, 5] (the case $c(t) > 0$), and [4] (the case when c is allowed to change its sign).

Proposition 6. *Suppose that one of the following conditions holds:*

(i) *$c(t) < 0$ for large t;*
(ii) *Both integrals J_r, J_c are convergent;*
(iii) *Assumptions of Proposition 4 or 5 are satisfied.*

Then a solution $\tilde{x}$ of (1) is principal if and only if

$$\lim_{t\to\infty} \frac{\tilde{x}(t)}{x(t)} = 0 \tag{14}$$

for every solution x of (1) linearly independent of $\tilde{x}$.

We complete this section by the following statement which is a new result so we present it including the proof.

Proposition 7. *Suppose that $c(t) > 0$ for large t and one of the following conditions holds:*

(i) *$J_r = \infty$ and both integrals in (12) are convergent;*
(ii) *$J_c = \infty$ and both integrals in (13) are convergent.*

If $\tilde{x}$ is the principal solution of (1), *then*

$$\int^{\infty} \frac{dt}{r^{q-1}|\tilde{x}(t)|^m} = \infty \tag{15}$$

for any $m > 1$. Conversely, if $\tilde{x}$ is a solution of (1) *satisfying* (15) *for some $m > 1$, then $\tilde{x}$ is the principal solution.*

Proof. First assume (i). By [5, Corolary 1], the principal solution $\tilde{x}$ tends to a nonzero constant and so (15) holds for any $m > 1$. By [6, Theorem 7] and [5, Corolary 1], any linearly independent solution x satisfies

$$\lim_{t\to\infty} \frac{x(t)}{\int^t r^{1-q}(s)ds} = c, \quad 0 < |c| < \infty,$$

thus

$$\int^{\infty} \frac{dt}{r^{q-1}(t)|x(t)|^m} < \infty \quad \Longleftrightarrow \quad \int^{\infty} \frac{r^{1-q}(t)}{(\int^t r^{1-q}(s)ds)^m} dt < \infty$$

for any $m > 1$. Concerning the case (ii), we proceed by a similar way using the fact that the principal solution $\tilde{x}$ and any other solution x satisfy

$$\lim_{t\to\infty} \frac{\tilde{x}(t)}{\int_t^{\infty} r^{1-q}(s)ds} = c, \quad 0 < |c| < \infty, \quad \lim_{t\to\infty} x(t) = d, \quad 0 < |d| < \infty.$$

□

Finally note that integral characterization (8) and limit characterization (14) have been studied for the first time in [14, Theorem 4] for perturbed half-linear Euler equations.

3. Auxiliary statements

Now we present some auxiliary results concerning certain functions of two and three variables which we will need later. An important role in our investigation is played by the functions

$$\begin{aligned} Q(u,v) &:= \frac{1}{p}|u|^q - \Phi^{-1}(u)v + \frac{1}{q}|v|^q, \\ W(u,v) &:= \frac{[\Phi^{-1}(v) - \Phi^{-1}(u)](v-u)}{Q(u,v)}, \end{aligned}$$

and

$$H(x,u,v) := \frac{1}{r^{q-1}(t)x^p} \frac{Q(u,v)}{(u-v)^2}.$$

Observe that for $p = 2$ we have

$$Q(u,v) = \frac{1}{2}(v-u)^2, \quad W(u,v) \equiv 2, \quad \text{and} \quad H(x,u,v) = \frac{1}{2r(t)x^2}.$$

Lemma 1. *There exist positive constants $K_1 < K_2$ such that*

$$0 < K_1 \le W(u,v) \le K_2 \tag{16}$$

for every $u, v \in \mathbb{R}$.

Proof. We have $W(0,v) = q$ for $v \in \mathbb{R}$, hence (16) holds for $u = 0$. If $u \neq 0$, let $z = v/u$. Then

$$W(u,v) = F(z) := \frac{|z|^q - \Phi^{-1}(z) - z + 1}{\frac{1}{q}|z|^q - z + \frac{1}{p}}$$

and it suffices to show that the function F satisfies the inequalities stated in the Lemma. This function is nonnegative since

$$|z|^q - \Phi^{-1}(z) - z + 1 = (\Phi^{-1}(z) - 1)(z-1) \ge 0$$

and $\frac{1}{q}|z|^q - z + \frac{1}{p} \ge 0$ (Young's inequality), and the required inequalities follow from the fact that at the only zero point $z = 1$ of the denominator, we have

$$\lim_{z \to 1} \frac{|z|^q - \Phi^{-1}(z) - z + 1}{\frac{1}{q}|z|^q - z + \frac{1}{p}} = \frac{q(q-1) + (q-1)(q-2)}{q(q-1)} = \frac{2(q-1)}{q},$$

i.e., this is a removable discontinuity, and from finiteness of the limit

$$\lim_{z \to \pm\infty} \frac{|z|^q - \Phi^{-1}(z) - z + 1}{\frac{1}{q}|z|^q - z + \frac{1}{p}} = q.$$

The proof is complete. □

The next statement is a technical result concerning a certain identity for solutions of (1) and of the associated Riccati equation (6).

Lemma 2. *Let x, y be nonoscillatory solutions of* (1) *and let w_x, w_y be the associated solutions of* (6). *Then*

$$\left(\frac{1}{|x(t)|^p (w_y(t) - w_x(t))} \right)' = pH(x(t), w_x(t), w_y(t)). \tag{17}$$

Proof. Without loss of generality we may suppose that $x(t) > 0$ for large t. Denote by $f(t) = x^p(t)(w_y(t) - w_x(t))$. Then, we have

$$\begin{aligned} f'(t) &= p\Phi(x)x'(w_y - w_x) + x^p(w_y' - w_x') \\ &= p\Phi(x)r^{1-q}\Phi^{-1}(w_x)x(w_y - w_x) - (p-1)x^p r^{1-q}(|w_y|^q - |w_x|^q) \\ &= x^p r^{1-q}\left[p\Phi^{-1}(w_x)(w_y - w_x) - (p-1)(|w_y|^q - |w_x|^q)\right], \end{aligned}$$

and hence

$$
\begin{aligned}
-\Big(\frac{1}{f(t)}\Big)' &= \frac{f'(t)}{f^2(t)} = \frac{x^p r^{1-q}\left[p\Phi^{-1}(w_x)(w_y-w_x)-(p-1)(|w_y|^q-|w_x|^q)\right]}{x^{2p}(w_y-w_x)^2} \\
&= -\frac{(p-1)|w_y|^q+pw_y\Phi^{-1}(w_x)-|w_x|^q}{r^{q-1}x^p(w_y-w_x)^2} \\
&= -\frac{p}{r^{q-1}x^p}\frac{Q(w_x,w_y)}{(w_y-w_x)^2} \\
&= -pH(x,w_x,w_y)
\end{aligned}
$$

and this completes the proof. □

4. Main result

Now we are in a position to formulate the main result of our paper. This result relates various quantities involving nonoscillatory solutions of (1) and associated solutions of the Riccati equation (6). As corollaries, it provides a unified view on the various characterizations of the principal solutions of (1) presented in Section 2.

Theorem 1. *Let x, y be a pair of independent solutions of* (1) *and let w_x, w_y be the associated solutions of* (6). *Then the following integral and limit relations are equivalent:*

(i)

$$\int^\infty H(x(t), w_x(t), w_y(t))\, dt = \infty;$$

(ii)

$$\int^\infty \frac{1}{r^{q-1}(t)|x(t)|^p}\frac{\Phi^{-1}(w_y(t))-\Phi^{-1}(w_x(t))}{w_y(t)-w_x(t)}\, dt = \infty;$$

(iii)

$$\lim_{t\to\infty} |x(t)|^p(w_y(t)-w_x(t)) = 0 \quad \textit{and } w_x(t) < w_y(t) \textit{ for large } t;$$

(iv)

$$\lim_{t\to\infty}\frac{x(t)}{y(t)} = 0.$$

Proof. According to the homogeneity of the solution space of (1), we can suppose that the relationship between x, y and w_x, w_y is

$$x(t) = \exp\left\{\int_T^t r^{1-q}(s)\Phi^{-1}(w_x(s))\,ds\right\}, \qquad (18)$$

$$y(t) = \exp\left\{\int_T^t r^{1-q}(s)\Phi^{-1}(w_y(s))\,ds\right\},$$

for some $T \in \mathbb{R}$.
(i) $\iff$ (ii): Using Lemma 1 we get

$$\frac{1}{K_2}\frac{\Phi^{-1}(v)-\Phi^{-1}(u)}{v-u} \le \frac{Q(u,v)}{(u-v)^2} \le \frac{1}{K_1}\frac{\Phi^{-1}(v)-\Phi^{-1}(u)}{v-u}$$

for every $u, v \in \mathbb{R}$. From this we get

$$\frac{1}{K_2}F(x(t),w_x(t),w_y(t)) \le H(x(t),w_x(t),w_y(t)) \le \frac{1}{K_1}F(x(t),w_x(t),w_y(t)),$$

where

$$F(x(t),w_x(t),w_y(t)) = \frac{\Phi^{-1}(w_y(t))-\Phi^{-1}(w_x(t))}{r^{q-1}(t)|x(t)|^p(w_y(t)-w_x(t))}.$$

Integrating these inequalities, the equivalency between (i) and (ii) follows.
(i) $\iff$ (iii): Using Lemma 2, we get by integration of (17) from T to t, $T < t$,

$$\left.\frac{1}{|x(s)|^p(w_y(s)-w_x(s))}\right|_T^t = p\int_T^t H(x(s),w_x(s),w_y(s))\,ds. \qquad (19)$$

Letting $t \to \infty$, we get the conclusion.
(i) $\iff$ (iv): According to (18), we have for some $T \in \mathbb{R}$

$$\frac{y(t)}{x(t)} = \exp\left\{\int_T^t r^{1-q}(s)\left[\Phi^{-1}(w_y(s))-\Phi^{-1}(w_x(s))\right]\,ds\right\}.$$

Hence, it suffices to show the equivalency between (i) and

$$\int_T^\infty r^{1-q}(t)\left[\Phi^{-1}(w_y(t))-\Phi^{-1}(w_x(t))\right]\,dt = \infty. \qquad (20)$$

First suppose that (i) holds. Using the L'Hospital rule in the form $\liminf(f/g) \geq \liminf(f'/g')$, we get

$$L := \liminf_{t\to\infty} \frac{\int_T^t r^{1-q}(s)\,[\Phi^{-1}(w_y(s)) - \Phi^{-1}(w_x(s))]\; ds}{\log\left(\int_T^t H(x(s), w_x(s), w_y(s))\, ds\right)}$$

$$\geq \liminf_{t\to\infty} \left(\frac{[\Phi^{-1}(w_y(t)) - \Phi^{-1}(w_x(t))]\,(w_y(t) - w_x(t))^2}{px^{-p}(t)\,Q(w_x(t), w_y(t))} \int_T^t H(x(s), w_x(s), w_y(s))\, ds \right)$$

$$= \frac{1}{p} \liminf_{t\to\infty} \left(x^p(t)[w_y(t) - w_x(t)] W(w_x(t), w_y(t)) \int_T^t H(x(s), w_x(s), w_y(s))\, ds \right).$$

Lemma 2 yields (19) and thus

$$x^p(t)\big(w_y(t) - w_x(t)\big) = \frac{1}{K + \int_T^t H(x(s), w_x(s), w_y(s))\, ds},$$

where K is a real constant. Taking into account (16), we have

$$L \geq \frac{1}{p} \liminf_{t\to\infty} \frac{\int_T^t H(x(s), w_x(s), w_y(s))\, ds}{K + \int_T^t H(x(s), w_x(s), w_y(s))\, ds} W(w_x(t), w_y(t)) \geq \frac{K_1}{p} > 0.$$

Consequently, (i) implies (20).

Conversely, suppose that (20) holds. Proceeding similarly as above,

$$\tilde{L} := \liminf_{t\to\infty} \frac{\log\left(\int_T^t H(x(s), w_x(s), w_y(s))\, ds\right)}{\int_T^t r^{1-q}(s)\,[\Phi^{-1}(w_y(s)) - \Phi^{-1}(w_x(s))]\; ds}$$

$$\geq p \liminf_{t\to\infty} \frac{K + \int_T^t H(x(s), w_x(s), w_y(s))\, ds}{\left(\int_T^t H(x(s), w_x(s), w_y(s))\, ds\right) W(w_x(t), w_y(t))}$$

$$\geq \frac{p}{K_2} \lim_{t\to\infty} \frac{K + \int_T^t H(x(s), w_x(s), w_y(s))\, ds}{\int_T^t H(x(s), w_x(s), w_y(s))\, ds} > 0.$$

Thus, (20) implies (i). □

As an immediate consequence of Theorem 1 we have the following statement.

Corollary 1. *Let x be a solution of* (1) *and w_x be the associated solution of* (6)*. If any of conditions* (i) – (iv) *of Theorem 1 holds for every solution y of* (1) *and the associated solution w_y of* (6)*, then x is the principal solution of* (1).

Remark 1. In the linear case, i.e. $p = 2$, the conditions (i) and (ii) of Theorem 1 are the same and reduces to (4).

Corollary 2. *Let any of assumptions* (i)-(iii) *of Proposition 6 hold, let $\tilde{x}$ be a principal solution of* (1)*, and $\tilde{w}$ be the associated minimal solution of* (6)*. Then*

$$\lim_{t\to\infty} |\tilde{x}(t)|^p (w(t) - \tilde{w}(t)) = 0$$

for any other solution w of (6)*.*

Based on the linear case, Corollary 2, and explicitly computable examples, we conjecture that the converse statement to Corollary 1 holds.

Conjecture 1. *Suppose that $\tilde{x}$ is the principal solution of* (1) *and $\tilde{w}$ is the associated solution of* (6)*. Then all conditions* (i) – (iv) *of Theorem 1 hold for any other solution w of* (6)*.*

In Section 2 we have presented various characterizations of the principal solutions of (1). Now, using condition (ii) from Theorem 1, we can relate these characterizations if we have at our disposal some additional information about asymptotic behavior of solutions of (6).

Corollary 3. *Let $\tilde{x}$ be a solution of* (1) *such that $\tilde{x}'(t) \neq 0$ for large t and $\tilde{w}$ be the associated solution of* (6)*. Suppose that there exists a constant $K > 0$ such that*

$$\limsup_{t\to\infty} \left| \frac{w(t)}{\tilde{w}(t)} \right| < K \tag{21}$$

for every solution w of (6)*. Then the following statements are valid:*

(i) (8) *holds if and only if* (10) *holds for any solution x of* (1) *linearly independent of $\tilde{x}$.*
(ii) *If* (8) *holds, then $\tilde{x}$ is the principal solution of* (1)*.*
(iii) *If $p \geq 2$, then $\tilde{x}$ is the principal solution of* (1) *if and only if* (8) *holds.*

Proof. By Theorem 1, it is sufficient to prove that condition (8) is equivalent to condition (ii) of this theorem for any solution w. We have

$$\frac{1}{r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}} = \frac{1}{r^{q-1}(t)|\tilde{x}(t)|^p} \frac{\Phi^{-1}(\tilde{w}(t))}{\tilde{w}(t)}$$

and

$$\frac{\Phi^{-1}(w(t)) - \Phi^{-1}(\tilde{w}(t))}{w(t) - \tilde{w}(t)} = \frac{\Phi^{-1}(\tilde{w}(t))}{\tilde{w}(t)} G\Big(\frac{w(t)}{\tilde{w}(t)}\Big)$$

where $G(t) = \frac{\Phi^{-1}(t)-1}{t-1}$. Since the function G is is positive, bounded and bounded away from zero on the interval $[-K, K]$, there exist positive constants m_1, m_2 such that

$$m_1 \leq G\Big(\frac{w(t)}{\tilde{w}(t)}\Big) \leq m_2.$$

Thus,

$$\frac{m_1}{r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}} \le \frac{1}{r^{q-1}(t)|\tilde{x}(t)|^p}\frac{\Phi^{-1}(\tilde{w}(t))}{\tilde{w}(t)}G\Big(\frac{w(t)}{\tilde{w}(t)}\Big) \le \frac{m_2}{r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}}.$$

Hence (ii) is equivalent to (8). Now the conclusion follows from Corollary 1 and the parts (ii)–(iv) of Theorem 1. □

Remark 2. The statement (i) of Corollary 3 does not hold without assumption (21), as Example 1 illustrates. More precisely, one can check that the minimal solution of the corresponding Riccati equation to (9) satisfies $\tilde{w}(t) \sim 3t^{-4}$. By [6, Theorems 4,7] any other solution w satisfies $w(t) \sim t^{-1/2}$, thus (21) is not satisfied.

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