Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **1**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

Growth of meromorphic solutions of higher-order linear differential equations

Wenjuan Chen[1] Junfeng Xu[2*]

1.Department of Mathematics, Shandong University, Jinan, Shandong 250100, P.R.China

2.Department of Mathematics, Wuyi University, Jiangmen, Guangdong 529020, P.R.China

Abstract. In this paper, we investigate the higher-order linear differential equations with meromorphic coefficients. We improve and extend a result of M.S. Liu and C.L. Yuan, by using the estimates for the logarithmic derivative of a transcendental meromorphic function due to Gundersen, and the extended Winman-Valiron theory which proved by J. Wang and H.X. Yi. In addition, we also consider the nonhomogeneous linear differential equations.

Keywords: Linear differential equation; meromorphic function; growth order; Nevanlinna theorey; iterated order
2000 AMS Subject Classifications: 34M10, 30D35

1 Introduction and main results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [11, 22]). The term "meromorphic function" will mean meromorphic in the whole complex plane $\mathbb{C}$.

For the second order linear differential equation

$$f'' + e^{-z} f' + B(z) f = 0, \tag{1.1}$$

where $B(z)$ is an entire function of finite order. It is well known that each solution f of (1.1) is an entire function, and that if f_1 and f_2 are any two linearly independent solutions of (1.1), then at least one of f_1, f_2 must have infinitely order([12]). Hence, "most" solutions of (1.1) will have infinite order. However, the equation (1.1) with $B(z) = -(1 + e^{-z})$ possesses a solution $f = e^z$ of finite order.

Thus a natural question is: what condition on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) will have infinite order? Frei, Ozawa, Amemiya and Langley, and Gunderson studied the question. For the case that $B(z)$ is a transcendental entire function, Gundersen [8] proved that if $\rho(B) \neq 1$, then for every solution $f \not\equiv 0$ of (1.1) has infinite order.

For the above question, there are many results for second order linear differential equations (see, for example [1, 4, 6, 7, 10, 15]). In 2002, Z. X. Chen considered the problem and obtained the following result in [4].

*Corresponding author.
E-mail: chenwenjuan@mail.sdu.edu.cn(W.J. Chen); xujunf@gmail.com(J.F. Xu)

Theorem 1.1. *Let a, b be nonzero complex numbers and $a \neq b$, $Q(z) \not\equiv 0$ be a nonconstant polynomial or $Q(z) = h(z)e^{bz}$, where $h(z)$ is a nonzero polynomial. Then every solution $f \not\equiv 0$ of the equation*

$$f'' + e^{bz}f' + Q(z)f = 0$$

has infinite order and $\sigma_2(f) = 1$.

In 2006, Liu and Yuan generalized Theorem 1.1 and obtained the following result.

Theorem 1.2 (see. [17, Theorem 1]). *Suppose that a, b are nonzero complex numbers, $h_j(j = 0, 1, \cdots, k-1)(h_0 \not\equiv 0)$ be meromorphic functions that have finite poles and $\sigma = \max\{\sigma(h_j) : j = 0, 1, \cdots, k-1\} < 1$. If $\arg a \neq \arg b$ or $a = cb(0 < c < 1)$, then every transcendental meromorphic solution f of the equation*

$$f^{(k)} + h_{k-1}f^{(k-1)} + \cdots + e^{az}f^{(s)} + \cdots + h_1f' + h_0e^{bz}f = 0. \tag{1.2}$$

have infinite order and $\sigma_2(f) = 1$.

It is natural to ask the following question: What can we say if we remove the condition $h_j(j = 0, 1, \cdots, k-1)$ have finite poles in Theorem 1.2. In this paper, we first investigate the problem and obtain the following result.

Theorem 1.3. *Let $P(z)$ and $Q(z)$ be a nonconstant polynomials such that*

$$P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0,$$

$$Q(z) = b_nz^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0$$

for some complex numbers $a_i, b_i(i = 0, 1, 2, \cdots, n)$ with $a_n \neq 0, b_n \neq 0$ and let $h_j(j = 0, 1, \cdots, k-1)(h_0 \not\equiv 0)$ be meromorphic functions and $\sigma = \max\{\sigma(h_j) : j = 0, 1, \cdots, k-1\} < n$. If $\arg a_n \neq \arg b_n$ or $a_n = cb_n(0 < c < 1)$, suppose that all poles of f are of uniformly bounded multiplicity. Then every transcendental meromorphic solution f of the equation

$$f^{(k)} + h_{k-1}f^{(k-1)} + \cdots + h_se^{P(z)}f^{(s)} + \cdots + h_1f' + h_0e^{Q(z)}f = 0 \tag{1.3}$$

have infinite order and $\sigma_2(f) = n$.

Next, we continue to investigate the problem and extend Theorem 1.2.

Theorem 1.4. *Let $P(z)$ and $Q(z)$ be a nonconstant polynomials as the above, for some complex numbers $a_i, b_i(i = 0, 1, 2, \cdots, n)$ with $a_n \neq 0, b_n \neq 0$ and let $h_j(j = 0, 1, \cdots, k-1)(h_0 \not\equiv 0)$ be meromorphic functions and $\sigma = \max\{\sigma(h_j) : j = 1, \cdots, k-1\} < n$. Suppose all poles of f are of uniformly bounded multiplicity. Then the following three statements hold:*

1. *If $a_n = b_n$, and $\deg(P - Q) = m \geq 1, \sigma < m$, then every transcendental meromorphic solution f of the equation (1.3) have infinite order and $m \leq \sigma_2(f) \leq n$.*

2. *If $a_n = cb_n$ with $c > 1$, and $\deg(P - Q) = m \geq 1, \sigma < m$, then every solution $f \not\equiv 0$ of the equation (1.3) is of infinite order, and $\sigma_2(f) = n$.*

3. If $\sigma < \sigma(h_0) < 1/2$, $a_n = cb_n$ with $c \geq 1$ and $P(z) - cQ(z)$ is a constant, then every solution $f \not\equiv 0$ of equation (1.3) is of infinite order, and $\sigma(h_0) \leq \sigma_2(f) \leq n$.

Remark 1.1 Setting $h_j (j = 1, 2, \ldots, k-1)$ be entire functions in Theorem 1.3 and Theorem 1.4, we get Theorem 1 in [17]."

Considering nonhomogeneous linear differential equations

$$f^{(k)} + h_{k-1} f^{(k-1)} + \cdots + h_s e^{P(z)} f^{(s)} + \cdots + h_1 f' + h_0 e^{Q(z)} f = F. \tag{1.4}$$

Corresponding to (1.3), we obtain the following result:

Theorem 1.5. *Let $k \geq 2$, $s \in \{1, \cdots, k-1\}$, $h_0 \not\equiv 0$, $h_1, \cdots, h_{k-1}$; $P(z)$,$Q(z)$ satisfy the hypothesis of Theorem 1.4; $F \not\equiv 0$ be an meromorphic function of finite order. Suppose all poles of f are of uniformly bounded multiplicity and if at least one of the three statements of Theorem 1.4 hold, then all solutions f of non-homogeneous linear differential equation (1.4) with at most one exceptional solution f_0 of finite order, satisfy*

$$\lambda(f) = \overline{\lambda}(f) = \sigma(f) = \infty, \quad \lambda_2(f) = \overline{\lambda}_2(f) = \sigma_2(f).$$

Futhermore, if such an exceptional solution f_0 of finite order of (1.4) exists, then we have

$$\sigma(f_0) \leq \max\{n, \sigma(F), \overline{\lambda}(f_0)\}.$$

Remark 1.2. Setting $h_j (j = 1, 2, \ldots, k-1)$ and $F(z)$ be entire functions in Theorem 1.5, we get Theorem 2 in [17].

2 Lemmas

The linear measure of a set $E \subset [0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\, dt$. The logarithmic measure of a set $E \subset [1, +\infty)$ is defined by $lm(E) = \int_1^{+\infty} \chi_E(t)/t\, dt$, where $\chi_E(t)$ is the characteristic function of E. The upper and lower densities of E are

$$\overline{\text{dens}} E = \limsup_{r \to +\infty} \frac{m(E \cap [0, r])}{r}, \quad \underline{\text{dens}} E = \liminf_{r \to +\infty} \frac{m(E \cap [0, r])}{r}.$$

Lemma 2.1 (see. [4]). *Let $f(z)$ be a entire function with $\sigma(f) = \infty$, and $\sigma_2(f) = \alpha < \infty$, let a set $E \subset [1, \infty)$ that has finite logarithmic measure. Then there exists $\{z_k = r_k e^{i\theta_k}\}$ such that $|f(z_k)| = M(r_k, f), \theta_k \in [0, 2\pi), \lim_{k \to \infty} \theta_k = \theta_0 \in [0, 2\pi), r_k \notin E, r_k \to \infty$, and for any given $\varepsilon > 0$, for a sufficiently large r_k, we have*

$$\limsup_{r \to \infty} \frac{\log \nu_f(r_k)}{\log r_k} = +\infty, \tag{2.1}$$

$$\exp\{r_k^{\alpha - \varepsilon}\} < \nu_f(r_k) < \exp\{r_k^{\alpha + \varepsilon}\} \tag{2.2}$$

Lemma 2.2 (see.[2, 14]). *Let $F(r)$ and $G(r)$ be monotone nondecreasing functions on $(0, \infty)$ such that (i) $F(r) \leq G(r) n.e.$ or (ii) for $r \neq H \cup [0, 1]$ having finite logarithmic measure, then for any constant $\alpha > 1$, there exists $r_0 > 0$ such that $F(r) \leq G(\alpha r)$ for all $r > r_0$.*

Lemma 2.3 (see. [9]). *Let f be a transcendental meromorphic function. Let $\alpha > 1$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Then the following two statements hold:*

(a) There exists a set $E_1 \subset (1, \infty)$ which has finite logarithmic measure, and a constant $C > 0$, such that for all z satisfying $|z| \notin E_1 \bigcup [0, 1]$, we have (with $r = |z|$)

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq C\Big[\frac{T(\alpha r, f)}{r}(\log r)^{\alpha} \log T(\alpha r, f)\Big]^{k-j}. \tag{2.3}$$

(b) There exists a set $E_2 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) - E_2$, then there is a constant $R = R(\theta) > 0$ such that (2.3) holds for all z satisfying $\arg z = \theta$ and $R \leq |z|$.

Lemma 2.4 ([18], pp. 253-255). *Let n be a positive integer, and let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with $a_n = \alpha_n e^{i\theta_n}, \alpha_n > 0$. For given $\varepsilon, 0 < \varepsilon < \pi/4n$, we introduce $2n$ closed angles*

$$D_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon \leq \theta \leq -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon (j = 0, 1, \cdots, 2n-1).$$

Then, there exists a positive number $R = R(\varepsilon)$ such that

$$Re\ P(z) > \alpha_n r^n (1 - \varepsilon) \sin n\varepsilon$$

if $|z| = r > R$ and $z \in D_j$, where j is even, while

$$Re\ P(z) < -\alpha_n r^n (1 - \varepsilon) \sin n\varepsilon$$

if $|z| = r > R$ and $z \in D_j$, where j is odd.

Lemma 2.5 ([4], Lemma 1). *Let $g(z)$ be a meormorphic function with $\sigma(g) = \beta < \infty$. Then for any $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ with $lmE < \infty$, such that for all z with $|z| = r \notin ([0, 1] \cup E), r \to \infty$, then*

$$|g(z)| \leq \exp\{r^{\beta+\varepsilon}\}.$$

Applying Lemma 2.5 to $1/g(z)$, we can obtain that for any given $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ with $lmE < \infty$, such that for all z with $|z| = r \notin ([0, 1] \cup E), r \to \infty$, then

$$\exp\{-r^{\beta+\varepsilon}\} \leq |g(z)| \leq \exp\{r^{\beta+\varepsilon}\}. \tag{2.4}$$

It is well known that the Wiman-Valiron theory (see, [14]) is an indispensable device while considering the growth of entire solution of a complex differential equation. In order to consider the growth of meromorphic function solutions of a complex differential equation, Wang and Yi [19] extended the Wiman-Valiron theory from entire functions to meromorphic functions. Here we give the special form where meromorphic function has infinite order:

Lemma 2.6 ([19, 20]). *Let $f(z) = g(z)/d(z)$ be the infinite order meromorphic function and $\sigma_2(f) = \sigma$, where $g(z)$ and $d(z)$ are entire function, $\sigma(d) < \infty$, there exists a sequence $r_j (r_j \to \infty)$ satisfying $z_j = r_j e^{i\theta_j}, \theta_j \in [0, 2\pi)$, $\lim_{j\to\infty} \theta_j = \theta_0 \in [0, 2\pi)$, $|g(z_j)| = M(r_j, g)$ and j is sufficient large, we have*

$$\frac{f^{(n)}(z_j)}{f(z_j)} = \Big(\frac{\nu_g(r_j)}{z_j}\Big)^n (1 + o(1))(n \in \mathbb{N}),$$

$$\limsup_{r\to\infty} \frac{\log\log \nu_g(r)}{\log r} = \sigma_2(g).$$

Lemma 2.7. *Let $k \geq 2$ and $A_0, A_1, \cdots, A_{k-1}$ are meromorphic function. Let $\sigma = \max\{\sigma(A_j), j = 0, 1, \cdots, k-1\}$ and all poles of f are of uniformly bounded multiplicity. Then every transcendental meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0, \tag{2.5}$$

satisfies $\sigma_2(f) \leq \sigma$.

Proof. Since $f \not\equiv 0$ is a transcendental meromorphic solution of the equation (2.5). If $\sigma(f) < \infty$, then $\sigma_2 = 0 \leq \sigma$. If $\sigma(f) = \infty$. We can rewrite (2.5) to

$$-\frac{f^{(k)}}{f} = A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0. \tag{2.6}$$

Obviously, the poles of f must be the poles of $A_j (j = 0, 1, \cdots, k-1)$, note that all poles of f are of uniformly bounded multiplicity, then $\lambda(1/f) \leq \sigma$. By Hadmard factorization theorem, we know f can be written to $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire function, and $\lambda(d) = \sigma(d) = \lambda(1/f) \leq \sigma$, $\sigma_2(f) = \sigma_2(g)$. By Lemma 2.5 and Lemma 2.6, for any small $\varepsilon > 0$, there exists a sequence $r_j (r_j \to \infty)$ satisfying $z_j = r_j e^{i\theta_j}, \theta_j \in [0, 2\pi)$, $\lim\limits_{j\to\infty} \theta_j = \theta_0 \in [0, 2\pi)$, $|g(z_j)| = M(r_j, g)$ and j is sufficient large, we have

$$\frac{f^{(n)}(z_j)}{f(z_j)} = \Big(\frac{\nu_g(r_j)}{z_j}\Big)^n (1 + o(1)), \quad (n \in \mathbb{N}), \tag{2.7}$$

$$\limsup_{r\to\infty} \frac{\log\log \nu_g(r)}{\log r} = \sigma_2(g), \tag{2.8}$$

$$|A_j(z)| \leq e^{r_j^{\sigma+\varepsilon}}, \quad (j = 1, 2, \cdots, k-1), \tag{2.9}$$

Substituting (2.7),(2.9) into (2.6), we obtain

$$v_g(r_j)(1 + o(1)) \leq 2r_j \exp\{r_j^{\sigma+\varepsilon_j}\}. \tag{2.10}$$

Then by (2.8), (2.10) and for the arbitrary ε, we can obtain $\sigma_2(f) \leq \sigma$. We complete the proof of the lemma. $\square$

Remark 3. Here we point out that the condition all poles of f are of uniformly bounded multiplicity in Theorem 1 of [3] and Theorem 1.3 of [20] was missing. Since the growth of the coefficients A_j gives only an estimate for the counting function of the distinct poles of f, but not for $N(r, f)$.

Lemma 2.8. *(see. [5]) Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ are finite order meromorphic function. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then f satisfies $\lambda(f) = \overline{\lambda}(f) = \sigma(f) = \infty$.

3 Proofs of main results

3.1 Proof of Theorem 1.3

Proof. Let $f \not\equiv 0$ be a transcendental solution of the equation (1.1). We consider two case:

Case 1: When $\arg a_n \neq \arg b_n$, by Lemma 2.4, there exist constants $c > 0$, $R_1 > 0$ and $\theta_1 < \theta_2$ such that for all $r \geq R_1$ and $\theta \in (\theta_1, \theta_2)$, we have

$$\begin{aligned} &\text{Re } P(re^{i\theta}) < 0, \\ &\text{Re } Q(re^{i\theta}) > br^n. \end{aligned} \tag{3.1}$$

Note that $\sigma = \max\{\sigma(h_j), j = 0, 1, \cdots, k-1\} < n$. Then by Lemma 2.5, for any $\varepsilon (0 < \varepsilon < (n-\sigma)/2)$, there exists a set $E_1 \subset (1, \infty)$ that has finite linear measure such that when $|z| = r \notin ([0,1] \cup E), r \to \infty$, we have

$$\left|\frac{h_j}{h_0}\right| \leq \exp\{r^{\frac{\sigma+n}{2}}\}, \quad (j = 0, 1, \cdots, k-1). \tag{3.2}$$

Since f is a transcendental meromorphic function, by Lemma 2.3, there exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure such that when $|z| = r \notin ([0,1] \cup E), r \to \infty$, we have

$$|\frac{f^{(j)}(z)}{f(z)}| \leq Br[T(2r, f)]^{j+1}, \quad (j = 0, 1, \cdots, k-1). \tag{3.3}$$

From the equation (1.1), we obtain

$$|e^Q| \leq \left|\frac{1}{h_0}\right| \left|\frac{f^{(k)}}{f}\right| + \left|\frac{h_{k-1}}{h_0}\right| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + \left|\frac{h_s}{h_0}\right| |e^P| \left|\frac{f^{(s)}}{f}\right| + \cdots + \left|\frac{h_1}{h_0}\right| \left|\frac{f'}{f}\right|. \tag{3.4}$$

Therefore, from (3.1)-(3.4), for $z = re^{i\theta}, \theta \in (\theta_1, \theta_2)$, $r \notin [0,1] \cup E_1 \cup E_2$, we have

$$\exp\{br^n\} \leq kAr \exp\{r^{\frac{\sigma+n}{2}}\}[T(2r, f)]^{k+1}.$$

Hence by Lemma 2.2, we obtain $\sigma(f) = \infty$ and $\sigma_2(f) \geq n$.

On the other hand, by Lemma 2.7, we have $\sigma_2(f) \leq n$, hence $\sigma_2(f) = n$.

Case 2: When $a_n = cb_n$ with $0 < c < 1$. Since $\deg Q = n > n - 1 = \deg(P - cQ)$, By Lemma 2.4, there exist constant $c > 0$, $R_2 > 0$ and $\theta_1 < \theta_2$ such that for all $r \geq R_2$ and $\theta \in (\theta_1, \theta_2)$, we have

$$\begin{aligned} &\text{Re } Q(re^{i\theta}) > br^n > 0, \\ &\text{Re } \{P(re^{i\theta}) - cQ(re^{i\theta})\} \leq M. \end{aligned} \tag{3.5}$$

From the equation (1.1), we obtain

$$\begin{aligned} &|e^{(1-c)Q}| \\ &\leq \left|\frac{1}{h_0}\right| |e^{-cQ}| \left|\frac{f^{(k)}}{f}\right| + \left|\frac{h_{k-1}}{h_0}\right| |e^{-cQ}| \left|\frac{f^{(k-1)}}{f}\right| \\ &+ \cdots + \left|\frac{h_s}{h_0}\right| |e^{P-cQ}| \left|\frac{f^{(s)}}{f}\right| + \cdots + \left|\frac{h_1}{h_0}\right| |e^{-cQ}| \left|\frac{f'}{f}\right|. \end{aligned}$$

Therefore, from this and (3.2),(3.3) and (3.5), for $z = re^{i\theta}, \theta \in (\theta_1, \theta_2)$, $r \notin [0,1] \cup E_1 \cup E_2$, we have

$$\exp\{b(1-c)r^n\} \leq (k+1)Br \exp\{r^{\frac{\sigma+n}{2}}\}[T(2r, f)]^{k+1}.$$

Hence by Lemma 2.2 again, we can obtain $\sigma(f)=\infty$ and $\sigma_2(f)\geq n$.

On the other hand, by Lemma 2.7, we have $\sigma_2(f)\leq n$, hence $\sigma_2(f)=n$, and the proof of Theorem 1.3 is completed. □

3.2 Proof of Theorem 1.4

Proof. We distinguish three cases:

(1) Suppose that $a_n=cb_n$ with $c\geq 1$, and $\deg(P-cQ)=m\geq 1$, $\sigma<m$. We claim that $\sigma(f)=\infty$ and $m\leq\sigma_2(f)\leq n$.

Since $\deg P(z)=n>m=\deg(Q-P/c)$, by Lemma 2.4, there exist a real number $b>0$ and a continuous curve Γ tending ∞ such that for all $z\in\Gamma$ with $|z|=r$, we have

$$\begin{aligned}&\mathrm{Re}\ P(z)=0,\\&\mathrm{Re}\ [Q(z)-\frac{1}{c}P(z)]\geq br^m.\end{aligned}\tag{3.6}$$

From the equation (1.3), we obtain

$$\begin{aligned}&|e^{Q-P/c}|\\&\leq\left|\frac{1}{h_0}\right||e^{-P/c}|\left|\frac{f^{(k)}}{f}\right|+\left|\frac{h_{k-1}}{h_0}\right||e^{-P/c}|\left|\frac{f^{(k-1)}}{f}\right|\\&+\cdots+\left|\frac{h_s}{h_0}\right||e^{(1-1/c)P}|\left|\frac{f^{(s)}}{f}\right|+\cdots+\left|\frac{h_1}{h_0}\right||e^{-P/c}|\left|\frac{f'}{f}\right|.\end{aligned}$$

Similar, we can get (3.2) and (3.3). Therefore, from this and (3.2),(3.3) and (3.6), for $z=re^{i\theta},\theta\in(\theta_1,\theta_2)$, $r\notin[0,1]\cup E_1\cup E_2$, we have

$$\exp\{br^m\}\leq(k+1)Br\exp\{r^{\frac{\sigma+n}{2}}\}[T(2r,f)]^{k+1}.$$

Hence by Lemma 2.2, from this we obtain $\sigma(f)=\infty$ and $\sigma_2(f)\geq m$. On the other hand, by Lemma 2.7, we have $\sigma_2(f)\leq n$, hence $m\leq\sigma_2(f)\leq n$.

(2) We shall verify that $\sigma_2(f)=n$. If it is not true, then it follows from the proof of Part (1) that $\sigma_2(f)=\alpha(m\leq\alpha<n)$, we shall arrive at a contradiction in the sequel.

Since $\sigma=\max\{\sigma(h_j):j=0,1,\cdots,k-1\}<m$, then by Lemma 2.5, for any given $\varepsilon(0<\varepsilon<\min\{\frac{m-\sigma}{3},\frac{n-\sigma}{3},\frac{\pi}{4n}\})$, there is a set $E_3\subset[1,\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r\notin E_3\cup[0,1]$, we have

$$\exp\{-r^{\sigma+\varepsilon}\}\leq|h_j(z)|\leq\exp\{r^{\sigma+\varepsilon}\},\quad(j=0,1,\cdots,k-1).\tag{3.7}$$

$$\exp\{-r^{m+\varepsilon}\}\leq|\exp\{(P(z)-cQ(z))\}|\leq\exp\{r^{m+\varepsilon}\}.\tag{3.8}$$

Let $f(z)=g(z)/d(z)$ be the infinite order meromorphic function and $\sigma_2(f)=\sigma$, where $g(z)$ and $d(z)$ are entire function, $\sigma(d)<\infty$, there exists a sequence $r_k(r_k\to\infty)$ satisfying $z_k=r_ke^{i\theta_k},\theta_k\in[0,2\pi)$, $\lim\limits_{k\to\infty}\theta_k=\theta_0\in[0,2\pi)$, $|g(z_k)|=M(r_k,g)$ and k is sufficient large, we have

$$\frac{f^{(j)}(z_k)}{f(z_k)}=(\frac{\nu_g(r_k)}{z_k})^j(1+o(1)),\quad(j=0,1,\cdots,k-1)\tag{3.9}$$

and

$$\exp\{r_k^{\sigma-\varepsilon}\} \le \nu_g(r_k) \le \exp\{r_k^{\sigma+\varepsilon}\}. \tag{3.10}$$

Let $Q(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$, where $b_n = |b_n|e^{i\theta}, |b_n| > 0, \theta_n \in [0, 2\pi)$. By Lemma 2.4, for the above ε, there are $2n$ opened angles

$$G_j : -\frac{\theta}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon (j = 0, 1, \cdots, 2n-1). \tag{3.11}$$

and a positive number $R = R(\varepsilon)$ such that

$$\text{Re } Q(z) > |b_n| r^n (1-\varepsilon)\sin n\varepsilon$$

if $|z| = r > R$ and $z \in D_j$, where j is even, while

$$\text{Re } Q(z) < -|b_n| r^n (1-\varepsilon)\sin n\varepsilon$$

if $|z| = r > R$ and $z \in D_j$, where j is odd.

For the above θ, if $\theta_0 \neq -\frac{\theta}{n} + (2j-1)\frac{\pi}{2n} (j = 0, 1, \cdots, 2n-1)$, then we may take ε sufficiently small, and there is some $G_j, j \in \{0, 1, \cdots, 2n-1\}$ such that $\theta_0 \in G_j$. Hence there are three cases: (i) $\theta_0 \in G_j$ for some odd number j; (ii) $\theta_0 \in G_j$ for some even number j; (iii) $\theta_0 = -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ for some $j \in \{0, 1, \cdots, 2n-1\}$.

Now we split this into three cases to prove:

Case (i): $\theta_0 \in G_j$ for some odd number j. Since G_j is an open set and $\lim_{k\to\infty} \theta_k = \theta_0$, there is a $K > 0$ such that $\theta_k \in G_j$ for $k > K$. By Lemma 2.4, we have

$$\text{Re } \{Q(r_k e^{i\theta_k})\} < -\sigma r_k^n (\sigma > 0), i.e., \text{Re } \{-Q(r_k e^{i\theta_k})\} > \sigma r_k^n (\sigma > 0). \tag{3.12}$$

Since $\deg(P - cQ) = m \ge 1$, from (3.12), we obtain that for a sufficiently large k,

$$\text{Re } \{P(z_k) - Q(z_k)\} = \text{Re } \{(c-1)Q + (P - cQ\} < -(c-1)\sigma r_k^n + d r_k^n < 0, \tag{3.13}$$

where $Re\{P(z_k) - Q(z_k)\} < d r_k^n$ for a sufficiently large k. Substituting (3.10) into (1.3), we get for $\{z_k = r_k e^{i\theta_k}\}$,

$$\begin{aligned}
&- e^{-Q(z_k)}[\nu_g^k(r_k)(1+o(1)) + z_k h_{k-1}\nu_g^{k-1}(r_k)(1+o(1)) + \cdots + \\
&z_k^{k-s-1} h_{s+1}(z_k)\nu_g^{s+1}(r_k)(1+o(1)) + z_k^{k-s+1} h_{s-1}(z_k)\nu_g^{s-1}(r_k)(1+o(1)) \\
&+ \cdots + z_k^{k-1} h_1(z_k)\nu_g(r_k)(1+o(1))] \\
&= z_k^{k-s} h_s(z_k) e^{P(z_k)-Q(z_k)}\nu_g^s(r_k)(1+o(1)) + z_k^k h_0(z_k).
\end{aligned} \tag{3.14}$$

Thus from (3.10) and (3.12), we obtain, for a sufficiently large k,

$$\begin{aligned}
&\big| - e^{-Q(z_k)}[\nu_g^k(r_k)(1+o(1)) + z_k h_{k-1}\nu_g^{k-1}(r_k)(1+o(1)) + \cdots + \\
&z_k^{k-s-1} h_{s+1}(z_k)\nu_g^{s+1}(r_k)(1+o(1)) + z_k^{k-s+1} h_{s-1}(z_k)\nu_g^{s-1}(r_k)(1+o(1)) \\
&+ \cdots + z_k^{k1} h_1(z_k)\nu_g(r_k)(1+o(1))]\big| \\
&> e^{\sigma r_k^n} e^{k r_k^{\sigma-\varepsilon}} \big[\frac{1}{2} - 2r_k|h_k(z_k)|/\nu_g(r_k) - \cdots - 2r_k^{k-1}|h_1(z_k)|/\nu_g^{k-1}(r_k)\big] > \frac{1}{4}e^{\sigma r_k^n}.
\end{aligned} \tag{3.15}$$

And from (3.7), (3.10) and (3.13), we have

$$\begin{aligned}|z_k^{k-s}h_s(z_k)e^{P(z_k)-Q(z_k)}\nu_g^{s+1}(r_k)(1+o(1))z_k^k h_0(z_k)|\\ \leq 2r_k^{k-s}e^{r_k^{\sigma+\varepsilon}}e^{sr_k^{\beta+\varepsilon}}+r_k^k e^{r_k^{\sigma+\varepsilon}} \leq e^{r_k^{\sigma+2\varepsilon}}.\end{aligned} \tag{3.16}$$

From (3.14) we see that (3.16) is in contradiction to (3.15).

Case (ii): $\theta_0 \in G_j$ where j is even. Since G_j is an open set and $\lim_{k\to\infty}\theta_k = \theta_0$, there is $K > 0$ such that $\theta_k \in G_j$ for $k > K$. By Lemma 2.4, we have

$$\begin{gathered}\text{Re }\{Q(r_k e^{i\theta_k})\} > \sigma r_k^n, \qquad \text{Re }\{-cQ(r_k e^{i\theta_k})\} < -c\sigma r_k^n,\\ \text{Re }\{(1-c)Q(r_k e^{i\theta_k})\} < (1-c)\sigma r_k^n.\end{gathered} \tag{3.17}$$

We may rewrite (3.14) to

$$\begin{aligned}&-z_k^{k-s}h_s(z_k)e^{P(z_k)-cQ(z_k)}\nu_g^s(r_k)(1+o(1)) = e^{-cQ(z_k)}[\nu_g^k(r_k)(1+o(1))\\ &+z_k h_{k-1}\nu_g^{k-1}(r_k)(1+o(1))+\cdots+z_k^{k-s-1}h_{s+1}(z_k)\nu_g^{s+1}(r_k)(1+o(1))\\ &+z_k^{k-s+1}h_{s-1}(z_k)\nu_g^{s-1}(r_k)(1+o(1))+\cdots+z_k^{k-1}h_1(z_k)\nu_g(r_k)(1+o(1))]\\ &+z_k^k h_0(z_k)e^{(1-c)Q(z_k)}.\end{aligned} \tag{3.18}$$

Thus from (3.7),(3.8),(3.10),(3.17) and (3.18), we have

$$\begin{aligned}e^{-r_k^{m+\varepsilon}} &< \frac{1}{2}r_k^{k-s}e^{-r_k^{\sigma+\varepsilon}}e^{-r_k^{m+\varepsilon}}e^{sr_k^{\sigma-\varepsilon}}\\ &\big|-z_k^{k-s}h_s(z_k)e^{P(z_k)-cQ(z_k)}\nu_g^s(r_k)(1+o(1))\big| < e^{\frac{1-c}{2}\sigma r_k^n}\end{aligned} \tag{3.19}$$

This is in contradiction to $n > m+\varepsilon$ and $c > 1$.

Case (iii). $\theta_0 = -\frac{\theta_n}{n}+(2j-1)\frac{\pi}{2n}$ for some $j \in \{0,1,\cdots,2n-1\}$. Since Re $\{Q(r_k e^{i\theta_k})\} = 0$ when r_k is sufficiently large and a ray $\arg z = \theta_0$ is an asymptotic line of $\{r_k e^{\theta_k}\}$, where is a $K > 0$ such that when $k > K$, we have

$$-1 < \text{Re }\{Q(r_k e^{i\theta_k})\} < 1. \tag{3.20}$$

Since $a_n = cb_n$, so the head terms of $P(z)$ and $Q(z)$ have the same argument, therefore by Lemma 2.4, Re $\{P(z)/c\}$ and Re $\{Q(z)\}$ possesses the same property in the above $G_j (j = 0,1,\cdots,2n-1)$, *i.e.*, when $k > K$, we have

$$-1 < \text{Re }\{P(r_k e^{i\theta_k})/c\} < 1. \tag{3.21}$$

Hence when $k > K$, we have

$$-2c < \text{Re }\{P(r_k e^{i\theta_k}) - cQ(r_k e^{i\theta_k})\} < 2c. \tag{3.22}$$

We may rewrite (3.14) to

$$\begin{aligned}&-e^{-cQ(z_k)}[\nu_g^k(r_k)(1+o(1))+z_k h_{k-1}\nu_g^{k-1}(r_k)(1+o(1))+\cdots+\\ &z_k^{k-s-1}h_{s+1}(z_k)\nu_g^{s+1}(r_k)(1+o(1))+z_k^{k-s+1}h_{s-1}(z_k)\nu_g^{s-1}(r_k)(1+o(1))\\ &+\cdots+z_k^{k-1}h_1(z_k)\nu_g(r_k)(1+o(1))]\\ &= z_k^{k-s}h_s(z_k)e^{P(z_k)-cQ(z_k)}\nu_g^s(r_k)(1+o(1))+z_k^k h_0(z_k)e^{(1-c)Q(z_k)}.\end{aligned} \tag{3.23}$$

Thus from (3.7), (3.10) and (3.21)-(3.23), we obtain, for a sufficiently large k,

$$\begin{aligned}\frac{1}{4}e^{-c}\nu_g^k(r_k) <& \big| - e^{-cQ(z_k)}[\nu_g^k(r_k)(1+o(1)) + z_k h_{k-1}\nu_g^{k-1}(r_k)(1+o(1)) + \cdots + \\ & z_k^{k-s-1}h_{s+1}(z_k)\nu_g^{s+1}(r_k)(1+o(1)) + z_k^{k-s+1}h_{s-1}(z_k)\nu_g^{s-1}(r_k)(1+o(1)) \\ & + \cdots + z_k^{k-1}h_1(z_k)\nu_g(r_k)(1+o(1))]\big| \\ & = \big|z_k^{k-s}h_s(z_k)e^{P(z_k)-cQ(z_k)}\nu_g^s(r_k)(1+o(1)) + z_k^k h_0(z_k)e^{(1-c)Q(z_k)}\big| \\ & \le 2r_k^{k-s}e^{r_k^{\sigma+\varepsilon}}\nu_g^k(r_k) + r_k^k e^{r_k^{\sigma+\varepsilon}}e^{r_k^{\sigma+\varepsilon}}e^{c-1} \le \nu_g^k(r_k)e^{r_k^{\sigma+2\varepsilon}}.\end{aligned} \tag{3.24}$$

This is in contradiction to $\nu_g(r_k) \ge \exp\{r_k^{\sigma-\varepsilon}\}$. Thus we complete the proof of Part (2) of Theorem 1.4.

(3). By using the same argument as in Theorem 2 (iv) of [13], we can prove part (3). Here we omit the detail. □

3.3 Proof of Theorem 1.5

Proof. Assume f_0 is a solution of finite order of (1.4). If there exists another solution $f_1(\not\equiv f_0)$ of finite order of (1.4), then $\sigma(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a solution of the corresponding homogeneous differential equation (1.3). However, by Theorem 1, we get that $\sigma(f_1 - f_0) = \infty$, which is in contradiction to $\sigma(f_1 - f_0) < \infty$. Hence all solutions f of non-homogeneous linear differential equation (1.4), with at most one exceptional solution f_0 of finite order, satisfy $\sigma(f) = \infty$.

Now suppose that f is a solution of infinite order of (1.4), then by Lemma 2.8, we obtain

$$\lambda(f) = \overline{\lambda}(f) = \sigma(f) = \infty.$$

In the following, we shall verify that every solution f of infinite order of (1.4) satisfy $\overline{\lambda}_2(f) = \sigma_2(f)$. In fact, by (1.4), it is easy to see that the zeros of f occurs at the poles of $h_j(z)(j = 1, \dots, k-1)$ or the zeros of $F(z)$. If f has a zero at z_0 of order n, $n > k$, then $F(z)$ must have a zero at z_0 of order $n-k$. Therefore we get by $F \not\equiv 0$ that

$$N(r, \frac{1}{f}) \le k\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{F}) + \sum_{j=0}^{k-1} N(r, h_j).$$

On the other hand, (1.4) may be rewritten as follows

$$\frac{1}{f} = \frac{1}{F}\Big[\frac{f^{(k)}}{f} + h_{k-1}\frac{f^{(k-1)}}{f} + \cdots + h_s e^P \frac{f^{(s)}}{f} \cdots + h_1\frac{f'}{f} + h_0 e^Q\Big].$$

So

$$m(r, \frac{1}{f}) \le m(r, \frac{1}{F}) + \sum_{j=0}^{k-1} m(r, h_j) + m(r, e^P) + m(r, e^Q) + \sum_{j=0}^{k-1} m(r, \frac{f^{(j)}}{f}) + O(1).$$

Hence by the logarithmic derivative lemma, there exists a set E having finite linear measure such that for all $r \notin E$, we have

$$T(r,f) = T(r,\frac{1}{f}) + O(1)$$
$$\leq T(r,\frac{1}{F}) + k\overline{N}(r,\frac{1}{f}) + \sum_{j=0}^{k-1} T(r,h_j) + m(r,e^P) + m(r,e^Q) + \sum_{j=0}^{k-1} m(r,\frac{f^{(j)}}{f}) + O(1)$$
$$\leq T(r,F) + \sum_{j=0}^{k-1} T(r,h_j) + C\log(rT(r,f)) + T(r,e^P) + T(r,e^Q) + k\overline{N}(r,\frac{1}{f}) + O(\log r),$$

where C is a positive constant. Since for any $\varepsilon > 0$ and sufficiently large r, we have

$$C\log(rT(r,f)) \leq \frac{1}{2}T(r,f), \quad T(r,F) \leq r^{\sigma(F)+\varepsilon}, \quad T(r,e^P) \leq r^{n+\varepsilon};$$

$$T(r,e^Q) \leq r^{n+\varepsilon}, \quad T(r,h_j) \leq r^{\sigma+\varepsilon}, j = 0,1,\cdots,k;$$

so that for $r \notin E$ and sufficiently large r, we have

$$T(r,f) \leq 2k\overline{N}(r,\frac{1}{f}) + (4k+5)r^{\sigma+\varepsilon} + 4r^{n+\varepsilon} + 2r^{\sigma(F)+\varepsilon}.$$

Hence by Lemma 2.2, we get that $\sigma_2(f) \leq \overline{\lambda_2}(f)$. It is obvious that $\lambda_2(f) \geq \overline{\lambda}_2(f) \geq \sigma_2(f)$, hence $\lambda_2(f) = \overline{\lambda}_2(f) = \sigma_2(f)$.

Finally, let f_0 be a solution of finite order of (1.4), then $f_0 \not\equiv 0$. Substitute it into (1.4), and rewrite it as follows

$$\frac{1}{f_0} = \frac{1}{F}\left[\frac{f_0^{(k)}}{f_0} + h_{k-1}\frac{f_0^{(k-1)}}{f_0} + \cdots + h_s e^P \frac{f_0^{(s)}}{f_0} + \cdots + h_1\frac{f_0'}{f_0} + h_0 e^Q\right].$$

Thus

$$m(r,\frac{1}{f_0}) \leq m(r,\frac{1}{F}) + \sum_{j=0}^{k-1} m(r,h_j) + m(r,e^P) + m(r,e^Q) + \sum_{j=0}^{k-1} m(r,\frac{f_0^{(j)}}{f_0}) + O(1).$$

It is easy to see that f_0 occurs at the poles of $h_j(z)(j = 1,\ldots,k-1)$ or the zeros of $F(z)$. If f_0 has a zero at z_0 of order n, $n > k$, then $F(z)$ must have a zero at z_0 of order $n-k$. Therefore we get by $F \not\equiv 0$ that

$$N(r,\frac{1}{f}) \leq k\overline{N}(r,\frac{1}{f}) + N(r,\frac{1}{F}) + N(r,h).$$

So by the logarithmic derivative lemma, and noting that $\sigma(f_0) < +\infty$, we can obtain that

$$T(r,f) = T(r,\frac{1}{f}) + O(1)$$
$$\leq T(r,F) + \sum_{j=0}^{k-1} T(r,h_j) + T(r,e^P) + T(r,e^Q) + k\overline{N}(r,\frac{1}{f}) + O(\log r).$$

Hence $\sigma(f_0) \leq \max\{n, \sigma(F), \overline{\lambda}(f_0)\}$, and this completes the proof of the theorem. □

Example 1. Consider the non-homogeneous linear differential equation

$$f^{(k)} + h_{k-1}f^{(k-1)} + \cdots + h_2 f'' + \frac{1}{z}e^{iz}f' - \frac{1}{z^2}e^{-iz}f = \frac{1}{z}2i\sin z,$$

where $h_2, \cdots, h_{k-1}$ are meromorphic functions. It has a solution $f_0(z) = z$ of finite order.

Example 2.(see. [16]) Consider the non-homogeneous linear differential

$$f''' + e^{z^2} f'' - f' + ze^{z^2-z} f = ze^{z^2} + e^{z^2+z}.$$

It has a solution $f_0(z) = e^z$ of finite order. $\sigma(f_0) = 1 < 2 = \max\{2, \sigma(F), \overline{\lambda}(f_0)\}$.

Acknowledgements

Project supported by the NSFC-RFBR and the NSFC (No. 10771121).

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(Received August 6, 2008)