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New results on the positive solutions of nonlinear second-order differential systems

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Abstract. In this paper, we study the three-point boundary value problems for systems of nonlinear second order ordinary differential equations of the form

$$\begin{cases} u'' = -f(t,v), & 0 < t < 1, \\ v'' = -g(t,u), & 0 < t < 1 \\ u(0) = v(0) = 0, \varsigma u(\zeta) = u(1), \varsigma v(\zeta) = v(1), \end{cases}$$

where $f : (0,1) \times [0,+\infty) \to [0,+\infty), g : [0,1] \times [0,+\infty) \to [0,+\infty), 0 < \zeta < 1, \varsigma > 0$, and $\varsigma\zeta < 1, f$ may be singular at $t = 0$ and/or $t = 1$. Under some rather simple conditions, by means of monotone iterative technique, a necessary and sufficient condition for the existence of positive solutions is established, a result on the existence and uniqueness of the positive solution and the iterative sequence of solution is given.

Keywords. Nonlinear second-order differential systems; Positive solutions; Cone; Necessary and sufficient condition

AMS (MOS) subject classification: 34B18.

1 Introduction

Recently an increasing interest has been observed in investigating the existence of positive solutions of boundary value problems for systems of differential equations. Singular differential systems arise in many branches of applied mathematics and physics such as gas dynamics, Newtonian fluid mechanics, nuclear physics. Such systems have been widely studied by many authors (see [1-8] and references therein). The study of positive radial solutions for elliptic problems in annular regions can usually be transformed into that of positive solutions of two-point boundary value problems for ordinary differential equations. The study of multi-point boundary value problems for

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linear second order ordinary differential equations was initiated by Il'in and Moiseev [5]. Since then, nonlinear multi-point boundary value problems have been studied by several authors using the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory, and fixed point theorem in cones. In particular, Zhou and Xu [6] gave some existence results for positive solutions for the following third-order boundary value problem

$$\begin{cases} u'' = -f(t,v), \quad 0 < t < 1, \\ v'' = -g(t,u), \quad 0 < t < 1 \\ u(0) = v(0) = 0, \varsigma u(\zeta) = u(1), \varsigma v(\zeta) = v(1), \end{cases} \tag{1.1}$$

by applying the fixed point index theory in cones. However, most of the above mentioned work is concerned only with sufficient conditions for solvability of systems. As far as we know, the study about a necessary and sufficient condition of positive solutions to differential system has been received much less attention. But to seek a necessary and sufficient condition of solutions for singular differential systems is also a important and interesting work.

Motivated by the work and the reasoning mentioned above, in this paper, by means of monotone iterative technique, we establish a necessary and sufficient condition of the existence of positive solutions for the above nonlinear singular differential system (1.1), where $f : (0,1) \times [0,+\infty) \to [0,+\infty), g : [0,1] \times [0,+\infty) \to [0,+\infty), 0 < \zeta < 1, \varsigma > 0$, and $\varsigma\zeta < 1, f$ may be singular at $t = 0$ and/or $t = 1$. At the same time, we also give the existence and uniqueness of solutions and the iterative sequence of solutions. To that end, two classes extending boundary value differential systems are discussed and some further results are obtained.

The following conditions are assumed to hold throughout the paper:

(A) $f \in C((0,1) \times [0,+\infty), [0,+\infty))$; for any fixed $t \in (0,1), f(t,y)$ is nondecreasing in y and $f(t,y) \not\equiv 0$; for any $c \in (0,1)$, there exists $\eta \in (0,1)$, such that, for all $(t,y) \in (0,1) \times [0,+\infty)$,

$$f(t,cy) \geq c^{\eta} f(t,y). \tag{1.2}$$

(B) $g \in C([0,1] \times [0,+\infty), [0,+\infty))$; for any fixed $t \in [0,1], g(t,y)$ is nondecreasing in y; for any $h \in (0,1)$, there exists $\lambda \in (0,1)$, such that, for all $(t,y) \in [0,1] \times [0,+\infty)$,

$$g(t,hy) \geq h^{\lambda} g(t,y). \tag{1.3}$$

Remark 1. It is easy to see that the last assumption in (A) is equivalent to the condition: for any $c \in [1, \infty)$, there exists an $\eta \in (0, 1)$, such that

$$f(t, cy) \leq c^{\eta} f(t, y) \ for\ all\ (t, y) \in (0, 1) \times [0, \infty). \tag{1.4}$$

And it is easy to see that the last assumption in (B) is equivalent to the condition: for any $h \in [1, \infty)$, there exists a $\lambda \in (0, 1)$, such that

$$g(t, hy) \leq h^{\lambda} g(t, y) \ for\ all\ (t, y) \in [0, 1] \times [0, \infty). \tag{1.5}$$

Definition. A pair $(u, v) \in \big(C[0, 1] \cap C^2[0, 1]\big) \times \big(C[0, 1] \cap C^2[0, 1]\big)$ is called a *positive solution* of (1.1) if (u, v) satisfies (1.1) and $u(t) > 0$ and $v(t) > 0$ for $t \in (0, 1)$. A positive solution of (1.1) is called a $C^1[0, 1] \times C^1[0, 1]$ positive solution if $u'(0+), u'(1 - 0)$ and $v'(0+), v'(1 - 0)$ both exist.

Throughout this paper, we shall work in the space $E = C[0, 1]$, which is a Banach space if it is endowed with the form $\|u\| = \max_{t \in [0,1]} |u(t)|$ for any $u \in E$. Let $P = \{u \in C[0, 1] : u(t) \geq 0, t \in [0, 1]\}$. Clearly P is a normal cone in the Banach space E.

Let $P_l = \{u \in C[0, 1] : \textit{there exist positive numbers } l, L \textit{ such that } lt \leq u \leq Lt\}$. For $u \in P_l$, we set $l_u := \sup\{\vartheta > 0 : u(t) \geq \vartheta t\}$, $L_u := \inf\{\kappa > 0 : u(t) \leq \kappa t\}$.

The following are the main results of this paper.

Theorem 1. Suppose that f satisfies (A) and g satisfies (B). Then

(1) The system (1.1) has a $C^1[0, 1] \times C^1[0, 1]$ positive solution if and only if

$$0 < \int_{0+}^{1-} f\left(s, \int_0^1 G(s, \tau) g(\tau, \tau) d\tau\right) ds < +\infty. \tag{\textbf{C}}$$

(2) If there exists a positive solution (u^*, v^*) in $C^1[0, 1] \times C^1[0, 1]$ to system (1.1), then (u^*, v^*) is unique, and for any initial value $u_0 \in P_l$, let

$$u_n(t) = \int_0^1 G(t, s) f\left(s, \int_0^1 G(s, \tau) g\big(\tau, u_{n-1}(\tau)\big) d\tau\right) ds, \quad n = 1, 2, \ldots,$$

$$v_n(t) = \int_0^1 G(t, s) g\big(s, u_n(s)\big) ds, \quad n = 1, 2, \ldots.$$

Then

$$u_n \rightarrow^{\|\cdot\|} u^*, \quad v_n \rightarrow^{\|\cdot\|} \int_0^1 G(t, s) g\big(s, u^*(s)\big) ds = v^*.$$

2 Preliminaries and lemmas

Lemma 1.([6,9,10]) *Let* $0 < \zeta < 1, 0 < \varsigma < \frac{1}{\zeta}, y(t) \in E$, *then the problem*

$$\begin{cases} u'' + y(t) = 0, & 0 < t < 1, \\ u(0) = 0, \varsigma u(\zeta) = u(1), \end{cases}$$

has a unique solution

$$\begin{aligned} u(t) &= -\int_0^t (t-s)y(s)ds - \frac{\varsigma t}{1-\varsigma\zeta}\int_0^\zeta (\zeta - s)y(s)ds + \frac{t}{1-\varsigma\zeta}\int_0^1 (1-s)y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds, \end{aligned}$$

where

$$G(t,s) = \begin{cases} \dfrac{t(1-s)}{1-\varsigma\zeta} - \dfrac{\varsigma t(\zeta - s)}{1-\varsigma\zeta} - (t-s), & 0 \le s \le t \le 1, and\ s \le \zeta \\ \dfrac{t(1-s)}{1-\varsigma\zeta} - \dfrac{\varsigma t(\zeta - s)}{1-\varsigma\zeta}, & 0 \le t \le s \le \zeta, \\ \dfrac{t(1-s)}{1-\varsigma\zeta}, & 0 \le t \le s \le 1, and\ s \ge \zeta, \\ \dfrac{t(1-s)}{1-\varsigma\zeta} - (t-s), & \zeta \le s \le t \le 1. \end{cases}$$

Remark 2. In fact, $G(t,s)$ can be rewritten as

$$G(t,s) = G_1(t,s) + \frac{\varsigma t}{1-\varsigma\zeta}G_1(\zeta, s),$$

where

$$G_1(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Lemma 2.([9])

$$G(t,s) \ge ns(1-s),\ \forall\ (t,s) \in [\zeta, 1] \times [0,1],$$

where $n = \frac{\min\{1,\varsigma\}\cdot\min\{1-\zeta,\zeta\}}{1-\varsigma\zeta}$.

Remark 3.

$$(i)\ G(t,s) \le mt,\ \forall\ (t,s) \in [0,1] \times [0,1],$$

(*ii*) $G(t,s) \geq nts(1-s),\ \forall\ (t,s) \in [\zeta,1]\times[0,1]$,

where $m = \frac{1-\varsigma\zeta^2}{1-\varsigma\zeta}$.

Proof. (i) For $\forall\ (t,s) \in [0,1]\times[0,1]$, we have

$$
\begin{aligned}
G(t,s) &= G_1(t,s) + \frac{\varsigma t}{1-\varsigma\zeta} G_1(\zeta,s) \leq t(1-t) + \frac{\varsigma t}{1-\varsigma\zeta}\zeta(1-\zeta) \\
&\leq t + \frac{\varsigma\zeta(1-\zeta)}{1-\varsigma\zeta} t = \left(1 + \frac{\varsigma\zeta(1-\zeta)}{1-\varsigma\zeta}\right) t = mt,
\end{aligned}
$$

(ii) By Lemma 2, (ii) is obviously for $(t,s) \in [\zeta,1]\times[0,1]$.

It is well known that $(u,v) \in (C[0,1]\cap C^2[0,1]) \times (C[0,1]\cap C^2[0,1])$ is a solution of the problem (1.1) if and only if $(u,v) \in C[0,1]\times C[0,1]$ is a solution of the following system of nonlinear integral equations:

$$
\begin{cases}
u(t) = \displaystyle\int_0^1 G(t,s) f(s, v(s)) ds, \\
v(t) = \displaystyle\int_0^1 G(t,s) g(s, u(s)) ds.
\end{cases}
$$

Moreover, the system can be written as the nonlinear integral equation

$$
u(t) = \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau\right) ds.
$$

Now let us define a nonlinear operator $F : P \to P$ by

$$
(Fu)(t) = \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau\right) ds.
$$

Thus the existence of solutions to system (1.1) is equivalent to the existence of fixed points of nonlinear operator F, i.e., if x is a fixed point of F in $C[0,1]$, then system (1.1) has a solution (u,v), which is given by

$$
\begin{cases}
u(t) = x(t), \\
v(t) = \displaystyle\int_0^1 G(t,s) g(s, x(s)) ds.
\end{cases}
\tag{2.1}
$$

Suppose (2.1) is a $C^1[0,1]\times C^1[0,1]$ positive solution of system (1.1). Clearly $x''(t) \leq 0$, which implies that x is a convex function on [0,1]. Combining with the boundary conditions, we assert that there exist constants $0 < \mu_1 < 1 < \mu_2$ such that

$$
\mu_1 t \leq x(t) \leq \mu_2 t. \tag{2.2}
$$

In the following, we show that (2.2) is right. In fact, there exists a $t_0 \in (0,1]$ such that $\|x\| = x(t_0)$. Since

$$\frac{G_1(t,s)}{G_1(t_0,s)} \geq t(1-t),$$

it is easy to check that

$$\frac{G(t,s)}{G(t_0,s)} = \frac{G_1(t,s) + \frac{\varsigma t}{1-\varsigma\zeta} G_1(\zeta,s)}{G_1(t_0,s) + \frac{\varsigma t_0}{1-\varsigma\zeta} G_1(\zeta,s)} \geq t(1-t).$$

Thus,

$$\begin{aligned} x(t) &= \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, x(\tau)) d\tau\right) ds \\ &= \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\geq t(1-t)\|x\|, \quad t \in [0,1]. \end{aligned}$$

For $t \in [0,1]$, by the concavity of x and the conditions of (1.1), we have that

$$x(t) \geq t x(1) = t\varsigma x(\zeta) \geq t\varsigma\zeta(1-\zeta)\|x\|, \quad t \in [0,1].$$

On the other hand,

$$x(t) = \int_0^t x'(s) ds \leq \max_{t\in[0,1]} \|x'(t)\| t, \quad t \in [0,1].$$

We can choose $\mu_1 = \varsigma\zeta(1-\zeta)\frac{\|x\|}{\|x\|+1} < 1, \mu_2 = \|x'(t)\| + 2 > 1$, then we get (2.2).

3 Proof of the Theorem 1

(1) (i)The necessity.

By (A)-(B) and (2.2) and the definition of a $C^1[0,1] \times C^1[0,1]$ positive solution of (1.1), we have

$$
\begin{aligned}
&\int_{0+}^{1-} f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds \\
&\qquad \le \int_{0+}^{1-} f\left(s, \int_0^1 G(s,\tau)g(\tau,\mu_1^{-1}x(\tau))d\tau\right) ds \\
&\qquad \le \int_{0+}^{1-} f\left(s, \mu_1^{-\lambda}\int_0^1 G(s,\tau)g(\tau,x(\tau))d\tau\right) ds \\
&\qquad \le \mu_1^{-\lambda\eta}\int_{0+}^{1-} f\left(s, \int_0^1 G(s,\tau)g(\tau,x(\tau))d\tau\right) ds \\
&\qquad = \mu_1^{-\lambda\eta}\int_{0+}^{1-} (-x''(s))ds \\
&\qquad \le \mu_1^{-\lambda\eta}[x'(0+) - x'(1-0)] \\
&\qquad < +\infty.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_{0+}^{1-} f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds \\
&\qquad \ge \int_{0+}^{1-} f\left(s, \int_0^1 G(s,\tau)g(\tau,\mu_2^{-1}x(\tau))d\tau\right) ds \\
&\qquad \ge \int_{0+}^{1-} f\left(s, \mu_2^{-\lambda}\int_0^1 G(s,\tau)g(\tau,x(\tau))d\tau\right) ds \\
&\qquad \ge \mu_2^{-\lambda\eta}\int_{0+}^{1-} f\left(s, \int_0^1 G(s,\tau)g(\tau,x(\tau))d\tau\right) ds \\
&\qquad = \mu_2^{-\lambda\eta}\int_{0+}^{1-} (-x''(s))ds \\
&\qquad \ge \mu_2^{-\lambda\eta}[x'(0+) - x'(1-0)] \\
&\qquad > 0.
\end{aligned}
$$

Therefore,

$$
0 < \int_{0+}^{1-} f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds < +\infty.
$$

(ii)The sufficiency.

For any $u \in P_l$, there exist l_u and L_u such that $l_u t \le u(t) \le L_u t$. By assumptions (A)-(B), for any $t \in [\zeta, 1]$, we have

$$
\begin{aligned}
(Fu)(t) &= \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau\right) ds \\
&\leq \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, L\tau) d\tau\right) ds \\
&\leq \begin{cases} L_u^{\lambda\eta} \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau,\tau) d\tau\right) ds, & L_u > 1, \\ \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau,\tau) d\tau\right) ds, & L_u \leq 1, \end{cases} \\
&\leq \max\{1, L_u^{\lambda\eta}\} \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau,\tau) d\tau\right) ds \\
&\leq \max\{1, L_u^{\lambda\eta}\} mt \int_0^1 f\left(s, \int_0^1 G(s,\tau) g(\tau,\tau) d\tau\right) ds,
\end{aligned}
$$

and

$$
\begin{aligned}
(Fu)(t) &= \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau\right) ds \\
&\geq \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, l\tau) d\tau\right) ds \\
&\geq \begin{cases} l_u^{\lambda\eta} \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau,\tau) d\tau\right) ds, & l_u < 1, \\ \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau,\tau) d\tau\right) ds, & l_u \geq 1, \end{cases} \\
&\geq \min\{1, l_u^{\lambda\eta}\} \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau,\tau) d\tau\right) ds \\
&\geq \min\{1, l_u^{\lambda\eta}\} nt \int_0^1 s(1-s) f\left(s, \int_0^1 G(s,\tau) g(\tau,\tau) d\tau\right) ds.
\end{aligned}
$$

The above facts imply that there exist positive numbers l_{Fu}, L_{Fu} such that $l_{Fu}t \leq Fu(t) \leq L_{Fu}t$, where $l_{Fu} = \sup\{\delta > 0 : Fu(t) \geq \delta t\}$ and $L_{Fu} = \inf\{\phi > 0 : Fu(t) \leq \phi t\}$. Thus $F : P_l \to P_l$. At the same time, in view of the monotonicity of f and g in the second variables, it is easy to see that F is also an increasing operator in E.

Taking

$$
\alpha_0 = \begin{cases} 1, & l_{Fu} \geq 1, \\ (l_{Fu})^{\frac{1}{1-\lambda\eta}}, & l_{Fu} < 1, \end{cases} \qquad \beta_0 = \begin{cases} 1, & L_{Fu} \leq 1, \\ (L_{Fu})^{\frac{1}{1-\lambda\eta}}, & L_{Fu} > 1, \end{cases}
$$

then, $0 < \alpha_0 \leq 1,\ \beta_0 \geq 1$.

Let

$$0<\alpha\le\alpha_0,\quad \beta=\beta_0,\quad u_0=\alpha t,\quad z_0=\beta t,$$

$$u_n=Fu_{n-1},\qquad z_n=Fz_{n-1},\qquad n=1,2,\dots.$$

Then, we have

$$\begin{aligned}u_1=Fu_0&=\int_0^1 G(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,\alpha\tau)d\tau\right)ds\\&\ge\alpha^{\lambda\eta}\int_0^1 G(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right)ds\\&=\alpha^{\lambda\eta}Ft\ge\alpha^{\lambda\eta}l_{Fu}t=\alpha\cdot\alpha^{\lambda\eta-1}l_{Fu}\cdot t\\&\ge\alpha t=u_0,\end{aligned}$$

and

$$\begin{aligned}z_1=Fz_0&=\int_0^1 G(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,\beta\tau)d\tau\right)ds\\&\le\beta^{\lambda\eta}\int_0^1 G(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right)ds\\&=\beta^{\lambda\eta}Ft\le\beta^{\lambda\eta}L_{Fu}t=\beta\cdot\beta^{\lambda\eta-1}L_{Fu}\cdot t\\&\le\beta t=z_0.\end{aligned}$$

Since $\alpha t=u_0\le z_0=\beta t$ and F is an increasing operator, we have

$$\alpha\beta^{-1}z_0=u_0\le u_1\le\cdots\le u_n\le\cdots\le z_n\le\cdots\le z_1\le z_0=\beta\alpha^{-1}u_0.\quad(3.1)$$

Noticing that

$$\begin{aligned}|(Fu_n)'(t)|=\Big|&\left[\int_0^1 G_1(t,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right)ds\right]'\\&+\left[\frac{\varsigma t}{1-\varsigma\zeta}\int_0^1 G_1(\zeta,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right)ds\right]'\Big|\\=\Big|\Big[&\int_0^t s(1-t)f\left(s,\int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right)ds\\&+\int_t^1 t(1-s)f\left(s,\int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right)ds\Big]'\\&+\left[\frac{\varsigma t}{1-\varsigma\zeta}\int_0^1 G_1(\zeta,s)f\left(s,\int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right)ds\right]'\Big|\end{aligned}$$

$$
\begin{aligned}
&= \Big| -\int_0^t sf\left(s, \int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right) ds \\
&\quad + \int_t^1 (1-s)f\left(s, \int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right) ds \\
&\quad + \frac{\varsigma}{1-\varsigma\zeta}\int_0^1 G_1(\zeta,s)f\left(s, \int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right) ds \Big| \\
&\le \left[2+\frac{\varsigma\zeta(1-\zeta)}{1-\varsigma\zeta}\right]\int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,u_{n-1}(\tau))d\tau\right) ds \\
&\le \left[2+\frac{\varsigma\zeta(1-\zeta)}{1-\varsigma\zeta}\right]\int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,z_0(\tau))d\tau\right) ds \\
&= \left[2+\frac{\varsigma\zeta(1-\zeta)}{1-\varsigma\zeta}\right]\int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,\beta\tau)d\tau\right) ds \\
&\le \left[2+\frac{\varsigma\zeta(1-\zeta)}{1-\varsigma\zeta}\right]\beta^{\lambda\eta}\int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds \\
&< +\infty,
\end{aligned}
$$

thus, $\{(Fu_n)(t) : n \in N\}$ is an equicontinuous set. Similarly, $\{(Fz_n)(t) : n \in N\}$ is also an equicontinuous set. It follows from (3.1) that $\{u_n\}$ and $\{z_n\}$ are relatively compact sets in E. Since P is normal, there exist $u^*, z^* \in E$ such that $u_n \to u^*, z_n \to z^*$ and $u^* \le z^*$. From $u_{n-1} \le u^* \le z^* \le z_{n-1}$, we get

$$Fu_{n-1} = u_n \le Fu^* \le Fz^* \le z_n = Fz_{n-1}.$$

Let $n \to +\infty$, we have

$$u^* \le Fu^* \le Fz^* \le z^*.$$

This and (3.1) imply that

$$u_0 \ge \alpha\beta^{-1}u^*, \quad z_0 \le \beta\alpha^{-1}z^*. \tag{3.2}$$

Let $\rho_n = \sup\{\xi > 0 : \xi u^* \le u_n,\ \xi z_n \le z^*\}$ and then

$$\rho_n u^* \le u_n \textit{ and } \rho_n z_n \le z^*, \qquad n = 1, 2, 3, \ldots, \tag{3.3}$$

and

$$\alpha\beta^{-1} \le \rho_0 \le \rho_1 \le \cdots \le \rho_n \le 1. \tag{3.4}$$

In what follows, we shall prove that

$$\lim_{n\to+\infty} \rho_n = \rho = 1. \tag{3.5}$$

In fact,

$$\begin{aligned} u_{n+1} &= F(u_n) \geq F(\rho_n u^*) \geq (\rho_n)^{\lambda\eta} F(u^*) = \Big(\frac{\rho_n}{\rho}\Big)^{\lambda\eta} \rho^{\lambda\eta} F(u^*) \\ &\geq \Big(\frac{\rho_n}{\rho}\Big)^{\lambda} \rho^{\lambda\eta} F(u^*) \\ &= \rho_n^\lambda \Big(\rho^\lambda\Big)^{(-(1-\eta))} F(u^*) \\ &\geq \rho_n \Big(\rho^\lambda\Big)^{(-(1-\eta))} u^*, \end{aligned}$$

From the definition of ρ_n, we have

$$\rho_{n+1} \geq \rho_n \Big(\rho^\lambda\Big)^{(-(1-\eta))}, \quad n = 1, 2, \ldots.$$

If $\rho < 1$, then

$$\rho_{n+1} \geq \rho_1 \bigg[\Big(\rho^\lambda\Big)^{(-(1-\eta))}\bigg]^n \to +\infty, \quad as \;\; n \to +\infty,$$

which contradicts with (3.4). Hence (3.5) holds.

Next we shall prove $u^* \geq z^*$. Let $\theta = \sup\{\xi > 0 : u^* \geq \xi z^*\}$. From $u^* \geq \alpha\beta^{-1} z_0 \geq \alpha\beta^{-1} z^*$, we know θ is well defined. We claim that $\theta \geq 1$. Otherwise, if $\theta < 1$, in view of (1.4) and (1.5), we have

$$Fz^* \leq F(\theta^{-1} u^*) \leq \theta^{-\lambda\eta} Fu^* \leq \theta^{-\eta} Fu^*. \tag{3.6}$$

It follows from (3.3) and (3.6) that

$$\rho_n^\lambda z_{n+1} = \rho_n^\lambda F z_n \leq \rho_n^\lambda F(\rho_n^{-1} z^*) \leq \rho_n^{\lambda(1-\eta)} F(z^*) \leq Fz^* \leq \theta^{-\eta} Fu^*$$

and

$$u_{n+1} = Fu_n \geq F(\rho_n u^*) \geq (\rho_n)^{\lambda\eta} F(u^*) \geq \rho_n^\lambda F u^*.$$

Then

$$\rho_n^\lambda z_{n+1} \leq \theta^{-\eta} \rho_n^{-\lambda} u_{n+1}.$$

Letting $n \to +\infty$, we have $z^* \leq \theta^{-\eta} u^*$, this contradicts the definition of θ. So $\theta \geq 1$, which implies $u^* \geq z^*$. Thus $u^* = z^*$ is the fixed point of F. Let

$$v^*(t) = \int_0^1 G(t,s) g(s, u^*(s)) ds, \quad t \in [0,1]$$

then system (1.1) has a positive solution (u^*, v^*) such that

$$\alpha t \leq u^* \leq \beta t, \quad rt \leq v^* \leq Rt,$$

where $r = n\alpha^\lambda \int_0^1 s(1-s)g(s,s)ds$, $R = m\beta^\lambda \int_0^1 g(s,s)ds$ and m, n, α, β as the above state.

In fact, by $\alpha t \le u^* \le \beta t$, in view of (1.3) and (1.5) and Remark 3, for any $t \in [\zeta, 1]$, we have

$$\begin{aligned} v^*(t) &= \int_0^1 G(t,s)g(s,u^*(s))ds \ge \int_0^1 nts(1-s)g(s,\alpha s)ds \\ &\ge tn\alpha^\lambda \int_0^1 s(1-s)g(s,s)ds = rt, \end{aligned}$$

and

$$v^*(t) = \int_0^1 G(t,s)g(s,u^*(s))ds \le \int_0^1 mtg(s,\beta s)ds \le tm\beta^\lambda \int_0^1 g(s,s)ds = Rt.$$

On the other hand,

$$\begin{aligned} \int_0^1 |(u^*)''(s)|ds &= \int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,u^*(\tau))d\tau\right) ds \\ &\le \int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,\beta\tau)d\tau\right) ds \\ &\le \beta^{\lambda\eta} \int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,\beta\tau)d\tau\right) ds \\ &< +\infty, \end{aligned}$$

and

$$\int_0^1 |(v^*)''(s)|ds = \int_0^1 g(s,u^*(s))ds \le \beta^\lambda \int_0^1 g(s,s)ds < +\infty.$$

Then both $(u^*)''(s)$ and $(v^*)''(s)$ are absolutely integrable, that is, $u^*, v^* \in C^1[0,1] \times C^1[0,1]$, which implies that (u^*, v^*) is a $C^1[0,1] \times C^1[0,1]$ positive solution to system (1.1).

(2) Let (u^*, v^*) be a $C^1[0,1] \times C^1[0,1]$ positive solution to system (1.1). If there is another $C^1[0,1] \times C^1[0,1]$ positive solution $(\overline{u}, \overline{v})$, then it follows from the proof of (2.2) that there exist positive numbers $0 < \mu_1 < 1 < \mu_2$ such that

$$\mu_1 t \le \overline{u(t)} \le \mu_2 t.$$

So $\overline{u} \in P_l$. Letting $\alpha \le \min\{\mu_1, \alpha_0\}$, $\beta \ge \max\{\mu_2, \beta_0\}$, then

$$u_0(t) \le \overline{u(t)} \le z_0(t).$$

Notice that F is a increasing operator and $F\overline{u} = \overline{u}$, one has $u_n(t) \le \overline{u(t)} \le z_n(t)$. Let $n \to +\infty$, then $u^* = \overline{u}$, and so $v^* = \overline{v}$. Hence the $C^1[0,1] \times C^1[0,1]$ positive solution to system (1.1) exists and is unique.

Finally, for any initial value $x_0 \in P_l$, there exist $0 < m_0 < 1 < \mu_0$ such that $m_0 t \leq x_0(t) \leq \mu_0 t$. Take $\alpha \leq \min\{m_0, \alpha_0\}, \beta \geq \max\{\mu_0, \beta_0\}$, then

$$u_0(t) \leq x_0(t) \leq z_0(t).$$

Since F is increasing, by mathematical induction, we have

$$u_n(t) \leq x_n(t) \leq z_n(t), n = 1, 2, \ldots.$$

Let $n \to \infty$, then $x_n(t) \to u^*(t)$, where

$$x_n(t) = \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, x_{n-1}(\tau)) d\tau\right) ds, n = 1, 2, \ldots.$$

If we let $v_n(t) = \int_0^1 G(t,s) g(s, x_n(s)) ds (n = 1, 2, \ldots)$. Then

$$v_n(t) \to v^*(t) = \int_0^1 G(t,s) g(s, x^*(s)) ds \quad as \quad n \to +\infty.$$

So we complete the proof of Theorem 1.

4 Further results

In the following, we consider the differential system

$$\begin{cases} u'' = -f(t, v, v''), \quad 0 < t < 1, \\ v'' = -g(t, u), \quad 0 < t < 1 \\ u(0) = v(0) = 0, \varsigma u(\zeta) = u(1), \varsigma v(\zeta) = v(1), \end{cases} \tag{4.1}$$

where $f \in C((0,1) \times [0, +\infty) \times (-\infty, 0], [0, +\infty)), g \in C([0,1] \times [0, +\infty), [0, +\infty))$, $\varsigma > 0, 0 < \zeta < 1$, and $\varsigma\zeta < 1$.

Theorem 2. Suppose that g satisfies (B) and the following condition is satisfied

(A1) For any fixed $t \in (0,1)$, $f(t, v, w)$ is nondecreasing in v and nonincreasing in w and $f(t, v, w) \not\equiv 0$; for any $c \in (0,1)$, there exists $\eta \in (0,1)$, such that, for all $(t, v, w) \in (0,1) \times [0, +\infty) \times (-\infty, 0]$,

$$f(t, cv, cw) \geq c^\eta f(t, v, w).$$

Then

(1) The system (4.1) has a $C^1[0,1] \times C^1[0,1]$ positive solution if and only if

$$0 < \int_{0+}^{1-} f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau, -g(s,s)\right) ds < +\infty.$$

(2) If there exists a positive solution (u^*, v^*) in $C^1[0,1] \times C^1[0,1]$ to system (1.1), then (u^*, v^*) is unique, and for any initial value $u_0 \in P_l$, let

$$u_n(t) = \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g\big(\tau, u_{n-1}(\tau)\big) d\tau, -g\big(s, u_{n-1}(s)\big)\right) ds, \ \ n = 1, 2, \ldots,$$

$$v_n(t) = \int_0^1 G(t,s) g\big(s, u_n(s)\big) ds, \quad n = 1, 2, \ldots.$$

Then

$$u_n \to u^*, \quad v_n \to \int_0^1 G(t,s) g\big(s, u^*(s)\big) ds = v^*.$$

Proof. In fact, the existence of positive solutions of (4.1) is equivalent to the existence of the fixed point of operator

$$(Fu)(t) = \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g\big(\tau, u(\tau)\big) d\tau, -g\big(s, u(s)\big)\right) ds.$$

Then, as the same as done in the proof of (1.1), we complete the proof.

Example. Consider the following differential system:

$$\begin{cases} u'' = -t^{-\alpha} v^{\alpha}, \ \ 0 < t < 1, \\ v'' = -t^{k} u^{\beta}, \ \ 0 < t < 1 \\ u(0) = v(0) = 0, \dfrac{1}{2} u(\dfrac{1}{2}) = u(1), \dfrac{1}{2} v(\dfrac{1}{2}) = v(1), \end{cases} \tag{4.2}$$

where $k > -\beta, 0 < \alpha, \beta < 1$, $f(t,v) = t^{-\alpha} v^{\alpha}$, $g(t,u) = t^k u^{\beta}$. Then, for any $c, h \in (0,1)$, we have

$$f(t, cv) = t^{-\alpha} (cv)^{\alpha} = c^{\alpha} t^{-\alpha} v^{\alpha} = c^{\alpha} f(t,v);$$

$$g(t, hu) = t^k (hu)^{\beta} = h^{\beta} t^k u^{\beta} = h^{\beta} g(t,u);$$

and

$$
\begin{aligned}
&\int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds\\
&\quad = \int_0^1 t^{-\alpha}\Big[(1-t)\int_0^t s^k s^{1+\beta}(1-s)^\beta ds + t\int_t^1 s^k s^\beta (1-s)^{1+\beta} ds\\
&\qquad + \frac{t}{3}\int_0^{1/2} s^k s^{1+\beta}(1-s)^\beta ds + \frac{t}{3}\int_{1/2}^1 s^k s^\beta (1-s)^{1+\beta} ds\Big]^\alpha dt\\
&\quad = \int_0^1 \Big[\frac{1-t}{t}\int_0^t s^k s^{1+\beta}(1-s)^\beta ds + \int_t^1 s^k s^\beta (1-s)^{1+\beta} ds\\
&\qquad + \frac{1}{3}\int_0^{1/2} s^k s^{1+\beta}(1-s)^\beta ds + \frac{1}{3}\int_{1/2}^1 s^k s^\beta (1-s)^{1+\beta} ds\Big]^\alpha dt\\
&\quad < \int_0^1 \Big[\frac{1-t}{t}\int_0^t s(1-s)^\beta ds + \int_t^1 (1-s)^{1+\beta} ds\\
&\qquad + \frac{1}{3}\int_0^{1/2} s(1-s)^\beta ds + \frac{1}{3}\int_{1/2}^1 (1-s)^{1+\beta} ds\Big]^\alpha dt\\
&\quad < (\frac{11}{6})^\alpha < +\infty;
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^1 f\left(s, \int_0^1 G(s,\tau)g(\tau,\tau)d\tau\right) ds\\
&\quad = \int_0^1 \Big[\frac{1-t}{t}\int_0^t s^k s^{1+\beta}(1-s)^\beta ds + \int_t^1 s^k s^\beta (1-s)^{1+\beta} ds\\
&\qquad + \frac{1}{3}\int_0^{1/2} s^k s^{1+\beta}(1-s)^\beta ds + \frac{1}{3}\int_{1/2}^1 s^k s^\beta (1-s)^{1+\beta} ds\Big]^\alpha dt\\
&\quad > \int_0^1 \Big[\frac{1-t}{t}\int_0^t s^{k+\beta+1}(1-s) ds + \int_t^1 s^{k+\beta}(1-s)^2 ds\\
&\qquad + \frac{1}{3}\int_0^{1/2} s^{k+\beta+1}(1-s) ds + \frac{1}{3}\int_{1/2}^1 s^{k+\beta}(1-s)^2 ds\Big]^\alpha dt\\
&\quad > 0.
\end{aligned}
$$

So (A),(B),(C) are satisfied. By Theorem 3.1 (1), the system (4.2) has positive solutions. Moreover, by Theorem 3.1 (2), if $(u^*(t), v^*(t))$ is a positive solution for the above system, then it is unique, and for any initial function $lt \leq u_0(t) \leq Lt$, we can construct iterative sequence (u_n, v_n) which converges to the unique solution (u^*, v^*) uniformly.

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