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Positive solutions for a multi-point eigenvalue problem involving the one dimensional p-Laplacian *†

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Abstract A multi-point boundary value problem involving the one dimensional p-Laplacian and depending on a parameter is studied in this paper and existence of positive solutions is established by means of a fixed point theorem for operators defined on Banach spaces with cones.

Keywords Positive solutions; Boundary value problems; One-dimensional p-Laplacian

1. Introduction

In this paper we study the existence of positive solutions to the boundary value problem (BVP) for the one-dimensional p-Laplacian

$$(\phi_p(u'))' + \lambda q(t)f(t,u) = 0,\ t \in (0,1), \tag{1.1}$$

$$u'(0) = \sum_{i=1}^{n-2} \alpha_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$ and α_i, β_i, f satisfy

(H_1) $\alpha_i, \beta_i \in [0,\infty)$ satisfy $0 < \sum_{i=1}^{n-2} \alpha_i < 1$, and $\sum_{i=1}^{n-2} \beta_i < 1$;

(H_2) $f \in C([0,1] \times [0,\infty), [0,\infty))$;

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(H_3) $q(t) \in C(0,1) \cap L^1[0,1]$ with $q > 0$ on $(0,1)$. The number λ is regarded as a parameter and is to be determined among positive numbers.

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Since then there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see [3,4,5,7]. The methods include the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory, and fixed point theorem in cones. For example,

In [6], R. Ma and N. Castaneda studied the following BVP

$$\begin{cases} x''(t) + a(t)f(x(t)) = 0, \quad 0 \le t \le 1, \\ x'(0) = \sum\limits_{i=1}^{m-2} \alpha_i x'(\xi_i), \qquad x(1) = \sum\limits_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases}$$

where $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $\alpha_i, \beta_i \ge 0$ with $0 < \sum_{i=1}^m \alpha_i < 1$ and $\sum_{i=1}^m \beta_i < 1$. They showed the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

The authors in [7] considered the multi-point BVP for one dimensional p-Laplacian

$$(\phi_p(u'))' + f(t,u) = 0, \; t \in (0,1),$$

$$\phi_p(u'(0)) = \sum_{i=1}^{n-2} \alpha_i \phi_p(u'(\xi_i)), \qquad u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i).$$

Using a fixed point theorem in a cone, we provided sufficient conditions for the existence of multiple positive solutions to the above BVP.

In paper [8], we investigated the following more general multi-point BVPs

$$(\phi_p(u'))' + q(t)f(t,u,u') = 0, \; t \in (0,1),$$

$$u(0) = \sum_{i=1}^{n-2} \alpha_i u(\xi_i), \qquad u'(1) = \sum_{i=1}^{n-2} \beta_i u'(\xi_i),$$

$$u'(0) = \sum_{i=1}^{n-2} \alpha_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i).$$

The main tool is a fixed point theorem due to Avery and Peterson[9], we provided sufficient conditions for the existence of multiple positive solutions.

In view of the common concern about multi-point boundary value problems as exhibited in [6,7,8] and their references, it is of interests to continue the investigation and study the problem (1.1) and (1.2).

Motivated by the works of [7] and [8], the aim of this paper is to show the existence of positive solutions u to BVP(1.1) and (1.2). For this purpose, we consider the Banach space $E = C[0,1]$ with the maximum norm $\|x\| = \max_{0\le t\le 1} |x(t)|$. By a positive solution of (1.1) and (1.2), we means a function $u \in C^1[0,1], (\phi_p(u'))'(t) \in C(0,1)\cap L[0,1]$ which is positive on $[0,1]$ and satisfies the differential equation (1.1) and the boundary conditions (1.2).

Our main results will depend on the following Guo-Krasnoselskii fixed-point theorem .

Theorem A[10][11]. Let E be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let

$$T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Tx\| \le \|x\|,\ x \in K \cap \partial\Omega_1$ and $\|Tx\| \ge \|x\|,\ x \in K \cap \partial\Omega_2$, or

(ii) $\|Tx\| \ge \|x\|,\ x \in K \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|,\ x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In section 3, we shall present some sufficient conditions with λ belonging to an open interval to ensure the existence of positive solutions to problems (1.1) and (1.2). To the author's knowledge, no one has studied the existence of positive solutions for problems (1.1) and (1.2) using the Guo-Krasnoselskii fixed-point theorem.

2. The preliminary lemmas

Let

$$C^+[0,1] = \{\omega \in C[0,1] : \omega(t) \ge 0, t \in [0,1]\}.$$

Lemma 2.1 Let $(H_1) - (H_3)$ hold. Then for $x \in C^+[0,1]$, the problem

$$(\phi_p(u'))' + \lambda q(t) f(t, x(t)) = 0,\ \ t \in (0,1), \tag{2.1}$$

$$u'(0) = \sum_{i=1}^{n-2} \alpha_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i), \tag{2.2}$$

has a unique solution

$$u(t) = B_x - \int_t^1 \phi_p^{-1}\left(A_x - \lambda \int_0^s q(\tau) f(\tau, x(\tau))d\tau\right) ds, \tag{2.3}$$

where A_x, B_x satisfy

$$\phi_p^{-1}(A_x) = \sum_{i=1}^{n-2} \alpha_i \phi_p^{-1}\left(A_x - \lambda \int_0^{\xi_i} q(s) f(s, x(s)) ds\right), \tag{2.4}$$

$$B_x = -\frac{1}{1 - \sum\limits_{i=1}^{n-2} \beta_i} \sum_{i=1}^{n-2} \beta_i \int_{\xi_i}^{1} \phi_p^{-1}\left(A_x - \lambda \int_0^{s} q(\tau) f(\tau, x(\tau)) d\tau\right) ds.$$

Lemma 2.2 Suppose $\beta_i, r_i \geq 0$ $(i = 1, 2, \cdots, n-2)$ and $\sum_{i=1}^{n-2} \beta_i < 1$, denote $r = \min\limits_{1 \leq i \leq n-2} r_i$, $R = \max\limits_{1 \leq i \leq n-2} r_i$, then there exists a unique

$$x \in \left[\frac{\phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)}{1 - \phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)} r, \frac{\phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)}{1 - \phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)} R\right]$$

such that

$$\phi_p(x) = \sum_{i=1}^{n-2} \beta_i \phi_p(x + r_i).$$

Proof. Define

$$H(x) = \phi_p(x) - \sum_{i=1}^{n-2} \beta_i \phi_p(x + r_i).$$

Then, $H(x) \in C((-\infty, +\infty), R)$.

Let

$$\underline{x} = \frac{\phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)}{1 - \phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)} \cdot r, \quad \overline{x} = \frac{\phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)}{1 - \phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)} \cdot R,$$

then we have

$$\begin{aligned}
H(\underline{x}) &= \phi_p(\underline{x}) - \sum_{i=1}^{n-2} \beta_i \phi_p(\underline{x} + r_i) \\
&\leq \phi_p(\underline{x}) - \sum_{i=1}^{n-2} \beta_i \phi_p(\underline{x} + r) \\
&= \phi_p(\underline{x}) - \phi_p\left[\phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)(\underline{x} + r)\right] \\
&= \phi_p(\underline{x}) - \phi_p(\underline{x}) = 0,
\end{aligned}$$

and

$$\begin{aligned}
H(\overline{x}) &= \phi_p(\overline{x}) - \sum_{i=1}^{n-2} \beta_i \phi_p(\overline{x} + r_i) \\
&\geq \phi_p(\overline{x}) - \sum_{i=1}^{n-2} \beta_i \phi_p(\overline{x} + R) \\
&= \phi_p(\overline{x}) - \phi_p\left[\phi_p^{-1}\left(\sum_{i=1}^{n-2} \beta_i\right)(\overline{x} + R)\right] \\
&= \phi_p(\overline{x}) - \phi_p(\overline{x}) = 0.
\end{aligned}$$

The zero point theorem guarantees that there exists an $x_0 \in [\underline{x},\ \overline{x}]$ such that $H(x_0) = 0$.

If there exist two constants $x_1, x_2 \in [\underline{x},\ \overline{x}]$ satisfying $H(x_1) = H(x_2) = 0$, then

Case 1. $x_1 = 0$.

i.e., $H(0) = 0,\ \sum_{i=1}^{n-2} \beta_i \phi_p(r_i) = 0$. So, $\beta_i \phi_p(r_i) = 0,\ (i = 1, 2, \cdots, n-2)$, then, $r_i \phi_p^{-1}(\beta_i) = 0,\ (i = 1, 2, \cdots, n-2)$. Therefore,

$$\begin{aligned} H(x) &= \phi_p(x) - \sum_{i=1}^{n-2} \beta_i \phi_p(x + r_i) \\ &= \phi_p(x) - \sum_{i=1}^{n-2} \phi_p\left(\phi_p^{-1}(\beta_i)x + \phi_p^{-1}(\beta_i) r_i\right) \\ &= \phi_p(x) - \sum_{i=1}^{n-2} \beta_i \phi_p(x) = \left(1 - \sum_{i=1}^{n-2} \beta_i\right) \phi_p(x). \end{aligned}$$

Obviously, there exists a unique $x = 0$ satisfying $H(x) = 0$. So, $x_1 = x_2 = 0$.

Case 2. $x_1 \neq 0$.

(i) If $x_1 \in (-\infty, 0)$, then

$$\begin{aligned} H(x_1) &= \phi_p(x_1) - \sum_{i=1}^{n-2} \beta_i \phi_p(x_1 + r_i) \\ &\leq \phi_p(x_1) - \sum_{i=1}^{n-2} \beta_i \phi_p(x_1) \\ &= \left(1 - \sum_{i=1}^{n-2} \beta_i\right) \phi_p(x_1) < 0. \end{aligned}$$

So, when, $x_1 \in (-\infty, 0), H(x_1) \neq 0$.

(ii) If $x_1 \in (0, +\infty)$, then

$$\begin{aligned} H(x_1) &= \phi_p(x_1) - \sum_{i=1}^{n-2} \beta_i \phi_p(x_1 + r_i) \\ &= \phi_p(x_1) \left[1 - \sum_{i=1}^{n-2} \beta_i \phi_p\left(1 + \frac{r_i}{x_1}\right)\right] \\ &= \phi_p(x_1) \widetilde{H}(x_1), \end{aligned}$$

where

$$\widetilde{H}(x) = 1 - \sum_{i=1}^{n-2} \beta_i \phi_p\left(1 + \frac{r_i}{x}\right).$$

As $H(x_1) = 0$ and $x_1 \neq 0$, so there must exist $i_0 \in \{1, 2, \cdots, n-2\}$, such that $\beta_{i_0} \phi_p(r_{i_0}) > 0$. Thus, we get $\widetilde{H}(x)$ is strictly increasing on $(0, +\infty)$.

If $H(x_1) = H(x_2) = 0$, then $\widetilde{H}(x_1) = \widetilde{H}(x_2) = 0$. So, $x_1 = x_2$ since $\widetilde{H}(x)$ is strictly increasing on $(0, +\infty)$.

Therefore, $H(x) = 0$ has a unique solution on $(0, +\infty)$. Combining case 1, case 2, we obtain $H(x) = 0$ has a unique solution on $(-\infty, +\infty)$ and $x \in [\underline{x}, \overline{x}]$.

Lemma 2.3 Let $k = \dfrac{\phi_p\left(\sum_{i=1}^{n-2}\alpha_i\right)}{1-\phi_p\left(\sum_{i=1}^{n-2}\alpha_i\right)}$, then there exists a unique real number A_x that satisfies (2.4). Furthermore, A_x is contained in the interval $\left[-\lambda k\int_0^1 q(s)f(s,x(s))ds, 0\right]$.

Proof The equation is equivalent to

$$\phi_p^{-1}(-A_x) = \sum_{i=1}^{n-2}\alpha_i\phi_p^{-1}\left(-A_x + \lambda\int_0^{\xi_i} q(s)f(s,x(s))ds\right).$$

By Lemma 2.2, we can easily obtain

$$\begin{aligned} -A_x &\in \left[\lambda k\int_0^{\xi_1} q(s)f(s,x(s))ds, \lambda k\int_0^{\xi_{n-2}} q(s)f(s,x(s))ds\right] \\ &\subset \left[0, \lambda k\int_0^1 q(s)f(s,x(s))ds\right]. \end{aligned}$$

So the conclusion is obvious.

Lemma 2.4 Let $(H_1)-(H_3)$ hold. If $x \in C^+[0,1]$ and $\lambda > 0$, then the unique solution of problem (2.1)-(2.2) satisfies $u(t) \geq 0$ for $t \in [0,1]$.

Proof: According to Lemmas 2.1 and 2.3, we first have

$$\begin{aligned} u(1) &= B_x \\ &= -\frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1 \phi_p^{-1}\left(A_x - \lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\ &= \frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1 \phi_p^{-1}\left(-A_x + \lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\ &\geq \frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1 \phi_p^{-1}\left(\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\ &\geq 0, \end{aligned}$$

and

$$\begin{aligned} u(0) &= B_x - \int_0^1 \phi_p^{-1}\left(A_x - \lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\ &= u(1) + \int_0^1 \phi_p^{-1}\left(-A_x + \lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\ &\geq u(1) + \int_0^1 \phi_p^{-1}\left(\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\ &\geq 0. \end{aligned}$$

If $t \in (0,1)$, we have

$$\begin{aligned} u(t) &= B_x - \int_t^1 \phi_p^{-1}\left(A_x - \lambda \int_0^s q(\tau) f(\tau, x(\tau)) d\tau\right) ds \\ &= u(1) + \int_t^1 \phi_p^{-1}\left(-A_x + \lambda \int_0^s q(\tau) f(\tau, x(\tau)) d\tau\right) ds \\ &\geq u(1) + \int_t^1 \phi_p^{-1}\left(\lambda \int_0^s q(\tau) f(\tau, x(\tau)) d\tau\right) ds \\ &\geq 0. \end{aligned}$$

So $u(t) \geq 0, t \in [0,1]$.

Lemma 2.5 Let $(H_1) - (H_3)$ hold. If $x \in C^+[0,1]$ and $\lambda > 0$, then the unique solution of problem (2.1)-(2.2) satisfies

$$\min_{t \in [0,1]} u(t) \geq \gamma \|u\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{n-2} \beta_i (1 - \xi_i)}{1 - \sum_{i=1}^{n-2} \beta_i \xi_i}.$$

Proof: Clearly $u'(t) = \phi_p^{-1}\left(A_x - \lambda \int_0^t q(s) f(s, x(s)) ds\right) = -\phi_p^{-1}\left(-A_x + \lambda \int_0^t q(s) f(s, x(s)) ds\right) \leq 0$. This implies that

$$\|u\| = u(0) \quad \text{and} \quad \min_{t \in [0,1]} u(t) = u(1).$$

Furthermore, it is easy to see that $u'(t_2) \leq u'(t_1)$ for any $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$. Hence $u'(t)$ is a decreasing function on [0,1]. This means that the graph of $u'(t)$ is concave down on (0,1). For each $i \in \{1, 2, \cdots, n-2\}$ we have

$$\frac{u(\xi_i) - u(1)}{1 - \xi_i} \geq \frac{u(0) - u(1)}{1},$$

i.e.,

$$u(\xi_i) - \xi_i u(1) \geq (1 - \xi_i) u(0)$$

so that

$$\sum_{i=1}^{n-2} \beta_i u(\xi_i) - \sum_{i=1}^{n-2} \beta_i \xi_i u(1) \geq \sum_{i=1}^{n-2} \beta_i (1 - \xi_i) u(0)$$

and, by means of the boundary condition $u(1) = \sum\limits_{i=1}^{n-2} \beta_i u(\xi_i)$, we have

$$u(1) \geq \frac{\sum_{i=1}^{n-2} \beta_i (1 - \xi_i)}{1 - \sum_{i=1}^{n-2} \beta_i \xi_i} u(0).$$

This completes the proof.

Now we define

$$K=\{\omega|\omega\in C^{+}[0,1],\ \min_{0\le t\le 1}\omega(t)\ge\gamma\|\omega\|\},$$

where γ is defined in Lemma 2.5. For any $\lambda>0$, define operator $T_\lambda : C^{+}[0,1]\to K$ by

$$\begin{aligned}(T_\lambda x)(t) &= -\frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^{1}\phi_p^{-1}\left(A_x-\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds\\ &\quad -\int_t^1\phi_p^{-1}\left(A_x-\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds. \qquad (2.5)\end{aligned}$$

By Lemmas 2.1 and 2.3, we know $T_\lambda x$ is well defined. Furthermore, we have the following result.

Lemma 2.6[Lemma 2.4, 7] $T : K\to K$ is completely continuous.

Let

$$\min f_\infty := \lim_{x\to\infty}\inf\min_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)},\qquad \max f_0 := \lim_{x\to 0^+}\sup\max_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)},$$
$$\min f_0 := \lim_{x\to 0^+}\inf\min_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)},\qquad \max f_\infty := \lim_{x\to\infty}\sup\max_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)}.$$

3. Main results

We now give our results on the existence of positive solutions of BVP(1.1) and (1.2).

Theorem 3.1 Let $(H_1)-(H_3)$ hold and $\min f_\infty>0, \max f_0<\infty$. If

$$\frac{1}{\gamma^{p-1}\min f_\infty\cdot N^{p-1}}<\lambda<\frac{1}{(k+1)\max f_0\cdot M^{p-1}}. \qquad (3.1)$$

where

$$\begin{aligned} M &= \frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i}{1-\sum_{i=1}^{n-2}\beta_i}\cdot\phi_p^{-1}\left(\int_0^1 q(s)ds\right),\\ N &= \frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^{1}\phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right)ds+\int_0^1\phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right)ds,\end{aligned}$$

then the problem (1.1)(1.2) has at least one positive solution.

Proof. Define the operator T_λ as (2.5). Under the condition (3.1), there exists an $\varepsilon>0$ such that

$$\frac{1}{\gamma^{p-1}(\min f_\infty-\varepsilon)\cdot N^{p-1}}\le\lambda\le\frac{1}{(k+1)(\max f_0+\varepsilon)\cdot M^{p-1}}. \qquad (3.2)$$

(i). Since $\max f_0<\infty$, there exists an $H_1>0$ such that for $x: 0\le x\le H_1$,

$$f(t,x)\le(\max f_0+\varepsilon)\phi_p(x). \qquad (3.3)$$

Let $\Omega_1 = \{x \in E : \|x\| < H_1\}$, then for $x \in K \cap \partial\Omega_1$, we have

$$\begin{aligned} -A_x + \lambda \int_0^s q(s)f(s,x(s))ds &\leq \lambda k \int_0^1 q(s)f(s,x(s))ds + \lambda \int_0^s q(s)f(s,x(s))ds \\ &\leq \lambda(k+1) \int_0^1 q(s)f(s,x(s))ds \\ &\leq \lambda(k+1)(\max f_0 + \varepsilon) \int_0^1 q(s)\phi_p(x(s))ds \\ &\leq \lambda(k+1)(\max f_0 + \varepsilon)\phi_p(\|x\|) \int_0^1 q(s)ds. \end{aligned}$$

So,

$$\phi_p^{-1}\left(-A_x + \lambda \int_0^s q(s)f(s,x(s))ds\right) \leq [\lambda(k+1)(\max f_0 + \varepsilon)]^{q-1}\, \phi_p^{-1}\left(\int_0^1 q(s)ds\right) \|x\|.$$

Therefore,

$$\begin{aligned} \|T_\lambda x\| = (T_\lambda x)(0) &= \frac{1}{1 - \sum\limits_{i=1}^{n-2}\beta_i} \sum_{i=1}^{n-2} \beta_i \int_{\xi_i}^1 \phi_p^{-1}\left(-A_x + \lambda \int_0^s q(\tau)f(\tau,x(\tau))d\tau\right) ds \\ &\quad + \int_0^1 \phi_p^{-1}\left(-A_x + \lambda \int_0^s q(\tau)f(\tau,x(\tau))d\tau\right) ds \\ &\leq \frac{1 - \sum_{i=1}^{n-2}\beta_i\xi_i}{1 - \sum_{i=1}^{n-2}\beta_i} [\lambda(k+1)(\max f_0 + \varepsilon)]^{q-1}\, \phi_p^{-1}\left(\int_0^1 q(s)ds\right) \|x\| \\ &= M[\lambda(k+1)(\max f_0 + \varepsilon)]^{q-1}\|x\| \\ &\leq \|x\|. \end{aligned}$$

Thus $\|T_\lambda x\| \leq \|x\|$.

(ii). Next, since $\min f_\infty > 0$, there exists an $\overline{H}_2 > 0$ such that for $x > \overline{H}_2$,

$$f(t,x) \geq (\min f_\infty - \varepsilon)\phi_p(x). \tag{3.4}$$

Take $H_2 = \max\{\overline{H}_2, 2H_1\}$ and $\Omega_2 = \{x \in E : \|x\| < H_2\}$. Then for $x \in K \cap \partial\Omega_2$, we have

$$\begin{aligned} -A_x + \lambda \int_0^s q(\tau)f(s,x(\tau))d\tau &\geq \lambda \int_0^s q(\tau)f(\tau,x(\tau))d\tau \\ &\geq \lambda(\min f_\infty - \varepsilon) \int_0^s q(\tau)\phi_p(x(\tau))d\tau \\ &\geq \lambda\phi_p(\gamma)(\min f_\infty - \varepsilon)\phi_p(\|x\|) \int_0^s q(\tau)d\tau. \end{aligned}$$

So,

$$\phi_p^{-1}\left(-A_x + \lambda \int_0^s q(s)f(s,x(s))ds\right) \geq [\lambda(\min f_\infty - \varepsilon)]^{q-1}\, \phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right) \gamma\|x\|.$$

Therefore,

$$\begin{aligned}
\|T_\lambda x\| = (T_\lambda x)(0) &= \frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1 \phi_p^{-1}\left(-A_x+\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\
&\quad +\int_0^1 \phi_p^{-1}\left(-A_x+\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\
&\geq [\lambda(\min f_\infty-\varepsilon)]^{q-1}\gamma\|x\|\frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1\phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right)ds \\
&\quad +[\lambda(\min f_\infty-\varepsilon)]^{q-1}\gamma\|x\|\int_0^1\phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right)ds \\
&= N\,[\lambda(\min f_\infty-\varepsilon)]^{q-1}\gamma\|x\| \\
&\geq \|x\|.
\end{aligned}$$

So $\|T_\lambda x\|\geq\|x\|$.

Therefore, by the first part of Theorem A, T_λ has a fixed point $x^*\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$ such that $H_1\leq\|x^*\|\leq H_2$. It is easily checked that $x^*(t)$ is a positive solution of problems (1.1) and (1.2). The proof is complete.

Theorem 3.2. Let $(H_1)-(H_3)$ hold and $\min f_0>0,\max f_\infty<\infty$. If

$$\frac{1}{\gamma^{p-1}\min f_0\cdot N^{p-1}}<\lambda<\frac{1}{(k+1)\max f_\infty\cdot M^{p-1}}. \tag{3.5}$$

then the problem (1.1)(1.2) has at least one positive solution.

Proof. Under the condition (3.5), there exists an $\varepsilon>0$ such that

$$\frac{1}{\gamma^{p-1}(\min f_0-\varepsilon)\cdot N^{p-1}}\leq\lambda\leq\frac{1}{(k+1)(\max f_\infty+\varepsilon)\cdot M^{p-1}}. \tag{3.6}$$

(i). Since $\min f_0>0$, there exists an $H_1>0$ such that for $x:0\leq x\leq H_1$,

$$f(t,x)\geq(\min f_0-\varepsilon)\phi_p(x). \tag{3.7}$$

Let $\Omega_1=\{x\in E:\|x\|<H_1\}$, then for $x\in K\cap\partial\Omega_1$, we have

$$\begin{aligned}
-A_x+\lambda\int_0^s q(\tau)f(s,x(\tau))d\tau &\geq \lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau \\
&\geq \lambda(\min f_0-\varepsilon)\int_0^s q(\tau)\phi_p(x(\tau))d\tau \\
&\geq \lambda\phi_p(\gamma)(\min f_0-\varepsilon)\phi_p(\|x\|)\int_0^s q(\tau)d\tau.
\end{aligned}$$

So,

$$\phi_p^{-1}\left(-A_x+\lambda\int_0^s q(s)f(s,x(s))ds\right)\ge[\lambda(\min f_0-\varepsilon)]^{q-1}\,\phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right)\gamma\|x\|.$$

Therefore,

$$\begin{aligned}
\|T_\lambda x\|=(T_\lambda x)(0) &= \frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1\phi_p^{-1}\left(-A_x+\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds\\
&\quad+\int_0^1\phi_p^{-1}\left(-A_x+\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds\\
&\ge [\lambda(\min f_0-\varepsilon)]^{q-1}\,\gamma\|x\|\frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1\phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right)ds\\
&\quad+[\lambda(\min f_0-\varepsilon)]^{q-1}\,\gamma\|x\|\int_0^1\phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right)ds\\
&= N\,[\lambda(\min f_0-\varepsilon)]^{q-1}\,\gamma\|x\|\\
&\ge \|x\|.
\end{aligned}$$

So $\|T_\lambda x\|\ge\|x\|$.

(ii). Since $\max f_\infty<\infty$, there exists an $\overline{H}_2>0$ such that for $x>\overline{H}_2$,

$$f(t,x)\le(\max f_\infty+\varepsilon)\phi_p(x). \tag{3.8}$$

There are two cases : Case 1, f is bounded, and Case 2, f is unbounded.

Case 1. Suppose that f is bounded, i.e., there exists $N>0$ such that $f(t,x)\le\phi_p(N)$ for $t\in[0,1]$ and $0\le x<\infty$. Define

$$H_2=\max\left\{2H_1,\frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i}{1-\sum_{i=1}^{n-2}\beta_i}\,[\lambda(k+1)]^{q-1}\,\phi_p^{-1}\left(\int_0^1 q(s)ds\right)N\right\},$$

and $\Omega_2=\{x\in E:\|x\|<H_2\}$. Then for $x\in K\cap\partial\Omega_2$, we have

$$\begin{aligned}
-A_x+\lambda\int_0^s q(s)f(s,x(s))ds &\le \lambda k\int_0^1 q(s)f(s,x(s))ds+\lambda\int_0^s q(s)f(s,x(s))ds\\
&\le \lambda(k+1)\int_0^1 q(s)f(s,x(s))ds\\
&\le \lambda(k+1)\phi_p(N)\int_0^1 q(s)ds.
\end{aligned}$$

So,

$$\phi_p^{-1}\left(-A_x+\lambda\int_0^s q(s)f(s,x(s))ds\right)\le[\lambda(k+1)]^{q-1}\,\phi_p^{-1}\left(\int_0^1 q(s)ds\right)N.$$

Therefore,

$$\begin{aligned}
\|T_\lambda x\| = (T_\lambda x)(0) &= \frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1 \phi_p^{-1}\left(-A_x+\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\
&\quad +\int_0^1 \phi_p^{-1}\left(-A_x+\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\
&\le \frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i}{1-\sum_{i=1}^{n-2}\beta_i}\left[\lambda(k+1)\right]^{q-1}\phi_p^{-1}\left(\int_0^1 q(s)ds\right)N \\
&\le H_2=\|x\|.
\end{aligned}$$

Case 2. We choose $H_2 > \max\{2H_1, \overline{H}_2\}$ such that $f(t,x)\le f(t,H_2)$ for $t\in[0,1]$ and $0<x<H_2$. Let $\Omega_2=\{x\in E : \|x\|<H_2\}$. Then for $x\in K\cap\partial\Omega_2$, we have

$$\begin{aligned}
-A_x+\lambda\int_0^s q(s)f(s,x(s))ds &\le \lambda k\int_0^1 q(s)f(s,x(s))ds+\lambda\int_0^s q(s)f(s,x(s))ds \\
&\le \lambda(k+1)\int_0^1 q(s)f(s,x(s))ds \\
&\le \lambda(k+1)\int_0^1 q(s)f(s,H_2)ds \\
&\le \lambda(k+1)(\max f_\infty+\varepsilon)\phi_p(H_2)\int_0^1 q(s)ds.
\end{aligned}$$

So,

$$\phi_p^{-1}\left(-A_x+\lambda\int_0^s q(s)f(s,x(s))ds\right)\le\left[\lambda(k+1)(\max f_\infty+\varepsilon)\right]^{q-1}\phi_p^{-1}\left(\int_0^1 q(s)ds\right)H_2.$$

Therefore,

$$\begin{aligned}
\|T_\lambda x\| = (T_\lambda x)(0) &= \frac{1}{1-\sum\limits_{i=1}^{n-2}\beta_i}\sum_{i=1}^{n-2}\beta_i\int_{\xi_i}^1 \phi_p^{-1}\left(-A_x+\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\
&\quad +\int_0^1 \phi_p^{-1}\left(-A_x+\lambda\int_0^s q(\tau)f(\tau,x(\tau))d\tau\right)ds \\
&\le \frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i}{1-\sum_{i=1}^{n-2}\beta_i}\left[\lambda(k+1)(\max f_\infty+\varepsilon)\right]^{q-1}\phi_p^{-1}\left(\int_0^1 q(s)ds\right)H_2 \\
&= M\left[\lambda(k+1)(\max f_\infty+\varepsilon)\right]^{q-1}H_2 \\
&\le H_2=\|x\|.
\end{aligned}$$

Therefore, by the second part of Theorem A, T_λ has a fixed point $x^*\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$ such that $H_1\le\|x^*\|\le H_2$. It is easily checked that $x^*(t)$ is a positive solution of problems (1.1) and (1.2).

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