TOTAL STABILITY IN ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY *

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1. INTRODUCTION

Recently, authors [2] have discussed some equivalent relations for ρ-uniform stabilities of a given equation and those of its limiting equations by using the skew product flow constructed by quasi-processes on a general metric space. In 1992, Murakami and Yoshizawa [6] pointed out that for functional differential equations with infinite delay on a fading memory space $\mathcal{B} = \mathcal{B}((-\infty, 0]; R^n)$ ρ-stability is a useful tool in the study of the existence of almost periodic solutions for almost periodic systems and they proved that ρ-total stability is equivalent to BC-total stability.

The purpose of this paper is to show that equivalent relations established by Murakami and Yoshizawa [6] holds even for functional differential equations with infinite delay on a fading memory space $\mathcal{B} = \mathcal{B}((-\infty, 0]; X)$ with a general Banach space X.

2. FADING MEMORY SPACES AND SOME DEFINITIONS

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Let X be a Banach space with norm $|\cdot|_X$. For any interval $J \subset R := (-\infty, \infty)$, we denote by $\mathrm{BC}(J;X)$ the space of all bounded and continuous functions mapping J into X. Clearly $\mathrm{BC}(J;X)$ is a Banach space with the norm $|\cdot|_{\mathrm{BC}(J;X)}$ defined by $|\phi|_{\mathrm{BC}(J;X)} = \sup\{|\phi(t)|_X : t \in J\}$. If $J = R^- := (-\infty, 0]$, then we simply write $\mathrm{BC}(J;X)$ and $|\cdot|_{\mathrm{BC}(J;X)}$ as BC and $|\cdot|_{\mathrm{BC}}$, respectively. For any function $u : (-\infty, a) \mapsto X$ and $t < a$, we define a function $u_t : R^- \mapsto X$ by $u_t(s) = u(t+s)$ for $s \in R^-$. Let $\mathcal{B} = \mathcal{B}(R^-; X)$ be a real Banach space of functions mapping R^- into X with a norm $|\cdot|_{\mathcal{B}}$. The space $\mathcal{B}$ is assumed to have the following properties:

(A1) There exist a positive constant N and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on $R^+ := [0, \infty)$ with the property that if $u : (-\infty, a) \mapsto X$ is continuous on $[\sigma, a)$ with $u_\sigma \in \mathcal{B}$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,
(i) $u_t \in \mathcal{B}$,
(ii) u_t is continuous in t (w.r.t. $|\cdot|_{\mathcal{B}}$),
(iii) $N|u(t)|_X \le |u_t|_{\mathcal{B}} \le K(t-\sigma)\sup_{\sigma \le s \le t}|u(s)|_X + M(t-\sigma)|u_\sigma|_{\mathcal{B}}$.

(A2) If $\{\phi^n\}$ is a sequence in $\mathcal{B} \cap \mathrm{BC}$ converging to a function ϕ uniformly on any compact intertval in R^- and $\sup_n |\phi^n|_{\mathrm{BC}} < \infty$, then $\phi \in \mathcal{B}$ and $|\phi^n - \phi|_{\mathcal{B}} \to 0$ as $n \to \infty$.

It is known [3, Proposition 7.1.1] that the space $\mathcal{B}$ contains BC and that there is a constant $\ell > 0$ such that

$$|\phi|_{\mathcal{B}} \le \ell|\phi|_{\mathrm{BC}}, \quad \phi \in \mathrm{BC}. \tag{1}$$

Set $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$ and define an operator $S_0(t) : \mathcal{B}_0 \mapsto \mathcal{B}_0$ by

$$[S_0(t)\phi](s) = \begin{cases} \phi(t+s) & \text{if } t+s \le 0, \\ \\ 0 & \text{if } t+s > 0 \end{cases}$$

for each $t \ge 0$. In virtue of (A1), one gets that the family $\{S_0(t)\}_{t\ge0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{B}_0$. We consider the following properties:

(A3) $\lim_{t\to\infty}|S_0(t)\phi|_{\mathcal{B}} = 0, \quad \phi \in \mathcal{B}_0.$

The space $\mathcal{B}$ is called *a fading memory space*, if it satisfies (A3) in addition to (A1) and (A2). It is known [3, Proposition 7.1.5] that the functions $K(\cdot)$ and $M(\cdot)$ in (A1) can be chosen as $K(t) \equiv \ell$ and $M(t) \equiv (1 + (\ell/N))\|S_0(t)\|$. Here and hereafter, we denote by $\|\cdot\|$ the operator norm of linear bounded operators. Note that (A3) implies $\sup_{t\ge0}\|S_0(t)\| < \infty$ by the Banach-Steinhaus theorem. Therefore, whenever $\mathcal{B}$ is a fading memory space, we can assume that the functions $K(\cdot)$ and $M(\cdot)$ in (A1) satisfy $K(\cdot) \equiv K$ and $M(\cdot) \equiv M$, constants.

We provide a typical example of fading memory spaces. Let $g : R^- \mapsto [1, \infty)$ be any continuous nonincreasing function such that $g(0) = 1$ and $g(s) \to \infty$ as $s \to -\infty$. We set

$$C_g^0 := C_g^0(X) = \{\phi : R^- \mapsto X \text{ is continuous with } \lim_{s \to -\infty} |\phi(s)|_X / g(s) = 0\}.$$

Then the space C_g^0 equipped with the norm

$$|\phi|_g = \sup_{s \le 0} \frac{|\phi(s)|_X}{g(s)}, \quad \phi \in C_g^0,$$

is a Banach space and it satisfies (A1)–(A3). Hence the space C_g^0 is a fading memory space. We note that the space C_g^0 is separable whenever X is separable.

Throughout the remainder of this paper, we assume that $\mathcal{B}$ is a fading memory space which is separable.

We now consider the following functional differential equation

$$\frac{du}{dt} = Au(t) + F(t, u_t), \tag{2}$$

where A is the infinitesimal generator of a compact semigroup $\{T(t)\}_{t \ge 0}$ of bounded linear operators on X and $F : R^+ \times \mathcal{B} \to X$ is continuous. We assume the following conditions on F:

(H1) $F(t, \phi)$ is uniformly continuous on $R^+ \times S$ for any compact set S in $\mathcal{B}$.

(H2) For any $H > 0$, there is an $L(H) > 0$ such that $|F(t, \phi)|_X \le L(H)$ for all $t \in R^+$ and $\phi \in \mathcal{B}$ such that $|\phi|_{\mathcal{B}} \le H$.

For any topological spaces $\mathcal{J}$ and $\mathcal{X}$, we denote by $C(\mathcal{J}; \mathcal{X})$ the set of all continuous functions from $\mathcal{J}$ into $\mathcal{X}$. By virture of (H1) and (H2), it follows that for any $(\sigma, \phi) \in R \times \mathcal{B}$, there exists a function $u \in C((-\infty, t_1); X)$ such that $u_\sigma = \phi$ and the following relation holds:

$$u(t) = T(t - \sigma)\phi(0) + \int_\sigma^t T(t - s)F(s, u_s)ds, \quad \sigma \le t < t_1,$$

(cf. [1, Theorem 1]). Such a function u is called a (mild) solution of (2) through (σ, ϕ) defined on $[\sigma, t_1)$ and denoted by $u(t) := u(t, \sigma, \phi, F)$.

In the above, t_1 can be taken as $t_1 = \infty$ if $\sup_{\sigma \le t < t_1} |u(t)|_X < \infty$(cf. [1, Corollary 2]). In the following, we always assume the following condition, too:

(H3) Equation (2) has a bounded solution $\bar{u}(t)$ defined on R^+ such that $\bar{u}_0 \in \mathrm{BC}$ and $|\bar{u}_t|_{\mathcal{B}} \leq C_1$ for all $t \in R^+$.

By virtue of [4, Lemma 2], we see that the set $\overline{\{\bar{u}(t) : t \in R^+\}}$ is compact in X, $\bar{u}(t)$ is uniformly continuous on R^+ and the set $\overline{\{\bar{u}_t : t \in R^+\}}$ is compact in $\mathcal{B}$.

Now we shall give the definition of BC-total stability.

Definition 1 *The bounded solution $\bar{u}(t)$ of* (2) *is said to be* BC-*totally stable* (BC-TS) *if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that $\sigma \in R^+, \phi \in \mathrm{BC}$ with $|\bar{u}_\sigma - \phi|_{\mathrm{BC}} < \delta(\varepsilon)$ and $h \in \mathrm{BC}([\sigma,\infty);\mathrm{X})$ with $\sup_{t\in[\sigma,\infty)} |h(t)|_X < \delta(\varepsilon)$ imply $|\bar{u}(t) - u(t,\sigma,\phi,F+h)|_X < \varepsilon$ for $t \geq \sigma$, where $u(\cdot,\sigma,\phi,F+h)$ denotes the solution of*

$$\frac{du}{dt} = Au(t) + F(t,u_t) + h(t), \quad t \geq \sigma, \tag{3}$$

through (σ,ϕ).

For any $\phi, \psi \in \mathrm{BC}$, we set

$$\rho(\phi,\psi) = \sum_{j=1}^{\infty} 2^{-j} |\phi - \psi|_j / \{1 + |\phi - \psi|_j\},$$

where $|\cdot|_j = |\cdot|_{[-j,0]}$. Then (BC,$\rho$) is a metric space. Furthermore, it is clear that $\rho(\phi^k,\phi) \to 0$ as $k \to \infty$ if and only if $\phi^k \to \phi$ compactly on R^-. Let U be a closed bounded subset of X whose interior U^i contains the set $\overline{\{\bar{u}(t) : t \in R\}}$, where $\bar{u}$ is the one in (H3). Whenever $\phi \in \mathrm{BC}$ satisfies $\phi(s) \in U$ for all $s \in R^-$, we write as $\phi(\cdot) \in U$, for simply.

We shall give the definition of ρ-total stability.

Definition 2 *The bounded solution $\bar{u}(t)$ of* (2) *is said to be ρ-totally stable with respect to U* (ρ-TS w.r.t.U) *if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that $\sigma \in R^+, \phi(\cdot) \in U$ with $\rho(\bar{u}_\sigma,\phi) < \delta(\varepsilon)$ and $h \in \mathrm{BC}([\sigma,\infty);X)$ with $\sup_{t\in[\sigma,\infty)} |h(t)|_X < \delta(\varepsilon)$ imply $\rho(\bar{u}_t, u_t(\sigma,\phi,F+h)) < \varepsilon$ for $t \geq \sigma$.*

In the above, if the term $\rho(u_t(\sigma,\phi,F+h),\bar{u}_t)$ is replaced by $|u(t,\sigma,F+h) - \bar{u}(t)|_X$, then we have another concept of ρ-total stability, which will be referred to as the (ρ,X)-total stability.

As was shown in [6, Lemma 2], these two concepts of ρ-total stability are equivalent.

3. EQUIVALENCE OF BC-TOTAL STABILITY AND ρ-TOTAL STABILITY

In this section, we shall discuss the equivalence between the BC-total stability and the ρ-total stability, and extend a result due to Murakami and Yoshizawa [6, Theorems 1]) for $X = R^n$.

We now state our main result of this section.

Theorem The solution $\bar{u}(t)$ of (2) is BC-TS if and only if it is ρ-TS w.r.t.U for any bounded set U in X such that $U^i \supset O_{\bar{u}} := \overline{\{\bar{u}(t) : t \in R\}}$.

In order to prove the theorem, we need a lemma. A subset $\mathcal{F}$ of $C(R^+; X)$ is said to be uniformly equicontinuous on R^+, if
$\sup\{|x(t+\delta) - x(t)|_X : t \in R^+, x \in \mathcal{F}\} \to 0$ as $\delta \to 0^+$. For any set $\mathcal{F}$ in $C(R^+; X)$ and any set S in $\mathcal{B}$, we set

$$R(\mathcal{F}) = \{x(t) : t \in R^+,\ \ x \in \mathcal{F}\}$$

$$W(S, \mathcal{F}) = \{x(\cdot) : R \mapsto X : x_0 \in S, x|_{R^+} \in \mathcal{F}\}$$

and

$$V(S, \mathcal{F}) = \{x_t : t \in R^+, x \in W(S, \mathcal{F})\}.$$

Lemma ([5, Lemma 1]) If S is a compact subset in $\mathcal{B}$ and if $\mathcal{F}$ is a uniformly equicontinuous set in $C(R^+; X)$ such that the set $R(\mathcal{F})$ is relatively compact in X, then the set $V(S, \mathcal{F})$ is relatively compact in $\mathcal{B}$.

Proof of Theorem The "if" part is easily shown by noting that $\rho(\phi, \psi) \le |\phi - \psi|_{\mathrm{BC}}$ for $\phi, \psi \in \mathrm{BC}$. We shall establish the "only if" part. We assume that the solution $\bar{u}(t)$ of (2) is BC-TS but not (ρ, X)-TS w.r.t.U; here $U \subset \{x \in X : |x|_X \le c\}$ for some $c > 0$. Since the solution $\bar{u}(t)$ of (2) is not (ρ, X)-TS w.r.t.U, there exist an $\varepsilon \in (0, 1)$, sequences $\{\tau_m\} \subset R^+, \{t_m\}(t_m > \tau_m), \{\phi^m\} \subset \mathrm{BC}$ with $\phi^m(\cdot) \in U, \{h_m\}$ with $h_m \in \mathrm{BC}([\tau_m, \infty))$, and solutions $\{u(t, \tau_m, \phi^m, F + h_m) := \hat{u}^m(t)\}$ of

$$\frac{du}{dt} = Au(t) + F(t, u_t) + h_m(t)$$

such that

$$\rho(\phi^m, \bar{u}_{\tau_m}) < 1/m \ \text{ and } \ |h_m|_{[\tau_m, \infty)} < 1/m \tag{4}$$

and that

$$|\hat{u}^m(t_m) - \bar{u}(t_m)|_X = \varepsilon \ \text{ and } \ |\hat{u}^m(t) - \bar{u}(t)|_X < \varepsilon \ \text{ on } \ [\tau_m, t_m) \tag{5}$$

for $m \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of all positive integers. For each $m \in \mathbf{N}$ and $r \in R^+$, we define $\phi^{m,r} \in \mathrm{BC}$ by

$$\phi^{m,r}(\theta) = \begin{cases} \phi^m(\theta) & \text{if } \ -r \leq \theta \leq 0, \\ \\ \phi^m(-r) + \bar{u}(\tau_m + \theta) - \bar{u}(\tau_m - r) & \text{if } \ \theta < -r. \end{cases}$$

We note that

$$\sup\{|\phi^{m,r} - \phi^m|_{\mathcal{B}} : m \in \mathbf{N}\} \to 0 \ \ as \ \ r \to \infty. \tag{6}$$

Indeed, if (6) is false, then there exist an $\varepsilon > 0$ and sequences $\{m_k\} \subset \mathbf{N}$ and $\{r_k\}, r_k \to \infty$ as $k \to \infty$, such that $|\phi^{m_k,r_k} - \phi^{m_k}|_{\mathcal{B}} \geq \varepsilon$ for $k = 1, 2, \cdots$. Put $\psi^k := \phi^{m_k,r_k} - \phi^{m_k}$. Clearly, $\{\psi^k\}$ is a sequence in BC which converges to zero function compactly on R^- and $\sup_k |\psi^k|_{\mathrm{BC}} < \infty$. Then Axiom (A2) yield that $|\psi^k|_{\mathcal{B}} \to 0$ as $k \to \infty$, a contradiction.

Next we shall show that the set $\{\phi^m, \phi^{m,r} : m \in \mathbf{N}, r \in R^+\}$ is relatively compact in $\mathcal{B}$. Since the set $\overline{\{\bar{u}_t : t \in R^+\}}$ is compact in $\mathcal{B}$ as noted in the preceding section, (4) and Axiom (A2) yield that any sequence $\{\phi^{m_j}\}_{j=1}^{\infty} (m_j \in \mathbf{N})$ has a convergent subsequence in $\mathcal{B}$. Therefore, it sufficies to show that any sequence $\{\phi^{m_j,r_j}\}_{j=1}^{\infty} (m_j \in \mathbf{N}, r_j \in R^+)$ has a convergent subsequence in $\mathcal{B}$. We assert that the sequence of functions $\{\phi^{m_j,r_j}(\theta)\}_{j=1}^{\infty}$ contains a subsequence which is equicontinuous on any compact set in R^-. If this is the case, then the sequence $\{\phi^{m_j,r_j}\}_{j=1}^{\infty}$ would have a convergent subsequence in $\mathcal{B}$ by Ascoli's theorem and Axiom (A2), as required. Now, notice that the sequence of functions $\{\bar{u}(\tau_{m_j} + \theta)\}$ is equicontinuous on any compact set in R^-. Then the assertion obviously holds true when the sequence $\{m_j\}$ is bounded. Taking a subsequence if necessary, it is thus sufficient to consider the case $m_j \to \infty$ as $j \to \infty$. In this case, it follows from (4) that $\phi^{m_j}(\theta) - \bar{u}(\tau_{m_j} + \theta) =: w^j(\theta) \to 0$ uniformly on any compact set in R^-. Consequently, $\{w^j(\theta)\}$ is equicontinuous on any compact set in R^-, and so is $\{\phi^{m_j}(\theta)\}$. Therefore the assertion immediately follows from this observation.

Now, for any $m \in \mathbf{N}$, set $u^m(t) = \hat{u}^m(t + \tau_m)$ if $t \leq t_m - \tau_m$ and $u^m(t) = u^m(t - \tau_m)$ if $t > t_m - \tau_m$. Moreover, set $u^{m,r}(t) = \phi^{m,r}(t)$ if $t \in R^-$ and $u^{m,r}(t) = u^m(t)$ if $t \in R^+$. In what follows, we shall show that $\{u^m(t)\}$ is a family of uniformly equicontinuous functions on R^+. To do this, we first prove that

$$\inf_m (t_m - \tau_m) > 0. \tag{7}$$

Assume that (7) is false. By taking a subsequence if necessary, we may assume that $\lim_{m\to\infty}(t_m - \tau_m) = 0$. If $m \geq 3$ and $0 \leq t \leq \min\{t_m - \tau_m, 1\}$, then

$$\begin{aligned}
|u^m(t) - \bar{u}(t+\tau_m)|_X &= |T(t)\phi^m(0) + \int_0^t T(t-s)\{F(s+\tau_m, u_s^m) + h_m(s+\tau_m)\}ds \\
&\qquad -T(t)\bar{u}(\tau_m) - \int_0^t T(t-s)F(s+\tau_m, \bar{u}_{s+\tau_m})ds|_X \\
&\leq C_2\{|\phi^m - \bar{u}_{\tau_m}|_1 + \int_0^t (2L(H)+1)ds\} \\
&\leq C_2\{2/(m-2) + t(2L(H)+1)\},
\end{aligned}$$

where $H = \sup\{|\bar{u}_s|_{\mathcal{B}},\ |u_s^m|_{\mathcal{B}} : 0 \leq s \leq t_m - \tau_m,\ m \in \mathbf{N}\}$ and $C_2 = \sup_{0\leq s\leq 1}||T(s)||$. Then (5) yields that

$$\varepsilon \leq C_2\{2/(m-2) + (t_m - \tau_m)(2L(H)+1)\} \to 0$$

as $m \to \infty$, a contradiction. We next prove that the set $O := \{u^m(t) : t \in R^+,\ m \in \mathbf{N}\}$ is relatively compact in X. To do this, for each η such that $0 < \eta < \inf_m(t_m - \tau_m)$ we consider the sets $O_\eta = \{u^m(t) : t \geq \eta,\ m \in \mathbf{N}\}$ and $\tilde{O}_\eta = \{u^m(t) : 0 \leq t \leq \eta,\ m \in \mathbf{N}\}$. Then $\alpha(O) = \max\{\alpha(O_\eta),\ \alpha(\tilde{O}_\eta)\}$, where $\alpha(\cdot)$ is the Kuratowski's measure of noncompactness of sets in X. For the details of the properties of $\alpha(\cdot)$, see [5; Section 1.4]. Let $0 < \nu < \min\{1, \eta\}$. If $\eta \leq t \leq t_m - \tau_m$, then

$$\begin{aligned}
u^m(t) &= T(t)\phi^m(0) + \int_0^t T(t-s)\{F(s+\tau_m, u_s^m) + h_m(s+\tau_m)\}ds \\
&= T(\nu)[T(t-\nu)u^m(0) + \int_0^{t-\nu} T(t-\nu-s)\{F(s+\tau_m, u_s^m) + h(s+\tau_m)\}ds] \\
&\quad + \int_{t-\nu}^t T(t-s)\{F(s+\tau_m, u_s^m) + h(s+\tau_m)\}ds \\
&= T(\nu)u^m(t-\nu) + \int_{t-\nu}^t T(t-s)\{F(s+\tau_m, u_s^m) + h(s+\tau_m)\}ds.
\end{aligned}$$

Since the set $T(\nu)\{u^m(t-\nu) : t \geq \eta,\ m \in \mathbf{N}\}$ is relatively compact in X because of the compactness of the semigroup $\{T(t)\}_{t\geq 0}$, it follows that

$$\alpha(O_\eta) \leq C_2\{L(H)+1\}\nu.$$

Letting $\nu \to 0$ in the above, we get $\alpha(O_\eta) = 0$ for all η such that $0 < \eta < \inf_m(t_m - \tau_m)$. Observe that the set $\{T(t)\phi^m(0) : 0 \leq t \leq \eta,\ m \in \mathbf{N}\}$ is relatively compact in X. Then

$$\begin{aligned}
\alpha(O) &= \alpha(\tilde{O}_\eta) \\
&= \alpha(\{T(t)\phi^m(0) + \int_0^t T(t-s)\{F(s+\tau_m, u_s^m) + h(s+\tau_m)\}ds : 0 \leq t \leq \eta,\ m \in \mathbf{N}\}) \\
&= \alpha(\{\int_0^t T(t-s)\{F(s+\tau_m, u_s^m) + h(s+\tau_m)\}ds : 0 \leq t \leq \eta,\ m \in \mathbf{N}\}) \\
&= C_2(L(H)+1)\eta
\end{aligned}$$

for all η such that $0 < \eta < \inf_m (t_m - \tau_m)$, which shows $\alpha(O) = 0$; consequently O must be relatively compact in X.

Now, in order to establish the uniform equicontinuity of the family $\{u^m(t)\}$ on R^+, let $\sigma \le s \le t \le s+1$ and $t \le t_m - \tau_m$. Then

$$\begin{aligned} |u^m(t) - u^m(s)|_X &\le |T(t-s)u^m(s) - u^m(s)|_X + |\int_s^t T(t-\tau)\{F(\tau+\tau_m, u^m_\tau) \\ &\quad + h(\tau+\tau_m)\}d\tau|_X \\ &\le \sup\{|T(t-s)z - z|_X : z \in O\} + C_2\{L(H)+1\}|t-s|. \end{aligned}$$

Since the set O is relatively compact in X, $T(\tau)z$ is uniformly continuous in $\tau \in [0,1]$ uniformly for $z \in O$. This leads to $\sup\{|u^m(t) - u^m(s)|_X : 0 \le s \le t \le s+1,\ m \in \mathbf{N}\} \to 0$ as $|t-s| \to 0$, which proves the uniform equicontinuity of $\{u^m\}$ on R^+.

Since $\{\phi^m, \phi^{m,r} : m \in \mathbf{N}, r \in R^+\}$ is relatively compact in $\mathcal{B}$, $|u^m_t|_{\mathcal{B}} \le K\{1 + |\bar{u}|_{[0,\infty)}\} + M|\phi^m|_{\mathcal{B}} \le K\{1 + |\bar{u}|_{[0,\infty)}\} + M\ell c$ by (1) and (A1-iii), and the family $\{u^m(t)\}$ is uniformly equicontinuous on R^+, it follows from Lemma that the set $W := \overline{\{u^{m,r}_t, u^m_t : m \in \mathbf{N}, t \in R^+, r \in R^+\}}$ is compact in $\mathcal{B}$. Hence $F(t,\phi)$ is uniformly continuous on $R^+ \times W$. Define a continuous function $q_{m,r}$ on R^+ by $q_{m,r}(t) = F(t + \tau_m, u^m_t) - F(t+\tau_m, u^{m,r}_t)$ if $0 \le t \le t_m - \tau_m$, and $q_{m,r}(t) = q_{m,r}(t_m - \tau_m)$ if $t > t_m - \tau_m$. Since $|u^{m,r}_t - u^m_t|_{\mathcal{B}} \le M|\phi^{m,r} - \phi^m|_{\mathcal{B}}$ $(t \in R^+, m \in \mathbf{N})$ by (A1-iii), it follows from (6) that $\sup\{|u^{m,r}_t - u^m_t|_{\mathcal{B}} : t \in R^+, m \in \mathbf{N}\} \to 0$ as $r \to \infty$; hence one can choose an $r = r(\varepsilon) \in \mathbf{N}$ in such a way that

$$\sup\{|q_{m,r}(t)|_X : m \in \mathbf{N}, t \in R^+\} < \delta(\varepsilon/2)/2,$$

where $\delta(\cdot)$ is the one for BC-TS of the solution $\bar{u}(t)$ of (2). Moreover, for this r, select an $m \in \mathbf{N}$ such that $m > 2^r(1+\delta(\varepsilon/2))/\delta(\varepsilon/2)$. Then $2^{-r}|\phi^m - \bar{u}_{t_m}|_r/[1+|\phi^m - \bar{u}_{t_m}|_r] \le \rho(\phi^m, \bar{u}_{t_m}) < 2^{-r}\delta(\varepsilon/2)/[1+\delta(\varepsilon/2)]$ by (4), which implies that

$$|\phi^m - \bar{u}_{t_m}|_r < \delta(\varepsilon/2) \ \text{ or } \ |\phi^{m,r} - \bar{u}_{t_m}|_{\mathrm{BC}} < \delta(\varepsilon/2).$$

The function $u^{m,r}$ satisfies $u^{m,r}_0 = \phi^{m,r}$ and

$$\begin{aligned} u^{m,r}(t) &= u^m(t) \\ &= T(t)\phi^{m,r}(0) + \int_0^t T(t-s)\{F(s+\tau_m, u^m_s) + h_m(s+\tau_m)\}ds \\ &= T(t)\phi^m(0) + \int_0^t T(t-s)\{F(s+\tau_m, u^{m,r}_s) + q_{m,r}(s) + h_m(s+\tau_m)\}ds \end{aligned}$$

for $t \in [0, t_m - \tau_m)$. Since $\bar{u}^m(t) = \bar{u}(t+\tau_m)$ is a BC-TS solution of

$$\frac{du}{dt} = Au(t) + F(t+\tau_m, u_t)$$

with the same $\delta(\cdot)$ as the one for $\bar{u}(t)$, from the fact that $\sup_{t\geq 0}|q_{m,r}(t)+h_m(\tau_m+t)|_X < \delta(\varepsilon/2)/2+1/m < \delta(\varepsilon/2)$ it follows that $|u^{m,r}(t)-\bar{u}(t+\tau_m)|_X < \varepsilon/2$ on $[0, t_m-\tau_m)$. In particular, we have $|u^{m,r}(t_m-\tau_m)-\bar{u}(t_m)|_X < \varepsilon$ or $|\hat{u}^m(t_m)-\bar{u}(t_m)|_X < \varepsilon$, which contradicts (5).

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