



Regularization of radial solutions of p -Laplace equations, and computations using infinite series

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Abstract. We consider radial solutions of equations with the p -Laplace operator in R^n . We introduce a change of variables, which in effect removes the singularity at $r = 0$. While solutions are not of class C^2 , in general, we show that solutions are C^2 functions of $r^{\frac{p}{2(p-1)}}$. Then we express the solution as an infinite series in powers of $r^{\frac{p}{p-1}}$, and give explicit formulas for its coefficients. We implement this algorithm, using *Mathematica*. *Mathematica's* ability to perform the exact computations turns out to be crucial.

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1 Introduction

Recently there has been an enormous interest in equations with the p -Laplace operator in R^n (with $p > 1$, $u = u(x)$, $x \in R^n$)

$$\operatorname{div} (|\nabla u|^{p-2} \nabla u) + f(u) = 0,$$


see e.g., a review by P. Drábek [1], and the important paper of B. Franchi et al. [2]. Radial solutions of this equation, with the initial data at $r = 0$, satisfy

$$\varphi(u')' + \frac{n-1}{r} \varphi(u') + f(u) = 0, \quad u(0) = \alpha > 0, \quad u'(0) = 0, \quad (1.1)$$

where $\varphi(v) = v|v|^{p-2}$, $p > 1$. To guess the form of the solution, let us drop the higher order term and consider

$$\frac{n-1}{r} \varphi(u') + f(u) = 0, \quad u(0) = \alpha. \quad (1.2)$$

This is a completely different equation, however, in case $p = 2$, it is easy to check that the form of solutions is the same: in both cases, it is a series $\sum_{n=0}^{\infty} a_n r^{2n}$ (with different coefficients), see P. Korman [4] or [5]. It is natural to guess that the form of solutions will be same for (1.1) and

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(1.2), in case $p \neq 2$ too. With that in mind, let us solve (1.2) in case $f(u) = e^u$. Since $\alpha > 0$, we see from (1.2) that $u'(r) < 0$ for all r . Then $\varphi(u') = -(-u')^{p-1}$, and we have

$$u' = - \left[\frac{r}{n-1} e^u \right]^{\frac{1}{p-1}} = - \frac{r^{\frac{1}{p-1}}}{(n-1)^{\frac{1}{p-1}}} e^{\frac{1}{p-1}u}.$$

Integrating, we get

$$u(r) = \alpha - (p-1) \ln \left(1 + \frac{e^{\frac{1}{p-1}\alpha}}{p(n-1)^{\frac{1}{p-1}}} r^{\frac{p}{p-1}} \right).$$

We see that $u(r)$ is a function of $r^{\frac{p}{p-1}}$, and, for r small, we can expand it as a series $u(r) = \sum_{n=0}^{\infty} b_n r^{n \frac{p}{p-1}}$, with some coefficients b_n . Motivated by this example, we make a change of variables $r \rightarrow z$ in (1.1), by letting $z^2 = r^{\frac{p}{p-1}}$. We expect solutions of (1.1) to be of the form $\sum_{n=0}^{\infty} c_n z^{2n}$, which is a real analytic function of z , if this series converges.

The following lemma provides the crucial change of variables.

Lemma 1.1. *Denote*

$$\bar{\alpha} = \frac{p}{2(p-1)}, \quad \beta = \frac{1}{\bar{\alpha}}(\bar{\alpha} - 1) = \frac{-p+2}{p}, \quad \gamma = \beta(p-1) = 3 - p - \frac{2}{p}, \quad (1.3)$$

$$a = \bar{\alpha}^p, \quad A = \bar{\alpha}^p \gamma + (n-1)\bar{\alpha}^{p-1}.$$

Then, for $p > 2$, the change of variables $z^2 = r^{\frac{p}{p-1}}$ transforms (1.1) into

$$au''(z) + \frac{A}{(p-1)z} u'(z) + \frac{z^{p-2}}{\varphi'(u'(z))} f(u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0. \quad (1.4)$$

Conversely, if the solution of (1.4) is of the form $u = v(z^2)$, with $v(t) \in C^1(\bar{\mathbb{R}}_+)$, then the same change of variables transforms (1.4) into (1.1), for any $p > 1$.

Proof. We have $z = r^\alpha$, $\frac{du}{dr} = \alpha \frac{du}{dz} r^{\alpha-1} = \alpha z^\beta \frac{du}{dz}$. By the homogeneity of φ , we have $(\varphi(cv) = c^{p-1}\varphi(v))$, for any $c > 0$)

$$\varphi(u'(r)) = \varphi(\alpha z^\beta u_z) = \alpha^{p-1} z^{\beta(p-1)} \varphi(u_z) = \alpha^{p-1} z^\gamma \varphi(u_z).$$

Then (1.1) becomes

$$\alpha z^\beta \frac{d}{dz} \left[\alpha^{p-1} z^\gamma \varphi(u_z) \right] + (n-1) z^{-2+2/p} \alpha^{p-1} z^\gamma \varphi(u_z) + f(u) = 0,$$

which simplifies to

$$az\varphi'(u'(z))u''(z) + A\varphi(u'(z)) + z^{1/\alpha-\gamma}f(u) = 0.$$

This implies (1.4), keeping in mind that $\varphi'(v) = (p-1)|v|^{p-2}$, $\varphi(v) = \frac{1}{p-1}v\varphi'(v)$, and that $1/\alpha - \gamma - 1 = p - 2$. Also, $\frac{du}{dz} = \frac{2(p-1)}{p} \frac{du}{dr} z^{\frac{p-2}{p}}$, so that $\frac{du}{dz}(0) = 0$. (It is only on the last step that we need $p > 2$.)

Conversely, our change of variables transforms the equation in (1.4) into the one in (1.1). Under our assumption, $u(r) = v(r^{\frac{p}{p-1}})$, so that $u'(0) = 0$ for any $p > 1$. \square

The change of variables $z^2 = r^{\frac{p}{p-1}}$ in effect removes the singularity at zero for p -Laplace equations. Indeed,

$$\lim_{z \rightarrow 0} \frac{z^{p-2}}{\varphi'(u'(z))} = \frac{1}{(p-1)|u''(0)|^{p-2}},$$

which lets us compute $u''(0)$ from the equation (1.4) (the existence of $u''(0)$ is proved later). Indeed, assuming that $f(\alpha) > 0$, we have

$$u''(0) = - \left[\frac{f(\alpha)}{a(p-1) + A} \right]^{\frac{1}{p-1}}. \quad (1.5)$$

(In case $f(\alpha) < 0$, we have $u''(0) = [-f(\alpha)/(a(p-1) + A)]^{\frac{1}{p-1}}$.) We prove that $u(z)$ is smooth, provided that $f(u)$ is smooth. It follows that the solution of p -Laplace problem (1.1) has the form $u(r^{\frac{p}{2(p-1)}})$, with smooth $u(z)$. We believe that our reduction of the p -Laplace equation (1.2) to the form (1.4) is likely to find other applications.

We express the solution of (1.1) in the form $u(r) = \sum_{k=0}^{\infty} a_k r^{k \frac{p}{p-1}}$, and present *explicit* formulas to compute the coefficients a_k . Interestingly, the coefficient a_1 turned out to be special, as it enters in two ways the formula for other a_k . Our formulas are easy to implement in *Mathematica*, and very accurate series approximations can be computed reasonably quickly. We utilize *Mathematica's* ability to perform the "exact computations", as we explain in Section 3.

2 Regularity of solutions in case $p > 2$

It is well known that solutions of p -Laplace equations are not of class C^2 , in general. In fact, rewriting the equation in (1.1) in the form

$$(p-1)u'' + \frac{n-1}{r}u' + |u'|^{2-p}f(u) = 0, \quad (2.1)$$

we see that in case $p > 2$, $u''(0)$ does not exist. We show that in this case the solution of (1.1) is a C^2 function of $r^{\frac{p}{2(p-1)}}$.

We rewrite the equation in (1.1) as

$$r^{n-1}\varphi(u'(r)) = - \int_0^r t^{n-1}f(u(t)) dt. \quad (2.2)$$

Observe that $\varphi^{-1}(t) = -(-t)^{\frac{1}{p-1}}$, for $t < 0$. If we assume that $f(\alpha) > 0$, then for small $r > 0$, we may express from (2.2)

$$-u'(r) = \frac{1}{r^{\frac{n-1}{p-1}}} \left[\int_0^r t^{n-1}f(u(t)) dt \right]^{\frac{1}{p-1}}. \quad (2.3)$$

Integrating

$$u(r) = \alpha - \int_0^r \frac{1}{t^{\frac{n-1}{p-1}}} \left[\int_0^t s^{n-1}f(u(s)) ds \right]^{\frac{1}{p-1}} dt. \quad (2.4)$$

We recall the following lemma from J. A. Iaiia [3].

Lemma 2.1. *Assume that $f(u)$ is Lipschitz continuous. Then one can find an $\epsilon > 0$, so that the problem (1.1) has a unique solution $u(r) \in C^1[0, \epsilon)$. In case $1 < p \leq 2$, $u(r) \in C^2[0, \epsilon)$.*

Proof. In the space $C[0, \epsilon)$ we denote $B_R^\epsilon = \{u \in C[0, \epsilon), \text{ such that } \|u - \alpha\| \leq R\}$, where $\|\cdot\|$ denotes the norm in $C[0, \epsilon)$. The proof of Lemma 2.1 involved showing that the map $T(u)$, defined by the right hand side of (2.4), is a contraction, taking B_R^ϵ into itself, for any $R > 0$, and ϵ sufficiently small (see [3], and also [6] for a similar argument). This argument provided a continuous solution of (2.4), which by (2.3) is in $C^1[0, \epsilon)$, and in case $1 < p \leq 2$, $u(r) \in C^2[0, \epsilon)$, by (2.1) (from (2.3) it follows that the limit $\lim_{r \rightarrow 0} \frac{u'(r)}{r} = u''(0) = 0$ exists). \square

In case $p > 2$, we have the following regularity result.

Theorem 2.2. *Assume that $p > 2$, $f(u)$ is Lipschitz continuous and $f(\alpha) > 0$. For $\epsilon > 0$ sufficiently small, the problem (1.1) has a solution of the form $u(r^{\frac{p}{2(p-1)}})$, where $u(z) \in C^2[0, \epsilon)$, and $u'(z) < 0$ on $(0, \epsilon)$. This solution is unique among all continuous functions satisfying (2.4). If, moreover, $f(u) \in C^k$, then $u(z) \in C^{k+2}[0, \epsilon)$.*

Proof. By Lemma 2.1 we have a unique solution of the problem (1.1), $u(r) \in C^1[0, \epsilon_1)$, for some $\epsilon_1 > 0$ small. By Lemma 1.1, this translates to a solution of the problem (1.4), $u(z) \in C^1[0, \epsilon_1)$. With $m = \frac{A}{a(p-1)}$, we multiply the equation in (1.4) by z^m , and rewrite it as

$$-\frac{u'(z)}{z} = \frac{1}{a(p-1)} \frac{\int_0^z \frac{t^{m+p-2}}{|u'(t)|^{p-2}} f(u(t)) dt}{z^{m+1}}.$$

Taking the limit as $z \rightarrow 0$, and denoting $L = \lim_{z \rightarrow 0} \frac{u'(z)}{z}$, we get

$$-L = \frac{f(\alpha)}{a(p-1)(m+1)|L|^{p-2}}.$$

It follows that this limit L exists, proving the existence of $u''(0)$, as given by (1.5). Observe that $u''(0) < 0$. It follows that $u'(z) < 0$ for small z , so that $\varphi'(u'(z)) < 0$, and then $u(z) \in C^2[0, \epsilon)$, from the equation (1.4).

Assume that $f(u) \in C^1$. Differentiate the equation (1.4)

$$au'''' + \frac{A}{p-1} \frac{u''z - u'}{z^2} + \frac{p-2}{p-1} \left(-\frac{z}{u'}\right)^{p-3} \frac{zu'' - u'}{u'^2} f(u) + \frac{1}{p-1} \left(-\frac{z}{u'}\right)^{p-2} f'u' = 0.$$

From here, $u(z) \in C^3(0, \epsilon)$. Letting $z \rightarrow 0$, and using that $\lim_{z \rightarrow 0} \frac{u''z - u'}{z^2} = \frac{1}{2}u''''(0)$, and $\lim_{z \rightarrow 0} \frac{zu'' - u'}{u'^2} = \frac{u'''(0)}{2u''(0)^2}$, we conclude that $u''''(0) = 0$ (the existence of $u''''(0)$ is proved as before). It follows that $u(z) \in C^3[0, \epsilon)$. Higher regularity is proved by taking further derivatives of the equation. \square

3 Representation of solutions using infinite series

We shall consider an auxiliary problem

$$au''(z) + \frac{A}{(p-1)z} u'(z) + \frac{|z|^{p-2}}{\varphi'(u'(z))} f(u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0. \quad (3.1)$$

Lemma 3.1. *Any solution of the problem (3.1) is an even function.*

Proof. Observe that the change of variables $z \rightarrow -z$ leaves (3.1) invariant. If solution $u(z)$ were not even, then $u(-z)$ would be another solution of (3.1). By Lemma 1.1, $u(z)$ and $u(-z)$ translate into two different solutions of the problem (1.1), contradicting the uniqueness part of Lemma 2.1. \square

It follows from the last lemma that any series solution of (3.1) must be of the form $\sum_{n=0}^{\infty} a_n z^{2n}$. The same must be true for the problem (1.4), since for $z > 0$ it agrees with (3.1). Numerically, we shall be computing the partial sums $\sum_{n=0}^k a_n z^{2n}$, which will provide us with the solution, up to the terms of order $O(z^{2k+2})$. Write the partial sum in the form

$$u(z) = \bar{u}(z) + a_k z^{2k}, \quad (3.2)$$

where $\bar{u}(z) = \sum_{n=0}^{k-1} a_n z^{2n}$. We regard $\bar{u}(z)$ as already computed, and the question is how to compute a_k . Using the constants defined in (1.3), we let

$$B_k = 2k(2k-1)a + \frac{2kA}{p-1}.$$

Theorem 3.2. *Assume that $\alpha > 0$, $f(u) \in C^\infty(\mathbb{R})$, and $f(\alpha) > 0$. The solution of the problem (1.4) in terms of a series of the form $\sum_{k=0}^{\infty} a_k z^{2k}$ is obtained by taking $a_0 = \alpha$, then*

$$a_1 = - \left[\frac{1}{(p-1)2^{p-2}B_1} f(\alpha) \right]^{\frac{1}{p-1}} < 0, \quad (3.3)$$

and for $k \geq 2$, we have (the following limits exist)

$$a_k = - \frac{1}{B_k C_k} \lim_{z \rightarrow 0} \frac{\frac{z^{p-2}}{\varphi'(\bar{u}'(z))} f(\bar{u}) + a \bar{u}''(z) + \frac{A}{(p-1)z} \bar{u}'(z)}{z^{2k-2}}, \quad (3.4)$$

where $\bar{u} = \sum_{n=0}^{k-1} a_n z^{2n}$ is the previously computed approximation, and

$$C_k = 1 + \frac{k(p-2)f(\alpha)}{(p-1)2^{p-2}B_k(-a_1)^{p-1}}.$$

Proof. Plugging $u = \alpha + a_1 z^2$ into the equation (1.4), gives

$$a_1 B_1 = - \frac{z^{p-2}}{(p-1)|2a_1 z|^{p-2}} f(\alpha + a_1 z^2).$$

Letting $z \rightarrow 0$,

$$a_1 B_1 = - \frac{1}{(p-1)|2a_1|^{p-2}} f(\alpha),$$

which implies that $a_1 < 0$, leading to (3.3). Of course, $u = \alpha + a_1 z^2$ is not a solution of (1.4). But the other terms of the solution $u(z) = \sum_{k=0}^{\infty} a_k z^{2k}$ produce a correction, which disappears in the limit. Indeed, plugging $u(z) = \alpha + a_1 z^2 + \sum_{k=2}^{\infty} a_k z^{2k}$ into (1.4), gives

$$a_1 B_1 + \sum_{k=2}^{\infty} a_k B_k z^{2k-2} = - \frac{z^{p-2}}{(p-1)|2a_1 z + \sum_{k=2}^{\infty} 2ka_k z^{2k-1}|^{p-2}} f\left(\alpha + a_1 z^2 + \sum_{k=2}^{\infty} a_k z^{2k}\right),$$

and going to the limit, with $z \rightarrow 0$, gives the same value of a_1 .

Plugging $u(z) = \bar{u}(z) + a_k z^{2k}$ into the equation (1.4), gives

$$-a_k B_k = \frac{\frac{z^{p-2}}{\varphi'(\bar{u}' + 2ka_k z^{2k-1})} f(\bar{u}(z) + a_k z^{2k}) + a \bar{u}''(z) + \frac{A}{(p-1)z} \bar{u}'(z)}{z^{2k-2}}. \quad (3.5)$$

We now expand the quotient in the first term in the numerator. In this expansion we do not need to show the terms that are of order $O(z^{2k-1})$ and higher, since $\lim_{z \rightarrow 0} O(z^{2k-1})/z^{2k-2} = 0$. For $z > 0$ and small, we have (observe that $u'(z) < 0$, and $u'(z) \sim 2a_1 z$, for z small)

$$\begin{aligned} (p-1) \frac{z^{p-2}}{\varphi'(\bar{u}' + 2ka_k z^{2k-1})} &= \frac{z^{p-2}}{(-\bar{u}' - 2ka_k z^{2k-1})^{p-2}} \\ &= \frac{z^{p-2}}{(-\bar{u}')^{p-2} \left(1 + 2ka_k \frac{z^{2k-1}}{\bar{u}'(z)}\right)^{p-2}} = \frac{z^{p-2}}{(-\bar{u}')^{p-2}} \left(1 - 2ka_k(p-2) \frac{z^{2k-1}}{\bar{u}'(z)}\right) + O(z^{2k-1}) \\ &= \frac{z^{p-2}}{(-\bar{u}')^{p-2}} \left(1 + 2ka_k(p-2) \frac{z^{2k-2}}{(-2a_1)}\right) + O(z^{2k-1}) \\ &= \frac{(p-1)z^{p-2}}{\varphi'(\bar{u}')} + 2ka_k(p-2) \frac{z^{2k-2}}{(-2a_1)(-2a_1)^{p-2}} + O(z^{2k-1}). \end{aligned}$$

(Observe that $\frac{z}{-\bar{u}'} = \frac{1}{-2a_1} + o(z)$, and $\frac{z^{p-2}}{(-\bar{u}')^{p-2}} = \frac{1}{(-2a_1)^{p-2}} + o(z)$.) Also

$$f(\bar{u}(z) + a_k z^{2k}) = f(\bar{u}(z)) + O(z^{2k}).$$

Using these expressions in (3.5), and taking the limit, we get

$$a_k = -\frac{1}{B_k} \lim_{z \rightarrow 0} \frac{\frac{z^{p-2}}{\varphi'(\bar{u}'(z))} f(\bar{u}) + a \bar{u}''(z) + \frac{A}{(p-1)z} \bar{u}'(z)}{z^{2k-2}} - \frac{k(p-2)f(\alpha)}{(p-1)B_k 2^{p-2} (-a_1)^{p-1}} a_k.$$

(By Theorem 2.2, $u(z) \in C^\infty[0, \epsilon]$. Hence, the limit representing $a_k = \frac{u^{(2k)}(0)}{(2k)!}$ exists.) Solving this equation for a_k , we conclude (3.4). Plugging $u(z) = \bar{u}(z) + a_k z^{2k} + \sum_{n=k+1}^{\infty} a_n z^{2n}$ into (1.4), produces the same formula for a_k . \square

With a_k 's computed as in this theorem, the series $\sum_{k=0}^{\infty} a_k r^{\frac{kp}{p-1}}$ gives the solution to the original problem (1.1). In case $p = 2$, we proved in [4] that when $f(u)$ is real analytic, the series $\sum_{k=0}^{\infty} a_k z^{2k}$ converges for small z , giving us a real analytic solution. It is natural to expect convergence for $p \neq 2$ too, so that the solution of (1.1) is a real analytic function of $r^{\frac{p}{p-1}}$.

4 Numerical computations

It is easy to implement our formulas for computing the solution in *Mathematica*. It is crucial that *Mathematica* can perform exact computations for fractions. If one tries floating point computations, the limits in (3.4) become infinite. All numbers must be entered as fractions. For example, one cannot enter $p = 4.1$, it should be $p = \frac{41}{10}$ instead. (*Mathematica* switches to floating point computations, once it sees a number entered as a floating point.)

Example 4.1. We solved

$$\varphi(u')' + \frac{n-1}{r} \varphi(u') + e^u = 0, \quad u(0) = 1, \quad u'(0) = 0, \quad (4.1)$$

with $\varphi(v) = v|v|^{p-2}$, and $p = \frac{41}{10}$. *Mathematica* calculated that the corresponding equation (1.4) is

$$a(p-1)u''(z) + \frac{A}{z}u'(z) + \frac{z^{p-2}}{(-u'(z))^{p-2}}e^{u(z)} = 0, \quad u(0) = 1, \quad u'(0) = 0, \quad (4.2)$$

with $a(p-1) = \frac{2825761 \sqrt[10]{\frac{41}{62}}}{4766560}$, $A = \frac{1309499 \sqrt[10]{\frac{41}{62}}}{4766560}$. When we computed the series solution of (4.2) up to a_5 , *Mathematica* returned (instantaneously)

$$u(z) = 1 - \frac{31}{41} \left(\frac{e}{3}\right)^{10/31} z^2 + \frac{4805 \cdot 3^{11/31} e^{20/31}}{225254} z^4 - \frac{326241241 \left(\frac{e}{3}\right)^{30/31}}{43314091660} z^6 \\ + \frac{51312765230579 e^{40/31}}{154203017487865920 \cdot 3^{9/31}} z^8 - \frac{13334484822273130589 e^{50/31}}{283500239799651287332500 \cdot 3^{19/31}} z^{10}.$$

The same solution using floating point numbers is

$$u(z) = 1 - 0.732424z^2 + 0.0600499z^4 - 0.00684643z^6 \\ + 0.000879009z^8 - 0.000120356z^{10}.$$

For the original equation (4.1), this implies (we have $\frac{p}{p-1} = \frac{41}{31}$, and $z^2 = r^{\frac{41}{31}}$)

$$u(r) = 1 - 0.732424r^{\frac{41}{31}} + 0.0600499r^{\frac{82}{31}} - 0.00684643r^{\frac{123}{31}} \\ + 0.000879009r^{\frac{164}{31}} - 0.000120356r^{\frac{205}{31}} + \dots$$

To check the accuracy of this computation, we denoted by $q(z)$ the left-hand side of (4.2) (with $u(z)$ being the above polynomial of tenth degree), and asked *Mathematica* to expand $q(z)$ into series about $z = 0$. *Mathematica* returned: $q(z) = O(z^{\frac{121}{10}})$. We have performed similar computations, with similar results, for other values of $u(0) = \alpha$. For larger values of α , e.g., for $\alpha = 2$, the computations take longer, but no more than several minutes.

We have obtained similar results for all other $f(u)$ and p that we tried (including the case $1 < p < 2$). We wish to stress that in all computations, when the solution of (1.4) was computed up to the order z^{2n} , the defect function $q(z)$ was at least of order $O(z^{2n+2})$ near $z = 0$. This heuristic result is consistent with the Theorem 3.2, but does not seem to follow from it.

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