Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **33**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

Existence of multiple positive solutions of higher order multi-point nonhomogeneous boundary value problem

Dapeng Xie[a,*], Yang Liu[a], Chuanzhi Bai[b]
[a] *Department of Mathematics, Hefei Normal University, Hefei, Anhui 230601, P R China*
[b] *School of Mathematical Sciences, Huaiyin Normal University, Huaian, Jiangsu 223300, P R China*

Abstract

In this paper, by using the Avery and Peterson fixed point theorem, we establish the existence of multiple positive solutions for the following higher order multi-point nonhomogeneous boundary value problem

$$u^{(n)}(t) + f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad t \in (0,1),$$
$$u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(0) = 0, \quad u^{(n-2)}(1) - \sum_{i=1}^{m} a_i u^{(n-2)}(\xi_i) = \lambda,$$

where $n \geq 3$ and $m \geq 1$ are integers, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$ are constants, $\lambda \in [0,\infty)$ is a parameter, $a_i > 0$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} a_i\xi_i < 1$, $f(t, u, u', \cdots, u^{(n-2)}) \in C([0,1] \times [0,\infty)^{n-1}, [0,\infty))$. We give an example to illustrate our result.

Keywords: Multi-point boundary value problem; Cone; Nonhomogeneous; Positive solution; Fixed point theorem.

1. Introduction

In this paper, we consider the existence of multiple positive solutions for the following higher order multi-point nonhomogeneous boundary value problem

$$\begin{cases} u^{(n)}(t) + f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(0) = 0, \quad u^{(n-2)}(1) - \sum_{i=1}^{m} a_i u^{(n-2)}(\xi_i) = \lambda, \end{cases} \tag{1.1}$$

where $n \geq 3$ and $m \geq 1$ are integers, $\lambda \in [0,\infty)$ is a parameter, and a_i, ξ_i, f satisfying

(H_1) $a_i > 0$ for $1 \leq i \leq m$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$ and $\sum_{i=1}^{m} a_i\xi_i < 1$;
(H_2) $f : [0,1] \times [0,\infty)^{n-1} \to [0,\infty)$ is continuous.

For the past few years, the existence of solutions for higher order ordinary differential equations has received a wide attention. We refer the reader to [2-6,8,15-20] and references therein. However, most of the above mentioned references only consider the cases in which f does not contain higher

**E-mail address*: xiedapeng9@yahoo.com.cn

order derivatives of u and the parameter $\lambda = 0$. This is because the presence of higher order derivatives in the nonlinear function f and the parameter $\lambda \neq 0$ make the study more difficult. For example, in [6], Graef and Yang obtained existence and nonexistence results for positive solutions of the following nth ordinary differential equation

$$\begin{cases} u^{(n)}(t) + \mu g(t) f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(0) = u^{(n-2)}(1) - \sum_{i=1}^{m} a_i u^{(n-2)}(\xi_i) = 0, \end{cases} \tag{1.2}$$

under the following assumptions:

(C_1) $f : [0,1] \to [0,\infty)$ is a continuous function and $\mu > 0$ is a parameter;
(C_2) $g : [0,1] \to [0,\infty)$ is a continuous function with $\int_0^1 g(t)dt > 0$;
(C_3) $n \geq 3$ and $m \geq 1$ are integers;
(C_4) $a_i > 0$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} a_i = 1$;
(C_5) $\frac{1}{2} \leq \xi_1 < \xi_2 < \cdots < \xi_m < 1$.

Obviously, condition (H_1) in this paper is weaker than the conditions (C_4) and (C_5).

Often when authors deal with higher order boundary value problems in which the nonlinear function f contains higher order derivatives, they transform the higher order equation into a second order equation, see [7,9,14,22] and references therein. For instance, in [22], by using the fixed-point principle in a cone and the fixed-point index theory for a strict-set-contraction operator, Zhang, Feng and Ge established the existence and nonexistence of positive solutions for nth-order three-point boundary value problems in Banach spaces

$$\begin{cases} u^{(n)}(t) + f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) = \theta, \quad t \in J, \\ u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(0) = \theta, \quad u^{(n-2)}(1) = \rho u^{(n-2)}(\eta), \end{cases} \tag{1.3}$$

where $J = [0,1]$, $f \in C(J \times P^{n-1}, P)$, P is a cone of real Banach space, $\rho \in (0,1)$, and θ is the zero element of the real Banach space.

Nonhomogeneous boundary value problems have received special attention from many authors in recent years (see [10-13,17]). Recently, in the case of $n = 3$ and $m = 1$, by employing the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem, Sun [17] established existence and nonexistence of positive solutions to the problem (1.1) when $f(t, u(t), u'(t)) = a(t)f(u(t))$, $\lambda \in (0,\infty)$ and the nonlinearity f is either superlinear or sublinear. However, our problem is more general than the problem of [2-4,6,8,15,17,22] and the aim of our paper is to investigate the existence of two or three positive solutions for the problem (1.1). The key tool in our approach is the Avery and Peterson fixed point theorem. We give an example to illustrate our result. To the best of our knowledge, no previous results are available for triple positive solutions for the nth-order multi-point boundary value problem with the higher order derivatives and the parameter λ by using the Avery and Peterson fixed point theorem. The goal of this paper is to fill this gap.

2. Preliminary Lemmas

Definition 2.1. *The map α is said to be a nonnegative continuous concave functional on a cone K of a real Banach space E provided that $\alpha : K \to [0, \infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),\ \forall x, y \in K,\ 0 \leq t \leq 1.$$

Similarly, we say the map β is a nonnegative continuous convex functional on a cone K of a real Banach space E provided that $\beta : K \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y),\ \forall x, y \in K,\ 0 \leq t \leq 1.$$

Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. Then for positive real numbers a, b, c and d, we define the following convex sets:

$$P(\gamma, d) = \{x \in K \mid \gamma(x) < d\},$$

$$P(\gamma, \alpha, b, d) = \{x \in K \mid b \leq \alpha(x), \gamma(x) \leq d\},$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in K \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$

$$Q(\gamma, \psi, a, d) = \{x \in K \mid a \leq \psi(x), \gamma(x) \leq d\}.$$

Lemma 2.1.([1]) *Let K be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(\mu x) \leq \mu\psi(x)$ for $0 \leq \mu \leq 1$, such that for some positive numbers M and d,*

$$\alpha(x) \leq \psi(x) \text{ and } \|x\| \leq M\gamma(x).$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b and c with $a < b$ such that

(i) *$\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;*

(ii) *$\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;*

(iii) *$0 \notin Q(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in Q(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then T has at least three fixed points x_1, x_2, $x_3 \in \overline{P(\gamma, d)}$, such that

$$\gamma(x_i) \leq d \text{ for } i = 1, 2, 3, \quad b < \alpha(x_1), \quad a < \psi(x_2) \text{ with } \alpha(x_2) < b, \quad \psi(x_3) < a.$$

Lemma 2.2. *Suppose that $\Delta =: \sum_{i=1}^{m} a_i\xi_i \neq 0$, then for $y(t) \in C[0, 1]$, the boundary value problem*

$$\left\{\begin{array}{l} u^{(n)}(t)+y(t)=0, \quad t\in(0,1), \\ u(0)=u'(0)=\cdots=u^{(n-3)}(0)=u^{(n-2)}(0)=0, \quad u^{(n-2)}(1)-\sum_{i=1}^m a_i u^{(n-2)}(\xi_i)=\lambda, \end{array}\right. \tag{2.1}$$

has a unique solution

$$u(t)=\int_0^1 G_1(t,s)y(s)ds+\frac{\sum_{i=1}^m a_i t^{n-1}}{(n-1)!(1-\Delta)}\int_0^1 G_2(\xi_i,s)y(s)ds+\frac{\lambda t^{n-1}}{(n-1)!(1-\Delta)}, \tag{2.2}$$

where

$$G_1(t,s)=\frac{1}{(n-1)!}\left\{\begin{array}{ll} t^{n-1}(1-s)-(t-s)^{n-1}, & 0\le s\le t\le 1, \\ (1-s)t^{n-1}, & 0\le t\le s\le 1, \end{array}\right.$$

and

$$G_2(t,s)=\frac{\partial^{n-2}}{\partial t^{n-2}}G_1(t,s)=\left\{\begin{array}{ll} s(1-t), & 0\le s\le t\le 1, \\ t(1-s), & 0\le t\le s\le 1. \end{array}\right.$$

Proof. To prove this, we let

$$u(t)=-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)ds+At^{n-1}+\sum_{i=1}^{n-2}A_i t^i+B.$$

Since $u^{(i)}(0)=0$ for $i=0,1,2,\cdots,n-2$, we get $B=0$ and $A_i=0$ for $i=1,2,\cdots,n-2$. Now we solve for A by $u^{(n-2)}(1)-\sum_{i=1}^m a_i u^{(n-2)}(\xi_i)=\lambda$, we see that

$$-\int_0^1(1-s)y(s)ds+(n-1)!A+\sum_{i=1}^m a_i\int_0^{\xi_i}(\xi_i-s)y(s)ds-(n-1)!A\cdot\Delta=\lambda,$$

which implies

$$A=\frac{1}{(n-1)!(1-\Delta)}\left(\int_0^1(1-s)y(s)ds-\sum_{i=1}^m a_i\int_0^{\xi_i}(\xi_i-s)y(s)ds+\lambda\right).$$

Therefore, the problem (2.1) has a unique solution

$$\begin{aligned} u(t)=&-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)ds+\frac{1}{(n-1)!(1-\Delta)}\left(\int_0^1(1-s)t^{n-1}y(s)ds\right. \\ &\left.-\sum_{i=1}^m a_i\int_0^{\xi_i}(\xi_i-s)t^{n-1}y(s)ds+\lambda t^{n-1}\right) \\ =&-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)ds+\frac{1}{(n-1)!}\int_0^1(1-s)t^{n-1}y(s)ds+\frac{\Delta}{(n-1)!(1-\Delta)} \\ &\times\int_0^1(1-s)t^{n-1}y(s)ds-\frac{\sum_{i=1}^m a_i}{(n-1)!(1-\Delta)}\int_0^{\xi_i}(\xi_i-s)t^{n-1}y(s)ds+\frac{\lambda t^{n-1}}{(n-1)!(1-\Delta)} \\ =&\frac{1}{(n-1)!}\int_0^t[(1-s)t^{n-1}-(t-s)^{n-1}]y(s)ds+\frac{1}{(n-1)!}\int_t^1(1-s)t^{n-1}y(s)ds+\frac{\sum_{i=1}^m a_i t^{n-1}}{(n-1)!(1-\Delta)} \\ &\times\int_0^1\xi_i(1-s)y(s)ds-\frac{\sum_{i=1}^m a_i t^{n-1}}{(n-1)!(1-\Delta)}\int_0^{\xi_i}(\xi_i-s)y(s)ds+\frac{\lambda t^{n-1}}{(n-1)!(1-\Delta)} \end{aligned}$$

$$= \frac{1}{(n-1)!}\int_0^t [(1-s)t^{n-1} - (t-s)^{n-1}]y(s)ds + \frac{1}{(n-1)!}\int_t^1 (1-s)t^{n-1}y(s)ds$$
$$+\frac{\sum_{i=1}^m a_i t^{n-1}}{(n-1)!(1-\Delta)}\left(\int_0^{\xi_i} s(1-\xi_i)y(s)ds + \int_{\xi_i}^1 \xi_i(1-s)y(s)ds\right) + \frac{\lambda t^{n-1}}{(n-1)!(1-\Delta)}$$
$$= \int_0^1 G_1(t,s)y(s)ds + \frac{\sum_{i=1}^m a_i t^{n-1}}{(n-1)!(1-\Delta)}\int_0^1 G_2(\xi_i,s)y(s)ds + \frac{\lambda t^{n-1}}{(n-1)!(1-\Delta)}.$$

Lemma 2.3. *Let $0 < \tau < \frac{1}{2}$. $G_1(t,s)$ and $G_2(t,s)$ have the following properties*

(i) *$G_2(t,t)G_2(s,s) \le G_2(t,s) \le G_2(s,s)$, for all $(t,s) \in [0,1]\times[0,1]$, and $\max\limits_{t\in[0,1]} \int_0^1 G_2(t,s)ds = \frac{1}{8}$;*

(ii) *$G_1(t,s) \ge 0$, for all $(t,s) \in [0,1]\times[0,1]$, $G_1(t,s) \ge \tau^{n-1}G_1(1,s)$, for all $(t,s) \in [\tau, 1-\tau]\times[0,1]$.*

Proof. It is obvious that (i) holds. Next we check (ii). For all $(t,s) \in [0,1]\times[0,1]$, if $s \le t$, we have

$$\begin{aligned} G_1(t,s) &= \frac{1}{(n-1)!}[(1-s)t^{n-1} - (t-s)^{n-1}] \\ &\ge \frac{1}{(n-1)!}[(1-s)t^{n-1} - (t-ts)^{n-1}] \\ &= \frac{t^{n-1}}{(n-1)!}[(1-s) - (1-s)^{n-1}]. \end{aligned} \tag{2.3}$$

If $t \le s$, we get

$$G_1(t,s) = \frac{t^{n-1}}{(n-1)!}(1-s) \ge \frac{t^{n-1}}{(n-1)!}[(1-s) - (1-s)^{n-1}]. \tag{2.4}$$

Therefore, it follows from (2.3) and (2.4) that

$$G_1(t,s) \ge 0, \text{ for all } (t,s) \in [0,1]\times[0,1],$$

and

$$G_1(t,s) \ge \frac{\tau^{n-1}}{(n-1)!}[(1-s) - (1-s)^{n-1}] = \tau^{n-1}G_1(1,s), \text{ for all } (t,s) \in [\tau, 1-\tau]\times[0,1].$$

Lemma 2.4. *We assume that conditions* (H_1) *and* (H_2) *hold, the unique solution $u(t)$ of the BVP* (1.1) *satisfies:*

$$u^{(i)}(t) \ge 0 \ (i = 0, 1, \cdots, n-2),\ \forall t \in [0,1]. \tag{2.5}$$

Proof. Let $v(t) = u^{(n-2)}(t)$, for $0 \le t \le 1$. Thus,

$$v''(t) = u^{(n)}(t) = -f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) \le 0, \quad \text{for } 0 \le t \le 1,$$

$$v(0) = 0,\ v(1) - \textstyle\sum_{i=1}^m a_i v(\xi_i) = \lambda.$$

In the following we will show that $v(1) \geq 0$. If otherwise, $v(1) < 0$. From $v(0) = 0$ and $v(t)$ is concave downward, we obtain

$$v(t) \geq tv(1), \text{ for } 0 \leq t \leq 1.$$

Hence,

$$\lambda = v(1) - \textstyle\sum_{i=1}^{m} a_i v(\xi_i) \leq v(1) - \sum_{i=1}^{m} a_i \xi_i v(1) = (1 - \sum_{i=1}^{m} a_i \xi_i) v(1) < 0,$$

which contradicts $\lambda \in [0, \infty)$, and so $v(1) \geq 0$.

Since $v(0) = 0$, $v(1) \geq 0$ and $v(t)$ is concave downward, we have

$$v(t) = u^{(n-2)}(t) \geq 0, \text{ for } 0 \leq t \leq 1. \tag{2.6}$$

Combining (2.6) with $u^{(i)}(0) = 0$ $(i = 0, 1, \cdots, n-3)$, we obtain

$$u^{(i)}(t) \geq 0 \ (i = 0, 1, \cdots, n-3), \text{ for all } t \in [0, 1]. \tag{2.7}$$

Therefore, we have by (2.6) and (2.7) that (2.5) holds.

Lemma 2.5. *We assume that conditions* (H_1) *and* (H_2) *hold, the unique solution* $u(t)$ *of the BVP* (1.1) *satisfies:*

(i) $u^{(n-2)}(t) \geq \tau(1-\tau)|u^{(n-2)}|_0, \quad \forall t \in [\tau, 1-\tau];$

(ii) $u(t) \geq \tau^{n-1}|u|_0, \quad \forall t \in [\tau, 1-\tau],$

where $|u^{(i)}|_0 = \max\limits_{t \in [0,1]} |u^{(i)}(t)|$ $(i = 0, 1, \cdots, n-2)$, τ *as in Lemma* 2.3.

Proof. (i) From (2.2) and Lemma 2.3 (i), we have

$$\begin{aligned} u^{(n-2)}(t) &= \int_0^1 G_2(t,s) f(s, u(s), \cdots, u^{(n-2)}(s)) ds \\ &\quad + \frac{\sum_{i=1}^m a_i t}{1-\Delta} \int_0^1 G_2(\xi_i, s) f(s, u(s), \cdots, u^{(n-2)}(s)) ds + \frac{\lambda t}{1-\Delta} \\ &\leq \int_0^1 G_2(s,s) f(s, u(s), \cdots, u^{(n-2)}(s)) ds \\ &\quad + \frac{\sum_{i=1}^m a_i}{1-\Delta} \int_0^1 G_2(\xi_i, s) f(s, u(s), \cdots, u^{(n-2)}(s)) ds + \frac{\lambda}{1-\Delta}, \end{aligned}$$

which implies

$$|u^{(n-2)}|_0 \leq \int_0^1 G_2(s,s) f(s, u(s), \cdots, u^{(n-2)}(s)) ds$$

$$+\frac{\sum_{i=1}^m a_i}{1-\Delta}\int_0^1 G_2(\xi_i,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds+\frac{\lambda}{1-\Delta}. \tag{2.8}$$

On the other hand, for each $t\in[\tau,1-\tau]$, we obtain by (2.2), (2.8) and Lemma 2.3 (i) that

$$\begin{aligned}
u^{(n-2)}(t) &= \int_0^1 G_2(t,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds\\
&\quad+\frac{\sum_{i=1}^m a_i t}{1-\Delta}\int_0^1 G_2(\xi_i,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds+\frac{\lambda t}{1-\Delta}\\
&\geq \tau(1-\tau)\int_0^1 G_2(s,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds\\
&\quad+\frac{\sum_{i=1}^m a_i \tau}{1-\Delta}\int_0^1 G_2(\xi_i,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds+\frac{\lambda \tau}{1-\Delta}\\
&\geq \tau(1-\tau)\bigg(\int_0^1 G_2(s,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds\\
&\quad+\frac{\sum_{i=1}^m a_i}{1-\Delta}\int_0^1 G_2(\xi_i,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds+\frac{\lambda}{1-\Delta}\bigg)\\
&\geq \tau(1-\tau)|u^{(n-2)}|_0.
\end{aligned}$$

(ii) It follows from (2.2), (2.5) and Lemma 2.3 (ii) that

$$\begin{aligned}
|u|_0 &= \max_{t\in[0,1]}|u(t)| = u(1) = \int_0^1 G_1(1,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds\\
&\quad+\frac{\sum_{i=1}^m a_i}{(n-1)!(1-\Delta)}\int_0^1 G_2(\xi_i,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds+\frac{\lambda}{(n-1)!(1-\Delta)}.
\end{aligned} \tag{2.9}$$

Thus, for any $t\in[\tau,1-\tau]$, in view of Lemma 2.3 (ii), (2.2) and (2.9), we get

$$\begin{aligned}
u(t) &\geq \tau^{n-1}\bigg(\int_0^1 G_1(1,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds\\
&\quad+\frac{\sum_{i=1}^m a_i}{(n-1)!(1-\Delta)}\int_0^1 G_2(\xi_i,s)f(s,u(s),\cdots,u^{(n-2)}(s))ds+\frac{\lambda}{(n-1)!(1-\Delta)}\bigg)\\
&\geq \tau^{n-1}|u|_0.
\end{aligned}$$

From the above discussion, the proof is complete.

From Lemma 2.4, Lemma 2.5 (i) and the proof of lemma 2.4 in [21], we can easily check that the following Lemma holds.

Lemma 2.6. *Suppose that* (H_1) *and* (H_2) *hold, then the unique solution* $u(t)$ *of the BVP* (1.1) *satisfies*

$$u(t)\leq u'(t)\leq\cdots\leq u^{(n-3)}(t)\leq |u^{(n-2)}|_0,\ \forall t\in[0,1],$$

and

$$u^{(n-3)}(t) \le \tfrac{1}{\tau(1-\tau)} u^{(n-2)}(t),\ \forall t \in [\tau, 1-\tau],$$

where τ is as in Lemma 2.3.

3. Main results

Let $C^{n-2}[0,1]$ be endowed with the norm $\|u\| = \max\{|u|_0, |u'|_0, \cdots, |u^{(n-2)}|_0\}$,

where $|u^{(i)}|_0 = \max_{t\in[0,1]} |u^{(i)}(t)|$ $(i = 0, 1, \cdots, n-2)$. Denote

$E = \{u \in C^{n-2}[0,1],\ u^{(i)}(t) \ge 0\ (i = 0, 1, \cdots, n-2)$ and $u^{(i)}(t) \le u^{(i+1)}(t) \le |u^{(n-2)}|_0\ (i = 0, 1, \cdots, n-4),\ \forall t \in [0,1]\}$,

$P = \{u \in E : u(t) \ge \tau^{n-1}|u|_0$ and $u^{(n-3)}(t) \le \tfrac{1}{\tau(1-\tau)} u^{(n-2)}(t),\ \forall t \in [\tau, 1-\tau]\}$.

It is obvious that P is a cone in $C^{(n-2)}[0,1]$.

We define the operator T on P by

$$\begin{aligned} Tu(t) = & \int_0^1 G_1(t,s) f(s, u(s), u'(s), \ldots, u^{(n-2)}(s))ds \\ & + \frac{\sum_{i=1}^m a_i t^{n-1}}{(n-1)!(1-\Delta)} \int_0^1 G_2(\xi_i, s) f(s, u(s), u'(s), \ldots, u^{(n-2)}(s))ds + \frac{\lambda t^{n-1}}{(n-1)!(1-\Delta)}, \end{aligned}$$

where $G_1(t,s)$ and $G_2(t,s)$ are given in Lemma 2.2. It is easy to see that the BVP (1.1) has a solution $u(t)$ if and only if $u(t)$ is a fixed point of the operator T.

In order to obtain the results, we define the nonnegative continuous functional α, the nonnegative continuous convex functional θ, γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\gamma(u) = \max_{t\in[0,1]} |u^{(n-2)}(t)|, \quad \psi(u) = \theta(u) = \max_{t\in[0,1]} |u(t)|, \quad \alpha(u) = \min_{t\in[\tau,1-\tau]} |u(t)|,$$

where τ as in Lemma 2.3. We observe here that, for all $u \in P$,

$$\tau^{n-1}\theta(u) \le \alpha(u) \le \theta(u) = \psi(u),\ \|u\| = \max\{|u|_0, |u'|_0, \cdots, |u^{(n-2)}|_0\} = \gamma(u). \tag{3.1}$$

We use the following notations. Let

$$M = \frac{1}{4} + \frac{2\sum_{i=1}^{i=m} a_i}{1-\Delta} \int_0^1 G_2(\xi_i, s)ds,$$

$$N = 2\int_0^1 G_1(1,s)ds + \frac{2\sum_{i=1}^{i=m} a_i}{(n-1)!(1-\Delta)} \int_0^1 G_2(\xi_i, s)ds,$$

$$R=\int_0^1 G_1(\tau,s)ds+\frac{\sum_{i=1}^{i=m}a_i\tau^{n-1}}{(n-1)!(1-\Delta)}\int_0^1 G_2(\xi_i,s)ds.$$

To present our main results, we assume there exist constants $0<a<b<\tau^{n-1}d$ and the following assumptions hold.

(H$_3$) $f(t,u,u',\cdots,u^{(n-2)})\le \frac{d}{M}$, for $(t,u,u',\cdots,u^{(n-2)})\in[0,1]\times[0,d]^{n-1}$;

(H$_4$) $f(t,u,u',\cdots,u^{(n-2)})>\frac{b}{R}$, for $(t,u,u',\cdots,u^{(n-2)})\in[\tau,1-\tau]\times[b,\frac{b}{\tau^{n-1}}]\times[b,d]^{n-3}\times[\tau(1-\tau)b,d]$;

(H$_5$) $f(t,u,u',\cdots,u^{(n-2)})<\frac{a}{N}$, for $(t,u,u',\cdots,u^{(n-2)})\in[0,1]\times[0,a]\times[0,d]^{n-2}$.

Theorem 3.1. *Assume that* (H$_1$) $-$ (H$_5$) *hold, in addition, suppose* λ *satisfy*

$$0\le\lambda\le\min\{d,(n-1)!a\}\frac{1-\Delta}{2}. \tag{3.2}$$

Then the BVP (1.1) *has at least two positive solutions* u_1, u_2 *and one nonnegative solution* u_3 *such that*

$$\max_{t\in[0,1]}|u_i^{(n-2)}(t)|\le d,\quad \text{for } i=1,2,3;\qquad \min_{t\in[\tau,1-\tau]}|u_1(t)|>b;$$

$$a<\max_{t\in[0,1]}|u_2(t)|,\ \text{with}\ \min_{t\in[\tau,1-\tau]}|u_2(t)|<b;\quad \max_{t\in[0,1]}|u_3(t)|<a.$$

Proof. First we show $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$ is a completely continuous operator.

If $u\in P$, then from Lemma 2.4 and Lemma 2.6, $(Tu)^{(i)}(t)\ge 0$ $(i=0,1,\cdots,n-2)$, $(Tu)^{(i)}(t)\le (Tu)^{(i+1)}(t)\le |(Tu)^{(n-2)}|_0$ $(i=0,1,\cdots,n-4)$, $\forall t\in[0,1]$ and $(Tu)^{(n-3)}(t)\le\frac{1}{\tau(1-\tau)}(Tu)^{(n-2)}(t)$, $\forall t\in[\tau,1-\tau]$, and by Lemma 2.5 (ii), $Tu(t)\ge\tau^{n-1}|Tu|_0$, $\forall t\in[\tau,1-\tau]$. This shows that $T:P\to P$. It can be shown that $T:P\to P$ is completely continuous by the Arzela-Ascoli theorem.

If $u\in\overline{P(\gamma,d)}$, then $\gamma(u)=\max_{t\in[0,1]}|u^{(n-2)}(t)|\le d$, and so $0\le u^{(i)}(t)\le d$ $(i=0,1,\cdots,n-2)$, for all $t\in[0,1]$, then assumption (H$_3$) implies $f(t,u,u',\cdots,u^{(n-2)})\le\frac{d}{M}$. Then, it follows by Lemma 2.3 (i) and (3.2) that

$$\begin{aligned}
\gamma(Tu)&=\max_{t\in[0,1]}|(Tu)^{(n-2)}(t)|=\max_{t\in[0,1]}(Tu)^{(n-2)}(t)\\
&=\max_{t\in[0,1]}\left\{\int_0^1 G_2(t,s)f(s,u(s),u'(s),\ldots,u^{(n-2)}(s))ds\right.\\
&\quad\left.+\frac{\sum_{i=1}^m a_it}{1-\Delta}\int_0^1 G_2(\xi_i,s)f(s,u(s),u'(s),\ldots,u^{(n-2)}(s))ds+\frac{\lambda t}{1-\Delta}\right\}\\
&\le\frac{d}{M}\cdot\left(\max_{t\in[0,1]}\int_0^1 G_2(t,s)ds+\frac{\sum_{i=1}^m a_i}{1-\Delta}\int_0^1 G_2(\xi_i,s)ds\right)+\frac{\lambda}{1-\Delta}\\
&\le\frac{d}{M}\cdot\left(\frac{1}{8}+\frac{\sum_{i=1}^m a_i}{1-\Delta}\int_0^1 G_2(\xi_i,s)ds\right)+\frac{d}{2}
\end{aligned}$$

$$= \frac{d}{2} + \frac{d}{2} = d.$$

Therefore, $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

Next, we show all the conditions of Lemma 2.1 are satisfied.

To check condition (i) of Lemma 2.1, we take $u(t) = \frac{b}{\tau^{n-1}}$, for $t \in [0,1]$. It is easy to see that $u(t) = \frac{b}{\tau^{n-1}} \in P(\gamma, \theta, \alpha, b, \frac{b}{\tau^{n-1}}, d)$ and $\alpha(u) = \alpha(\frac{b}{\tau^{n-1}}) > b$, and so $\{u \in P(\gamma, \theta, \alpha, b, \frac{b}{\tau^{n-1}}, d) | \alpha(u) > b\} \neq \emptyset$. For $u \in P(\gamma, \theta, \alpha, b, \frac{b}{\tau^{n-1}}, d)$, we have

$$b \le u(t) \le \tfrac{b}{\tau^{n-1}}, \text{ for } t \in [\tau, 1-\tau], \text{ and } \max_{t \in [0,1]} |u^{(n-2)}(t)| \le d,$$

then,

$$b \le u^{(i)}(t) \le d \ (i = 1, \cdots, n-3), \text{ for } t \in [\tau, 1-\tau],$$

and so

$$\tau(1-\tau)b \le \tau(1-\tau)u^{(n-3)}(t) \le u^{(n-2)}(t) \le d, \text{ for } t \in [\tau, 1-\tau].$$

Thus, assumption (H_4) implies $f(t, u, u', \cdots, u^{(n-2)}) > \frac{b}{R}$, and by the definitions of α and the cone P, we have

$$\begin{aligned}
\alpha(Tu) &= \min_{t \in [\tau, 1-\tau]} |Tu(t)| = Tu(\tau) = \int_0^1 G_1(\tau, s) f(s, u(s), u'(s), \ldots, u^{(n-2)}(s)) ds \\
&\quad + \frac{\sum_{i=1}^m a_i \tau^{n-1}}{(n-1)!(1-\Delta)} \int_0^1 G_2(\xi_i, s) f(s, u(s), u'(s), \ldots, u^{(n-2)}(s)) ds + \frac{\lambda \tau^{n-1}}{(n-1)!(1-\Delta)} \\
&> \frac{b}{R} \cdot \left(\int_0^1 G_1(\tau, s) ds + \frac{\sum_{i=1}^m a_i \tau^{n-1}}{(n-1)!(1-\Delta)} \int_0^1 G_2(\xi_i, s) ds \right) = b.
\end{aligned}$$

So, condition (i) of Lemma 2.1 is satisfied.

Secondly, we show (ii) of Lemma 2.1 is satisfied. From (3.1) and $b \le \tau^{n-1} d$, we get

$$\alpha(Tu) \ge \tau^{n-1} \theta(Tu) > \tau^{n-1} \tfrac{b}{\tau^{n-1}} = b, \text{ for all } u \in P(\gamma, \alpha, a, d) \text{ with } \theta(Tu) > \tfrac{b}{\tau^{n-1}}.$$

Finally, we show condition (iii) of Lemma 2.1 is also satisfied. Obviously, as $\psi(0) = 0 < a$, there holds $0 \notin Q(\alpha, \psi, a, d)$. Suppose $u \in Q(\alpha, \psi, a, d)$ with $\psi(u) = a$. Then we have by (3.2), the assumption (H_5), the definitions of ψ and the cone P that

$$\begin{aligned}
\psi(Tu) &= \max_{t \in [0,1]} |Tu(t)| = Tu(1) \\
&= \int_0^1 G_1(1, s) f(s, u(s), u'(s), \ldots, u^{(n-2)}(s)) ds \\
&\quad + \frac{\sum_{i=1}^m a_i}{(n-1)!(1-\Delta)} \int_0^1 G_2(\xi_i, s) f(s, u(s), u'(s), \ldots, u^{(n-2)}(s)) ds + \frac{\lambda}{(n-1)!(1-\Delta)}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{a}{N} \cdot \left(\int_0^1 G_1(1,s)ds + \frac{\sum_{i=1}^m a_i}{(n-1)!(1-\Delta)} \int_0^1 G_2(\xi_i, s)ds \right) + \frac{a}{2} \\
&= \frac{a}{2} + \frac{a}{2} = a.
\end{aligned}
$$

This shows condition (iii) of Lemma 2.1 is also satisfied. Therefore, the hypotheses of Lemma 2.1 are satisfied and there exist two positive solutions u_1, u_2 and one nonnegative solution u_3 for the BVP (1.1) such that

$$
\begin{aligned}
&\max_{t\in[0,1]} |u_i^{(n-2)}(t)| \le d, \quad \text{for } i = 1,2,3; \qquad \min_{t\in[\tau,1-\tau]} |u_1(t)| > b; \\
&a < \max_{t\in[0,1]} |u_2(t)|, \text{ with } \min_{t\in[\tau,1-\tau]} |u_2(t)| < b; \quad \max_{t\in[0,1]} |u_3(t)| < a.
\end{aligned}
$$

Remark 3.2. By Theorem 3.1, there are three non-negative solutions, two positive and a third u_3 which may be zero.

4. Example

In this section, in order to illustrate our main result, we consider an example.

Example 4.1. Consider the boundary value problem

$$
\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \quad u'(1) - \frac{1}{4}u'(\frac{1}{3}) - \frac{3}{4}u'(\frac{2}{3}) = \lambda, \end{cases} \tag{4.1}
$$

where

$$
f(t,u,v) = \begin{cases} \sin \pi t + u^6 + \dfrac{\sqrt{v}}{30}, & 0 \le t \le 1, 0 \le u < 2, v \ge 0, \\ \sin \pi t + 64 + \dfrac{15}{2}\sqrt{u-2} + \dfrac{\sqrt{v}}{30}, & 0 \le t \le 1, 2 \le u < 18, v \ge 0, \\ \sin \pi t + 94 + \sqrt{u-18} + \dfrac{\sqrt{v}}{30}, & 0 \le t \le 1, u \ge 18, v \ge 0. \end{cases}
$$

To show the problem (4.1) has at least two positive solutions and one nonnegative solution, we apply Theorem 3.1 with $n = 3$, $m = 2$, $a_1 = \frac{1}{4}$, $a_2 = \frac{3}{4}$, $\xi_1 = \frac{1}{3}$ and $\xi_2 = \frac{2}{3}$. Clearly (H_1) and (H_2) are satisfied. We take $\tau = \frac{1}{3}$. After some simple calculations, we get $M = \frac{47}{60}$, $N = \frac{13}{30}$ and $R = \frac{59}{1620}$. If we take $a = 1$, $b = 2$ and $d = 81$, we obtain

$$
\begin{aligned}
&f(t,u,v) \le 103.237 < 103.404 = \frac{d}{M}, \quad \text{for } 0 \le t \le 1, 0 \le u \le 81, 0 \le v \le 81, \\
&f(t,u,v) \ge 64.888 > 54.915 = \frac{b}{R}, \quad \text{for } \tfrac{1}{3} \le t \le \tfrac{2}{3}, 2 \le u \le 18, \tfrac{4}{9} \le v \le 81, \\
&f(t,u,v) \le 2.3 < 2.308 = \frac{a}{N}, \quad \text{for } 0 \le t \le 1, 0 \le u \le 1, 0 \le v \le 81.
\end{aligned}
$$

Thus, for $0 \le \lambda \le \min\{d, (n-1)!a\}\frac{1-\Delta}{2} = \frac{1}{2}$, by Theorem 3.1, the problem (4.1) has at least two positive solutions u_1, u_2 and one nonnegative solution u_3 such that

$$\max_{t\in[0,1]} |u_i^{(n-2)}(t)| \leq 81, \quad \text{for } i=1,2,3; \quad \min_{t\in[\frac{1}{3},\frac{2}{3}]} |u_1(t)| > 2;$$

$$1 < \max_{t\in[0,1]} |u_2(t)|, \text{ with } \min_{t\in[\frac{1}{3},\frac{2}{3}]} |u_2(t)| < 2; \quad \max_{t\in[0,1]} |u_3(t)| < 1.$$

Since 0 is not a solution for any $\lambda \in [0, 1/2]$, it follows that u_3 is also a positive solution.

Acknowledgements

The authors are very grateful to the referees for careful reading of the original manuscript and for valuable suggestions on improving this paper. Project supported by the National Natural Science Foundation of China (10771212) and the Natural Science Foundation of Jiangsu Education Office (06KJB110010).

References

[1] R. Avery, A.C. Peterson, Three positive fixed points of nonlinear operators on ordered Banach spaces, Comput. Math. Appl. 42 (2001), 313-322.

[2] P.W. Eloe, B. Ahmad, Positive solutions of a nonlinear nth boundary value problems with nonlocal conditions, Appl. Math. Lett. 18 (2005), 521-527.

[3] P.W. Eloe, J. Henderson, Positive solutions for higher order ordinary differential equations, Electron. J. Diff. Equ. 1995 (1995), no. 03, 1-8.

[4] J.R. Graef, J. Henderson and B. Yang, Positive solutions of a nonlinear higher order boundary value problems, Electron. J. Diff. Equ. 2007 (2007), no. 45, 1-10.

[5] J.R. Graef, L.J. Kong, B. Yang, Existence of solutions for a higher order multi-point boundary value problem, Results Math. 53 (2009), no. 1-2, 77-101.

[6] J.R. Graef, B. Yang, Positive solutions to a multi-point higher order boundary-vale problem, J. Math. Anal. Appl. 316 (2006), 409-421.

[7] Y.P. Guo, W.G. Ge and Y. Gao, Twin positive solutions for higher order m-point boundary value problems with sign changing nonlinearities, Appl. Math. Comput. 146 (2003), 299-311.

[8] Y.P. Guo, Y.D. Ji and J.H. Zhang, Three positive solutions for a nonlinear nth order m-point boundary value problems, Nonlinear Anal. 68 (2008), 3485-3492.

[9] Y.P. Guo, J.W. Tian, Positive solutions of m-point boundary value problems for higher order ordinary differential equations, Nonlinear Anal. 66 (2007), 1573-1586.

[10] L.J. Kong, Q.K. Kong, Higher order boundary value problems with nonhomogeneous boundary conditions, Nonlinear Anal. 72 (2010), no. 1, 240-261.

[11] L.J. Kong, Q.K. Kong, Second-order boundary value problems with nonhomogeneous boundary conditions. II, J. Math. Anal. Appl. 330 (2007), no. 2, 1393-1411.

[12] L.J. Kong, Q.K. Kong, Second-order boundary value problems with nonhomogeneous boundary conditions. I, Math. Nachr. 278 (2005), no. 1-2, 173-193.

[13] L.J. Kong, J.S.W. Wong, Positive solutions for multi-point boundary value problems with nonhomogeneous boundary conditions, J. Math. Anal. Appl. 367 (2010), 588-611.

[14] L.S. Lib, X.G. Zhang and Y.H. Wu, Existence positive solutions for singular higher-order differential equations, Nonlinear Anal. 68 (2008), 3948-3961.

[15] C.C. Pang, W. Dong and Z.L. Wei, Green's function and positive solutions of nth order m-point boundary value problem, Appl. Math. Comput. 182 (2006), 1231-1239.

[16] H. Su, B. Wang, Z. Wei and X. Zhang, Positive solutions of four-point boundary value problems for higher-order p-Laplacian operator, J. Math. Anal. Appl. 330 (2007), no. 2, 836-851.

[17] Y.P. Sun, Positive solutions for third-order there-point nonhomogeneous boundary value problems, Appl. Math. Lett. 22 (2009), 45-51.

[18] J.R.L. Webb, Positive solutions of some higher order nonlocal boundary value problems, Electron. J. Qual. Theory Differ. Equ. 2009, Special Edition I, No. 29, 1-15.

[19] J.R.L. Webb, Infante, Gennaro, Non-local boundary value problems of arbitrary order, J. Lond. Math. Soc. 79 (2009), no. 1, 238-258.

[20] J.R.L. Webb, Nonlocal conjugate type boundary value problems of higher order, Nonlinear Anal. 71 (2009), 1933-1940.

[21] Y.M. Zhou, H. Su, Positive solutions of four-point boundary value problems for higher-order with p-laplacian operator, Electron. J. Diff. Equ. 2007 (2007), no. 05, 1-14.

[22] X.M. Zhang, M.Q. Feng and W.G. Ge, Existence and nonexistence of positive solutions for a class of nth-order three-point boundary value problems in Banach spaces, Nonlinear Anal. 70 (2009), 584-597.

(Received April 8, 2010)