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Homoclinic solutions for a class of non-periodic second order Hamiltonian systems [1]

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Abstract: We study the existence of homoclinic solutions for the second order Hamiltonian system $\ddot{u}+V_u(t,u)=f(t)$. Let $V(t,u)=-K(t,u)+W(t,u)\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ be T-periodic in t, where K is a quadratic growth function and W may be asymptotically quadratic or super-quadratic at infinity. One homoclinic solution is obtained as a limit of solutions of a sequence of periodic second order differential equations.

Key words: Hamiltonian systems; Homoclinic solutions; Condition(C); Asymptotically quadratic; Super-quadratic.

AMS Subject Classification (2000): 37K05 .

1 Introduction and the main result

In this paper, we consider the existence of homoclinic solutions for the second order Hamiltonian system:

$$\ddot{u}(t)+V_u(t,u(t))=f(t), \tag{1.1}$$

where $f\in L^2(\mathbb{R},\mathbb{R}^n)$ is continuous and bounded, $V(t,u)=-K(t,u)+W(t,u)\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ is T-periodic in $t, T>0$.

Let us recall that a solution $u(t)$ of (1.1) is homoclinic to 0 if $u(t)\not\equiv 0, u(t)\to 0$ and $\dot{u}(t)\to 0$ as $t\to\pm\infty$.

In recent years, the existence of homoclinic solutions for (1.1) has been studied extensively by variational methods(see, for instance, [6,7,11-13,15]). Most of them considered (1.1) with W satisfying the Ambrosetti-Rabinowitz condition, that is, there exists $\mu>2$ such that

$$0<\mu W(t,u)\le (W_u(t,u),u) \text{ for all } t\in\mathbb{R} \text{ and } u\in\mathbb{R}^n\setminus\{0\}.$$

It is well known that the major difficulty is to check the Palais-Smale (PS) condition (a compactness condition) when one considers (1.1) on the whole space $\mathbb{R}$ via variational methods. Recall that a sequence $\{u_n\}$ is said to be a (PS) sequence of φ provided that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n)\to 0$. φ satisfies the (PS) condition if any (PS) sequence possesses a convergent subsequence. Using the Ambrosetti-Rabinowitz condition, one can easily establish the boundedness of (PS) sequences, which is crucial to check the (PS) condition. Later some authors managed to weaken this condition (see, e.g., [4] and [10]). Many authors also treated some new growth conditions. For example, [1,9] considered the sub-quadratic case and [2, 14] dealt with the asymptotically quadratic case.

If $L(t)$ and $W(t,u)$ are either independent of t or periodic in t, the problem seems a little simple and there are many results. In [8,10], the authors considered

$$\ddot{u}(t)+V_u(t,u(t))=0$$

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with $V(t,u) = -\frac{1}{2}(Lu,u) + W(u)$ being independent of t, i.e., the system is autonomous. They obtained one homoclinic solution as a limit of solutions of a certain sequence of periodic systems. By this method, [5] considered the case that $L(t)$ and $W(t,u)$ are periodic in t. It also assumed that $L(t)$ is positive definite and symmetric, $W(t,u)$ satisfies the Ambrosetti-Rabinowitz growth condition. Without periodicity condition, the problem is quite different from the ones just described for the lack of compactness of the Sobolev embedding. Using the method in [5, 8, 10], [3] considered (1.1). The authors replaced $\frac{1}{2}(L(t)u,u)$ by $K(t,u)$ which satisfies the following condition

$$\exists b_1, b_2 > 0 \text{ such that for all } (t,u) \in \mathbb{R}\times\mathbb{R}^n,\ b_1|u|^2 \le K(t,u) \le b_2|u|^2$$

and

$$K(t,u) \le (u, K_u(t,u)) \le 2K(t,u).$$

Given W satisfied the Ambrosetti-Rabinowitz condition, they obtained one homoclinic solution.

We note that the nonlinearity W in all the papers mentioned above are nonnegative, which ensures the variational functional possesses some good properties. If the nonlinearity is allowed to be negative, new difficulties arise and there haven't been too many results. In this paper, we consider this case. we study homoclinic solutions of (1.1) on the whole space $\mathbb{R}$. (1.1) is not periodic in nature. We will consider some new nonlinearity W. Precisely, W is allowed to be negative near the origin and satisfies asymptotically quadratic or super-quadratic growth condition at infinity. It is obvious that W doesn't satisfy the Ambrosetti-Rabinowitz condition. We will approximate a solution of (1.1) by a limit of solutions of a sequence of periodic systems.

We make the following assumptions:
$(\mathbf{A_1})$ $V(t,u) = -K(t,u) + W(t,u)$ is a C^1 function on $\mathbb{R}\times\mathbb{R}^n$, T-periodic in $t, T > 0$, and $V_u(t,u) \to 0$ as $|u| \to 0$ uniformly in $t \in \mathbb{R}$;
$(\mathbf{A_2})$ For all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$, $0 \le (u, K_u(t,u)) \le 2K(t,u)$, and there exist $0 < \underline{b} \le \overline{b}$ such that $\underline{b}|u|^2 \le K(t,u) \le \overline{b}|u|^2$;
$(\mathbf{A_3})$ There exist constants $d_1 > 0, r \ge 2$ such that $W(t,u) \le d_1|u|^r$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$;
$(\mathbf{A_4})$ There exist constants $d_2 > 0$, μ with $r \ge \mu > r-1$ and $\beta \in L^1(\mathbb{R}, \mathbb{R}^+)$ such that

$$(W_u(t,u), u) - 2W(t,u) \ge d_2|u|^\mu - \beta(t)$$

for all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$;
$(\mathbf{A_5})$ There exist positive constants $R, \rho_0 > 0$ such that $\frac{W(t,u)}{|u|^2} \ge \frac{2\pi^2}{T^2}$ as $|u| > R$, $t \in [-T,T]$ and $W(t,u) \le 0$ as $|u| \le \rho_0$, $t \in [-T,T]$.

For each $k \in \mathbb{N}$, let L^2_{2kT} be the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ equipped with the norm

$$\|u\|_{L^2_{2kT}} = \Big(\int_{-kT}^{kT} |u(t)|^2 dt\Big)^{\frac{1}{2}},$$

and L^∞_{2kT} be the space of $2kT$-periodic essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_{L^\infty_{2kT}} := ess\sup\{|u(t)| : t \in [-kT, kT]\}.$$

Denote $E_k := W^{1,2}_{2kT}$ be the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|u\|_{E_k} := \Big(\int_{-kT}^{kT} (|\dot{u}(t)|^2 + |u(t)|^2)\,\mathrm{d}t\Big)^{\frac{1}{2}}.$$

By Rabinowitz in [5], we have

Proposition 1 *There is a constant $C_1 > 0$ such that for each $k \in \mathbb{N}$ and $u \in E_k$ the following inequality holds:*

$$\|u\|_{L^\infty_{2kT}} \le C_1 \|u\|_{E_k}.$$

We also make the following assumption:

($\mathbf{A_6}$) f is nonzero continuous and bounded, there is a constant C_0 such that $\|f\|_{L^2(\mathbb{R},\mathbb{R}^n)} = \Big(\int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t\Big)^{\frac{1}{2}} \le C_0 := \min\{\frac{1}{2}, \underline{b}\}\frac{\rho_0}{2C_1}$.

Our main result reads as follows.

Theorem 1 *If assumptions $(A_1) - (A_6)$ are satisfied, then (1.1) possesses at least one homoclinic solution $u \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$.*

Remark 1.1 *(A_5) shows that W may be either asymptotically quadratic or super-quadratic growth at infinity.*

Remark 1.2 *There are functions which satisfy assumptions $(A_1) - (A_5)$. For example,*

$$K(t,x) = \begin{cases} (1 + \frac{1}{1+x^2})x^2 & x \ge 0, \\ (1 + \frac{2}{1+x^2})x^2 & x < 0 \end{cases}$$

and

$$W(t,x) = -2x^2 + x^4 \quad (\textit{the super} - \textit{quadratic case})$$

or

$$W(t,x) = x^2 - 2x^{\frac{4}{3}} \qquad (\textit{the asymptotically quadratic case})$$

Let $f_k : \mathbb{R} \to \mathbb{R}^n$ be a $2kT$-periodic extension of the restriction of f to the interval $[-kT, kT]$ and u_k be a $2kT$-periodic solution of

$$\ddot{u}(t) + V_u(t, u(t)) = f_k(t) \tag{1.2}$$

obtained by the Mountain Pass Theorem. We will show that the sequence $\{u_k\}$ possesses a subsequence which converges to a homoclinic solution of (1.1).

The main difficulties in treating (1.1) are caused by the fact that in order to get appropriate convergence of the sequence of approximative functions $\{u_k\}$ we need the sequence $\{\|u_k\|_{E_k}\}$ to be bounded uniformly in $k \in \mathbb{N}$ and the constants ρ and α appearing in the condition (3) of the Mountain Pass Theorem (see Theorem 2) to be independent of k.

2 Proof of the main result

For each $k \in \mathbb{N}$, we consider the second order system (1.2) on E_k.

Define

$$\varphi_k(u) = \int_{-kT}^{kT} \Big[\frac{1}{2}|\dot{u}(t)|^2 - V(t,u(t)) + (f_k(t),u(t))\Big]dt.$$

It is clear that $\varphi_k \in C^1(E_k,\mathbb{R})$ and

$$\varphi_k'(u)v = \int_{-kT}^{kT} \Big[(\dot{u}(t),\dot{v}(t)) - (V_u(t,u(t)),v(t)) + (f_k(t),v(t))\Big]dt.$$

We all know that critical points of φ_k are classical $2kT$-periodic solutions of (1.2).

Lemma 2.1 *Under $(A_1)-(A_6)$, for each $k \in \mathbb{N}$, (1.2) possesses a $2kT$-periodic solution.*

We will prove this lemma via the Mountain Pass Theorem by Rabinowitz in [14]. We state this theorem as follows.

Theorem 2 *Let E be a real Banach space and $\varphi : E \to \mathbb{R}$ be a C^1 function. If φ satisfies the following conditions:*
(1) $\varphi(0) = 0$;
(2) *φ satisfies the (PS) condition on E;*
(3) *There exist constants $\rho, \alpha > 0$ such that $\varphi|_{S_\rho} \geq \alpha$;*
(4) *There exists $e \in E\backslash B_\rho$ such that $\varphi(e) \leq 0$,*
where B_ρ is a closed ball in E of radius ρ centred at 0 and S_ρ is the boundary of B_ρ. Then φ possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi(g(s)),$$

where

$$\Gamma = \{g \in C([0,1],E) : g(0) = 0, g(1) = e\}.$$

Instead of the (PS) condition, we use condition (C). Recall a function φ satisfies condition (C) on E if any sequence $\{u_j\} \subset E$ such that $\{\varphi(u_j)\}$ is bounded and $(1 + \|u_j\|)\|\varphi'(u_j)\| \to 0$ has a convergent subsequence. The Mountain Pass Theorem still holds true under condition (C).

Proof of Lemma 2.1 From our assumptions, it is easy to see that $\varphi_k(0) = 0$.
Step 1. φ_k satisfies condition (C).

Suppose $\{u_j\} \subset E_k$, $\{\varphi_k(u_j)\}$ is bounded and $(1+\|u_j\|_{E_k})\|\varphi_k'(u_j)\| \to 0$ as $j \to \infty$. Then there is a constant $M_k > 0$ such that

$$\varphi_k(u_j) \leq M_k, \quad (1 + \|u_j\|_{E_k})\|\varphi_k'(u_j)\| \leq M_k \tag{2.3}$$

for all $j \in \mathbb{N}$.

By (2.3), (A_2), and (A_4),

$$\begin{aligned} 3M_k &\geq 2\varphi_k(u_j) - \varphi_k'(u_j)u_j \\ &= \int_{-kT}^{kT} \Big[(V_u(t,u_j(t)),u_j(t)) - 2V(t,u_j(t))\Big]\,\mathrm{d}t + \int_{-kT}^{kT} (f_k(t),u_j(t))\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \int_{-kT}^{kT} \Big[2K(t,u_j(t)) - \Big(K_u(t,u_j(t)),u_j(t)\Big)\Big]\,\mathrm{d}t \\
&\quad + \int_{-kT}^{kT} \Big[\Big(W_u(t,u_j(t)),u_j(t)\Big) - 2W(t,u_j(t))\Big]\,\mathrm{d}t + \int_{-kT}^{kT} (f_k(t),u_j(t))\,\mathrm{d}t \\
&\geq d_2 \int_{-kT}^{kT} |u_j(t)|^{\mu}\,\mathrm{d}t - \int_{-kT}^{kT} \beta(t)\,\mathrm{d}t - \|f_k\|_{L^2_{2kT}}\|u_j\|_{E_k} \\
&\geq d_2 \int_{-kT}^{kT} |u_j(t)|^{\mu}\,\mathrm{d}t - \beta_0 - C_0\|u_j\|_{E_k},
\end{aligned}
$$

where $\beta_0 = \int_{-kT}^{kT} \beta(t)\,\mathrm{d}t$.

Therefore,

$$
\int_{-kT}^{kT} |u_j(t)|^{\mu}\,\mathrm{d}t \leq \frac{1}{d_2}(3M_k + \beta_0 + C_0\|u_j\|_{E_k}). \tag{2.4}
$$

By (A_2) and (A_3),

$$
\begin{aligned}
\frac{1}{2}\|\dot{u}_j\|^2_{L^2_{2kT}} &= \varphi_k(u_j) - \int_{-kT}^{kT} K(t,u_j(t))\,\mathrm{d}t + \int_{-kT}^{kT} W(t,u_j(t))\,\mathrm{d}t \\
&\quad - \int_{-kT}^{kT} (f_k(t),u_j(t))\,\mathrm{d}t \\
&\leq M_k - \underline{b}\int_{-kT}^{kT} |u_j(t)|^2\,\mathrm{d}t + d_1 \int_{-kT}^{kT} |u_j(t)|^r\,\mathrm{d}t \\
&\quad + C_0\|u_j\|_{E_k}.
\end{aligned}
$$

Then one has

$$
\frac{1}{2}\|\dot{u}_j\|^2_{L^2_{2kT}} + \underline{b}\|u_j\|^2_{L^2_{2kT}} \leq M_k + d_1 \int_{-kT}^{kT} |u_j(t)|^r\,\mathrm{d}t + C_0\|u_j\|_{E_k}.
$$

Therefore, by (2.4),

$$
\begin{aligned}
\min\{\frac{1}{2},\underline{b}\}\|u_j\|^2_{E_k} &\leq M_k + d_1 \int_{-kT}^{kT} |u_j(t)|^r\,\mathrm{d}t + C_0\|u_j\|_{E_k} \\
&\leq M_k + d_1\|u_j\|^{r-\mu}_{L^\infty_{2kT}} \int_{-kT}^{kT} |u_j(t)|^{\mu}\,\mathrm{d}t + C_0\|u_j\|_{E_k} \\
&\leq M_k + d_1 C_1^{r-\mu}\|u_j\|^{r-\mu}_{E_k} \int_{-kT}^{kT} |u_j(t)|^{\mu}\,\mathrm{d}t + C_0\|u_j\|_{E_k} \\
&\leq M_k + \frac{d_1}{d_2} C_1^{r-\mu}\|u_j\|^{r-\mu}_{E_k}(3M_k + \beta_0 + C_0\|u_j\|_{E_k}) + C_0\|u_j\|_{E_k}.
\end{aligned}
$$

Since $r - \mu < 1$, we get $\{\|u_j\|_{E_k}\}$ is bounded. Going if necessary to a subsequence, we can assume that there exists $u \in E_k$ such that $u_j \rightharpoonup u$ in E_k as $j \to +\infty$, which implies $u_j \to u$ uniformly on $[-kT, kT]$.

Therefore

$$
(\varphi_k'(u_j) - \varphi_k'(u))(u_j - u) \to 0,
$$

$$
\|u_j - u\|_{L^2_{2kT}} \to 0
$$

and

$$
\int_{-kT}^{kT} \Big(V_u(t,u_j(t)) - V_u(t,u(t)), u_j(t) - u(t)\Big)\,\mathrm{d}t \to 0
$$

as $j \to +\infty$.

By an easy computation, we can see that

$$(\varphi_k'(u_j) - \varphi_k'(u))(u_j - u) = \|\dot{u}_j - \dot{u}\|^2_{L^2_{2kT}} - \int_{-kT}^{kT} \Big(V_u(t, u_j(t)) - V_u(t, u(t)), u_j(t) - u(t)\Big)\,\mathrm{d}t.$$

Hence we have $\|\dot{u}_j - \dot{u}\|^2_{L^2_{2kT}} \to 0$, and so $u_j \to u$ in E_k.

Step 2. There are constants $\rho > 0, \alpha > 0$ independent of k, such that $\varphi_k|_{S_\rho} \geq \alpha$, where $S_\rho = \{u \in E_k | \|u\|_{E_k} = \rho\}$.

Choose $\rho = \frac{\rho_0}{C_1}$, then for $u \in S_\rho$ we have $\|u\|_{L^\infty_{2kT}} \leq \rho_0$. Therefore, $|u| \leq \rho_0$ for all $t \in [-kT, kT]$, and then by (A_5), $W(t, u) \leq 0$. Together with (A_2), we obtain

$$\begin{aligned}
\varphi_k(u) &= \int_{-kT}^{kT} \Big[\frac{1}{2}|\dot{u}(t)|^2 + K(t, u(t)) - W(t, u(t))\Big]\,\mathrm{d}t + \int_{-kT}^{kT} (f_k(t), u(t))\,\mathrm{d}t \\
&\geq \frac{1}{2}\int_{-kT}^{kT} |\dot{u}(t)|^2\,\mathrm{d}t + \underline{b}\int_{-kT}^{kT} |u(t)|^2\,\mathrm{d}t - \|f_k\|_{L^2_{2kT}}\|u(t)\|_{E_k} \\
&\geq \min\{\frac{1}{2}, \underline{b}\}\|u(t)\|^2_{E_k} - C_0\|u\|_{E_k} \\
&= \min\{\frac{1}{2}, \underline{b}\}\rho^2 - C_0\rho \\
&= \min\{\frac{1}{2}, \underline{b}\}\frac{{\rho_0}^2}{2{C_1}^2} := \alpha.
\end{aligned}$$

Step 3. For the ρ defined as above, there exists $e_k \in E_k$ such that $\|e_k\|_{E_k} > \rho, \varphi_k(e_k) \leq 0$.

By (A_5),

$$\frac{W(t, u)}{|u|^2} \geq \frac{2\pi^2}{T^2}$$

for all $|u| > R$ and $t \in [-T, T]$.

Let $\delta = \max_{\{t \in [-T,T], |u| \leq R\}} |W(t, u)|$, we obtain

$$W(t, u) \geq \frac{2\pi^2}{T^2}(|u|^2 - R^2) - \delta \tag{2.5}$$

for all $u \in \mathbb{R}^n, t \in \mathbb{R}$.

Set

$$\bar{e}_k(t) = \begin{cases} s\sin(\omega t)e, & t \in [-T, T] \\ 0, & t \in [-kT, kT]\backslash[-T, T] \end{cases} \tag{2.6}$$

where $\omega = \frac{\pi}{T}, e = (1, 0, \cdots, 0)$. Let e_k be the 2kT-periodic extension of $\bar{e}_k$, then $e_k \in E_k$, and $\|e_k\|_{E_k} \to \infty$ as $s \to \infty$. We assume s is large enough such that $\|e_k\|_{E_k} \geq \rho$. Combining (A_2) and (2.5), we obtain

$$\begin{aligned}
\varphi_k(e_k(t)) &= \int_{-kT}^{kT} \Big[\frac{1}{2}|\dot{e}_k(t)|^2 + K(t, e_k(t)) - W(t, e_k(t)) + (f_k(t), e_k(t))\Big]\,\mathrm{d}t \\
&\leq \int_{-T}^{T} \frac{1}{2}|\dot{e}_k(t)|^2\mathrm{d}t + \bar{b}\int_{-T}^{T} |e_k(t)|^2\mathrm{d}t - \frac{2\pi^2}{T^2}\int_{-T}^{T} |e_k(t)|^2\mathrm{d}t \\
&\quad + \Big(\frac{2\pi^2R^2}{T^2} + \delta\Big)2T + C_0\Big(\int_{-T}^{T} |e_k(t)|^2\,\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\leq \frac{1}{2}s^2\omega^2\int_{-T}^{T} |\cos(\omega t)|^2\,\mathrm{d}t + \Big(\bar{b}s^2 - \frac{2\pi^2s^2}{T^2}\Big)\int_{-T}^{T} |\sin(\omega t)|^2\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+C_0 s\Big(\int_{-T}^{T} |\sin(\omega t)|^2\,\mathrm{d}t\Big)^{\frac{1}{2}} + 2T\Big(\delta + \frac{2\pi^2 R^2}{T^2}\Big)\\
= &\Big(\frac{1}{2}\omega^2 + \bar{b} - \frac{2\pi^2}{T^2}\Big)s^2 T + C_0 s T^{\frac{1}{2}} + 2T\Big(\delta + \frac{2\pi^2 R^2}{T^2}\Big)\\
\le &-\frac{\pi^2}{T}s^2 + C_0 s T^{\frac{1}{2}} + 2T\Big(\delta + \frac{2\pi^2 R^2}{T^2}\Big) \to -\infty
\end{aligned}
$$

as $s \to \infty$. So for all $k \in \mathbb{N}$, we can choose an s large enough such that e_k defined as above satisfies $\|e_k\|_{E_k} > \rho$ and $\varphi_k(e_k) \le 0$.

Therefore, by the Mountain Pass Theorem, φ_k possesses a critical value c_k defined by

$$c_k = \inf_{g\in\Gamma_k} \max_{s\in[0,1]} \varphi_k(g(s))$$

satisfying $c_k \ge \alpha$, where

$$\Gamma_k = \{g \in C([0,1], E_k) : g(0) = 0, g(1) = e_k\}.$$

By Lemma 2.1, we know for each $k \in \mathbb{N}$, there exists $u_k \in E_k$ such that

$$\varphi_k(u_k) = c_k, \qquad \varphi_k'(u_k) = 0.$$

Consequently u_k is a classical $2kT$-periodic solution of (1.2). Moreover, since $c_k \ge \alpha > 0, u_k$ is a nontrivial solution.

In the following, we will show that there exists a subsequence of $\{u_k\}$ which almost uniformly converges to a C^1 function. We denote $C^p_{loc}(\mathbb{R}, \mathbb{R}^n)(p \in \mathbb{N}\cup\{0\})$, the space of C^p functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the topology of almost uniformly convergence of functions and all derivatives up to the order p. We have

Lemma 2.2 *Let $\{u_k\}_{k\in\mathbb{N}}$ be the sequence given as above. Then it possesses a subsequence also denoted by $\{u_k\}$ and a C^1 function $u_0 : \mathbb{R} \to \mathbb{R}^n$ such that $u_k \to u_0$ in $C^1_{loc}(\mathbb{R}, \mathbb{R}^n)$, as $k \to +\infty$.*

Proof We will prove this lemma by the Arzela-Ascoli Theorem. We first show that the sequence $\{c_k\}_{k\in\mathbb{N}}$ and $\{\|u_k\|_{E_k}\}_{k\in\mathbb{N}}$ are bounded.

For each $k \in \mathbb{N}$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined as above. Then $g_k \in \Gamma_k$ and $\varphi_k(g_k(s)) = \varphi_1(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0,1]$.

Hence

$$c_k = \inf_{g\in\Gamma_k} \max_{s\in[0,1]} \varphi_k(g_k(s)) \le \max_{s\in[0,1]} \varphi_k(g_k(s)) = \max_{s\in[0,1]} \varphi_1(g_1(s)) := M_0$$

independently of $k \in \mathbb{N}$.

Therefore

$$\varphi_k(u_k) = c_k \le M_0$$

for all $k \in \mathbb{N}$.

As $\varphi_k'(u_k) = 0$, we have

$$(1 + \|u_k\|_{E_k})\|\varphi_k'(u_k)\| = 0 \le M_0$$

for all $k \in \mathbb{N}$.

Along the proof of Step 1. in Lemma 2.1, it is easy to prove that$\{\|u_k\|_{E_k}\}$ is bounded uniformly in $k \in \mathbb{N}$, which means there exists a constant $M_1 > 0$ independent of k such that

$$\|u_k\|_{E_k} \le M_1.$$

In order to show that $\{u_k\}_{k\in\mathbb{N}}$ and $\{\dot{u}_k\}_{k\in\mathbb{N}}$ are equicontinuous, we first prove $\{u_k\}_{k\in\mathbb{N}}$, $\{\dot{u}_k\}_{k\in\mathbb{N}}$ and $\{\ddot{u}_k\}_{k\in\mathbb{N}}$ are uniformly bounded in L^∞_{2kT}.

By Proposition 1,

$$\|u_k\|_{L^\infty_{2kT}} \le C_1 M_1 := M_2 \tag{2.7}$$

for all $k \in \mathbb{N}$.

By (1.2) and the definition of f_k, it is clear that

$$|\ddot{u}_k(t)| \le |f_k(t)| + |V_u(t, u_k(t))| = |f(t)| + |V_u(t, u_k(t))|$$

for $t \in [-kT, kT]$.

Together with (2.7), (A_1) and (A_6), we obtain there exists $M_3 > 0$ independent of k such that

$$\|\ddot{u}_k\|_{L^\infty_{2kT}} \le M_3. \tag{2.8}$$

Since for each $k \in \mathbb{N}$ and $t \in \mathbb{R}$ there exists $\xi \in [t-1, t]$ such that

$$u_k(t) - u_k(t-1) = \dot{u}_k(\xi),$$

we can see

$$\begin{aligned} |\dot{u}_k(t)| &= \left| \int_\xi^t \ddot{u}_k(s)\,\mathrm{d}s + \dot{u}_k(\xi) \right| \\ &\le \int_{t-1}^t |\ddot{u}_k(s)|\,\mathrm{d}s + |u_k(t) - u_k(t-1)| \\ &\le M_3 + 2M_2 := M_4. \end{aligned}$$

Thus we have

$$\|\dot{u}_k\|_{L^\infty_{2kT}} \le M_4.$$

Therefore

$$|u_k(t) - u_k(t')| = \left| \int_{t'}^t \dot{u}_k(s)\,\mathrm{d}s \right| \le \int_{t'}^t |\dot{u}_k(s)|\,\mathrm{d}s \le M_4|t - t'|.$$

That is $\{u_k\}_{k\in\mathbb{N}}$ are equicontinuous.

Analogously, $\{\dot{u}_k\}_{k\in\mathbb{N}}$ are also equicontinuous.

By the Arzela-Ascoli Theorem, there is a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, which converges to a C^1 function u_0 in $C^1_{loc}(\mathbb{R}, \mathbb{R}^n)$.

Now we are coming to the point of proving Theorem 1. We need the following results from [3].

Proposition 2 *Let $u : \mathbb{R} \to \mathbb{R}^n$ be a continuous mapping. If a weak derivative $\dot{u} : \mathbb{R} \to \mathbb{R}^n$ is continuous at t_0, then u is differential at t_0 and*

$$\lim_{t\to t_0} \frac{u(t) - u(t_0)}{t - t_0} = \dot{u}(t_0).$$

Proposition 3 *Let $u : \mathbb{R} \to \mathbb{R}^n$ be a continuous mapping such that $\dot{u} \in L^2_{loc}(\mathbb{R}, \mathbb{R}^n)$ (the space of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ locally square integrable). For every $t \in \mathbb{R}$ the following inequality holds:*

$$|u(t)| \le \sqrt{2} \left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|u(s)|^2 + |\dot{u}(s)|^2)\,\mathrm{d}s \right)^{\frac{1}{2}} \tag{2.9}$$

Proof of Theorem 1 We prove u_0 is exactly our desired homoclinic solution of (1.1).

Arguing just as Lemma 2.9 in [3], for each $k \in \mathbb{N}$ and $t \in \mathbb{R}$, u_k satisfies

$$\ddot{u}_k(t) = f_k(t) - V_u(t, u_k(t)). \tag{2.10}$$

Since $u_k \to u_0$ and $f_k \to f$ almost uniformly on $\mathbb{R}$, we obtain

$$\ddot{u}_k \to f(t) - V_u(t, u_0(t)).$$

For any finite interval $[a, b]$, there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $t \in [a, b]$, (2.10) becomes

$$\ddot{u}_k(t) = f(t) - V_u(t, u_k(t)).$$

So $\ddot{u}_k(t)$ is continuous on $[a, b]$ for each $k \geq k_0$. By Proposition 2, $\ddot{u}_k(t)$ is a derivative of $\dot{u}_k(t)$ in (a, b) for each $k \geq k_0$.

Combining $\ddot{u}_k \to f(t) - V_u(t, u_0(t))$ and $\dot{u}_k \to \dot{u}_0$ almost uniformly on $\mathbb{R}$, we obtain

$$\ddot{u}_0(t) = f(t) - V_u(t, u_0(t))$$

in (a, b) and then in $\mathbb{R}$. So u_0 satisfies (1.1).

We now prove $u_0(t) \to 0$ as $t \to \pm\infty$.

Obviously, for each $i \in \mathbb{N}$ there is $k_i \in \mathbb{N}$ such that for all $k \geq k_i$,

$$\int_{-iT}^{iT} (|u_k(t)|^2 + |\dot{u}_k(t)|^2)\,\mathrm{d}t \leq \|u_k\|_{E_k}^2 \leq M_1^2.$$

Letting $k \to +\infty$, we obtain

$$\int_{-iT}^{iT} (|u_0(t)|^2 + |\dot{u}_0(t)|^2)\,\mathrm{d}t \leq M_1^2.$$

As $i \to \infty$, we have

$$\int_{-\infty}^{+\infty} (|u_0(t)|^2 + |\dot{u}_0(t)|^2)\,\mathrm{d}t \leq M_1^2.$$

Hence we get

$$\int_{|t|\geq\rho} (|u_0(t)|^2 + |\dot{u}_0(t)|^2)\,\mathrm{d}t \to 0 \tag{2.11}$$

as $\rho \to +\infty$.

By Proposition 3 and (2.11), for all $t > \rho + \frac{1}{2}$ we have

$$\begin{aligned}
|u_0(t)| &\leq \sqrt{2}\left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|u_0(s)|^2 + |\dot{u}_0(s)|^2)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\left(\int_{\rho}^{+\infty} (|u_0(s)|^2 + |\dot{u}_0(s)|^2)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\left(\int_{|t|\geq\rho} (|u_0(s)|^2 + |\dot{u}_0(s)|^2)\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, as $\rho \to +\infty, t \to +\infty$, we get $u_0(t) \to 0$.

Using the same method, we can obtain $u_0(t) \to 0$ as $t \to -\infty$.

In the following we prove that $\dot{u}_0(t) \to 0$ as $t \to \pm\infty$.

Since $u_0(s) \to 0$ as $s \to \pm\infty$, and $\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 \, ds \to 0$ as $t \to \pm\infty$, by (A_1), we have

$$\begin{aligned}\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{u}_0(s)|^2 \, ds \ = \ & \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|V_u(s, u_0(s))|^2 + |f(s)|^2) \, ds \\ & -2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (V_u(s, u_0(s)), f(s)) \, ds \to 0 \qquad (2.12)\end{aligned}$$

as $t \to \pm\infty$.

From Proposition 3, we get

$$|\dot{u}_0(t)|^2 \leq 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|u_0(s)|^2 + |\dot{u}_0(s)|^2) \, ds + 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{u}_0(s)|^2 \, ds.$$

Together with (2.11) and (2.12), we obtain $\dot{u}(t) \to 0$ as $t \to \pm\infty$.

Since f is nonzero, we know $u_0 \not\equiv 0$. So u_0 is a homoclinic solution of (1.1).

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