



Dynamics of a Beddington–DeAngelis-type predator–prey system with constant rate harvesting

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Abstract. In the paper, a predator–prey system with Beddington–DeAngelis functional response and constant rate harvesting is considered. Various dynamical behaviors of the system including saddle–node points and a cusp of codimension 2 are investigated by using the analysis of qualitative method and bifurcation theory. Also, it is shown that the system undergoes several kinds of bifurcation such as the saddle–node bifurcation, the subcritical and supercritical Hopf bifurcation, Bogdanov–Takens bifurcation by choosing the death rate of the predator and the harvesting rate of the prey as the bifurcation parameters. Some numerical examples are illustrated in order to substantiate our theoretical results. These results unveil far richer dynamics compared to the system without harvesting.

Keywords: Beddington–DeAngelis-type predator–prey system, harvesting rate, saddle–node point, cusp, Hopf bifurcation, Bogdanov–Takens bifurcation.

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1 Introduction

In population dynamics, a lot of mathematical models to investigate the relationship between preys and predators have been suggested by ecologists and mathematicians [2, 16]. In fact, there are well-known predator–prey models such as Lotka–Volterra models [12, 13], Holling-type models [7, 17], Michaelis–Menten-type models (so called ratio-dependent predator–prey models) [1, 5], Ivlev-type models [10] and so on. Especially, many researchers are interested in the following Beddington–DeAngelis predator–prey model

$$\begin{cases} x'(t) = rx(1 - \frac{x}{K}) - \frac{\gamma xy}{\beta y + x + \alpha}, \\ y'(t) = -dy + \frac{\delta xy}{\beta y + x + \alpha}, \\ (x(0^+), y(0^+)) = (x_0, y_0), \end{cases} \quad (1.1)$$

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where x, y represent the population density of the prey and the predator at time t , respectively. Usually, K is called the carrying capacity of the prey. The constant r is called the intrinsic growth rate of the prey. The constants δ, d are the conversion rate and the death rate of the predator, respectively. The term βy measures the mutual interference between predators. The reason for the model is that if one take the parameter β close to 0 then the model can be regarded as a Holling-type II predator–prey model and also one take the parameter α close to 0 then the model can be thought of a ratio-dependent predator–prey model. The dynamics of system (1.1) has been studied extensively in [3,4,6,8,9]. Research on this system (1.1) revealed various dynamics such as deterministic extinction, existence of multiple attractors and stable limit cycles, etc.

For mathematical simplicity, we need to rewrite system (1.1) as a dimensionless system with following scaling

$$rt \rightarrow t, \quad \frac{x}{K} \rightarrow x, \quad \frac{\beta}{\alpha}y \rightarrow y.$$

Then we can obtain the following dimensionless system

$$\begin{cases} x'(t) = x(1-x) - \frac{bxy}{y+ax+1}, \\ y'(t) = -Dy + \frac{exy}{y+ax+1}, \end{cases} \quad (1.2)$$

where $a = \frac{K}{\alpha}$, $b = \frac{\gamma}{r\beta}$, $D = rd$ and $e = \frac{\delta K}{r\alpha}$ are positive constants.

From the point of view of human needs, the exploitation of biological resources and the harvesting of populations are commonly practiced in fishery, forestry, and wildlife management. Concerning the conservation for the long-term benefits of humanity, there is a wide range of interest in the use of bioeconomic models to gain insight into the scientific management of renewable resources like fisheries and forestries [18]. For the reason, it is important to deal with mathematical models with the harvesting of populations. Thus, by taking into account the above reasons, in the paper, we will focus on the Beddington–DeAngelis-type predator–prey system with constant rate harvesting as follows;

$$\begin{cases} x'(t) = x(1-x) - \frac{bxy}{y+ax+1} - h, \\ y'(t) = -Dy + \frac{exy}{y+ax+1}. \end{cases} \quad (1.3)$$

The organization of this paper is as follows. In next section, we study the existence of the equilibria and their stability in the small neighborhood for system (1.3). In Section 3 we show that system (1.3) can display bifurcation phenomena such as a saddle-node bifurcation, supercritical and subcritical Hopf bifurcations and Bogdanov–Takens bifurcation for $a = 3$, $b = 1$. Finally, we give a conclusion in Section 5.

2 Equilibria and stabilities

In this section, we investigate dynamical behaviors of system (1.3) around equilibria. From biological point of view, we shall assume that the dynamics of system (1.3) is on the positive cone $\mathbb{R}_+^2 = \{(x, y) : x \geq 0, y \geq 0\}$. Also, we shall consider the biologically meaningful initial condition $x(0) \geq 0, y(0) \geq 0$. Since the right-hand side two equations of system (1.3) are continuous and Lipschitzian in the close first quadrant \mathbb{R}_+^2 , it is easy to see that solution of

system (1.3) exists and is unique for positive initial conditions. Also, we can see that the x -axis is invariant under the flow. However, this is not the case on the y -axis. In fact, all solutions to the system (1.3) touching the y -axis cross out of the first quadrant due to the harvesting rate h of the prey. Thus the first quadrant is no longer positively invariant under the flow generated by system (1.3), which makes the qualitative analysis of system (1.3) difficult.

In order to find out equilibria of system (1.3), we must consider the following two equations

$$f_1(x, y) = 0, \quad f_2(x, y) = 0, \quad (2.1)$$

where

$$\begin{cases} x(1-x) - \frac{bxy}{y+ax+1} - h = f_1(x, y), \\ -Dy + \frac{exy}{y+ax+1} = f_2(x, y). \end{cases} \quad (2.2)$$

It is obvious that equation (2.1) has at most four equilibria as follows;

$$(x_i, y_i) \quad \text{and} \quad (x_i^*, y_i^*) (i = 1, 2),$$

where, for $i = 1, 2$,

$$\begin{cases} x_i = \frac{1 + (-1)^i \sqrt{1-4h}}{2} \quad \text{and} \quad y_i = 0, \\ x_i^* = \frac{A + (-1)^i \sqrt{A^2 - 4B}}{2} \quad \text{and} \quad y_i^* = \frac{e-aD}{D} x_i^* - 1, \end{cases} \quad (2.3)$$

$A = 1 - \frac{b(e-aD)}{e}$ and $B = h - \frac{bD}{e}$.

For simplicity, let $(x_0, y_0) = (\frac{1}{2}, 0)$ and $(x_0^*, y_0^*) = (\frac{A}{2}, \frac{e-aD}{D} x_0^* - 1)$ when $h = \frac{1}{4}$ and $A^2 = 4B$, respectively.

Now, we summarize conditions for the existence of the above equilibria.

Lemma 2.1. Let $A = 1 - \frac{b(e-aD)}{e}$, $B = h - \frac{bD}{e}$, $h_0 = \frac{A^2}{4} + \frac{bD}{e}$ and $h_1 = \frac{De-(a+1)D^2}{(e-aD)^2}$.

(i) System (1.3) has no equilibria in \mathbb{R}_+^2 if $h > \max\{\frac{1}{4}, h_0\}$.

(ii) System (1.3) has a unique equilibrium (x_0, y_0) in \mathbb{R}_+^2 if $h_0 < h = \frac{1}{4}$.

(iii) System (1.3) has two equilibria (x_1, y_1) and (x_2, y_2) in \mathbb{R}_+^2 if one of the following conditions holds:

(iii.1) $0 < h < \frac{1}{4}$ and $e \leq aD$;

(iii.2) $\frac{bD}{e} \leq h < \min\{\frac{1}{4}, h_0\}$ and $be \geq abD + e$;

(iii.3) $h_0 < h < \frac{1}{4}$.

(iv) System (1.3) has three equilibria (x_0^*, y_0^*) , (x_1, y_1) and (x_2, y_2) in \mathbb{R}_+^2 if $h = h_0 < \frac{1}{4}$, $be < abD + e$ and $(e-aD)(e+abD-be) > 2eD$.

(v) System (1.3) has four equilibria, (x_1^*, y_1^*) , (x_2^*, y_2^*) , (x_1, y_1) and (x_2, y_2) in \mathbb{R}_+^2 if $\max\{h_1, \frac{bD}{e}\} < h < \min\{h_0, \frac{1}{4}\}$, $be < abD + e$ and $(e-aD)(abD+e-be) > 2eD$.

Proof. From equations (2.1), we have the equations $x^2 - x + h = 0$ for $y = 0$ and $x^2 - Ax + B = 0$ for $y = \frac{e-aD}{D}x - 1$, where $A = 1 - \frac{b(e-aD)}{e}$ and $B = h - \frac{bD}{e}$. Since the discriminants of $x^2 - x + h = 0$ and $x^2 - Ax + B = 0$ are $1 - 4h$ and $A^2 - 4B$, respectively, there is no equilibrium of system (1.3) if $h > \max\{\frac{1}{4}, h_0\}$. In addition, the only equilibrium $(x_0, y_0) = (\frac{1}{2}, 0)$ exists when $h_0 < h = \frac{1}{4}$. In order to discuss the equilibria of the equation $x^2 - Ax + B = 0$ for $y = \frac{e-aD}{D}x - 1$ we must consider the nonnegative discriminant, the positiveness of the roots of $x^2 - Ax + B = 0$ satisfying $y = \frac{e-aD}{D}x - 1 > 0$. In fact, $x^2 - Ax + B = 0$ has the double root (x_0^*, y_0^*) if $h = h_0 < \frac{1}{4}$ and it is positive if $be < abD + e$ and $(e - aD)(e + abD - be) > 2eD$. Moreover, the equation $x^2 - Ax + B = 0$ has two distinct roots (x_i^*, y_i^*) , $i = 1, 2$ if $h < \min\{h_0, \frac{1}{4}\}$ and these roots are positive if $\max\{h_1, \frac{bD}{e}\} < h$, $be < abD + e$ and $(e - aD)(abD + e - be) > 2eD$. Therefore, (i)~(v) hold. \square

The Jacobian matrix of system (1.3) is defined by

$$J(x, y) = \begin{pmatrix} 1 - 2x - \frac{by(y+1)}{(y+ax+1)^2} & -\frac{bx(ax+1)}{(y+ax+1)^2} \\ \frac{ey(y+1)}{(y+ax+1)^2} & -D + \frac{ex(ax+1)}{(y+ax+1)^2} \end{pmatrix}, \quad (2.4)$$

where x and y are coordinates of equilibria, respectively. Thus elementary calculations yield, for $i = 1, 2$,

$$J(x_i, y_i) = \begin{pmatrix} 1 - 2x_i & -\frac{bx_i}{ax_i+1} \\ 0 & -D + \frac{ex_i}{ax_i+1} \end{pmatrix} \quad (2.5)$$

and

$$J(x_i^*, y_i^*) = \begin{pmatrix} 1 - 2x_i^* - \frac{bD(e-aD)}{e^2} \frac{y_i^*}{x_i^*} & -\frac{bD^2}{e^2} \left(a + \frac{1}{x_i^*}\right) \\ \frac{D(e-aD)}{e} \frac{y_i^*}{x_i^*} & -D + \frac{D^2}{e} \left(a + \frac{1}{x_i^*}\right) \end{pmatrix}, \quad (2.6)$$

if equilibria exist.

Theorem 2.2. *If $h_0 < h = \frac{1}{4}$, then system (1.3) has a unique equilibrium $(x_0, y_0) = (\frac{1}{2}, 0)$. Moreover,*

- (i) (x_0, y_0) is a saddle if $e = D(a+2)$;
- (ii) (x_0, y_0) is a saddle-node if $e \neq D(a+2)$, and it is attracting (repelling) if $e < D(a+2)$ ($e > D(a+2)$).

Proof. From Lemma 2.1, system (1.3) has a unique equilibrium (x_0, y_0) if $h_0 < h = \frac{1}{4}$. From (2.5), we get

$$J(x_0, y_0) = \begin{pmatrix} 0 & -\frac{b}{a+2} \\ 0 & -D + \frac{e}{a+2} \end{pmatrix} \quad (2.7)$$

and hence the determinant of the matrix $J(x_0, y_0)$ is zero. Thus the equilibrium (x_0, y_0) is degenerate. In order to determine the dynamics of system (1.3) in the neighborhood of the equilibrium (x_0, y_0) , we need to transform the equilibrium (x_0, y_0) of system (1.3) to the origin

and expand the right hand side of system (1.3) as a Taylor series. Then system (1.3) can be written as

$$\begin{cases} x'(t) = -\frac{b}{2+a}y - x^2 - \frac{4b}{(2+a)^2}xy + \frac{2b}{(2+a)^2}y^2 \\ \quad + \frac{8ab}{(2+a)^3}x^2y - \frac{4b(a-2)}{(2+a)^3}xy^2 - \frac{4b}{(2+a)^3}y^3 + P_1(x, y), \\ y'(t) = (-D + \frac{e}{2+a})y + \frac{4e}{(2+a)^2}xy - \frac{2e}{(2+a)^2}y^2 \\ \quad - \frac{8ae}{(2+a)^3}x^2y + \frac{4e(a-2)}{(2+a)^3}xy^2 + \frac{4e}{(2+a)^3}y^3 + Q_1(x, y), \end{cases} \quad (2.8)$$

where $P_1(x, y)$ and $Q_1(x, y)$ are C^∞ functions of order at least four in (x, y) .

In order to show the conclusion (i), suppose that the condition $e = D(a+2)$ holds. Then both eigenvalues of the matrix $J(x_0, y_0)$ are zero. We use the procedure used in [22] to reduce system via the normal form method. First, let $\tau = -\frac{b}{2+a}t$ in system (2.8). Then system (2.8) becomes

$$\begin{cases} x'(\tau) = y + P_2(x, y), \\ y'(\tau) = Q_2(x, y), \end{cases} \quad (2.9)$$

where $P_2(x, y)$ and $Q_2(x, y)$ are C^∞ functions of order at least two in (x, y) . From $y + P_2(x, y) = 0$, we can obtain the implicit function

$$y = \phi(x) = -\frac{2+a}{b}x^2 + \frac{4}{b}x^3 + R_1(x), \quad (2.10)$$

where $R_1(x)$ is a C^∞ function of order at least four. And we have

$$\psi(x) \equiv Q(x, \phi(x)) = \frac{4e}{b^2}x^3 + \frac{2(4+4a+a^2-8b)e}{(2+a)b^3}x^4 + R_2(x), \quad (2.11)$$

where $R_2(x)$ is a C^∞ function of order at least five. Using the notation of Theorem 7.2 of Chapter 2 in [22], we can find $k = 2m + 1 = 3$, $m = 1$, and $a_k = \frac{4e}{b^2} > 0$. Thus the equilibrium $(0, 0)$ of system is a saddle point. Therefore (x_0, y_0) is a saddle if $e = D(a+2)$.

In order to prove the conclusion (ii), assume that $e \neq D(a+2)$ hold. In fact, since the trace of the matrix $J(x_0, y_0)$ is not zero, one of the eigenvalues of the matrix $J(x_0, y_0)$ is zero and the other is nonzero. By letting $x = X - \frac{b}{2+a}Y$ and $y = (-D + \frac{e}{2+a})Y$, system (2.8) can be written as

$$\begin{cases} X'(t) = -X^2 + \frac{2b(2+a+2D)}{(2+a)^2}XY \\ \quad - \frac{b(b(4+4a+a^2+4e) - 2(e - (2+a)D)^2)}{(2+a)^4}Y^2 + P_3(X, Y), \\ Y'(t) = (-D + \frac{e}{2+a})Y + \frac{4e}{(2+a)^2}XY - \frac{2e(2b - (2+a)D + e)}{(2+a)^3}Y^2 + Q_3(X, Y), \end{cases} \quad (2.12)$$

where $P_3(X, Y)$ and $Q_3(X, Y)$ are C^∞ functions of order at least three in (X, Y) . Again, let $\tau = (-D + \frac{e}{2+a})t$. Then $d\tau = (-D + \frac{e}{2+a})dt$ and hence system (2.12) becomes

$$\begin{cases} X'(\tau) = \frac{2+a}{(2+a)D - e}X^2 - \frac{2b(2+a+2D)}{(2+a)((2+a)D - e)}XY \\ \quad + \frac{b(-2(-(2+a)D + e)^2 + b(4+4a+a^2+4e))}{(2+a)^3((2+a)D - e)}Y^2 + P_4(X, Y), \\ Y'(\tau) = Y - \frac{4e}{(2+a)((2+a)D - e)}XY + \frac{2e(2b - (2+a)D + e)}{(2+a)^2((2+a)D - e)}Y^2 + Q_4(X, Y), \end{cases} \quad (2.13)$$

where $P_4(X, Y)$ and $Q_4(X, Y)$ are C^∞ functions of order at least three in (X, Y) . Now using Theorem 7.1 of Chapter 2 in [22], we can obtain that (x_0, y_0) is attracting(repelling) if $e < D(a+2)$ ($e > D(a+2)$). \square

Theorem 2.3. *Suppose that system (1.3) has only two equilibria (x_1, y_1) and (x_2, y_2) . Then the dynamics of system (1.3) has the following properties.*

- (i) (x_1, y_1) is a hyperbolic saddle and (x_2, y_2) is a hyperbolic stable node if $0 < h < \frac{1}{4}$ and $e \leq aD$.
- (ii) Assume that conditions $\frac{bD}{e} \leq h < \min\{\frac{1}{4}, h_0\}$ and $be \geq abD + e$ hold.
 - (ii.1) (x_1, y_1) and (x_2, y_2) are hyperbolic saddles if $h < h_1$;
 - (ii.2) (x_1, y_1) is a hyperbolic saddle and (x_2, y_2) is a hyperbolic stable node if $h > h_1$ and $e \leq (a+2)D$;
 - (ii.3) (x_1, y_1) is a hyperbolic unstable node and (x_2, y_2) is a hyperbolic saddle if $h > h_1$ and $(a+2)D < e$.
- (iii) Assume that the conditions $h_0 < h < \frac{1}{4}$ and $abD < be < abD + e$ hold.
 - (iii.1) (x_1, y_1) and (x_2, y_2) are hyperbolic saddles if $h < h_1$;
 - (iii.2) (x_1, y_1) is a hyperbolic saddle and (x_2, y_2) is a hyperbolic stable node if $h > h_1$ and $aD < e \leq (a+2)D$;
 - (iii.3) (x_1, y_1) is a hyperbolic unstable node and (x_2, y_2) is a hyperbolic saddle if $h > h_1$ and $(a+2)D < e$.

Proof. It is easy to show the existence of equilibria under given conditions due to Lemma 2.1. Thus we have only to consider their stabilities. It follows from (2.5) that the eigenvalues of the Jacobian matrix J at the equilibria (x_1, y_1) and (x_2, y_2) are $1 - 2x_i$ and $-D + \frac{ex_i}{ax_i+1}$ ($i = 1, 2$). Since $1 - 2x_1 = \sqrt{1 - 4h} > 0$ and $1 - 2x_2 = -\sqrt{1 - 4h} < 0$ if $h < \frac{1}{4}$, the dynamics of system (1.3) depends on the sign of the eigenvalues $-D + \frac{ex_i}{ax_i+1}$ for $i = 1, 2$. Note that $-D + \frac{ex_1}{ax_1+1} = -D + \frac{e(1-\sqrt{1-4h})}{2+a(1-\sqrt{1-4h})} \equiv E_1$ and $-D + \frac{ex_2}{ax_2+1} = -D + \frac{e(1+\sqrt{1-4h})}{2+a(1+\sqrt{1-4h})} \equiv E_2$.

(i) Since $e \leq aD$, $E_1 = \frac{-2D+(e-aD)H^-}{2+aH^-} < 0$ and $E_2 = \frac{-2D+(e-aD)H^+}{2+aH^+} < 0$, where $H^\pm = 1 \pm \sqrt{1-4h}$. Thus, by linear stability analysis, (x_1, y_1) is a hyperbolic saddle and (x_2, y_2) is a hyperbolic stable node.

Now consider the following cases.

Case (I) : $aD < e \leq (a+2)D$

If $aD < e \leq (a+2)D$, then the eigenvalue $E_1 = -D + \frac{e(1-\sqrt{1-4h})}{2+a(1-\sqrt{1-4h})} = \frac{e-(a+2)D-(e-aD)\sqrt{1-4h}}{2+a(1-\sqrt{1-4h})}$ is negative. On the other hand, the sign of the eigenvalue E_2 depends on that of $e - (a+2)D + (e-aD)\sqrt{1-4h}$. Elementary calculations yield that $E_2 > 0 (< 0)$ if $h < h_1 (h > h_1)$, where $h_1 = \frac{De-(a+1)D^2}{(e-aD)^2}$.

Case (II) : $(a+2)D < e$

If $(a+2)D < e$, then the eigenvalue $E_2 = -D + \frac{e(1+\sqrt{1-4h})}{2+a(1+\sqrt{1-4h})} = \frac{e-(a+2)D+(e-aD)\sqrt{1-4h}}{2+a(1+\sqrt{1-4h})}$ is positive. In addition, if $h < h_1 (h > h_1)$ then $e - (a+2)D - (e-aD)\sqrt{1-4h} < 0 (> 0)$ and hence $E_1 < 0 (> 0)$.

(ii) Note that the condition $be \geq abD + e$ implies $e \geq abD$.

(ii.1) From cases (I) and (II), we know that the sign of the multiplication of the eigenvalues E_1 and E_2 is always negative if $h < h_1$. Thus the equilibria (x_1, y_1) and (x_2, y_2) are hyperbolic saddles.

(ii.2) By case (I), $E_1 < 0$ when $e \leq (a + 2)D$ and $E_2 < 0$ when $h > h_1$ and $e \leq (a + 2)D$. Therefore (x_1, y_1) is a hyperbolic saddle and (x_2, y_2) is a hyperbolic stable node.

(ii.3) Similar to (ii.2), by using case (II), we have that (x_1, y_1) is a hyperbolic unstable node and (x_2, y_2) is a hyperbolic saddle if $h > h_1$ and $(a + 2)D < e$.

(iii) Since $aD > e$, we get that $h_0 > \frac{1}{4}$. Thus if $h_0 < h < \frac{1}{4}$, $aD < e$. Therefore, similar to the proof of (ii), we can know that (iii.1), (iii.2) and (iii.3) hold. \square

Theorem 2.4. *If $h = h_0 < \frac{1}{4}$, $be < abD + e$ and $(e - aD)(e + abD - be) > 2eD$, then system (1.3) has three equilibria $(x_0^*, y_0^*) = (\frac{A}{2}, \frac{e - aD}{D}x_0^* - 1)$, (x_1, y_1) and (x_2, y_2) where $A = 1 - \frac{b(e - aD)}{e}$. Moreover,*

(i) (x_0^*, y_0^*) is a saddle-node if $a^3b^2D^2 + abDe(a(1 - 2b - D) + 2(1 + e)) + be^2(a(b - e) - 2 - e) - De(2 + a) \neq 0$;

(ii) (x_0^*, y_0^*) is a cusp of codimension 2 if $a^3b^2D^2 - a^2bDe(-1 + 2b + D) + ae(2bD + e(-D + b(-1 + b + 2D))) + e^2(-2D + e - b(2 + e)) = 0$.

(iii) (x_1, y_1) is a hyperbolic unstable node and (x_2, y_2) is a hyperbolic saddle if $h_1 < h$ and $(a + 2)D < e < aD + \frac{e}{b}$.

Proof. It follows from Lemma 2.1 that three equilibria (x_0^*, y_0^*) , (x_1, y_1) and (x_2, y_2) of system (1.3) exist if $h = h_0 < \frac{1}{4}$, $be < abD + e$ and $(e - aD)(e + abD - be) > 2eD$. In addition, similarly to the proof of Theorem 2.3 we can easily show that (iii) holds.

It is easy to see that the determinant of the matrix $J(x_0^*, y_0^*)$ is zero. Thus the equilibrium (x_0^*, y_0^*) is degenerate and at least one of the eigenvalues of the matrix $J(x_0^*, y_0^*)$ is zero. In fact, from elementary calculation, we obtain the eigenvalues are $E_1 = 0$ and

$$E_2 = \frac{D}{e^2(-abD - e + be)} \left[a^3b^2D^2 - a^2bDe(-1 + 2b + D) + ae(2bD + e(-D + b(-1 + b + 2D))) + e^2(-2D + e - b(2 + e)) \right]. \quad (2.14)$$

In order to determine the dynamics of system (1.3) in the neighborhood of the equilibrium (x_0^*, y_0^*) , we need to transform the equilibrium (x_0^*, y_0^*) of system (1.3) to the origin and expand the right hand side of system (1.3) as a Taylor series. Then system (1.3) can be written as

$$\begin{cases} x'(t) = B_1x + B_2y + B_3x^2 + B_4xy + B_5y^2 + g_1(x, y), \\ y'(t) = C_1x + C_2y + C_3x^2 + C_4xy + C_5y^2 + g_2(x, y), \end{cases} \quad (2.15)$$

where $g_1(x, y)$ and $g_2(x, y)$ are C^∞ functions of at least the third order with respect to (x, y)

and

$$\begin{aligned}
B_1 &= \frac{b(aD - e)(a^2bD^2 + 2De + aDe - abDe)}{e^2(-abD - e + be)}, \\
B_2 &= \frac{a^2b^2D^3 + 2bD^2e + abD^2e - ab^2D^2e}{e^2(-abD - e + be)}, \\
B_3 &= -1 - \frac{2bD(-a^2D + ae)(a^2bD^2 + 2De + aDe - 2abDe - e^2 + be^2)}{e^2(-abD - e + be)^2}, \\
B_4 &= \frac{4b(a^3bD^4 + 2aD^3e + a^2D^3e - 2a^2bD^3e - D^2e^2 - aD^2e^2 + abD^2e^2)}{e^2(-abD - e + be)^2}, \\
B_5 &= \frac{2D^3(a^2b^2D + 2be + abe - ab^2e)}{e^2(-abD - e + be)^2},
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
C_1 &= \frac{(e - aD)(a^2bD^2 + 2De + aDe - 2abDe - e^2 + be^2)}{e(-abD - e + be)}, \\
C_2 &= -\frac{D(a^2bD^2 + 2De + aDe - 2abDe - e^2 + be^2)}{e(-abD - e + be)}, \\
C_3 &= -\frac{2D(a^2D - ae)(a^2bD^2 + 2De + aDe - 2abDe - e^2 + be^2)}{(-abD - e + be)^2}, \\
C_4 &= -\frac{4(a^3bD^4 + 2aD^3e + a^2D^3e - 2a^2bD^3e - D^2e^2 - aD^2e^2 + abD^2e^2)}{e(-abD - e + be)^2} \text{ and} \\
C_5 &= -\frac{2D^3(a^2bD + 2e + ae - abe)}{e(-abD - e + be)^2}.
\end{aligned} \tag{2.17}$$

Now, assume that the condition of (i) holds. Then it follows from (2.14) that the eigenvalue E_2 is nonzero, which means that the trace of $J(x_0^*, y_0^*)$ is nonzero. Thus $B_1 + C_2 \neq 0$. Since the determinant of the matrix $J(x_0^*, y_0^*)$ is zero, $B_1C_2 - B_2C_1 = 0$. Consider C^∞ changes of coordinates in a small neighborhood of $(0, 0)$ as follows;

$$\begin{aligned}
x &= x_1, & y &= \frac{1}{B_2}(y_1 - B_1x_1) \quad \text{and} \\
x_1 &= x_2 + y_2, & y_1 &= (B_1 + C_2)y_2.
\end{aligned} \tag{2.18}$$

Then system (2.15) can be transformed into

$$\begin{cases}
x_2'(t) = \left(F_1 - \frac{G_2}{G_1}\right)x_2^2 + \left(2F_1 + F_2G_1 - \frac{2G_2}{G_1} - G_3\right)x_2y_2 \\
\quad + \left(F_1 + F_2G_1 + F_3G_1^2 - \frac{G_2}{G_1} - G_3 - G_1G_4\right)y_2^2 + k_1(x_2, y_2), \\
y_2'(t) = G_1y_2 + \frac{G_2}{G_1}x_2^2 + \left(\frac{2G_2}{G_1} + G_3\right)x_2y_2 + \left(\frac{G_2}{G_1} + G_3 + G_1G_4\right)y_2^2 + k_2(x_2, y_2),
\end{cases} \tag{2.19}$$

where $k_1(x_2, y_2)$ and $k_2(x_2, y_2)$ are C^∞ functions with at least the third order with respect to (x_2, y_2) and

$$\begin{aligned}
F_1 &= B_3 - \frac{B_1B_4}{B_2} + \frac{B_1^2B_5}{B_2^2}, & F_2 &= \frac{B_4}{B_2} - \frac{2B_1B_5}{B_2^2}, & F_3 &= \frac{B_5}{B_2^2}, \\
G_1 &= B_1 + C_2, & G_2 &= B_1B_3 - \frac{B_1^2B_4}{B_2} + \frac{B_1^3B_5}{B_2^2} + B_2C_3 - B_1C_4 + \frac{B_1^2C_5}{B_2}, \\
G_3 &= \frac{B_1B_4}{B_2} - \frac{2B_1^2B_5}{B_2^2} + C_4 - \frac{2B_1C_5}{B_2}, & G_4 &= \frac{B_1B_5}{B_2^2} + \frac{C_5}{B_2}.
\end{aligned}$$

Now, let $\tau = G_1 t$. Then system (2.19) becomes

$$\begin{cases} x_2'(\tau) = \frac{F_1 G_1 - G_2}{G_1^2} x_2^2 + \frac{2F_1 G_1 + F_2 G_1^2 - 2G_2 - G_1 G_3}{G_1^2} x_2 y_2 \\ \quad + \left(F_2 + F_3 G_1 - G_4 + \frac{F_1 G_1 - G_2 - G_1 G_3}{G_1^2} \right) y_2^2 + l_1(x_2, y_2), \\ y_2'(\tau) = y_2 + \frac{G_2}{G_1^2} x_2^2 + \frac{2G_2 + G_1 G_3}{G_1^2} x_2 y_2 + \left(\frac{G_2}{G_1^2} + \frac{G_3}{G_1} + G_4 \right) y_2^2 + l_2(x_2, y_2), \end{cases} \quad (2.20)$$

where $l_1(x_2, y_2)$ and $l_2(x_2, y_2)$ are C^∞ functions with at least the third order with respect to (x_2, y_2) .

Let $\varphi(x_2, y_2) = \frac{F_1 G_1 - G_2}{G_1^2} x_2^2 + \frac{2F_1 G_1 + F_2 G_1^2 - 2G_2 - G_1 G_3}{G_1^2} x_2 y_2 + (F_2 + F_3 G_1 - G_4 + \frac{F_1 G_1 - G_2 - G_1 G_3}{G_1^2}) y_2^2 + l_1(x_2, y_2)$. Since x_0^* is a double root of equation (2.1), the coefficient $\frac{F_1 G_1 - G_2}{G_1^2}$ of x_2^2 in the equation $\varphi(x_2, 0) = \frac{F_1 G_1 - G_2}{G_1^2} x_2^2 + l_1(x_2, 0)$ is a nonzero constant depending on the parameter (a, b, D, e, h) and $l_1(x_2, 0)$ is a C^∞ function with at least the third order with respect to x_2 . Thus it follows from Theorem 7.1 of Chapter 2 in [22] that the equilibrium (x_0^*, y_0^*) is a saddle-node.

For the conclusion (ii), assume that $a^3 b^2 D^2 + abDe(a(1 - 2b - D) + 2(1 + e)) + be^2(a(b - e) - 2 - e) - De(2 + a) = 0$ is satisfied. Then the trace of the matrix $J(x_0^*, y_0^*)$ is zero. i.e. $B_1 + C_2 = 0$ in (2.15). Thus system (2.15) can be transformed into

$$\begin{cases} X'(t) = Y + \left(B_3 - \frac{B_1 B_4}{B_2} + \frac{B_1^2 B_5}{B_2^2} \right) X^2 + K_3(X, Y), \\ Y'(t) = \left(B_1 B_3 - \frac{B_1^2 B_4}{B_2} + \frac{B_1^3 B_5}{B_2^2} + B_2 C_3 - B_1 C_4 + \frac{B_1^2 C_5}{B_2} \right) X^2 + K_4(X, Y), \end{cases} \quad (2.21)$$

where $K_3(X, Y)$ and $K_4(X, Y)$ are C^∞ functions with at least the second order with respect to (X, Y) . By taking $X_1 = X$ and $Y_1 = Y + (B_3 - \frac{B_1 B_4}{B_2} + \frac{B_1^2 B_5}{B_2^2}) X^2 + K_3(X, Y)$, system (2.21) can be changed into

$$\begin{cases} X_1'(t) = Y_1, \\ Y_1'(t) = H_1 X_1^2 + H_2 X_1 Y_1 + K_5(X_1, Y_1), \end{cases} \quad (2.22)$$

where $H_1 = B_1 B_3 - \frac{B_1^2 B_4}{B_2} + \frac{B_1^3 B_5}{B_2^2} + B_2 C_3 - B_1 C_4 + \frac{B_1^2 C_5}{B_2}$, $H_2 = 2(B_3 - \frac{B_1 B_4}{B_2} + \frac{B_1^2 B_5}{B_2^2})$ and $K_5(X_1, Y_1)$ is C^∞ function with at least the third order with respect to (X_1, Y_1) . Elementary calculations yield that $H_2 = -2$ and

$$\begin{aligned} H_1 = & -\frac{1}{(e^3(-abD + (-1 + b)e)^3)} bD(aD - e)(2a^5 b^2 D^5(-1 + e) + a^4 b D^3(b^2 + 4D(-1 + e) \\ & - 6bD(-1 + e))e + 2(-1 + b)^2 e^4 + a^3 D^2 e(8bD(D - e)(-1 + e) - 3b^3 e \\ & + 3b^2(1 + 2D(-1 + e))e + 2D(-1 + e)e) + ae^2(8D^3(-1 + e) - 4(-1 + b)bDe \\ & + 4(-1 + b)D^2(-1 + e)e - (-1 + b)^3 e^2) + a^2 De^2(2D(4D - e)(-1 + e) + 3b^3 e \\ & + b(-12D^2(-1 + e) + 3e + 4D(-1 + e)e) + 2b^2(-3e + D(1 + e - e^2))). \end{aligned}$$

If $H_1 = 0$, then $be = abD + e$, which contradicts to the condition $be < abD + e$. Hence H_1 is a nonzero constant. Using the notation of Theorem 7.3 of Chapter 2 in [22], we can find $k = 2m = 2$, $m = 1$, and $a_k = H_1 \neq 0$ and $b_1 = H_2 \neq 0$. Hence the equilibrium $(0, 0)$ of system (1.3) is a cusp of codimension 2. Therefore (x_0^*, y_0^*) is a cusp of codimension 2. \square

Theorem 2.5. *If $\max\{h_1, \frac{bD}{e}\} < h < \min\{h_0, \frac{1}{4}\}$, $be < abD + e$ and $(e - aD)(abD + e - be) > 2eD$, then system (1.3) has four equilibria as shown in Lemma 2.1 and the dynamics of system (1.3) are as follows:*

- (i) (x_1, y_1) is a hyperbolic saddle and (x_2, y_2) is a hyperbolic stable node if $aD < e \leq (a + 2)D$;
- (ii) (x_1, y_1) is a hyperbolic unstable node and (x_2, y_2) is a hyperbolic saddle if $(a + 2)D < e < aD + \frac{e}{b}$;
- (iii) (x_1^*, y_1^*) is a hyperbolic saddle;
- (iv) (x_2^*, y_2^*) is a focus (or a center or a node). In particular, if $h^* = \frac{bD}{e} + \Phi(A - \Phi)$, then
 - (iv.1) (x_2^*, y_2^*) is a hyperbolic stable focus (or a node) if $h < h^*$,
 - (iv.2) (x_2^*, y_2^*) is a weak focus (or a center) if $h = h^*$,
 - (iv.3) (x_2^*, y_2^*) is a hyperbolic unstable focus (or a node) if $h^* < h$,

where $A = 1 - \frac{b(e-aD)}{e}$, $\Phi = \frac{1}{4e^2}(\theta + \sqrt{\theta^2 + 8e^2(bD(e-aD) + D^2e)})$ and $\theta = -a^2bD^2 + aD(2b + D)e - (-1 + b + D)e^2$.

Proof. By Theorem 2.4, (i) and (ii) can be easily obtained. Next, let us think about the sign of the determinant of the matrix $J(x_i^*, y_i^*)$ ($i = 1, 2$) in equation (2.6). Since $y_i^* = \frac{e-aD}{D}x_i^* - 1$ ($i = 1, 2$), straightforward computation shows that for $i = 1, 2$,

$$\det J(x_i^*, y_i^*) = \frac{D((e-aD)x_i^* - D)(2ex_i^* + eb - e - abD)}{e^2x_i^*}.$$

From the hypotheses, we have $y_i^* > 0$ ($i = 1, 2$) which implies $x_i^* > \frac{D}{e-aD} > 0$, $i = 1, 2$. Thus the sign of the determinant $\det J(x_i^*, y_i^*)$ depends on the value $2ex_i^* + eb - e - abD$. By the proof of Lemma 2.1 we know that x_1^* and x_2^* are roots of equation $F(x) = x^2 - Ax + B = 0$, where $A = 1 - \frac{b(e-aD)}{e} > 0$ and $B = h - \frac{bD}{e} > 0$. Since $x_1^* < x_2^*$, $(-1)^i F'(x_i^*) > 0$, $i = 1, 2$. Now, from simple calculations, we obtain $F'(x_i^*) = 2x_i^* - A = \frac{2ex_i^* - e + b(e-aD)}{e}$, $i = 1, 2$. Thus the inequalities $2ex_1^* + eb - e - abD < 0$ and $2ex_2^* + eb - e - abD > 0$ hold, which implies that $\det(J(x_1^*, y_1^*)) < 0$ and $\det(J(x_2^*, y_2^*)) > 0$. Therefore, (x_1^*, y_1^*) is a hyperbolic saddle and (x_2^*, y_2^*) is a focus (or a center or a nod). One can easily check that $\text{tr}(J(x_2^*, y_2^*)) = 0$ if $h = h^*$. Thus we can get (iii) and (iv) from linear stability analysis. \square

3 Bifurcations of system (1.3)

3.1 Saddle-node bifurcation

From Lemma 2.1 and Theorem 2.2 and 2.3, we can easily see that

$$SN_1 = \{(a, b, D, e) \mid h = \frac{1}{4}, e < aD, a > 0, b > 0, D > 0, e > 0\} \quad (3.1)$$

is a saddle-node bifurcation surface. In fact, the number of equilibria of system (1.3) changes from zero to two, and the two equilibria which are axial equilibrium points are the hyperbolic saddle and node when the parameters pass through the surface SN . Ecologically speaking, the system collapses for $h > \frac{1}{4}$ and the prey population becomes extinct if $h = \frac{1}{4}$, but the prey

population does not go to extinction for some initial value if $0 < h < \frac{1}{4}$. From Theorem 2.4 and 2.5, we can know that the surface

$$\begin{aligned} SN_2 = \{ & (a, b, D, e) \mid h = h_0 < \frac{1}{4}, (e - aD)(abD + e - be) > 2eD, \\ & be < abD + e, a > 0, b > 0, D > 0, e > 0, \\ & a^3b^2D^2 + abDe(a(1 - 2b - D) + 2(1 + e)) + be^2(a(b - e) - 2 - e) - De(2 + a) \neq 0 \} \end{aligned} \quad (3.2)$$

is also saddle-node bifurcation surface. This saddle–node bifurcation yields two positive equilibrium points. This means that the predator goes either extinct or out of \mathbb{R}_+^2 in finite time when $h > h_0$, and both the predator and prey populations coexist in the form of a positive interior equilibrium for some initial values when $h < h_0$, $be < abD + e$, $(e - aD)(abD + e - be) > 2eD$, $a^3b^2D^2 + abDe(a(1 - 2b - D) + 2(1 + e)) + be^2(a(b - e) - 2 - e) - De(2 + a) \neq 0$.

3.2 Hopf bifurcation

It follows from Theorem 2.5 that system (1.3) has two equilibria (x_1^*, y_1^*) and (x_2^*, y_2^*) , (x_1^*, y_1^*) is a hyperbolic saddle and (x_2^*, y_2^*) is a weak focus or a center if $\max\{h_1, \frac{bD}{e}\} < h = h^* < \min\{h_0, \frac{1}{4}\}$, $be < abD + e$ and $(e - aD)(abD + e - be) > 2eD$, where h^* is given in Theorem 2.5. Hence, system (1.3) may undergo Hopf bifurcation. Thus, we discuss conditions under which the stability of the equilibrium (x_2^*, y_2^*) will change in such a way that system (x_2^*, y_2^*) undergoes an Hopf bifurcation. For simplicity, let $(x_2^*, y_2^*) = (x_*, y_*)$.

In order to determine the stability of the equilibrium (x_*, y_*) , we need to compute the Lyapunov coefficients of the equilibrium (x_*, y_*) . For this, we first translate (x_*, y_*) of system (1.3) to the origin by making a transformation of $X = x - x_*$, $Y = y - y_*$. And then, using Taylor expansions, system (1.3) can be rewritten as

$$\begin{aligned} X'(t) = & a_{10}X + a_{01}Y + a_{20}X^2 + a_{11}XY + a_{02}Y^2 \\ & + a_{30}X^3 + a_{21}X^2Y + a_{12}XY^2 + a_{03}Y^3 + O_1(X, Y), \\ Y'(t) = & b_{10}X + b_{01}Y + b_{20}X^2 + b_{11}XY + b_{02}Y^2 \\ & + b_{30}X^3 + b_{21}X^2Y + b_{12}XY^2 + b_{03}Y^3 + O_2(X, Y), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} a_{10} = & 1 - 2x_* - \frac{by_*(1 + y_*)}{(1 + ax_* + y_*)^2}, & a_{01} = & -\frac{bx_*(1 + ax_*)}{(1 + ax_* + y_*)^2}, & a_{20} = & -1 + \frac{aby_* + aby_*^2}{(1 + ax_* + y_*)^3}, \\ a_{11} = & -\frac{b(1 + ax_* + y_* + 2ax_*y_*)}{(1 + ax_* + y_*)^3}, & a_{02} = & \frac{bx_*(1 + ax_*)}{(1 + ax_* + y_*)^3}, & a_{30} = & \frac{-a^2by_* - a^2by_*^2}{(1 + ax_* + y_*)^4}, \\ a_{21} = & \frac{ab + a^2bx_* + 2a^2bx_*y_* - aby_*^2}{(1 + ax_* + y_*)^4}, & a_{12} = & \frac{b(1 - a^2x_*^2 + y_* + 2ax_*y_*)}{(1 + ax_* + y_*)^4}, \\ a_{03} = & -\frac{bx_*(1 + ax_*)}{(1 + ax_* + y_*)^4}, & b_{10} = & \frac{ey_*(1 + y_*)}{(1 + ax_* + y_*)^2}, & b_{01} = & -D + \frac{ex_*(1 + ax_*)}{(1 + ax_* + y_*)^2}, \\ b_{20} = & \frac{-aey_* - aey_*^2}{(1 + ax_* + y_*)^3}, & b_{11} = & \frac{e(1 + ax_* + y_* + 2ax_*y_*)}{(1 + ax_* + y_*)^3}, & b_{02} = & -\frac{ex_*(1 + ax_*)}{(1 + ax_* + y_*)^3}, \\ b_{30} = & \frac{a^2ey_* + a^2ey_*^2}{(1 + ax_* + y_*)^4}, & b_{21} = & \frac{-ae - a^2ex_* - 2a^2ex_*y_* + aey_*^2}{(1 + ax_* + y_*)^4}, \\ b_{12} = & -\frac{e(1 - a^2x_*^2 + y_* + 2ax_*y_*)}{(1 + ax_* + y_*)^4}, & b_{03} = & \frac{ex_*(1 + ax_*)}{(1 + ax_* + y_*)^4}, \end{aligned} \quad (3.4)$$

and $O_k(X, Y)$ are C^∞ functions with at least the fourth order with respect to (X, Y) for $k = 1, 2$. It is clear from Theorem 2.5 that $a_{10}b_{01} - a_{01}b_{10} \neq 0$ and $a_{10} + b_{01} = 0$ if $\max\{h_1, \frac{bD}{e}\} < h = h^* < \min\{\frac{1}{4}, h_0\}$ and $be < abD + e$. Hence using the formula of the first Lyapunov number σ at the origin of (3.3) in [15, p. 353], we have, after a tedious computation,

$$\begin{aligned} \sigma = -\frac{3\pi}{2a_{01}\Delta^{\frac{3}{2}}} & \left\{ [a_{10}b_{10}(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) \right. \\ & + b_{10}^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}(b_{02}^2 - a_{20}a_{02}) - 2a_{10}a_{01}(a_{20}^2 - b_{20}b_{02}) \\ & - a_{01}^2(2a_{20}b_{20} + b_{11}b_{20}) + (a_{01}b_{10} - 2a_{10}^2)(b_{11}b_{02} - a_{11}a_{20})] \\ & - (a_{10}^2 + a_{01}b_{10})[3(b_{10}b_{03} - a_{01}a_{30}) \\ & \left. + 2a_{10}(a_{21} + b_{12}) + (b_{10}a_{12} - a_{01}b_{21})] \right\}, \end{aligned} \quad (3.5)$$

where

$$\Delta = \frac{D((e - aD)x_* - D)(2ex_* + eb - e - abD)}{e^2x_*}. \quad (3.6)$$

With the aid of numerical calculation, we can see that the sign of σ is not determined. For instance, when $(a, b, D, e) = (1.1, 0.8, 0.1, 3.1)$, $\sigma = 1608.6$. On the other hand, when $(a, b, D, e) = (3.2, 0.2, 0.1, 3.2)$, $\sigma = -366.0$. Therefore, there exist surfaces H_b and H_p in the parameter space (a, b, D, e, h) which satisfy

$$\begin{aligned} H_b = & \left\{ (a, b, D, e, h) \mid \max\{h_1, \frac{bD}{e}\} < h = h^* < \min\{h_0, \frac{1}{4}\}, be < abD + e, \right. \\ & \left. (e - aD)(abD + e - be) > 2eD, \sigma > 0 \right\}, \\ H_p = & \left\{ (a, b, D, e, h) \mid \max\{h_1, \frac{bD}{e}\} < h = h^* < \min\{h_0, \frac{1}{4}\}, be < abD + e, \right. \\ & \left. (e - aD)(abD + e - be) > 2eD, \sigma < 0 \right\}. \end{aligned} \quad (3.7)$$

Thus, if the parameter (a, b, D, e, h) is in $H_b(H_p)$, the equilibrium (x_*, y_*) of system (1.3) is a weak focus of multiplicity 1 and is unstable(stable) (see [11, 19–21]). In this case, the surfaces H_b and H_p are called *the subcritical and supercritical Hopf bifurcation surface* of system (1.3), respectively.

Now, assume that conditions $\max\{h_1, \frac{bD}{e}\} < h^* < \min\{\frac{1}{4}, h_0\}$, $be < abD + e$ and $(e - aD)(abD + e - be) > 2eD$ hold. Then, from Theorem 2.5, we know that the equilibrium (x_*, y_*) is a hyperbolic stable focus if $\max\{h_1, \frac{bD}{e}\} < h < h^*$ and is a hyperbolic unstable focus if $h^* < h < \min\{\frac{1}{4}, h_0\}$.

Summarizing the above discussion, we have the following theorem.

Theorem 3.1. *Suppose that the conditions $\max\{h_1, \frac{bD}{e}\} < h^* < \min\{\frac{1}{4}, h_0\}$, $be < abD + e$ and $(e - aD)(abD + e - be) > 2eD$ hold and let $V_b = \{(a, b, D, e) \mid \max\{h_1, \frac{bD}{e}\} < h < \min\{h_0, \frac{1}{4}\}, be < abD + e, (e - aD)(abD + e - be) > 2eD, \sigma > 0\}$ and $V_p = \{(a, b, D, e) \mid \max\{h_1, \frac{bD}{e}\} < h < \min\{h_0, \frac{1}{4}\}, be < abD + e, (e - aD)(abD + e - be) > 2eD, \sigma < 0\}$. Then (i) system (1.3) has at least one unstable limit cycle if $\max\{h_1, \frac{bD}{e}\} < h < h^*$, $|h - h^*| \ll 1$ and (a, b, D, e) in V_b and (ii) system (1.3) has at least one stable limit cycle if $h^* < h < \min\{\frac{1}{4}, h_0\}$, $|h - h^*| \ll 1$ and (a, b, D, e) in V_p .*

3.3 Bogdanov–Takens bifurcation

From Theorem 2.4, we can see that system (1.3) could have a positive equilibrium which is a saddle-node or a cusp. Especially, the part (ii) of Theorem 2.4 shows that the positive equilibrium (x_0^*, y_0^*) is a cusp of codimension 2 under some conditions, which implies that there may exist a Bogdanov–Takens bifurcation. In order to discuss such bifurcation phenomena of system (1.3) we need to fix some values of parameters for system (1.3) since the condition given in Theorem 2.4 are too complicated to analyze the number of codimension of the unique positive equilibrium. For the reason, in this section, we fix $a = 3, b = 1$ to investigate a Bogdanov–Takens bifurcation of system (1.3) in the parameter space (D, e, h) . Although we will deal with the special case $a = 3, b = 1$, the procedure of investigating for bifurcations of system (1.3) can be used to discuss bifurcations for general cases.

It follows from Theorem 2.4 that $(x_0^*, y_0^*) = (\frac{3D}{2e}, \frac{e-9D}{2e})$ is a cusp of codimension 2 if $h = h_0 < \frac{1}{4}$, $e > 9D$ and $D = D_0 > 0$, where $D_0 = \frac{e(e-3)+e\sqrt{e^2-78e+225}}{18(e-3)}$. Thus there exists a parameter space

$$BT = \{(D, e, h) \mid h = h_0 < \frac{1}{4}, e > 9D \text{ and } D = D_0 > 0\}.$$

In order to show that system (1.3) undergoes a Bogdanov–Takens bifurcation the values D and h can be chosen as bifurcation parameters. We need to find the universal unfolding of (x_0^*, y_0^*) . To do this let $h = h_0 - \mu_1$ and $D = D_0 - \mu_2$ in system (1.3), where μ_1 and μ_2 are very small parameters. Then for $(D, e, h) \in BT$ we can have the following system

$$\begin{cases} x'(t) = x(1-x) - \frac{xy}{y+3x+1} - h_0 + \mu_1, \\ y'(t) = -(D_0 - \mu_2)y + \frac{exy}{y+3x+1}, \end{cases} \quad (3.8)$$

where $D_0 = \frac{e(e-3)+e\sqrt{e^2-78e+225}}{18(e-3)}$, $h_0 = \frac{D_0(9D_0+4e)}{4e^2}$ and $e > 9D_0$. Translating (x_0^*, y_0^*) to the origin by letting $x_1 = x - x_0^*$ and $y_1 = y - y_0^*$ in (3.8). Using the Taylor expansion system (3.8) can be written as

$$\begin{cases} x_1'(t) = \alpha_0 + \alpha_1 x_1 + \alpha_2 y_1 + \alpha_3 x_1^2 + \alpha_4 x_1 y_1 + \alpha_5 y_1^2 + p_1(x_1, y_1), \\ y_1'(t) = \beta_0 + \beta_1 x_1 + \beta_2 y_1 + \beta_3 x_1^2 + \beta_4 x_1 y_1 + \beta_5 y_1^2 + q_1(x_1, y_1), \end{cases} \quad (3.9)$$

where p_1 and q_1 are C^∞ functions of x_1, y_1 of order of at least three and

$$\begin{aligned} \alpha_0 &= \mu_1, & \alpha_1 &= \frac{2}{3} - \frac{9D_0^2}{e^2} + \frac{D_0}{e}, & \alpha_2 &= -\frac{3D_0^2}{e^2} - \frac{2D_0}{3e}, & \alpha_3 &= -\frac{1}{3} + \frac{18D_0^2}{e^2} - \frac{8D_0}{e}, \\ \alpha_4 &= -\frac{4}{9} + \frac{12D_0^2}{e^2} - \frac{4D_0}{3e}, & \alpha_5 &= \frac{2D_0^2}{e^2} + \frac{4D_0}{9e}, & \beta_0 &= \mu_2 \left(\frac{1}{2} - \frac{9D_0}{2e} \right), \\ \beta_1 &= -4D_0 + \frac{9D_0^2}{e} + \frac{e}{3}, & \beta_2 &= -\frac{D_0}{3} + \frac{3D_0^2}{e} + \mu_2, & \beta_3 &= 8D_0 - \frac{18D_0^2}{e} - \frac{2e}{3}, \\ \beta_4 &= \frac{4D_0}{3} - \frac{12D_0^2}{e} + \frac{4e}{9}, & \text{and } \beta_5 &= -\frac{4D_0}{9} - \frac{2D_0^2}{e}. \end{aligned}$$

Consider C^∞ changes of coordinates in a small neighborhood of $(0, 0)$ as follows;

$$\begin{aligned} x_2 &= x_1, & y_2 &= \alpha_1 x_1 + \alpha_2 y_1 + \alpha_3 x_1^2 + \alpha_4 x_1 y_1 + \alpha_5 y_1^2 & \text{and} \\ x_3 &= x_2, & y_3 &= \alpha_0 + y_2. \end{aligned} \quad (3.10)$$

Then system (3.9) can be written as

$$\begin{cases} x'_3(t) = y_3 + p_3(x_3, y_3), \\ y'_3(t) = \delta_0 + \delta_1 x_3 + \delta_2 y_3 + \delta_3 x_3^2 + \delta_4 x_3 y_3 + \delta_5 y_3^2 + q_3(x_3, y_3), \end{cases} \quad (3.11)$$

where p_3 and q_3 are C^∞ functions of x_3, y_3 of order at least three and $\delta_0 = \gamma_1 - \alpha_0 \gamma_2 + \alpha_0^2 \gamma_5$, $\delta_1 = \gamma_1 - \alpha_0 \gamma_4$, $\delta_2 = \gamma_2 - 2\alpha_0 \gamma_5$, $\delta_3 = \gamma_3$, $\delta_4 = \gamma_4$, $\delta_5 = \gamma_5$,

$$\begin{aligned} \gamma_0 &= \alpha_0 \alpha_1 - \frac{\alpha_0 \alpha_2 \alpha_4}{\alpha_5} - \alpha_2 \beta_0 + \frac{\alpha_2^2 \beta_2}{\alpha_5} - \frac{\alpha_2^3 \beta_5}{\alpha_5^2}, \\ \gamma_1 &= 2\alpha_0 \alpha_3 + \frac{\alpha_0 \alpha_1 \alpha_4}{\alpha_2} - \frac{\alpha_0 \alpha_4^2}{\alpha_5} - \alpha_4 \beta_0 + \frac{2\alpha_1 \alpha_5 \beta_0}{\alpha_2} - \alpha_2 \beta_1 - 3\alpha_1 \beta_2 + \frac{2\alpha_2 \alpha_4 \beta_2}{\alpha_5} - \frac{3\alpha_2^2 \alpha_4 \beta_5}{\alpha_5^2} \\ &\quad + \frac{\alpha_2^2 \beta_4}{\alpha_5} + \frac{4\alpha_1 \alpha_2 \beta_5}{\alpha_5}, \\ \gamma_2 &= \alpha_1 - \frac{\alpha_0 \alpha_4}{\alpha_2} - \frac{\alpha_2 \alpha_4}{\alpha_5} - \frac{2\alpha_5 \beta_0}{\alpha_2} + 3\beta_2 - \frac{4\alpha_2 \beta_5}{\alpha_5}, \\ \gamma_3 &= \frac{\alpha_0 \alpha_3 \alpha_4^2}{\alpha_2} - \frac{\alpha_0 \alpha_1 \alpha_4^2}{\alpha_2^2} + \frac{\alpha_0 \alpha_1^2 \alpha_4 \alpha_5}{\alpha_2^3} + \frac{2\alpha_3 \alpha_5 \beta_0}{\alpha_2} \\ &\quad - \frac{2\alpha_1 \alpha_4 \alpha_5 \beta_0}{\alpha_2^2} + \frac{2\alpha_1^2 \alpha_5^2 \beta_0}{\alpha_2^3} - \alpha_4 \beta_1 + \frac{2\alpha_1 \alpha_5 \beta_1}{\alpha_2} - 3\alpha_3 \beta_2 + \frac{\alpha_4^2 \beta_2}{\alpha_5} - \frac{\alpha_1^2 \alpha_5 \beta_2}{\alpha_2^2} - \alpha_2 \beta_3 - 3\alpha_1 \beta_4 \\ &\quad + \frac{2\alpha_2 \alpha_4 \beta_4}{\alpha_5} - \frac{\alpha_1^2 \beta_5}{\alpha_2} - \frac{3\alpha_2 \alpha_4^2 \beta_5}{\alpha_5^2} + \frac{4\alpha_2 \alpha_3 \beta_5}{\alpha_5} + \frac{4\alpha_1 \alpha_4 \beta_5}{\alpha_5}, \\ \gamma_4 &= 2\alpha_3 + \frac{\alpha_1 \alpha_4}{\alpha_2} + \frac{\alpha_0 \alpha_4^2}{\alpha_2^2} - \frac{\alpha_4^2}{\alpha_5} - \frac{2\alpha_0 \alpha_1 \alpha_4 \alpha_5}{\alpha_2^3} + \frac{2\alpha_4 \alpha_5 \beta_0}{\alpha_2^2} - \frac{4\alpha_1 \alpha_5^2 \beta_0}{\alpha_2^3} - \frac{2\alpha_5 \beta_1}{\alpha_2} + \frac{2\alpha_1 \alpha_5 \beta_2}{\alpha_2^2} \\ &\quad + 3\beta_4 + \frac{2\alpha_1 \beta_5}{\alpha_2} - \frac{4\alpha_4 \beta_5}{\alpha_5}, \\ \gamma_5 &= -\frac{\alpha_4}{\alpha_2} + \frac{\alpha_0 \alpha_4 \alpha_5}{\alpha_2^3} + \frac{2\alpha_5^2 \beta_0}{\alpha_2^3} - \frac{\alpha_5 \beta_2}{\alpha_2^2} - \frac{\beta_5}{\alpha_2}. \end{aligned}$$

Next, by letting $s = \frac{t}{1 - \delta_5 x_3}$, $x_4 = x_3$ and $y_4 = (1 - \delta_5 x_3)y_3$ in (3.11) and rewriting s as t , system (3.11) can be transformed into

$$\begin{cases} x'_4(t) = y_4 + p_4(x_4, y_4), \\ y'_4(t) = \lambda_0 + \lambda_1 x_4 + \lambda_2 y_4 + \lambda_3 x_4^2 + \lambda_4 x_4 y_4 + q_4(x_4, y_4), \end{cases} \quad (3.12)$$

where p_4 and q_4 are C^∞ functions of x_4, y_4 of order at least three and $\lambda_0 = \delta_0$, $\lambda_1 = \delta_1 - 2\delta_0 \delta_5$, $\lambda_2 = \delta_2$, $\lambda_3 = \delta_3 - 2\delta_1 \delta_5 + \delta_0 \delta_5^2$ and $\lambda_4 = \delta_4 - \delta_2 \delta_5$. Let $\tau_0 = \lambda_0 - \frac{\lambda_1^2}{2\lambda_3} + \frac{\lambda_1^3}{4\lambda_3^2}$, $\tau_1 = \lambda_5 - \frac{\lambda_1 \lambda_4}{2\lambda_3}$, $\tau_2 = \lambda_3$ and $\tau_3 = \lambda_4$. Taking the change of variables

$$\begin{aligned} x_5 &= x_4 + \frac{\lambda_1}{2\lambda_3}, \quad y_5 = y_4 + p_4(x_4, y_4), \\ x_6 &= \frac{\tau_3}{\tau_2} x_5, \quad y_6 = \frac{\tau_3}{\tau_2} y_5, \quad \bar{t} = \frac{\tau_3}{\tau_2} t, \end{aligned} \quad (3.13)$$

we obtain the following system by setting \bar{t} as t ;

$$\begin{cases} x'_6(t) = y_6, \\ y'_6(t) = \omega_1(\mu_1, \mu_2) + \omega_2(\mu_1, \mu_2)y_1 + x_6^2 + x_6 y_6 + q_6(x_6, y_6), \end{cases} \quad (3.14)$$

where q_6 is a C^∞ function of x_6, y_6 of order at least three and $\omega_1(\mu_1, \mu_2) = \frac{\tau_0 \tau_3^4}{\tau_2^3}$ and $\omega_2(\mu_1, \mu_2) = \frac{\tau_1 \tau_3}{\tau_2}$. Therefore, we have the following theorem.

Theorem 3.2. *If $a = 3, b = 1, h_0 < \frac{1}{4}, e > 9D, 0 < |h - h_0| \ll 1$ and $0 < |D - D_0| \ll 1$, then system (1.3) undergoes a Bogdanov–Takens bifurcation. Hence there exist values of the parameters (D, e, h) such that system (1.3) has a unique unstable limit cycle for some parameter values, and system (1.3) has an unstable homoclinic loop for other parameter values.*

4 Numerical examples

In this section, we will illustrate some numerical examples to substantiate the validity of our theoretical results obtained in the previous sections.

Example 4.1. First let parameters a, b and e in system (1.3) be as follows;

$$a = 3, \quad b = 1 \quad \text{and} \quad e = 76. \quad (4.1)$$

If we take the value $D = \frac{e(e-3) - e\sqrt{e^2 - 78e + 225}}{18(e-3)}$, then it is not hard to see that system (1.3) satisfies the hypotheses of Theorem 2.4 and the condition $a^3 b^2 D^2 - a^2 b D e(-1 + 2b + D) + a e(2bD + e(-D + b(-1 + b + 2D))) + e^2(-2D + e - b(2 + e)) = 0$ hold. Also we can obtain $D \approx 3.7280$ and $h_0 \approx 0.0013$. Thus it follows from Theorem 2.4 that the equilibrium $(x_0^*, y_0^*) \approx (0.0736, 0.2797)$ is a cusp of codimension 2. Now, in order to simulate this phenomena numerically, we use two initial values $(x_0^* + 0.00001, y_0^* - 0.015) \approx (0.07361, 0.2643)$ and $(x_0^* + 0.00001, y_0^* - 0.01) \approx (0.07361, 0.2693)$. By means of Runge–Kutta method of order 4, we can draw Figure 4.1 numerically.

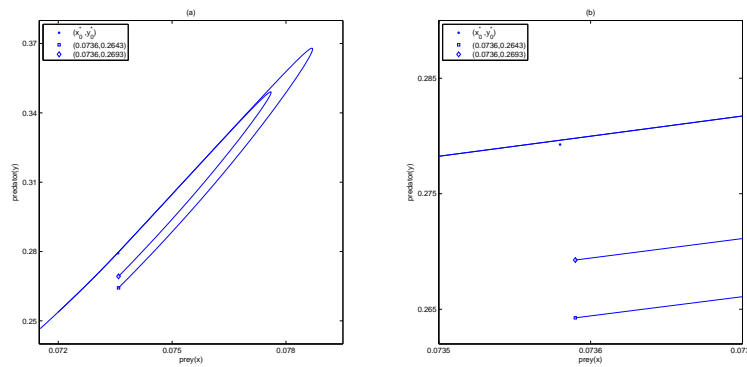


Figure 4.1: Phase portraits of system (1.3) when $a = 3, b = 1, e = 76$ and $D \approx 3.7280, h \approx 0.0013$. Here $(x_0^*, y_0^*) \approx (0.0736, 0.2797)$. (a) Two trajectories of system (1.3) starting with $(0.07361, 0.2643)$ and $(0.07361, 0.2693)$. (b) The enlarged part of (a) for $0.0735 \leq x \leq 0.0737$.

Example 4.2. In order to ascertain the existence of a center equilibrium point mentioned in Theorem 2.5, we will set some parameters in system (1.3) as

$$a = 0.8, \quad b = 0.4, \quad D = 0.2 \quad \text{and} \quad e = 1. \quad (4.2)$$

Then we can easily check that the hypotheses of Theorem 2.5 are satisfied. Moreover, we can obtain $(x_2^*, y_2^*) \approx (0.4067, 0.7081)$ and $h^* \approx 0.1846$. If the value of h is set to be as $h^* - 0.001 \approx$

0.1836, then it follows from Theorem 2.5 (iv) that (x_2^*, y_2^*) is a stable focus which is shown in Figure 4.2(a). On the other hand, if we take $h = h^*$, then the equilibrium (x_2^*, y_2^*) becomes a center shown in Figure 4.2(b).

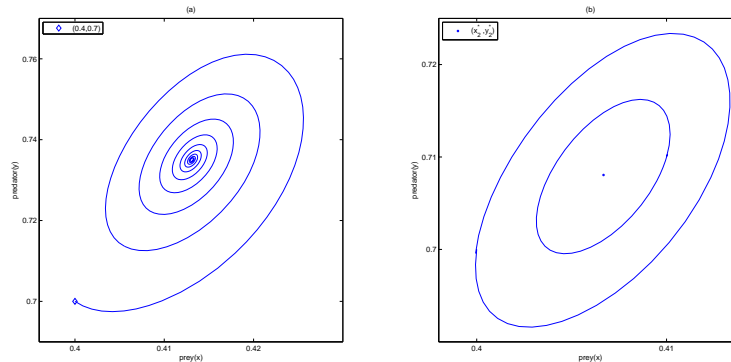


Figure 4.2: Phase portraits of system (1.3) when $a = 0.8, b = 0.4, D = 0.2$ and $e = 1.0$. (a) A trajectory of system (1.3) when $h = h^* - 0.001$ and an initial value $(0.4, 0.7)$. (b) Two trajectories of system (1.3) when $h = h^*$ and two initial values $(0.4, 0.7)$ and $(0.41, 0.71)$

Example 4.3. In the example, we will present a numerical evidence of the existence of a limit cycle stated in Theorem 3.1. For this, we set parameters a, b, D and e in system (1.3) as

$$a = 1.1, \quad b = 0.8, \quad D = 0.1 \quad \text{and} \quad e = 3.1. \quad (4.3)$$

Then from tedious calculation, we obtain the value $h^* \approx 0.0363$ and $\sigma \approx 1608.62933 > 0$. Also we can easily see that the hypotheses of Theorem 3.1 (i) hold. Thus we know that there exists a unstable limit cycle of system (1.3) for a suitable value h . By using Runge–Kutta 4th order method with negative step size, we illustrate such a limit cycle in Figure 4.3 when $h = h^* - 0.0005 \approx 0.0358$ and two initial values $(x_2^* + 0.01, y_2^* + 0.01) \approx (0.1796, 4.0816)$ and $(x_2^* + 0.1, y_2^* + 0.1) \approx (0.2696, 4.1716)$.

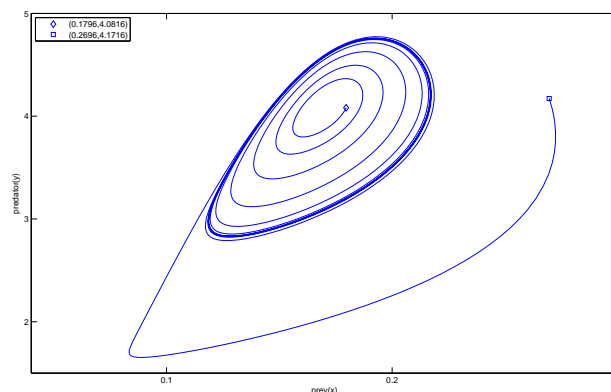


Figure 4.3: A phase portrait of a unstable limit cycle of system (1.3) when $a = 1.1, b = 0.8, D = 0.1, e = 3.1, h = h^* - 0.0005$ and two initial values $(0.1796, 4.0816)$ and $(0.2696, 4.1716)$.

5 Conclusion and discussion

In summary, we have analyzed dynamical behaviors of the Beddington–DeAngelis predator–prey system with constant harvesting rate. We have shown that there exist some parameter regions in which both predator and prey species can be existed or extinct simultaneously. Also, it has been obtained some sufficient conditions for the existence of limit cycles of the system via Hopf bifurcation and Bogdanov–Takens bifurcation. Finally, we have displayed some numerical examples to testify our theoretical results.

In this paper, following the work of L. Ling and W. Wang in [10], M. Haque in [6], J. Xia, Z. Liu, R. Yuan and S. Ruan in [18], D. Xiao and L. S. Jennings in [19], and D. Xiao, W. Li and M. Han in [20] we continue investing the harvesting effect on the dynamics of a predator–prey system with Beddington–DeAngelis functional response. As a result we figure out that harvesting may play a significant role in dynamics of a predator–prey system.

According to conditions of parameter values, we have investigated the existence of equilibrium points and the stabilities of them. In Lemma 2.1 we have considered parameter values in almost all cases, but some parameter values are not dealt with because they are very minor cases and their classification method is similar to Lemma 2.1 as shown in Lemma 5.1.

Lemma 5.1. *Let $A = 1 - \frac{b(e-aD)}{e}$, $B = h - \frac{bD}{e}$ and $h_0 = \frac{A^2}{4} + \frac{bD}{e}$, and assume that $be < abD + e$ and $(e - aD)(e + abD - be) > 2eD$. Then*

- (i) *system (1.3) has a unique equilibrium (x_0^*, y_0^*) in \mathbb{R}_+^2 if $\frac{1}{4} < h = h_0$,*
- (ii) *system (1.3) has two equilibria (x_1^*, y_1^*) and (x_2^*, y_2^*) in \mathbb{R}_+^2 if $\frac{1}{4} < h < h_0$,*
- (iii) *system (1.3) has three equilibria (x_0, y_0) , (x_1^*, y_1^*) and (x_2^*, y_2^*) in \mathbb{R}_+^2 if $\frac{1}{4} = h < h_0$.*

Thus using Lemma 5.1 and mathematical methods used in this paper, one can acquire similar results to those obtained in the paper. However we will not mention it since it could be very routine and tedious.

As mentioned in the abstract, by letting the parameters $\alpha = 0$ and $\beta = 0$, and adding the harvesting rate on the prey species in system (1.1), we can obtain the following two kinds of predator–prey systems (5.1) and (5.2) which are called a ratio-dependent system with the harvesting rate and a Holling-type II system with the harvesting rate, respectively.

$$\begin{cases} x'(t) = rx \left(1 - \frac{x}{K}\right) - \frac{\gamma xy}{\beta y + x} - h, \\ y'(t) = -dy + \frac{\delta xy}{\beta y + x}, \end{cases} \quad (5.1)$$

$$\begin{cases} x'(t) = rx \left(1 - \frac{x}{K}\right) - \frac{\gamma xy}{\beta y + \alpha} - h, \\ y'(t) = -dy + \frac{\delta xy}{\beta y + \alpha}. \end{cases} \quad (5.2)$$

Also, in [6], in order to account for the influence of intra-species competition among the predator population in system (1.1), the author considers the following predator–prey system.

$$\begin{cases} x'(t) = rx \left(1 - \frac{x}{K}\right) - \frac{\gamma xy}{\beta y + x + \alpha}, \\ y'(t) = -dy + \frac{\delta xy}{\beta y + x + \alpha} - hy^2. \end{cases} \quad (5.3)$$

Systems	Existence	LAS	SNB	HB	TB	BTB	References
(1.1)	Yes	Yes	No	Yes	No	No	[6,8]
(1.3)	Yes	Yes	Yes	Yes	No	Yes	–
(5.1)	Yes	Yes	Yes	Yes	No	Yes	[19]
(5.2)	Yes	Yes	Yes	Yes	No	Yes	[14,18]
(5.3)	Yes	Yes	Yes	Yes	No	Yes	[6]

Table 5.1: This table lists the stability and bifurcation of all systems (1.1), (1.3), (5.1), (5.2) and (5.3).

Although system (5.3) is a predator–prey system with competition effect on the predator, we may regard the system as a predator–prey system with harvesting term since the predator population could decrease due to the term hy^2 . Now we will compare the dynamics of these systems around equilibrium points. We will focus on a positive equilibrium state among equilibria since the existence of a positive equilibrium state could give rise to the coexistence of all species, which is a biologically significant fact from the point of view of biodiversity. In fact, all systems which mentioned above have at least one positive equilibrium like (x_i^*, y_i^*) , $i = 0, 1, 2$. Now we summarize the dynamical behavior of systems around positive equilibrium points in Table 5.1 by letting LAS, SNB, HB, TB and BTB stand for locally asymptotically stable, and saddle-node bifurcation, Hopf bifurcation, transcritical bifurcation and Bogdanov–Takens bifurcations, respectively. Comparing the first row to the second and fifth row of Table 5.1, we figure out that predator–prey systems with harvesting term exhibit much richer dynamics than the system without harvesting. From the second, third and fourth rows, we assert that predator–prey systems with harvesting rate on the prey species have similar dynamical behaviors around its positive equilibrium points even if their functional responses are different from each other. From Table 5.1, we conclude that the harvest of any species in a predator–prey system can make various dynamical behaviors regardless of functional responses.

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