Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **8**, 1–9; doi: 10.14232/ejqtde.2017.1.8 http://www.math.u-szeged.hu/ejqtde/

On the solvability of a boundary value problem for p-Laplacian differential equations

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Received 13 September 2016, appeared 27 January 2017
Communicated by Paul Eloe

Abstract. Using barrier strip conditions, we study the existence of $C^2[0,1]$-solutions of the boundary value problem $(\phi_p(x'))' = f(t,x,x')$, $x(0) = A$, $x'(1) = B$, where $\phi_p(s) = s|s|^{p-2}$, $p > 2$. The question of the existence of positive monotone solutions is also affected.

Keywords: boundary value problem, second order differential equation, p-Laplacian, existence, sign conditions.

2010 Mathematics Subject Classification: 34B15, 34B18.

1 Introduction

This paper is devoted to the solvability of the boundary value problem (BVP)

$$(\phi_p(x'))' = f(t,x,x'), \quad t \in [0,1], \tag{1.1}$$
$$x(0) = A, \quad x'(1) = B. \tag{1.2}$$

Here $\phi_p(s) = s|s|^{p-2}$, $p > 2$, the scalar function $f(t,x,y)$ is defined for $(t,x,y) \in [0,1] \times D_x \times D_y$, where the sets $D_x, D_y \subseteq \mathbf{R}$ may be bounded, and $B \geq 1$. Besides, f is continuous on a suitable subset of its domain.

The solvability of various singular and nonsingular BVPs with p-Laplacian has been studied, for example, in [1–5, 7–12, 14]. Conditions used in these works or do not allow the main nonlinearity to change sign, [2, 11], or impose a growth restriction on it, [3, 9, 11], or require the existence of upper and lower solutions, [1, 3, 5, 8, 9, 12]; other type conditions have been used in [7], where the main nonlinearity may changes its sign. As a rule, the obtained results guarantee the existence of positive solutions.

Another type of conditions have been used in [10] for studying the solvability of (1.1), (1.2) in the case $p \in (1,2)$. The existence of at least one positive and monotone $C^2[0,1]$-solution is established therein under the following barrier condition:

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H. There are constants L_i, F_i, $i = 1, 2$, and a sufficiently small $\sigma > 0$ such that

$$F_1 \geq F_2 + \sigma, \quad F_1 - \sigma > 0, \quad L_2 - \sigma \geq L_1,$$

$$[A - \sigma, L + \sigma] \subseteq D_x, \quad [F_2, L_2] \subseteq D_y, \quad \text{where } L = L_1 + |A|,$$

$$f(t, x, y) \geq 0 \quad \text{for } (t, x, y) \in [0, 1] \times D_x \times [L_1, L_2], \tag{1.3}$$

$$f(t, x, y) \leq 0 \quad \text{for } (t, x, y) \in [0, 1] \times D_A \times [F_2, F_1], \tag{1.4}$$

where the constants m and M are, respectively, the minimum and the maximum of $f(t, x, p)$ on $[0, 1] \times [A - \sigma, L + \sigma] \times [F_1 - \sigma, L_1 + \sigma]$ and $D_A = (-\infty, L] \cap D_x$.

Let us recall, the strips $[0, 1] \times [L_1, L_2]$ and $[0, 1] \times [F_2, F_1]$ are called "barrier" because they limit the values of the first derivatives of all $C^2[0, 1]$-solution of (1.1), (1.2) between themselves. Recently, it was shown in [13] that conditions of form (1.3) and (1.4) guarantee $C^1[0, 1]$-solutions to the ϕ-Laplacian equation

$$(\phi(x'))' = f(t, x, x'), \qquad t \in (0, 1),$$

with boundary conditions (1.2), where $\phi : \mathbf{R} \to \mathbf{R}$ is an increasing homeomorphism and $f : [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ is continuous.

It turned out that the cases $1 < p < 2$ and $p > 2$ require different technical approaches for the use of **H** for studying the solvability of (1.1), (1.2). So, in the present paper we show that **H** with the additional requirement

$$B - M \geq F_1 \tag{1.5}$$

guarantees the existence of at least one monotone, and positive in the case $A \geq 0$, $C^2[0, 1]$-solution to (1.1), (1.2) with $p > 2$. In fact, our main result is the following.

Theorem 1.1. *Let* **H** *and* (1.5) *hold, and $f(t, x, y)$ be continuous on the set $[0, 1] \times [A - \sigma, L + \sigma] \times [F_1 - \sigma, L_1 + \sigma]$. Then BVP* (1.1), (1.2) *has at least one strictly increasing solution in $C^2[0, 1]$ for each $p \in (2, \infty)$.*

The paper is organized as follows. In Section 2 we present preliminaries needed to formulate the Topological Transversality Theorem, which is our basic tool, and prove auxiliary results. In Section 3 we give the proof of Theorem 1.1, formulate a corollary and give an example.

2 Fixed point theorem, auxiliary results

Let K be a convex subset of a Banach space E and $U \subset K$ be open in K. Let $\mathbf{L}_{\partial U}(\overline{U}, K)$ be the set of compact maps from $\overline{U}$ to K which are fixed point free on ∂U; here, as usual, $\overline{U}$ and ∂U are the closure of U and boundary of U in K.

A map F in $\mathbf{L}_{\partial U}(\overline{U}, K)$ is essential if every map G in $\mathbf{L}_{\partial U}(\overline{U}, K)$ such that $G/\partial U = F/\partial U$ has a fixed point in U. It is clear, in particular, every essential map has a fixed point in U.

The following fixed point theorem due to A. Granas et al. [6].

Theorem 2.1 (Topological transversality theorem)**.** *Suppose:*

(i) $F, G : \overline{U} \to K$ *are compact maps;*

(ii) $G \in \mathbf{L}_{\partial U}(\overline{U}, K)$ *is essential;*

(iii) $H(x, \lambda), \lambda \in [0, 1]$, *is a compact homotopy joining G and F, i.e.* $H(x, 0) = G(x)$ *and* $H(x, 1) = F(x)$;

(iv) $H(x, \lambda), \lambda \in [0, 1]$, *is fixed point free on* ∂U.

Then $H(x, \lambda), \lambda \in [0, 1]$, *has at least one fixed point in U and in particular there is a* $x_0 \in U$ *such that* $x_0 = F(x_0)$.

The following results is important for our consideration. It can be found also in [6].

Theorem 2.2. *Let* $l \in U$ *be fixed and* $F \in \mathbf{L}_{\partial U}(\overline{U}, K)$ *be the constant map* $F(x) = l$ *for* $x \in \overline{U}$. *Then F is essential.*

Further, we need the following fact.

Proposition 2.3. *Let the constants B and M be such that* $B \geq 1$ *and* $B > M > 0$. *Then*

$$(B - M)^r \leq B^r - M \quad \textit{for } r \in [1, \infty).$$

Proof. The inequality is evident for $r = 1$. For $M \in (0, B)$ consider the function $g(r) = (B - M)^r - B^r + M$, $r \in (1, \infty)$. First, let $B - M \in (0, 1)$. Then $\ln(B - M) < 0$ and so

$$g'(r) = (B - M)^r \ln(B - M) - B^r \ln B < 0 \quad \text{for } r \in \mathbf{R}.$$

Next, assume $B - M = 1$. Now we get

$$g'(r) = -(1 + M)^r \ln(1 + M) < 0 \quad \text{for } r \in \mathbf{R}.$$

Finally, let $B - M \in (1, \infty)$. In this case from $B > B - M > 0$ we have $B^r \geq (B - M)^r$ for $r \in [0, \infty)$ and so

$$g'(r) \leq B^r \ln(B - M) - B^r \ln B = B^r \ln \frac{B - M}{B} < 0 \quad \text{for } r \in [0, \infty).$$

In summary, we have proved that $g'(r) < 0$ for each $r \in [0, \infty)$. Then, the result follows from the fact that $g(1) = 0$. □

Let us emphasize explicitly that we conduct the rest consideration of this section for an arbitrary fixed $p > 2$.

For $\lambda \in [0, 1]$ consider the family of BVPs

$$\begin{cases} (\phi_p(x'))' = \lambda f(t, x, x'), & t \in [0, 1], \\ x(0) = A, \ x'(1) = B, & B \geq 1, \end{cases} \tag{2.1}$$

where $f : [0, 1] \times D_x \times D_y \to \mathbf{R}$, $D_x, D_y \subseteq \mathbf{R}$. Since

$$\phi_p(s) = s|s|^{p-2} = \begin{cases} s^{p-1}, & s \geq 0, \\ -(-s)^{p-1}, & s < 0, \end{cases}$$

we have

$$\phi_p'(s) = \begin{cases} (p-1)s^{p-2}, & s \ge 0 \\ (p-1)(-s)^{p-2}, & s < 0 \end{cases} = (p-1)|s|^{p-2}$$

and $(\phi_p(x'(t)))' = (p-1)|x'(t)|^{p-2}x''(t)$, if $x''(t)$ exists. So, we can write (2.1) as

$$\begin{cases} (p-1)|x'(t)|^{p-2}x''(t) = \lambda f(t,x,x'),\ t \in [0,1], \\ x(0) = A,\ x'(1) = B. \end{cases} \tag{2.1'}$$

For convenience set

$$m_p = \frac{m}{(p-1)(F_1-\sigma)^{p-2}} \quad \text{and} \quad M_p = \frac{M}{(p-1)(F_1-\sigma)^{p-2}},$$

where F_1, σ, m and M are as in **H**.

The next result gives a priori bounds for the $C^2[0,1]$-solutions of family (2.1′) (as well as of (2.1)).

Lemma 2.4. *Let* **H** *hold and $x \in C^2[0,1]$ be a solution to family* (2.1′)*. Then*

$$A \le x(t) \le L,\ F_1 \le x'(t) \le L_1 \ \textit{and}\ \ m_p \le x''(t) \le M_p \ \textit{for}\ \ t \in [0,1].$$

Proof. The proof of the bounds for x and x' is the same as the corresponding part of the proof of [10, Lemma 3.1], but we will state it for completeness. So, assume on the contrary that

$$x'(t) \le L_1 \quad \text{for } t \in [0,1] \tag{2.2}$$

is not true. Then, $x'(1) = B \le L_1$ together with $x' \in C[0,1]$ implies that

$$S_+ = \{t \in [0,1] : L_1 < x'(t) \le L_2\}$$

is not empty. Moreover, there exists an interval $[\alpha, \beta] \subset S_+$ with the property

$$x'(\alpha) > x'(\beta). \tag{2.3}$$

Then, by the fundamental theorem of calculus applied to x', (2.3) implies that there is a $\gamma \in (\alpha, \beta)$ such that

$$x''(\gamma) < 0.$$

We have $(\gamma, x(\gamma), x'(\gamma)) \in S_+ \times D_x \times (L_1, L_2]$, which yields

$$f(\gamma, x(\gamma), x'(\gamma)) \ge 0,$$

by (1.3). Then,

$$0 > (p-1)|x'(\gamma)|^{p-2}x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) \ge 0 \quad \text{for } \lambda \in [0,1],$$

a contradiction. Thus, (2.2) is true.

By the mean value theorem, for each $t \in (0,1]$ there exists $\xi \in (0,t)$ such that $x(t) - x(0) = x'(\xi)t$, which yields

$$x(t) \le L \quad \text{for } t \in [0,1].$$

Arguing as above and using (1.4), we establish $x'(t) \ge F_1$ for all $t \in [0,1]$ and, as a consequence, $x(t) \ge A$ on $[0,1]$.

To reach the bounds for $x''(t)$ from

$$x'(t) > F_1 - \sigma > 0, \qquad t \in [0,1],$$

we obtain firstly

$$0 < \frac{1}{(p-1)(x'(t))^{p-2}} \leq \frac{1}{(p-1)(F_1-\sigma)^{p-2}}.$$

Next, multiplying both sides of this inequality by $\lambda M \geq 0$ and $\lambda m \leq 0$, for $t \in [0,1]$ obtain respectively

$$\frac{\lambda M}{(p-1)(x'(t))^{p-2}} \leq \frac{\lambda M}{(p-1)(F_1-\sigma)^{p-2}} \leq \frac{M}{(p-1)(F_1-\sigma)^{p-2}} = M_p,$$

and

$$\frac{\lambda m}{(p-1)(x'(t))^{p-2}} \geq \frac{\lambda m}{(p-1)(F_1-\sigma)^{p-2}} \geq \frac{m}{(p-1)(F_1-\sigma)^{p-2}} = m_p;$$

from $f(t,x,L_1) \geq 0$ for $(t,x) \in [0,1] \times [A-\sigma, L+\sigma]$ and $f(t,x,F_1) \leq 0$ for $(t,x) \in [0,1] \times [A-\sigma, L+\sigma]$, it follows that $M \geq 0$ and $m \leq 0$.

On the other hand,

$$m \leq f(t,x(t),x'(t)) \leq M \quad \text{for } t \in [0,1],$$

since $(x(t),x'(t)) \in [A,L] \times [F_1,L_1]$ for each $t \in [0,1]$. Multiplying the last inequality by $\lambda(p-1)^{-1}(x'(t))^{2-p} \geq 0$, $\lambda, t \in [0,1]$, we arrive to

$$m_p \leq \frac{\lambda m}{(p-1)(x'(t))^{p-2}} \leq \frac{\lambda f(t,x(t),x'(t))}{(p-1)(x'(t))^{p-2}} \leq \frac{\lambda M}{(p-1)|x'(t)|^{p-2}} \leq M_p$$

for all $\lambda, t \in [0,1]$, from where, keeping in mind that $x'(t) > 0$ on $[0,1]$, we get

$$m_p \leq \frac{\lambda f(t,x(t),x'(t))}{(p-1)|x'(t)|^{p-2}} \leq M_p \quad \text{for all } \lambda, t \in [0,1],$$

which yields the required bounds for $x''(t)$. □

Now, introduce sets

$$C^1_+[0,1] = \{x \in C^1[0,1] : x(t) > 0 \text{ on } [0,1],\ x(1) = \phi_p(B)\}$$

and, in case that **H** holds,

$$V = \{x \in C^1[0,1] : A-\sigma \leq x \leq L+\sigma,\ F_1-\sigma \leq x' \leq L_1+\sigma\}.$$

Introduce also the map $\Lambda_\lambda : V \to C^1_+[0,1]$ defined by

$$\Lambda_\lambda x = \lambda \int_1^t f(s,x(s),x'(s))ds + \phi_p(B) \quad \text{for } \lambda \in [0,1].$$

Lemma 2.5. *Let* **H** *hold and*

$$f(t,x,y) \in C\Big([0,1] \times [A-\sigma, L+\sigma] \times [F_1-\sigma, L_1+\sigma]\Big). \tag{2.4}$$

Then Λ_λ, $\lambda \in [0,1]$, is well defined and continuous.

Proof. Clearly, because of (2.4), $(\Lambda_\lambda x)'(t) = \lambda f(t, x(t), x'(t))$, $x \in V$, is continuous on $[0,1]$ for each $\lambda \in [0,1]$. Next, observe that for each $x \in V$ we have

$$\lambda f(t, x(t), x'(t)) \leq \lambda M \leq M \quad \text{for} \;\; \lambda, t \in [0,1].$$

Integrating this inequality from 1 to t, $t \in [0,1)$, we get

$$\lambda \int_1^t f(s, x(s), x'(s)) ds \geq M(t-1), \qquad t \in [0,1],$$

from where it follows

$$\lambda \int_1^t f(s, x(s), x'(s)) ds \geq -M, \qquad t \in [0,1],$$

and

$$-M + \phi_p(B) \leq (\Lambda_\lambda x)(t), \qquad t \in [0,1].$$

By (1.5) and Proposition 2.3, we have

$$0 < (F_1 - \sigma)^{p-1} < (B - M)^{p-1} \leq -M + B^{p-1} = -M + \phi_p(B)$$

and then,

$$0 < (F_1 - \sigma)^{p-1} < (\Lambda_\lambda x)(t), \qquad t \in [0,1].$$

Obviously, $(\Lambda_\lambda x)(1) = \phi_p(B)$. Finally, (2.4) implies that the map Λ_λ, $\lambda \in [0,1]$, is continuous on V. □

Further, introduce the sets

$$C^2_{BC}[0,1] = \{x \in C^2[0,1] : x(0) = A, \; x'(1) = B\},$$
$$K = \{x \in C^2_{BC}[0,1] : x'(t) > 0 \text{ on } [0,1]\}$$

and the map $\Phi_p : K \to C^1_+[0,1]$ defined by $\Phi_p x = \phi_p(x')$.

Lemma 2.6. *The map Φ_p is well defined and continuous.*

Proof. For each $x \in K$ we have $x'(t) > 0$, $t \in [0,1]$. Then,

$$(\Phi_p x)(t) = x'(t)|x'(t)|^{p-2} = x'(t)^{p-1} > 0 \quad \text{on } [0,1] \tag{2.5}$$

and, obviously, $(\Phi_p x)'(t) = (p-1)(x'(t))^{p-2} x''(t)$ is continuous on $[0,1]$. Also, $(\Phi_p x)(1) = x'(1)|x'(1)|^{p-2} = \phi_p(B)$. So, $\Phi_p x \in C^1_+[0,1]$. The continuity of Φ_p follows from $x' \in C[0,1]$ and (2.5). □

It is well known that the inverse function of $\phi_p(s)$ is $\phi_q(s) = s|s|^{q-2}$, $q^{-1} + p^{-1} = 1$, $p > 1$. Using it, we introduce the map $\Phi_q : C^1_+[0,1] \to K$, defined by

$$(\Phi_q y)(t) = \int_0^t \phi_q(y(s)) ds + A, \qquad t \in [0,1].$$

But, for $y \in C^1_+[0,1]$ we have $y(t) > 0$ on $[0,1]$ and so

$$(\Phi_q y)(t) = \int_0^t (y(s))^{\frac{1}{p-1}} ds + A, \qquad t \in [0,1].$$

Lemma 2.7. *The map $\Phi_q : C^1_+[0,1] \to K$ is well defined, the inverse map of Φ_p and continuous.*

Proof. For each fixed $y \in C^1_+[0,1]$ we get a unique $x(t) = (\Phi_q y)(t) = \int_0^t (y(s))^{\frac{1}{p-1}} ds + A$. In fact, to establish the veracity of the first two assertions, we have to show that $x \in K$ or, what is the same, to show that x is a unique $C^2[0,1]$-solution to the BVP

$$x'|x'|^{p-2} = y, \quad t \in [0,1], \qquad x(0) = A, \quad x'(1) = B \tag{2.6}$$

with $x'(t) > 0$ on $[0,1]$.

The last follows immediately from $x'(t) = (y(t))^{\frac{1}{p-1}}$ on $[0,1]$. Then, $x'|x'|^{p-2} = (x'(t))^{p-1} = y(t)$ for $t \in [0,1]$. Besides, $x'(1) = (y(1))^{\frac{1}{p-1}} = (\phi_p(B))^{\frac{1}{p-1}} = B$ and $x(0) = A$. Now, the continuity of $y'(t)$ and $y(t) > 0$ on $[0,1]$ imply that

$$x''(t) = \frac{1}{p-1}(y(t))^{\frac{2-p}{p-1}} y'(t)$$

exists and is continuous on $[0,1]$. Thus, $x(t)$ is a solution to (2.6) and is in $C^2[0,1]$.

To complete the proof we just have to observe that the continuity of Φ_q follows from the continuity of $y^{1/(p-1)}(t)$ on $[0,1]$. □

3 Proof of main result

Proof of Theorem 1.1. We will prove the assertion for an arbitrary fixed $p > 2$. Introduce the set

$$U = \{x \in K : A - \sigma < x < L + \sigma,\ F_1 - \sigma < x' < L_1 + \sigma,\ m_p - \sigma < x''(t) < M_p + \sigma\}$$

and consider the homotopy

$$H_\lambda : \overline{U} \times [0,1] \to K$$

defined by $H_\lambda(x) := \Phi_q \Lambda_\lambda j$, where $j : \overline{U} \to C^1[0,1]$ is the embedding $jx = x$. To show that all assumptions of Theorem 2.1 are fulfilled observe firstly that U is an open subset of K, and K is a convex subset of the Banach space $C^2[0,1]$. For the fixed points of H_λ, $\lambda \in [0,1]$, we have

$$\Phi_q \Lambda_\lambda j(x) = x$$

and

$$\Phi_p x = \Lambda_\lambda j(x),$$

which is the operator form of the family

$$\begin{cases} \phi_p(x') = \lambda \int_1^t f(s, x(s), x'(s)) ds + \phi_p(B),\ t \in (0,1), \\ x(0) = A,\ x'(1) = B. \end{cases} \tag{3.1}$$

Thus, the fixed points of H_λ coincide with the $C^2[0,1]$-solutions of (3.1). But, it is obvious that each $C^2[0,1]$-solution of (3.1) is a $C^2[0,1]$-solution of (2.1). So, all conclusions of Lemma 2.4 are valid in particular and for the $C^2[0,1]$-solutions of (3.1) which allow us to conclude that the $C^2[0,1]$-solutions of (3.1) do not belong to ∂U and so the homotopy is fixed point free on ∂U. On the other hand, it is well known that j is completely continuous, that is, it maps each bounded set to a compact one. Thus, $j(\overline{U})$ is a compact set. Besides, it is clear that $j(\overline{U}) \subset V$. Then, according to Lemma 2.5, $\Lambda_\lambda(j(\overline{U})) \subseteq C^1_+[0,1]$ is compact. Finally, the set

$\Phi_q(\Lambda_\lambda(j(\overline{U})) \subset K$ is compact, by Lemma 2.7. So, the homotopy is compact. Now, since for $x \in \overline{U}$ we have $\Lambda_0 j(x) = \phi_p(B) = B^{p-1}$, the map H_0 maps each $x \in \overline{U}$ to the unique solution $l = Bt + A \in K$ to the BVP

$$x' = B, \quad t \in (0,1),$$
$$x(0) = A, \quad x'(1) = B,$$

i.e., it is a constant map and so is essential, by Theorem 2.2. So, we can apply Theorem 2.1. It infers that the map $H_1(x)$ has a fixed point in U. It is easy to see that it is a $C^2[0,1]$-solution of the BVPs of families (3.1) and (2.1) obtained for $\lambda = 1$ and, what is the same, of (1.1), (1.2). □

An elementary consequence of the just proved theorem is the following.

Corollary 3.1. *Let $A \geq 0$, **H** and (1.5) hold, and $f(t,x,y)$ be continuous for $(t,x,y) \in [0,1] \times [A-\sigma, L+\sigma] \times [F_1 - \sigma, L_1 + \sigma]$. Then for each $p > 2$ BVP (1.1), (1.2) has at least one strictly increasing solution in $C^2[0,1]$ with positive values on $(0,1]$.*

We illustrate this result by the following example.

Example 3.2. Consider the BVP

$$(\phi_p(x'))' = \frac{(2x'-1)(x'-10)}{\sqrt{x+1}+100}, \quad t \in (0,1),$$
$$x(0) = 2, \quad x'(1) = 5,$$

where $p > 2$ is fixed.

It is easy to check that **H** holds for $F_2 = 1$, $F_1 = 2.1$, $L_1 = 11.9$, $L_2 = 13$ and $\sigma = 0.1$; moreover, we can take $L = 14$, $m = -0.5$ and $M = 0.5$. The function $f(t,x,y) = \frac{(2y-1)(y-10)}{\sqrt{x+1}+100}$ is continuous for $(t,x,y) \in [0,1] \times [2,14] \times [2.1, 11.9]$. Thus, we can apply Corollary 3.1 to conclude that this BVP has a positive strictly increasing solution in $C^2[0,1]$.

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