Higher-order generalized Cahn–Hilliard equations

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Abstract. Our aim in this paper is to study higher-order (in space) anisotropic generalized Cahn–Hilliard models. In particular, we obtain well-posedness results, as well as the existence of the global attractor. Such models can have applications in biology, image processing, etc. We also give numerical simulations which illustrate the effects of the higher-order terms on the anisotropy.

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1 Introduction

The Cahn–Hilliard equation,

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0, \tag{1.1}$$

plays an essential role in materials science and describes important qualitative features of two-phase systems related with phase separation processes, assuming isotropy and a constant temperature. This can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the apparition of nuclides in the material) or a total nucleation, the so-called spinodal decomposition: the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening, occurs at a slower time scale and is less understood, these microstructures coarsen. Such phenomena play an essential role in the mechanical properties of the material, e.g., strength. We refer the reader to, e.g., [8, 9, 16, 20, 29, 30, 32, 33, 38, 39] for more details.

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Here, u is the order parameter (e.g., a density of atoms) and f is the derivative of a double-well potential F. A thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model:

$$F(s) = \frac{\theta_c}{2}(1 - s^2) + \frac{\theta}{2}\left[(1 - s)\ln\left(\frac{1-s}{2}\right) + (1 + s)\ln\left(\frac{1+s}{2}\right)\right], \qquad s \in (-1, 1),\ 0 < \theta < \theta_c,$$
(1.2)

i.e.,

$$f(s) = -\theta_c s + \frac{\theta}{2}\ln\frac{1+s}{1-s},$$
(1.3)

although such a function is very often approximated by regular ones, typically,

$$F(s) = \frac{1}{4}(s^2 - 1)^2,$$
(1.4)

i.e.,

$$f(s) = s^3 - s.$$
(1.5)

Now, it is interesting to note that the Cahn–Hilliard equation and some of its variants are also relevant in other phenomena than phase separation. We can mention, for instance, population dynamics (see [18]), tumor growth (see [4] and [26]), bacterial films (see [27]), thin films (see [41] and [44]), image processing (see [5, 6, 10, 12, 19]) and even the rings of Saturn (see [45]) and the clustering of mussels (see [31]).

In particular, several such phenomena can be modeled by the following generalized Cahn–Hilliard equation:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(x, u) = 0.$$
(1.6)

We studied in [35] and [36] (see also [4, 12, 17, 21]) this equation.

The Cahn–Hilliard equation is based on the so-called Ginzburg–Landau free energy,

$$\Psi_{\mathrm{GL}} = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u)\right)dx,$$
(1.7)

where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^n$, $n = 1$, 2 or 3, with boundary Γ). In particular, in (1.7), the term $|\nabla u|^2$ models short-ranged interactions. It is however interesting to note that such a term is obtained by truncation of higher-order ones (see [9]); it can also be seen as a first-order approximation of a nonlocal term accounting for long-ranged interactions (see [22] and [23]).

G. Caginalp and E. Esenturk recently proposed in [7] (see also [11]) higher-order phase-field models in order to account for anisotropic interfaces (see also [28, 42, 47] for other approaches which, however, do not provide an explicit way to compute the anisotropy). More precisely, these authors proposed the following modified free energy, in which we omit the temperature:

$$\Psi_{\mathrm{HOGL}} = \int_\Omega \left(\frac{1}{2}\sum_{i=1}^k \sum_{|\alpha|=i} a_\alpha |\mathcal{D}^\alpha u|^2 + F(u)\right)dx, \qquad k \in \mathbb{N},$$
(1.8)

where, for $\alpha = (k_1, \ldots, k_n) \in (\mathbb{N} \cup \{0\})^n$,

$$|\alpha| = k_1 + \cdots + k_n$$

and, for $\alpha \neq (0, \ldots, 0)$,

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$$

(we agree that $\mathcal{D}^{(0,\dots,0)}v = v$). The corresponding higher-order Cahn–Hilliard equation then reads

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k}(-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha}u - \Delta f(u) = 0. \tag{1.9}$$

We studied in [13] and [14] the corresponding isotropic model which reads

$$\frac{\partial u}{\partial t} - \Delta P(-\Delta)u - \Delta f(u) = 0, \tag{1.10}$$

where

$$P(s) = \sum_{i=1}^{k} a_i s^i, \quad a_k > 0, \ k \in \mathbb{N}, \ s \in \mathbb{R}.$$

The anisotropic model (1.9) is treated in [15].

Our aim in this paper is to study the higher-order generalized Cahn–Hilliard model

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k}(-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha}u - \Delta f(u) + g(x,u) = 0. \tag{1.11}$$

In particular, we study the well-posedness and the regularity of solutions. We also prove the dissipativity of the corresponding solution operators, as well as the existence of the global attractor. We finally give numerical simulations which show the effects of the higher-order terms on the anisotropy.

2 Setting of the problem

We consider the following initial and boundary value problem, for $k \in \mathbb{N}$, $k \geq 2$ (the case $k = 1$ can be treated as in [35]):

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k}(-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha}u - \Delta f(u) + g(x,u) = 0, \tag{2.1}$$

$$\mathcal{D}^\alpha u = 0 \quad \text{on } \Gamma, \quad |\alpha| \leq k, \tag{2.2}$$

$$u|_{t=0} = u_0. \tag{2.3}$$

We assume that

$$a_\alpha > 0, \qquad |\alpha| = k, \tag{2.4}$$

and we introduce the elliptic operator A_k defined by

$$\langle A_k v, w \rangle_{H^{-k}(\Omega), H_0^k(\Omega)} = \sum_{|\alpha|=k} a_\alpha ((\mathcal{D}^\alpha v, \mathcal{D}^\alpha w)), \tag{2.5}$$

where $H^{-k}(\Omega)$ is the topological dual of $H_0^k(\Omega)$. Furthermore, $((\cdot, \cdot))$ denotes the usual L^2-scalar product, with associated norm $\| \cdot \|$. More generally, we denote by $\| \cdot \|_X$ the norm on the Banach space X; we also set $\| \cdot \|_{-1} = \|(-\Delta)^{-\frac{1}{2}} \cdot \|$, where $(-\Delta)^{-1}$ denotes the inverse minus Laplace operator associated with Dirichlet boundary conditions. We can note that

$$(v,w) \in H_0^k(\Omega)^2 \mapsto \sum_{|\alpha|=k} a_\alpha ((\mathcal{D}^\alpha v, \mathcal{D}^\alpha w))$$

is bilinear, symmetric, continuous and coercive, so that

$$A_k : H_0^k(\Omega) \to H^{-k}(\Omega)$$

is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order $2k$ (see [1–3]) that A_k is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(A_k) = H^{2k}(\Omega) \cap H_0^k(\Omega),$$

where, for $v \in D(A_k)$,

$$A_k v = (-1)^k \sum_{|\alpha|=k} a_\alpha \mathcal{D}^{2\alpha} v.$$

We further note that $D\big(A_k^{\frac{1}{2}}\big) = H_0^k(\Omega)$ and, for $(v,w) \in D\big(A_k^{\frac{1}{2}}\big)^2$,

$$\left(\left(A_k^{\frac{1}{2}} v, A_k^{\frac{1}{2}} w\right)\right) = \sum_{|\alpha|=k} a_\alpha ((\mathcal{D}^\alpha v, \mathcal{D}^\alpha w)).$$

We finally note that (see, e.g., [43]) $\|A_k \cdot \|$ (resp., $\big\|A_k^{\frac{1}{2}} \cdot \big\|$) is equivalent to the usual H^{2k}-norm (resp., H^k-norm) on $D(A_k)$ (resp., $D\big(A_k^{\frac{1}{2}}\big)$).

Similarly, we can define the linear operator $\overline{A}_k = -\Delta A_k$,

$$\overline{A}_k : H_0^{k+1}(\Omega) \to H^{-k-1}(\Omega)$$

which is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(\overline{A}_k) = H^{2k+2}(\Omega) \cap H_0^{k+1}(\Omega),$$

where, for $v \in D(\overline{A}_k)$,

$$\overline{A}_k v = (-1)^{k+1} \Delta \sum_{|\alpha|=k} a_\alpha \mathcal{D}^{2\alpha} v.$$

Furthermore, $D\big(\overline{A}_k^{\frac{1}{2}}\big) = H_0^{k+1}(\Omega)$ and, for $(v,w) \in D\big(\overline{A}_k^{\frac{1}{2}}\big)^2$,

$$\left(\left(\overline{A}_k^{\frac{1}{2}} v, \overline{A}_k^{\frac{1}{2}} w\right)\right) = \sum_{|\alpha|=k} a_\alpha ((\nabla \mathcal{D}^\alpha v, \nabla \mathcal{D}^\alpha w)).$$

Besides, $\|\overline{A}_k \cdot \|$ (resp., $\big\|\overline{A}_k^{\frac{1}{2}} \cdot \big\|$) is equivalent to the usual H^{2k+2}-norm (resp., H^{k+1}-norm) on $D(\overline{A}_k)$ (resp., $D\big(\overline{A}_k^{\frac{1}{2}}\big)$).

We finally consider the operator $\tilde{A}_k = (-\Delta)^{-1} A_k$, where

$$\tilde{A}_k : H_0^{k-1}(\Omega) \to H^{-k+1}(\Omega);$$

note that, as $-\Delta$ and A_k commute, then the same holds for $(-\Delta)^{-1}$ and A_k, so that $\tilde{A}_k = A_k(-\Delta)^{-1}$.

We have the following lemma.

Lemma 2.1. *The operator $\tilde{A}_k$ is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain*

$$D(\tilde{A}_k) = H^{2k-2}(\Omega) \cap H_0^{k-1}(\Omega),$$

where, for $v \in D(\tilde{A}_k)$,

$$\tilde{A}_k v = (-1)^k \sum_{|\alpha|=k} a_\alpha \mathcal{D}^{2\alpha}(-\Delta)^{-1}v.$$

Furthermore, $D\big(\tilde{A}_k^{\frac{1}{2}}\big) = H_0^{k-1}(\Omega)$ and, for $(v,w) \in D\big(\tilde{A}_k^{\frac{1}{2}}\big)^2$,

$$\left(\!\left(\tilde{A}_k^{\frac{1}{2}}v, \tilde{A}_k^{\frac{1}{2}}w\right)\!\right) = \sum_{|\alpha|=k} a_\alpha \left(\!\left(\mathcal{D}^\alpha(-\Delta)^{-\frac{1}{2}}v, \mathcal{D}^\alpha(-\Delta)^{-\frac{1}{2}}w\right)\!\right).$$

Besides, $\|\tilde{A}_k \cdot \|$ (resp., $\big\|\tilde{A}_k^{\frac{1}{2}} \cdot \big\|$) is equivalent to the usual H^{2k-2}-norm (resp., H^{k-1}-norm) on $D(\tilde{A}_k)$ (resp., $D\big(\tilde{A}_k^{\frac{1}{2}}\big)$).

Proof. We first note that $\tilde{A}_k$ clearly is linear and unbounded. Then, since $(-\Delta)^{-1}$ and A_k commute, it easily follows that $\tilde{A}_k$ is selfadjoint.

Next, the domain of $\tilde{A}_k$ is defined by

$$D(\tilde{A}_k) = \left\{v \in H_0^{k-1}(\Omega), \ \tilde{A}_k v \in L^2(\Omega)\right\}.$$

Noting that $\tilde{A}_k v = f$, $f \in L^2(\Omega)$, $v \in D(\tilde{A}_k)$, is equivalent to $A_k v = -\Delta f$, where $-\Delta f \in H^2(\Omega)'$, it follows from the elliptic regularity results of [1], [2] and [3] that $v \in H^{2k-2}(\Omega)$, so that $D(\tilde{A}_k) = H^{2k-2}(\Omega) \cap H_0^{k-1}(\Omega)$.

Noting then that $\tilde{A}_k^{-1}$ maps $L^2(\Omega)$ onto $H^{2k-2}(\Omega)$ and recalling that $k \geq 2$, we deduce that $\tilde{A}_k$ has compact inverse.

We now note that, considering the spectral properties of $-\Delta$ and A_k (see, e.g., [43]) and recalling that these two operators commute, $-\Delta$ and A_k have a spectral basis formed of common eigenvectors. This yields that, $\forall s_1, s_2 \in \mathbb{R}$, $(-\Delta)^{s_1}$ and $A_k^{s_2}$ commute.

Having this, we see that $\tilde{A}_k^{\frac{1}{2}} = (-\Delta)^{-\frac{1}{2}} A_k^{\frac{1}{2}}$, so that $D\big(\tilde{A}_k^{\frac{1}{2}}\big) = H_0^{k-1}(\Omega)$, and, for $(v,w) \in D\big(\tilde{A}_k^{\frac{1}{2}}\big)^2$,

$$\left(\!\left(\tilde{A}_k^{\frac{1}{2}}v, \tilde{A}_k^{\frac{1}{2}}w\right)\!\right) = \sum_{|\alpha|=k} a_\alpha \left(\!\left(\mathcal{D}^\alpha(-\Delta)^{-\frac{1}{2}}v, \mathcal{D}^\alpha(-\Delta)^{-\frac{1}{2}}w\right)\!\right).$$

Finally, as far as the equivalences of norms are concerned, we can note that, for instance, the norm $\big\|\tilde{A}_k^{\frac{1}{2}} \cdot \big\|$ is equivalent to the norm $\big\|(-\Delta)^{-\frac{1}{2}} \cdot \big\|_{H^k(\Omega)}$ and, thus, to the norm $\big\|(-\Delta)^{\frac{k-1}{2}} \cdot \big\|$. $\qquad\square$

Having this, we rewrite (2.1) as

$$\frac{\partial u}{\partial t} - \Delta A_k u - \Delta B_k u - \Delta f(u) + g(x,u) = 0, \tag{2.6}$$

where

$$B_k v = \sum_{i=1}^{k-1}(-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha}v.$$

As far as the nonlinear term f is concerned, we assume that

$$f \in C^2(\mathbb{R}), \quad f(0) = 0, \tag{2.7}$$

$$f' \geq -c_0, \quad c_0 \geq 0, \tag{2.8}$$

$$f(s)s \geq c_1 F(s) - c_2 \geq -c_3, \quad c_1 > 0, \ c_2, c_3 \geq 0, \ s \in \mathbb{R}, \tag{2.9}$$

$$F(s) \geq c_4 s^4 - c_5, \quad c_4 > 0, \ c_5 \geq 0, \ s \in \mathbb{R}, \tag{2.10}$$

where $F(s) = \int_0^s f(\xi) \, d\xi$. In particular, the usual cubic nonlinear term $f(s) = s^3 - s$ satisfies these assumptions.

Furthermore, as far as the function g is concerned, we assume that

$$g(\cdot, s) \text{ is measurable, } \forall s \in \mathbb{R}, \ g(x, \cdot) \text{ is of class } C^1, \text{ a.e. } x \in \Omega, \tag{2.11}$$

$$\frac{\partial g}{\partial s}(\cdot, s) \text{ is measurable,} \quad \forall s \in \mathbb{R};$$

$$|g(x, s)| \leq h(s), \quad \text{a.e. } x \in \Omega, \ s \in \mathbb{R}, \tag{2.12}$$

where $h \geq 0$ is continuous and satisfies

$$\|h(v)\|\|v\| \leq \varepsilon \int_\Omega F(v) \, dx + c_\varepsilon, \qquad \forall \varepsilon > 0, \tag{2.13}$$

$\forall v \in L^2(\Omega)$ such that $\int_\Omega F(v) \, dx < +\infty$, and

$$|h(s)|^2 \leq c_6 F(s) + c_7, \qquad c_6, \ c_7 \geq 0, \quad s \in \mathbb{R}; \tag{2.14}$$

$$\left|\frac{\partial g}{\partial s}(x, s)\right| \leq l(s), \quad \text{a.e. } x \in \Omega, \quad s \in \mathbb{R}, \tag{2.15}$$

where $l \geq 0$ is continuous.

Example 2.2. We assume that $f(s) = s^3 - s$. Assumptions (2.11)–(2.15) are satisfied in the following cases.

(i) Cahn–Hilliard–Oono equation (see [34], [40] and [46]). In that case,

$$g(x, s) = g(s) = \beta s, \qquad \beta > 0.$$

This function was proposed in [40] in order to account for long-ranged (i.e., nonlocal) interactions, but also to simplify numerical simulations.

(ii) Proliferation term. In that case,

$$g(x, s) = g(s) = \beta s(s - 1), \qquad \beta > 0.$$

This function was proposed in [26] in view of biological applications and, more precisely, to model wound healing and tumor growth (in one space dimension) and the clustering of brain tumor cells (in two space dimensions); see also [4] for other quadratic functions.

(iii) Fidelity term. In that case,

$$g(x,s) = \lambda_0 \chi_{\Omega \setminus D}(x)(s - \varphi(x)), \qquad \lambda_0 > 0, \ D \subset \Omega, \ \varphi \in L^2(\Omega),$$

where χ denotes the indicator function. This function was proposed in [5] and [6] in view of applications to image inpainting. Here, φ is a given (damaged) image and D is the inpainting (i.e., damaged) region. Furthermore, the fidelity term $g(x,u)$ is added in order to keep the solution close to the image outside the inpainting region. The idea in this model is to solve the equation up to steady state to obtain an inpainted (i.e., restored) version $u(x)$ of $\varphi(x)$.

Throughout the paper, the same letters c, c' and c'' denote (generally positive) constants which may vary from line to line. Similarly, the same letters Q and Q' denote (positive) monotone increasing and continuous (with respect to each argument) functions which may vary from line to line.

3 A priori estimates

Proposition 3.1. *Any sufficiently regular solution to* (2.1)–(2.3) *satisfies the following estimates:*

$$\|u(t)\|_{H^k(\Omega)}^2 \leq ce^{-c't}\left(\|u_0\|_{H^k(\Omega)}^2 + \int_\Omega F(u_0)\,dx\right) + c'', \qquad c' > 0, \ t \geq 0, \tag{3.1}$$

$$\int_t^{t+r}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 ds \leq ce^{-c't}\left(\|u_0\|_{H^k(\Omega)}^2 + \int_\Omega F(u_0)\,dx\right) + c'', \tag{3.2}$$

$$c' > 0, \ t \geq 0, \ r > 0 \text{ given,}$$

and

$$\|u(t)\|_{H^{2k}(\Omega)} \leq Q(e^{-ct}Q'(\|u_0\|_{H^k(\Omega)}) + c'), \qquad c > 0, \ t \geq 1, \tag{3.3}$$

where the continuous and monotone increasing function Q is of the form $Q(s) = cse^{c's}$.

Proof. The estimates below will be formal, but they can easily be justified within, e.g., a standard Galerkin scheme.

We multiply (2.6) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and integrate over Ω and by parts. This gives

$$\frac{d}{dt}\left(\left\|A_k^{\frac{1}{2}}u\right\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\,dx\right) + 2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 = -\left(\left(g(\cdot,u),(-\Delta)^{-1}\frac{\partial u}{\partial t}\right)\right),$$

where

$$B_k^{\frac{1}{2}}[u] = \sum_{i=1}^{k-1}\sum_{|\alpha|=i} a_\alpha \|\mathcal{D}^\alpha u\|^2$$

(note that $B_k^{\frac{1}{2}}[u]$ is not necessarily nonnegative). This yields, owing to (2.12) and (2.14),

$$\frac{d}{dt}\left(\left\|A_k^{\frac{1}{2}}u\right\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\,dx\right) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 \leq c\int_\Omega F(u)\,dx + c'. \tag{3.4}$$

We can note that, owing to the interpolation inequality

$$\|v\|_{H^i(\Omega)} \leq c(i)\|v\|_{H^m(\Omega)}^{\frac{i}{m}}\|v\|^{1-\frac{i}{m}}, \tag{3.5}$$

$$v \in H^m(\Omega), \ i \in \{1, \ldots, m-1\}, \ m \in \mathbb{N}, \ m \geq 2,$$

there holds

$$\left| B_k^{\frac{1}{2}}[u] \right| \leq \frac{1}{2} \left\| A_k^{\frac{1}{2}} u \right\|^2 + c\|u\|^2.$$

This yields, employing (2.10),

$$\left\| A_k^{\frac{1}{2}} u \right\|^2 + B_k^{\frac{1}{2}}[u] + 2 \int_\Omega F(u)\, dx \geq \frac{1}{2} \left\| A_k^{\frac{1}{2}} u \right\|^2 + \int_\Omega F(u)\, dx + c\|u\|^4_{L^4(\Omega)} - c'\|u\|^2 - c'',$$

whence

$$\left\| A_k^{\frac{1}{2}} u \right\|^2 + B_k^{\frac{1}{2}}[u] + 2 \int_\Omega F(u)\, dx \geq c \left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx \right) - c', \qquad c > 0, \qquad (3.6)$$

noting that, owing to Young's inequality,

$$\|u\|^2 \leq \varepsilon \|u\|^4_{L^4(\Omega)} + c_\varepsilon, \qquad \forall \varepsilon > 0. \qquad (3.7)$$

We then multiply (2.6) by $(-\Delta)^{-1}u$ and have, owing to (2.9), (2.12), (2.13) and the interpolation inequality (3.5),

$$\frac{d}{dt} \|u\|^2_{-1} + c \left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx \right) \leq c' \|u\|^2 + \varepsilon \int_\Omega F(u)\, dx + c''_\varepsilon, \qquad \forall \varepsilon > 0,$$

hence, proceeding as above and employing, in particular, (2.10),

$$\frac{d}{dt} \|u\|^2_{-1} + c \left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx \right) \leq c', \qquad c > 0. \qquad (3.8)$$

Summing δ_1 times (3.4) and (3.8), where $\delta_1 > 0$ is small enough, we obtain a differential inequality of the form

$$\frac{dE_1}{dt} + c \left(E_1 + \left\| \frac{\partial u}{\partial t} \right\|^2_{-1} \right) \leq c', \qquad c > 0, \qquad (3.9)$$

where

$$E_1 = \delta_1 \left(\left\| A_k^{\frac{1}{2}} u \right\|^2 + B_k^{\frac{1}{2}}[u] + 2 \int_\Omega F(u)\, dx \right) + \|u\|^2_{-1}$$

satisfies, owing to (3.6),

$$E_1 \geq c \left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx \right) - c', \qquad c > 0. \qquad (3.10)$$

Note indeed that

$$E_1 \leq c\|u\|^2_{H^k(\Omega)} + 2 \int_\Omega F(u)\, dx$$

$$\leq c \left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\, dx \right) - c', \qquad c > 0, \ c' \geq 0.$$

Estimates (3.1)–(3.2) then follow from (3.9)–(3.10) and Gronwall's lemma.

Multiplying next (2.6) by $\tilde{A}_k u$, we find, owing to (2.12) and the interpolation inequality (3.5),

$$\frac{d}{dt} \left\| \tilde{A}_k^{\frac{1}{2}} u \right\|^2 + c\|u\|^2_{H^{2k}(\Omega)} \leq c \left(\|u\|^2 + \|f(u)\|^2 + \|h(u)\|^2 \right). \qquad (3.11)$$

It follows from the continuity of f, F and h, the continuous embedding $H^k(\Omega) \subset \mathcal{C}(\overline{\Omega})$ (recall that $k \geq 2$) and (3.1) that

$$\|u\|^2 + \|f(u)\|^2 + \|h(u)\|^2 \leq Q(\|u\|_{H^k(\Omega)}) \leq e^{-ct} Q'(\|u_0\|_{H^k(\Omega)}) + c', \quad c > 0,\ t \geq 0, \quad (3.12)$$

so that

$$\frac{d}{dt} \left\| \tilde{A}_k^{\frac{1}{2}} u \right\|^2 + c\|u\|_{H^{2k}(\Omega)}^2 \leq e^{-c't} Q(\|u_0\|_{H^k(\Omega)}) + c'', \qquad c, c' > 0,\ t \geq 0. \quad (3.13)$$

Summing (3.9) and (3.13), we have a differential inequality of the form

$$\frac{dE_2}{dt} + c\left(E_2 + \|u\|_{H^{2k}(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 \right) \leq e^{-c't} Q(\|u_0\|_{H^k(\Omega)}) + c'', \qquad c, c' > 0,\ t \geq 0, \quad (3.14)$$

where

$$E_2 = E_1 + \|\tilde{A}_k^{\frac{1}{2}} u\|^2$$

satisfies

$$E_2 \geq c\left(\|u\|_{H^k(\Omega)}^2 + \int_\Omega F(u)\,dx \right) - c', \qquad c > 0. \quad (3.15)$$

We now multiply (2.6) by $\frac{\partial u}{\partial t}$ and obtain, noting that f is of class $\mathcal{C}^2$, so that

$$\|\Delta f(u)\| \leq Q(\|u\|_{H^k(\Omega)}),$$

and proceeding as above,

$$\frac{d}{dt}\left(\left\| \overline{A}_k^{\frac{1}{2}} u \right\|^2 + \overline{B}_k^{\frac{1}{2}}[u] \right) + \left\| \frac{\partial u}{\partial t} \right\|^2 \leq e^{-c't} Q(\|u_0\|_{H^k(\Omega)}) + c'', \qquad c, c' > 0, \quad (3.16)$$

where

$$\overline{B}_k^{\frac{1}{2}}[u] = \sum_{i=1}^{k-1} \sum_{|\alpha|=i} a_\alpha \|\nabla \mathcal{D}^\alpha u\|^2.$$

Summing finally (3.14) and (3.16), we find a differential inequality of the form

$$\frac{dE_3}{dt} + c\left(E_3 + \|u\|_{H^{2k}(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) \leq e^{-c't} Q(\|u_0\|_{H^k(\Omega)}) + c'', \qquad c, c' > 0,\ t \geq 0, \quad (3.17)$$

where

$$E_3 = E_2 + \left\| \overline{A}_k^{\frac{1}{2}} u \right\|^2 + \overline{B}_k^{\frac{1}{2}}[u]$$

satisfies, proceeding as above,

$$E_3 \geq c\left(\|u\|_{H^{k+1}(\Omega)}^2 + \int_\Omega F(u)\,dx \right) - c', \qquad c > 0. \quad (3.18)$$

In particular, it follows from (3.17)–(3.18) that

$$\|u(t)\|_{H^{k+1}(\Omega)} \leq e^{-ct} Q(\|u_0\|_{H^{k+1}(\Omega)}) + c', \qquad c > 0,\ t \geq 0. \quad (3.19)$$

We then rewrite (2.6) as an elliptic equation, for $t > 0$ fixed,

$$A_k u = -(-\Delta)^{-1} \frac{\partial u}{\partial t} - B_k u - f(u) - (-\Delta)^{-1} g(x, u), \quad \mathcal{D}^\alpha u = 0 \text{ on } \Gamma, \quad |\alpha| \leq k - 1. \quad (3.20)$$

Multiplying (3.20) by $A_k u$, we have, owing to (2.12) and the interpolation inequality (3.5),

$$\|A_k u\|^2 \le c\left(\|u\|^2 + \|f(u)\|^2 + \|h(u)\|^2 + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 \right), \tag{3.21}$$

hence, proceeding as above (employing, in particular, (3.12)),

$$\|u\|_{H^{2k}(\Omega)}^2 \le c\left(e^{-c't}Q(\|u_0\|_{H^k(\Omega)}) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 \right) + c'', \qquad c' > 0. \tag{3.22}$$

In a next step, we differentiate (2.6) with respect to time and obtain

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta A_k \frac{\partial u}{\partial t} - \Delta B_k \frac{\partial u}{\partial t} - \Delta\left(f'(u)\frac{\partial u}{\partial t} \right) + \frac{\partial g}{\partial s}(x, u)\frac{\partial u}{\partial t} = 0, \tag{3.23}$$

$$\mathcal{D}^\alpha \frac{\partial u}{\partial t} = 0 \quad \text{on } \Gamma, \quad |\alpha| \le k. \tag{3.24}$$

We multiply (3.23) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and find, owing to (2.8), (2.15), the interpolation inequality (3.5) and the continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$,

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + c\left\|\frac{\partial u}{\partial t}\right\|_{H^k(\Omega)}^2 \le c'\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \|l(u)\|\left\|\frac{\partial u}{\partial t}\right\|\left\|(-\Delta)^{-1}\frac{\partial u}{\partial t}\right\|_{L^\infty(\Omega)} \right)$$

$$\le c'\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \|l(u)\|\left\|\frac{\partial u}{\partial t}\right\|^2 \right), \qquad c > 0,$$

which yields, employing the interpolation inequality

$$\|v\|^2 \le c\|v\|_{-1}\|v\|_{H^1(\Omega)}, \qquad v \in H_0^1(\Omega), \tag{3.25}$$

and proceeding as above (note that l is continuous), the differential inequality

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + c\left\|\frac{\partial u}{\partial t}\right\|_{H^k(\Omega)}^2 \le c'(e^{-c''t}Q(\|u_0\|_{H^k(\Omega)}) + 1)\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2, \qquad c, c'' > 0. \tag{3.26}$$

In particular, this yields, owing to (3.2) and employing the uniform Gronwall's lemma (see, e.g., [43]),

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1} \le \frac{1}{r^{\frac{1}{2}}}Q(e^{-ct}Q'(\|u_0\|_{H^k(\Omega)}) + c'), \qquad c > 0,\ t \ge r,\ r > 0 \text{ given.} \tag{3.27}$$

Finally, (3.3) follows from (3.22) and (3.27) (for $r = 1$).   □

Remark 3.2. If we assume that $u_0 \in H^{2k+1}(\Omega) \cap H_0^k(\Omega)$, we deduce from (3.22), (3.26) and Gronwall's lemma an H^{2k}-estimate on u on $[0, 1]$ which, combined with (3.3), gives an H^{2k}-estimate on u, for all times. This is however not satisfactory, in particular, in view of the study of attractors.

Remark 3.3. We assume that, for simplicity, $g(x, s) = g(s)$ and we further assume that f is of class $\mathcal{C}^{k+1}$ and g is of class $\mathcal{C}^{k-1}$. Multiplying (2.6) by $\tilde{A}_k \frac{\partial u}{\partial t}$, we have

$$\frac{1}{2}\frac{d}{dt}(\|A_k u\|^2 + ((A_k u, B_k u))) + \left\|\tilde{A}_k^{\frac{1}{2}}\frac{\partial u}{\partial t}\right\|^2 = -\left(\left(\overline{A}_k^{\frac{1}{2}}f(u), \tilde{A}_k^{\frac{1}{2}}\frac{\partial u}{\partial t} \right) \right) - \left(\left(\tilde{A}_k^{\frac{1}{2}}g(u), \tilde{A}_k^{\frac{1}{2}}\frac{\partial u}{\partial t} \right) \right),$$

which yields, noting that $\left\|\overline{A}_k^{\frac{1}{2}}f(u)\right\|^2 + \left\|\tilde{A}_k^{\frac{1}{2}}g(u)\right\|^2 \leq Q(\|u\|_{H^{k+1}(\Omega)})$ and owing to (3.19),

$$\frac{d}{dt}(\|A_k u\|^2 + ((A_k u, B_k u))) \leq e^{-ct}Q(\|u_0\|_{H^{k+1}(\Omega)}) + c', \qquad c > 0,\ t \geq 0. \qquad (3.28)$$

Combining (3.28) with (3.17), it follows from (3.18) and the interpolation inequality (3.5) that

$$\|u(t)\|_{H^{2k}(\Omega)} \leq Q(\|u_0\|_{H^{2k}(\Omega)}), \qquad t \in [0,1],$$

so that, owing to (3.3),

$$\|u(t)\|_{H^{2k}(\Omega)} \leq Q(e^{-ct}Q'(\|u_0\|_{H^{2k}(\Omega)}) + c'), \qquad c > 0,\ t \geq 0. \qquad (3.29)$$

4 The dissipative semigroup

We first give the definition of a weak solution to (2.1)–(2.3).

Definition 4.1. We assume that $u_0 \in L^2(\Omega)$. A weak solution to (2.1)–(2.3) is a function u such that, for any given $T > 0$,

$$u \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^k(\Omega)),$$

$$u(0) = u_0 \quad \text{in } L^2(\Omega)$$

and

$$\frac{d}{dt}(((-\Delta)^{-1}u, v)) + \sum_{i=1}^{k}\sum_{|\alpha|=i} a_i((\mathcal{D}^\alpha u, \mathcal{D}^\alpha v)) + ((f(u), v))$$

$$+ (((-\Delta)^{-1}g(x,u), v)) = 0, \qquad \forall v \in H_0^k(\Omega),$$

in the sense of distributions.

We have the following theorem.

Theorem 4.2.

(i) We assume that $u_0 \in H_0^k(\Omega)$. Then, (2.1)–(2.3) possesses a unique weak solution u such that, $\forall T > 0$,

$$u \in L^\infty(\mathbb{R}^+; H_0^k(\Omega)) \cap L^2(0,T; H^{2k}(\Omega) \cap H_0^k(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,T; H^{-1}(\Omega)).$$

(ii) If we further assume that $u_0 \in H^{k+1}(\Omega) \cap H_0^k(\Omega)$, then, $\forall T > 0$,

$$u \in L^\infty(\mathbb{R}^+; H^{k+1}(\Omega) \cap H_0^k(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega)).$$

(iii) If we further assume that f is of class $\mathcal{C}^{k+1}$, $g(x,s) = g(s)$, g is of class $\mathcal{C}^{k-1}$ and $u_0 \in H^{2k}(\Omega) \cap H_0^k(\Omega)$, then

$$u \in L^\infty(\mathbb{R}^+; H^{2k}(\Omega) \cap H_0^k(\Omega)).$$

Proof. The proofs of existence and regularity in (i), (ii) and (iii) follow from the a priori estimates derived in the previous section and, e.g., a standard Galerkin scheme. Indeed, we can note that, since the operators $-\Delta$, A_k, $\overline{A}_k$ and $\tilde{A}_k$ are linear, selfadjoint and strictly positive operators with compact inverse which commute, they have a spectral basis formed of common eigenvectors. We then take this spectral basis as Galerkin basis, so that all the a priori estimates derived in the previous section are justified within the Galerkin scheme.

Let now u_1 and u_2 be two solutions to (2.1)–(2.2) with initial data $u_{0,1}$ and $u_{0,2}$, respectively. We set $u = u_1 - u_2$ and $u_0 = u_{0,1} - u_{0,2}$ and have

$$\frac{\partial u}{\partial t} - \Delta A_k u - \Delta B_k u - \Delta(f(u_1) - f(u_2)) + g(x, u_1) - g(x, u_2) = 0, \tag{4.1}$$

$$\mathcal{D}^\alpha u = 0 \quad \text{on } \Gamma, \quad |\alpha| \leq k, \tag{4.2}$$

$$u|_{t=0} = u_0. \tag{4.3}$$

Multiplying (4.1) by $(-\Delta)^{-1}u$, we obtain, owing to (2.8), (2.15), (3.1) and the interpolation inequalities (3.5) and (3.25),

$$\frac{d}{dt}\|u\|_{-1}^2 + c\|u\|_{H^k(\Omega)}^2 \leq Q\|u\|_{-1}^2, \qquad c > 0, \tag{4.4}$$

where

$$Q = Q(\|u_{0,1}\|_{H^k(\Omega)}, \|u_{0,2}\|_{H^k(\Omega)}).$$

Here, we have used the fact that, owing to (2.15) and (3.1),

$$\|g(x, u_1) - g(x, u_2)\| \leq Q(\|u_1\|_{H^k(\Omega)}, \|u_2\|_{H^k(\Omega)})\|u\|$$

$$\leq Q(\|u_{0,1}\|_{H^k(\Omega)}, \|u_{0,2}\|_{H^k(\Omega)})\|u\|.$$

It follows from (4.4) and Gronwall's lemma that

$$\|u(t)\|_{-1}^2 \leq e^{Qt}\|u_0\|_{-1}^2, \qquad t \geq 0, \tag{4.5}$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the H^{-1}-norm. $\square$

It follows from Theorem 4.2 that we can define the family of solving operators

$$S(t) : \Phi \to \Phi, \; u_0 \mapsto u(t), \qquad t \geq 0,$$

where $\Phi = H_0^k(\Omega)$. This family of solving operators forms a semigroup which is continuous with respect to the H^{-1}-topology. Finally, it follows from (3.1) that we have the following theorem.

Theorem 4.3. *The semigroup $S(t)$ is dissipative in Φ, in the sense that it possesses a bounded absorbing set $\mathcal{B}_0 \subset \Phi$ (i.e., $\forall B \subset \Phi$ bounded, $\exists t_0 = t_0(B) \geq 0$ such that $t \geq t_0 \implies S(t)B \subset \mathcal{B}_0$).*

Remark 4.4.

(i) Actually, it follows from (3.3) that we have a bounded absorbing set $\mathcal{B}_1$ which is compact in Φ and bounded in $H^{2k}(\Omega)$. This yields the existence of the global attractor $\mathcal{A}$ which is compact in Φ and bounded in $H^{2k}(\Omega)$.

(ii) We recall that the global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system. We refer the reader to, e.g., [37] and [43] for more details and discussions on this.

(iii) We can also prove, based on standard arguments (see, e.g., [37] and [43]) that $\mathcal{A}$ has finite dimension, in the sense of covering dimensions such as the Hausdorff and the fractal dimensions. The finite-dimensionality means, very roughly speaking, that, even though the initial phase space has infinite dimension, the reduced dynamics can be described by a finite number of parameters (we refer the interested reader to, e.g., [37] and [43] for discussions on this subject).

Remark 4.5. In the numerical simulations given in the next section below, the equations will be endowed with periodic boundary conditions. From a mathematical point of view, these boundary conditions are much more delicate to handle, since we have to estimate the spatial average of the order parameter $\langle u \rangle = \frac{1}{\text{Vol}(\Omega)} \int_\Omega u\, dx$ (see [12], [16] and [21]). When $g \equiv 0$, this is straightforward, since we have the conservation of mass, namely,

$$\langle u(t) \rangle = \langle u_0 \rangle, \qquad \forall t \geq 0.$$

However, when g does not vanish, we are not able to estimate this quantity in general.

5 Numerical simulations

We give in this section several numerical simulations in order to illustrate the effects of the higher-order terms on the anisotropy. The computations presented below are performed with the software FreeFem++ (see [24]), for $k = 2$. We also take Ω bi-dimensional and rectangular. Finally, the system is associated with periodic boundary conditions.

The problem can be written as, for $k = 2$,

$$\begin{cases} \frac{\partial u}{\partial t} + \Delta w + \frac{1}{\varepsilon} g(x, u) = 0, \\ w + a_{20}\varepsilon \frac{\partial^4 u}{\partial x^4} + a_{02}\varepsilon \frac{\partial^4 u}{\partial y^4} + a_{11}\varepsilon \frac{\partial^4 u}{\partial x^2 \partial y^2} - a_{10}\varepsilon \frac{\partial^2 u}{\partial x^2} - a_{01}\varepsilon \frac{\partial^2 u}{\partial y^2} + \frac{1}{\varepsilon} f(u) = 0, \\ u, w \quad \text{are} \quad \Omega\text{-periodic}, \\ u(0, x, y) = u_0(x, y), \end{cases}$$

where $\varepsilon > 0$ is introduced to take into account the diffuse interface thickness. Setting

$$\frac{\partial^2 u}{\partial x^2} = p, \qquad \frac{\partial^2 u}{\partial y^2} = q, \qquad \frac{\partial^4 u}{\partial x^2 \partial y^2} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2} + \frac{1}{2}\frac{\partial^2 q}{\partial x^2},$$

we have the variational formulation: find $(u, w, p, q) \in H^1_{\text{per}}(\Omega)^4$ such that

$$\begin{cases} \left(\left(\frac{\partial u}{\partial t}, v_1\right)\right) - ((\nabla w, \nabla v_1)) + \frac{1}{\varepsilon}((g(x, u), v_1)) = 0, \\ ((w, v_2)) - a_{20}\varepsilon\left(\left(\frac{\partial p}{\partial x}, \frac{\partial v_2}{\partial x}\right)\right) - a_{02}\varepsilon\left(\left(\frac{\partial q}{\partial y}, \frac{\partial v_2}{\partial y}\right)\right) - \frac{a_{11}\varepsilon}{2}\left(\left(\frac{\partial p}{\partial y}, \frac{\partial v_2}{\partial y}\right)\right) \\ \qquad - \frac{a_{11}\varepsilon}{2}\left(\left(\frac{\partial q}{\partial x}, \frac{\partial v_2}{\partial x}\right)\right) - a_{10}\varepsilon((p, v_2)) - a_{01}\varepsilon((q, v_2)) + \frac{1}{\varepsilon}((f(u), v_2)) = 0, \\ ((p, v_3)) + \left(\left(\frac{\partial u}{\partial x}, \frac{\partial v_3}{\partial x}\right)\right) = 0, \\ ((q, v_4)) + \left(\left(\frac{\partial u}{\partial y}, \frac{\partial v_4}{\partial y}\right)\right) = 0, \end{cases}$$

where the test functions v_1, v_2, v_3, v_4 all belong to $H^1_{\text{per}}(\Omega)$.

The mesh is obtained by dividing Ω into 149^2 rectangles, each rectangle being divided along the same diagonal into two triangles. The computations in Fig. 5.2, 5.3, 5.4 are based on a P_1 finite element method for the space discretization, while we used a P_2 finite element

method for Fig. 5.5, 5.6, 5.7. The time discretization uses a semi-implicit Euler scheme (implicit for the linear terms and explicit for the nonlinear ones).

We give numerical results concerning a higher-order Cahn–Hilliard–Oono equation (Fig. 5.2), a higher order phase-field crystal equation (Fig. 5.3, 5.4; see also [25])) and a higher-order Cahn–Hilliard equation with a mass source for tumor growth (Fig. 5.5, 5.6, 5.7; see also [4]). These results show that the anisotropy is strongly influenced by the choice of the coefficients in the higher-order terms. In particular, we can clearly see the anisotropy in the x, y and cross-directions. For instance, Fig. 5.5, Column 1, corresponds to a tumor growth simulated with the classical Cahn–Hilliard model (analogous simulations were performed in [4]). With very small coefficients for the sixth-order terms, the tumor evolves similarly, although the x, y and cross-directions are clearly noticeable (see Fig. 5.6). With larger coefficients, the tumor spreads and evolves faster; the anisotropy directions also become obvious (see Fig. 5.5, Column 2, for an isotropic situation and Fig. 5.7 for anisotropic ones).

(i) Cahn–Hilliard-Oono equation. (See Fig. 5.2.)

$$\begin{cases} f(u) = u^3 - u, \quad g(x,u) = 0.5u, \quad \varepsilon = 0.05, \\ u_0^{(1)} \text{ randomly distributed between } -1 \text{ and } 1, \\ \Omega = [0,1] \times [0,1], \quad \text{step size } \Delta t = 5 \times 10^{-8}, \\ \text{coefficients } a_{ij} \text{ in Table 5.1.} \end{cases}$$

(ii) Phase-field crystal equation. (See Fig. 5.3.)

$$\begin{cases} f(u) = u^3 + (1 - 0.025)u, \quad g(x,u) = 2u, \quad \varepsilon = 1, \\ u_0^{(2)} \text{ randomly distributed between } -0.2 \text{ and } 0.3, \\ \Omega = [-10,10] \times [-10,10], \quad \Delta t = 10^{-4}, \\ \text{coefficients } a_{ij} \text{ in Table 5.2.} \end{cases}$$

(iii) Phase-field crystal equation. (See Fig. 5.4.)

$$\begin{cases} f(u) = u^3 + (1 - 0.025)u, \quad g(x,u) = 2u, \quad \varepsilon = 1, \\ u_0^{(3)} = 0.07 - 0.02 \cos \frac{2\pi(x-12)}{32} \sin \frac{2\pi(y-1)}{32} \\ \quad + 0.02 \cos^2 \frac{\pi(x+10)}{32} \cos^2 \frac{\pi(y+3)}{32} \\ \quad - 0.01 \sin^2 \frac{4\pi x}{32} \sin^2 \frac{4\pi(y-6)}{32}, \\ \Omega = [0,32] \times [0,32], \quad \Delta t = 10^{-3}, \\ \text{coefficients } a_{ij} \text{ in Table 5.3.} \end{cases}$$

(iv) Tumor proliferation term. (See Fig. 5.5, 5.6, 5.7.)

$$\begin{cases} f(u) = u^3 - u, \quad \Omega = [-0.7,1.7] \times [-1.7,0.7], \quad \Delta t = 10^{-6} \\ g(x,u) = 46(u+1) - 280(u-1)^2(u+1)^2, \quad \varepsilon = 0.0125, \\ u_0^{(4)} = -\tanh\left(\frac{1}{\sqrt{2}\varepsilon}\left(\sqrt{2(x-0.5)^2 + 0.25(y+0.5)^2} - 0.1\right)\right) \in [-1,1], \\ \text{coefficients } a_{ij} \text{ in Tables 5.4, 5.5, 5.6.} \end{cases}$$

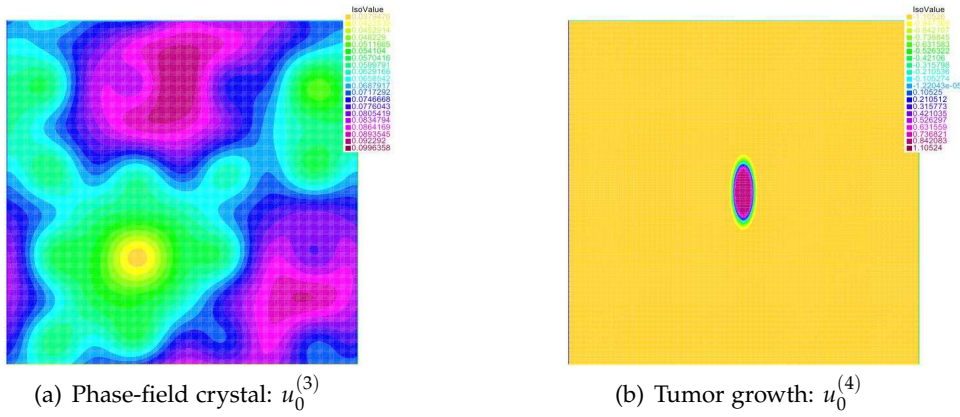The initial conditions $u_0^{(3)}$ and $u_0^{(4)}$ are shown in Fig. 5.1.

(a) Phase-field crystal: $u_0^{(3)}$

(b) Tumor growth: $u_0^{(4)}$

Figure 5.1: Initial conditions $u_0^{(3)}$ and $u_0^{(4)}$.

(a) $t = 10^{-6}$

(b) $t = 10^{-6}$

(c) $t = 10^{-6}$

(d) $t = 10^{-6}$

(e) $t = 5 \times 10^{-6}$

(f) $t = 5 \times 10^{-6}$
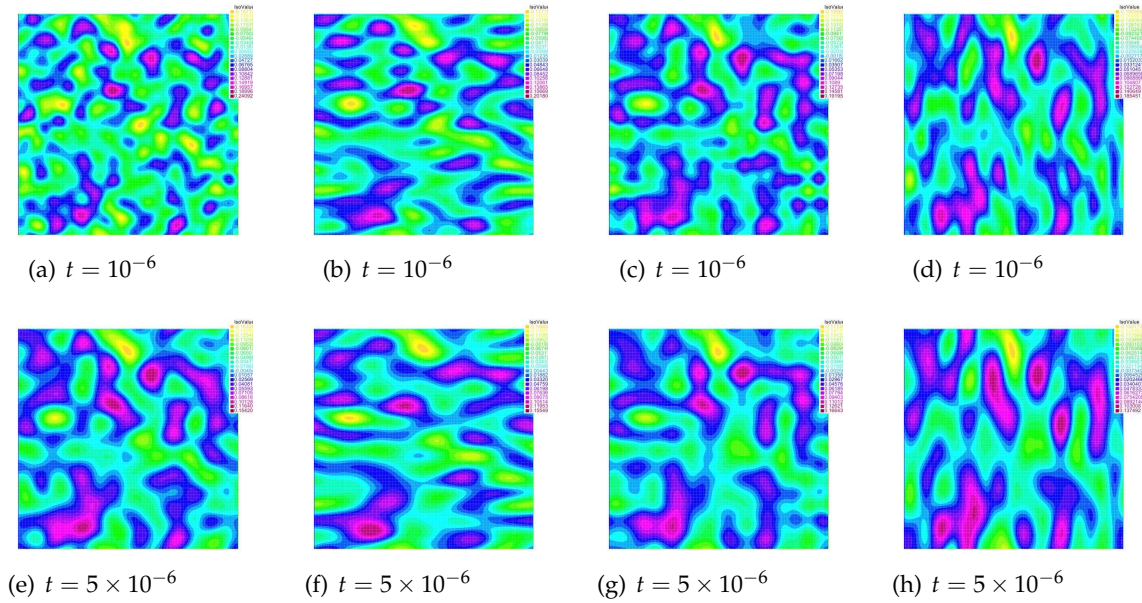

(g) $t = 5 \times 10^{-6}$

(h) $t = 5 \times 10^{-6}$

Figure 5.2: Cahn–Hilliard–Oono. Initial condition $u_0^{(1)}$, $f = u^3 - u$, $g = 0.5u$, $\varepsilon = 0.05$, $\Delta t = 5 \times 10^{-8}$.

Table 5.1: Coefficients a_{ij} for Fig. 5.2.

column	a_{20}	a_{11}	a_{02}	a_{10}	a_{01}	Remark
1	0	0	0	1	1	Cahn–Hilliard–Oono
2	1e-2	1e-4	1e-4	1e-4	1e-4	x-direction
3	1e-4	1e-2	1e-4	1e-4	1e-4	cross-direction
4	1e-4	1e-4	1e-2	1e-4	1e-4	y-direction

(a) $t = 2.5 \times 10^{-2}$ (b) $t = 2.5 \times 10^{-2}$ (c) $t = 2.5 \times 10^{-2}$ (d) $t = 2.5 \times 10^{-2}$

(e) $t = 5 \times 10^{-2}$ (f) $t = 5 \times 10^{-2}$ (g) $t = 5 \times 10^{-2}$ (h) $t = 5 \times 10^{-2}$

(i) $t = 7.5 \times 10^{-2}$ (j) $t = 7.5 \times 10^{-2}$ (k) $t = 7.5 \times 10^{-2}$ (l) $t = 7.5 \times 10^{-2}$

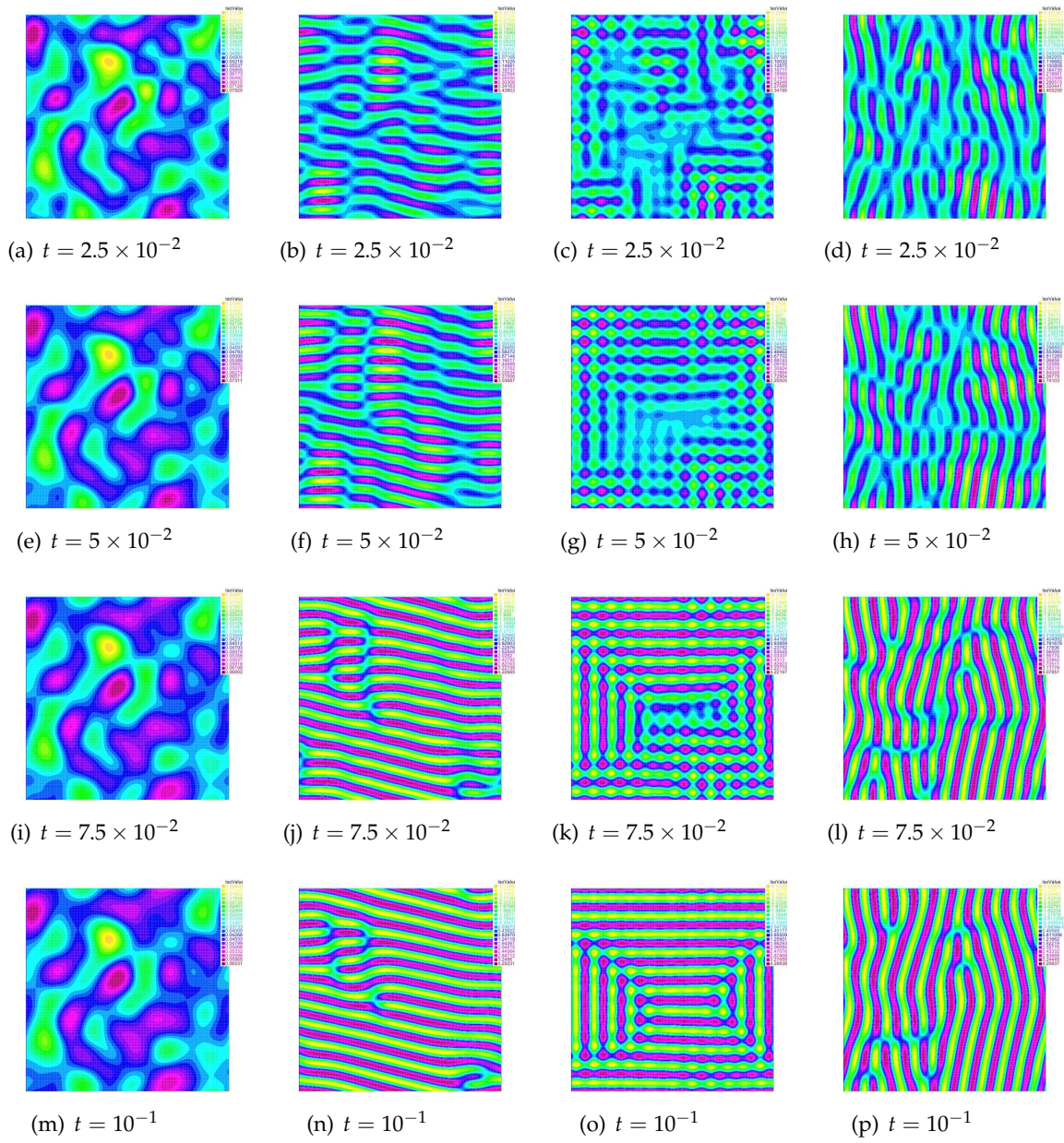(m) $t = 10^{-1}$ (n) $t = 10^{-1}$ (o) $t = 10^{-1}$ (p) $t = 10^{-1}$

Figure 5.3: Phase-field crystal. Initial condition $u_0^{(2)}$, $f = u^3 + (1 - 0.025)u$, $g = 2u$, $\varepsilon = 1$, $\Delta t = 10^{-4}$.

Table 5.2: Coefficients a_{ij} for Fig. 5.3.

Column	a_{20}	a_{11}	a_{02}	a_{10}	a_{01}	Remark
1	1	1	1	-2	-2	Phase-field crystal
2	1	0.1	0.1	-2	-2	x-direction
3	0.1	1	0.1	-2	-2	cross-direction
4	0.1	0.1	1	-2	-2	y-direction

(a) $t = 1.5$ (b) $t = 1.5$ (c) $t = 1.5$

(d) $t = 3$ (e) $t = 3$ (f) $t = 3$

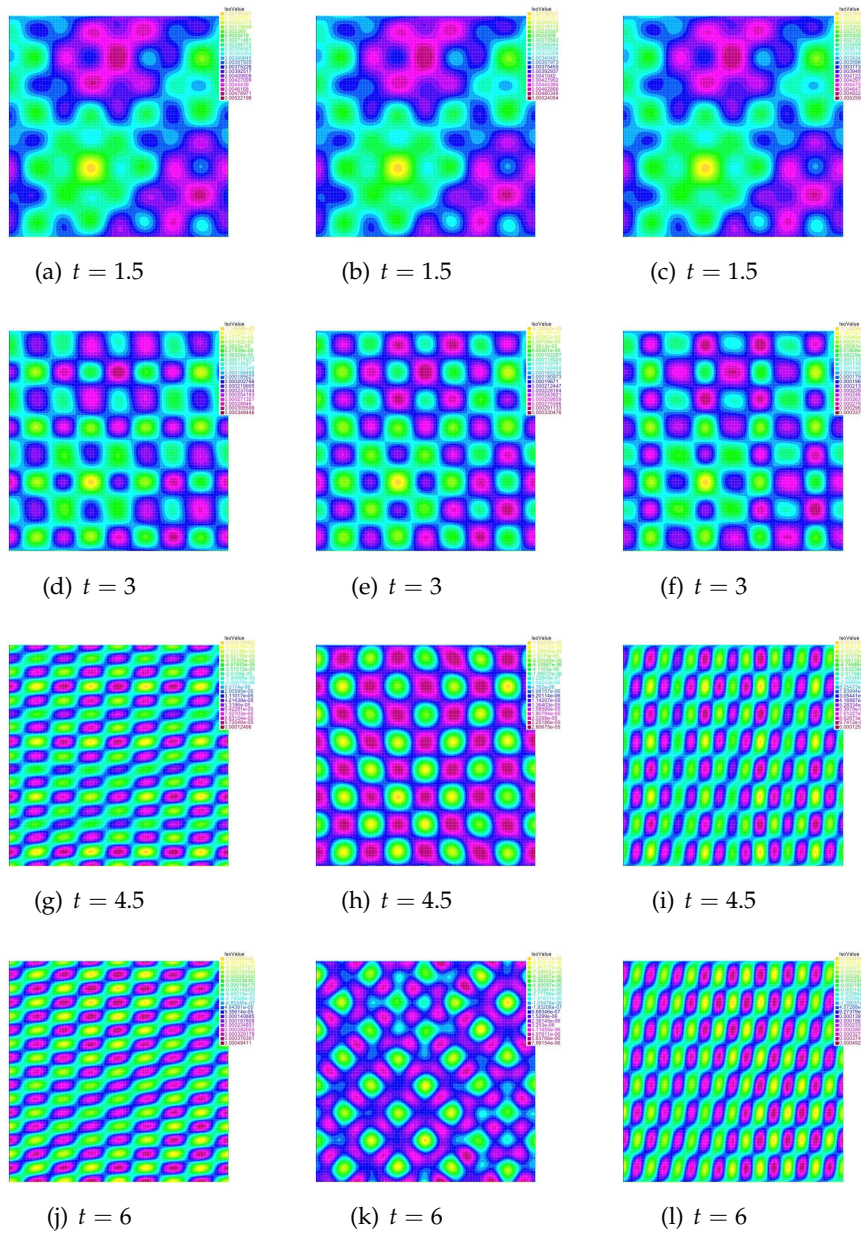(g) $t = 4.5$ (h) $t = 4.5$ (i) $t = 4.5$

(j) $t = 6$ (k) $t = 6$ (l) $t = 6$

Figure 5.4: Phase-field crystal. Initial condition $u_0^{(3)}$, $f = u^3 + (1 - 0.025)u$, $g = 2u$, $\varepsilon = 1$, $\Delta t = 10^{-3}$.

Table 5.3: Coefficients a_{ij} for Fig. 5.4.

Column	a_{20}	a_{11}	a_{02}	a_{10}	a_{01}	Remark
1	1	0.5	0.5	-2	-2	x-direction
2	0.5	1	0.5	-2	-2	cross-direction
3	0.5	0.5	1	-2	-2	y-direction

(a) $t = 1.5 \times 10^{-2}$ (b) $t = 1.5 \times 10^{-2}$
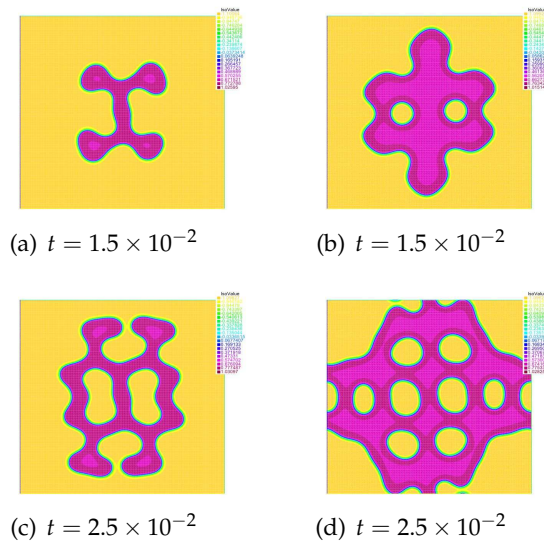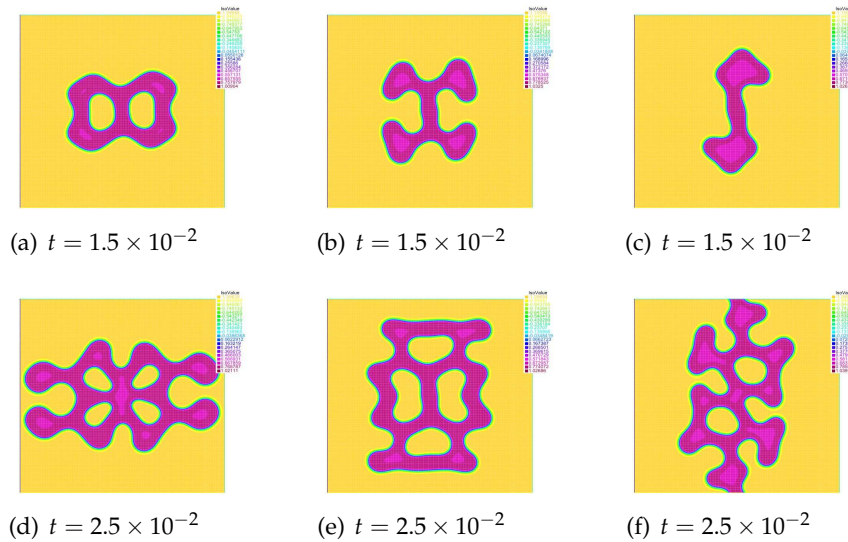
(c) $t = 2.5 \times 10^{-2}$ (d) $t = 2.5 \times 10^{-2}$

Figure 5.5: Tumor growth. Initial condition $u_0^{(4)}$, $f = u^3 - u$, $g = 46(u+1) - 280(u-1)^2(u+1)^2$, $\varepsilon = 0.0125$, $\Delta t = 10^{-6}$.

Table 5.4: Coefficients a_{ij} for Fig. 5.5.

Column	a_{20}	a_{11}	a_{02}	a_{10}	a_{01}	Remark
1	0	0	0	1	1	Cahn–Hilliard
2	5e-5	5e-5	5e-5	1	1	isotropy

(a) $t = 1.5 \times 10^{-2}$ (b) $t = 1.5 \times 10^{-2}$ (c) $t = 1.5 \times 10^{-2}$

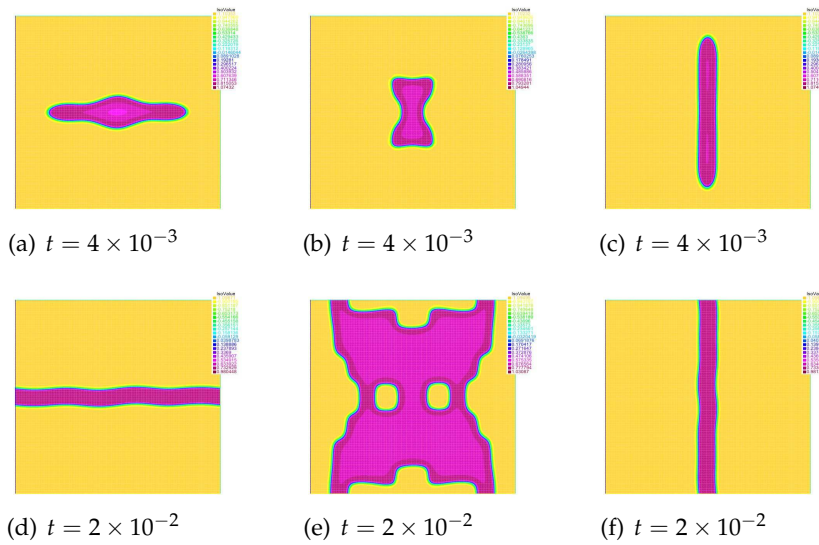(d) $t = 2.5 \times 10^{-2}$ (e) $t = 2.5 \times 10^{-2}$ (f) $t = 2.5 \times 10^{-2}$

Figure 5.6: Tumor growth. Initial condition $u_0^{(4)}$, $f = u^3 - u$, $g = 46(u+1) - 280(u-1)^2(u+1)^2$, $\varepsilon = 0.0125$, $\Delta t = 10^{-6}$

.

Table 5.5: Coefficients a_{ij} for Fig. 5.6.

Column	a_{20}	a_{11}	a_{02}	a_{10}	a_{01}	Remark
1	1.8e-5	5e-6	5e-6	1	1	x-direction
2	5e-6	1.8e-5	5e-6	1	1	cross-direction
3	5e-6	5e-6	1.8e-5	1	1	y-direction

(a) $t = 4 \times 10^{-3}$ (b) $t = 4 \times 10^{-3}$ (c) $t = 4 \times 10^{-3}$

(d) $t = 2 \times 10^{-2}$ (e) $t = 2 \times 10^{-2}$ (f) $t = 2 \times 10^{-2}$

Figure 5.7: Tumor growth. Initial condition $u_0^{(4)}$, $f = u^3 - u$, $g = 46(u+1) - 280(u-1)^2(u+1)^2$, $\varepsilon = 0.0125$, $\Delta t = 10^{-6}$.

Table 5.6: Coefficients a_{ij} for Fig. 5.7.

Column	a_{20}	a_{11}	a_{02}	a_{10}	a_{01}	Remark
1	5e-4	5e-6	5e-6	1	1	x-direction
2	5e-6	5e-4	5e-6	1	1	cross-direction
3	5e-6	5e-6	5e-4	1	1	y-direction

Acknowledgments

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