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Centre bifurcations of periodic orbits for some special three dimensional systems

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Abstract. In this paper, the bifurcated limit cycles from centre for a special three dimensional quadratic polynomial system and the Lü system are studied. For a given centre, the cyclicity is bounded from below by considering the linear parts of the corresponding Liapunov quantities of the perturbed system. We show that five limit cycles and two limit cycles can bifurcate from the centres for the three dimensional system and the Lü system respectively.

Keywords: centre bifurcation, periodic orbits, Lü system, Liapunov quantities.

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1 Introduction

We consider an analytic system of differential equations

$$\dot{u} = f(u), \qquad u \in \mathbb{R}^3 \tag{1.1}$$

has an isolated equilibrium point at the origin, the linear part df at the origin has one non-zero and two pure imaginary eigenvalues and that the components of f are quadratic polynomial functions.

A sufficient condition for a Hopf bifurcation in the three dimensional systems (1.1) (it possess two pure imaginary and one non-zero real eigenvalue) is illustrated bellow: let

$$\lambda^3 - T\lambda^2 - K\lambda - D = 0 \tag{1.2}$$

be the characteristic polynomial for system (1.1) where

$$T = \sum_{i=1}^{3} a_{i,i} \qquad \text{(trace of the Jacobian matrix of system (1.1) at the origin)},$$
$$D = \text{determinant of the Jacobian matrix of system (1.1) at the origin},$$
$$K = -(A_{1,1} + A_{2,2} + A_{3,3});$$

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where $A_1 = a_{2,2}a_{3,3} - a_{2,3}a_{3,2}$, $A_2 = a_{1,1}a_{3,3} - a_{1,3}a_{3,1}$ and $A_3 = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ and $a_{i,j}$, $i, j = 1, 2, 3$ are elements of the Jacobian matrix of system (1.1) at the origin.

Then the Hopf bifurcation occurs at a point (which is called Hopf point) on the surface

$$TK + D = 0; \quad K < 0 \quad \text{and} \quad T \neq 0. \tag{1.3}$$

We use an invertible linear change of coordinates and a rescaling of time, system (1.1) which satisfies (1.3) can be written into:

$$\begin{aligned} \dot{x}_1 &= -x_2 + F_1(x_1, x_2, x_3), \\ \dot{x}_2 &= x_1 + F_2(x_1, x_2, x_3), \\ \dot{x}_3 &= \lambda x_3 + F_3(x_1, x_2, x_3), \end{aligned} \tag{1.4}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, λ is a non-zero real number, F_1, F_2 and F_3 are real analytic functions on the neighborhood of the origin in $\mathbb{R}^3$, and with their derivatives vanish at the origin, the set of all parameters in F_1, F_2 and F_3 is denoted by Λ and $\mathbf{K}$ is the corresponding parameter space.

In this paper, we choose a special case of (1.1) as follows:

$$\begin{aligned} \dot{x_1} &= \lambda x_1 - x_2 + a(x_1^2 + x_2^2) + (cx_1 + dx_2)x_3, \\ \dot{x_2} &= x_1 + \lambda x_2 + b(x_1^2 + x_2^2) + (ex_1 + fx_2)x_3, \\ \dot{x_3} &= -x_3 + S(x_1^2 + x_2^2) + (Tx_1 + Ux_2)x_3, \end{aligned} \tag{1.5}$$

where a, b, c, d, e, f, S, T and U are real parameters. When $\lambda = 0$, this system has studied by Edneral *et al.* [5] and they investigated the nature of the local flow on the local center manifold at the origin. The following results are obtained (for their proof see [5]).

Proposition 1.1. *A system of the form* (1.5) *for which* $S = 0$ *and* $\lambda = 0$ *has a center on the local center manifold at the origin.*

Theorem 1.2. *A system of the form* (1.5) *for which* $a = b = c + f = 0, \lambda = 0$, *and* $S = 1$ *has a center on the local center manifold at the origin if and only if at least one of the following two sets of conditions holds:*

1. $8c + T^2 - U^2 = 4(e - d) - T^2 - U^2 = 2(e + d) + TU = 0$;
2. $c = d + e = 0$.

Theorem 1.3. *A system of the form* (1.5) *for which* $d + e = c = f = 0, \lambda = 0$ *and* $S = 1$ *has a center on the local center manifold at the origin if and only if at least one of the following three sets of conditions holds:*

1. $a = b = 0$;
2. $T - 2a = U - 2b = 0$;
3. $d = e = 0$.

If we perturb the parameters, how many periodic orbits can bifurcate from the origin? To answer this, we apply a new technique examining centre bifurcations to estimate the cyclicity of system (1.5) satisfying the conditions of Proposition 1.1, Theorem 1.2 and Theorem 1.3, which is explained in section three. Based on [4], the technique can be applied to other differential systems in $\mathbb{R}^3$ and we hope that it will be useful for a wider audience. For the

first time in three dimensional systems, such a technique was used by Salih [8] to prove that four limit cycles can bifurcate from the three dimensional Lotka–Volterra systems. In two dimensional systems, such a technique was used by Christopher [4] to show that at least eleven and seventeen limit cycles can bifurcate from a cubic centre and a quadratic non-degenerate centre, respectively with at least twenty-two limit cycles for another quadratic system globally.

The paper is organized as follows. In Section 2, the used technique which is used to estimate the bifurcated periodic orbits from centres is studied. The procedure for bifurcating limit cycles from centre and five bifurcating limit cycles from centre for a special three dimensional quadratic polynomial system are explained in Section 3. In the last section, we apply the same technique to the Lü system and show that only one limit cycle can be bifurcated from the centre.

2 A useful technique to examine the cyclicity bifurcating from centre

Bifurcation of limit cycles from critical points is the current area research in the bifurcation theory. A limit cycle is obtained by perturbing a focus or centre. One common approach is the centre bifurcation which is used to estimate the cyclicity and also to study the bifurcation of limit cycles from the centre (see Bautin [2] and Yu [9]).

Christopher in [4] investigated a technique to examine the cyclicity bifurcating from centre in two dimensional systems by linearizing the Liapunov quantities. Salih [8] generalized the technique to three dimensional systems to estimate the cyclicity of the centre. He applied the technique to the three dimensional Lotka–Volterra systems. The idea of the technique used here to estimate the cyclicity in three dimensional differential system can be illustrated by the following steps. Firstly, a point on a centre variety will be chosen, after that, the Liapunov quantities about this point will be linearized. If the codimension of the point that was chosen on a centre variety is r provided that the first r linear terms of Liapunov quantities are linearly independent, then $r-1$ is the cyclicity. That is, we can bifurcate $r-1$ limit cycles by a small perturbation.

Constructing the Liapunov function and calculating its focal values is a classical way to determine the number of limit cycles and their stability. In this method we seek a function of the form

$$F(x_1,x_2,x_3)=x_1^2+x_2^2+\sum_{k=3}^{\infty}F_k(x_1,x_2,x_3), \tag{2.1}$$

where $F_k=\sum_{i=0}^{k}\sum_{j=0}^{i}C_{k-i,i-j,j}x_1^{k-i}x_2^{i-j}x_3^{j}$ for system (1.1) and the coefficients of F_k satisfy

$$\mathcal{X}(F)=L_1(x_1^2+x_2^2)+L_2(x_1^2+x_2^2)^2+L_3(x_1^2+x_2^2)^3+\cdots, \tag{2.2}$$

where L_i, $i=1,2,\ldots$ are polynomials in the parameters of the system and the L_i is called the i^{th} Liapunov constant (focal value).

Explaining the technique in more detail, it is assumed that the centre critical point of (1.4) corresponds to $0\in\mathbf{K}$, by using a perturbation technique in parameters. This can be written:

$$\begin{aligned}\mathcal{X}&=\mathcal{X}_o+\mathcal{X}_1+\cdots,\\ F&=F_o+F_1+\cdots,\\ L_i&=L_{i0}+L_{i1}+\cdots,\qquad i=1,2,\ldots,\end{aligned} \tag{2.3}$$

where $\mathcal{X}_o, F_o$ and L_{0i} are calculated at the unperturbed parameters and $\mathcal{X}_1, F_1$ and L_{1i} are obtained at a perturbed parameters of first order (they contain the terms of degree one in Λ), and so forth. The Liapunov function F_i and the Liapunov quantity L_i have degree i in parameters. Putting equation (2.3) into equation (2.2) and we obtain:

$$\mathcal{X}_o F_o = 0, \quad \mathcal{X}_0 F_1 + \mathcal{X}_1 F_o = L_{11}(x_1^2 + x_2^2) + L_{21}(x_1^2 + x_2^2)^2 + \cdots , \tag{2.4}$$

and more general,

$$\mathcal{X}_o F_i + ... + \mathcal{X}_i F_o = L_{1i}(x_1^2 + x_2^2) + L_{2i}(x_1^2 + x_2^2)^2 + \cdots \tag{2.5}$$

The linear terms of the Liapunov quantities L_k (modulo the L_i, $i < k$) would be obtained by solving the pair equations (2.4) simultaneously by linear algebra. Equation (2.5) is used to generate the higher order terms of the Liapunov quantities.

3 Centre bifurcation of a quadratic three dimensional system

In this section, to examine the cyclicity bifurcating from centre at the origin of system (1.5) where the parameters satisfy the conditions of Proposition 1.1, Theorem 1.2 and Theorem 1.3 we apply the above technique which is described in Section 2. The main results of this section are the following theorems.

Theorem 3.1. *Suppose* $S = 1, \lambda = 0$ *and* $d + e = c = f = 0$ *for system* (1.5) *the following results are obtained.*

1. *If the parameters in system* (1.5) *satisfy the first set of conditions of Theorem 1.3, then five limit cycles can bifurcate from the origin.*

2. *If the parameters in system* (1.5) *satisfy the second set of conditions of Theorem 1.3, then five limit cycles can bifurcate from the origin.*

3. *If the parameters in system* (1.5) *satisfy the third set of conditions of Theorem 1.3, then four limit cycles can bifurcate from the origin.*

Proof. 1. When the conditions hold, system (1.5) reduce to

$$\begin{aligned} \dot{x_1} &= -x_2 + dx_2x_3, \\ \dot{x_2} &= x_1 - dx_1x_3, \\ \dot{x_3} &= -x_3 + x_1^2 + x_2^2 + (Tx_1 + Ux_2)x_3. \end{aligned}$$

We choose a point, $(\lambda, a, b, c, d, e, f, S, T, U) = (0, 0, 0, 0, 1, -1, 0, 1, 1, 0)$ on centre variety and it is easy to check that $\mathcal{X}_o F_o = 0$ where

$$\begin{aligned} \mathcal{X}_o &= \left((-x_2 + x_2x_3)\frac{\partial}{x_1}, (x_1 - x_1x_3)\frac{\partial}{x_2}, (-x_3 + x_1^2 + x_2^2 + x_1x_3)\frac{\partial}{x_3} \right), \\ F_o &= x_1^2 + x_2^2 + \sum_{k=3}^{N} \sum_{i=0}^{k} \sum_{j=0}^{i} C_{k-i,i-j,j} x_1^{k-i} x_2^{i-j} x_3^j. \end{aligned}$$

We let $(\lambda, a, b, c, d, e, f, S, T, U) = (0+\lambda_1, 0+a_1, 0+b1, 0+c_1, 1+d_1, -1+e_1, 0+f_1, 1+S_1, 1+T_1, 0+U_1)$, then the perturbed vector field and the perturbed Liapunov function of first order are defined by

$$\begin{aligned}\mathcal{X}_1 = &\Big((\lambda_1 x_1 + a_1(x_1^2+x_2^2) + (c_1x_1+d_1x_2)x_3)\frac{\partial}{x_1}, (\lambda_1 x_2 + b_1(x_1^2+x_2^2)\\ &+(e_1x_1+f_1x_2)x_3)\frac{\partial}{x_2}, (S_1(x_1^2+x_2^2)+(T_1x_1+U_1x_2)x_3)\frac{\partial}{x_3}\Big),\\ F_1 = &\sum_{k=3}^{N}\sum_{i=0}^{k}\sum_{j=0}^{i} D_{k-i,i-j,j}x_1^{k-i}x_2^{i-j}x_3^j\end{aligned}$$

Using computer algebra package MAPLE, $\mathcal{X}_0F_1 + \mathcal{X}_1F_o$ in equation (2.4) give us the following linearly independent terms of Liapunov quantities.

1. $L_1 = 2\lambda_1$.
2. $L_2 = f_1 + c_1$.
3. $L_3 = \frac{1}{40}(3d_1 + 20a_1 + 11f_1 + 3e_1 + 20b_1 + 9c_1)$.
4. $L_4 = \frac{1}{400}(153d_1 + 430a_1 + 206f_1 + 153e_1 + 260b_1 + 224)$.
5. $L_5 = \frac{1}{544000}(527253d_1 + 1007080a_1 + 330421f_1 + 527253e_1 + 507960b_1 + 676659)$.
6. $L_6 = \frac{1}{1202240000}\big(2460349388d_1 + 4091910030a_1 + 823850861f_1 + 2460349388e_1 + 2033633160b_1 + 3268059169\big)$.

The origin of system (1.5) is weak focus of order 5 if and only if

1. $\lambda_1 = 0$.
2. $c_1 = -f_1$.
3. $d_1 = \frac{1}{3}(-20a_1 - 2f_1 - 3e_1 - 20b_1)$.
4. $a_1 = \frac{-1}{59}(12f_1 + 76b_1)$.
5. $f_1 = \frac{659345}{553569}b_1;\ b_1 \neq 0$.

Since

$$J = \begin{vmatrix} \frac{\partial L_1}{\partial \lambda_1} & \frac{\partial L_1}{\partial c_1} & \frac{\partial L_1}{\partial d_1} & \frac{\partial L_1}{\partial a_1} & \frac{\partial L_1}{\partial f_1} \\ \frac{\partial L_2}{\partial \lambda_1} & \frac{\partial L_2}{\partial c_1} & \frac{\partial L_2}{\partial d_1} & \frac{\partial L_2}{\partial a_1} & \frac{\partial L_2}{\partial f_1} \\ \frac{\partial L_3}{\partial \lambda_1} & \frac{\partial L_3}{\partial c_1} & \frac{\partial L_3}{\partial d_1} & \frac{\partial L_3}{\partial a_1} & \frac{\partial L_3}{\partial f_1} \\ \frac{\partial L_4}{\partial \lambda_1} & \frac{\partial L_4}{\partial c_1} & \frac{\partial L_4}{\partial d_1} & \frac{\partial L_4}{\partial a_1} & \frac{\partial L_4}{\partial f_1} \\ \frac{\partial L_5}{\partial \lambda_1} & \frac{\partial L_5}{\partial c_1} & \frac{\partial L_5}{\partial d_1} & \frac{\partial L_5}{\partial a_1} & \frac{\partial L_5}{\partial f_1} \end{vmatrix} = \frac{1660707}{21760000} \neq 0,$$

then by suitable perturbation of the coefficients of Liapunov quantities, five limit cycles can be bifurcated from the origin of system (1.5) in the neighborhood of the origin. □

Remark 3.2. By the same way, we can prove the second and third part of the above theorem as well as the below theorems.

Theorem 3.3. *Suppose* $S = 1$, $\lambda = 0$ *and* $a = b = c + f = 0$ *for system* (1.5) *the following results are obtained.*

1. *If the parameters in system* (1.5) *satisfy the first set of conditions of Theorem* 1.2*, then three limit cycles can bifurcate from the origin.*
2. *If the parameters in system* (1.5) *satisfy the second set of conditions of Theorem* 1.2*, then five limit cycles can bifurcate from the origin.*

Theorem 3.4. *If the parameters in system* (1.5) *satisfy the condition of Proposition* 1.1*, i.e.* $S = 0$ *and* $\lambda = 0$*, then only one limit cycle can bifurcate from the origin.*

4 Centre bifurcation of the Lü system

In this section, we consider the three dimensional Lü system:

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= cy - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{4.1}$$

where a, b and c are real parameters. Besides the origin, system (4.1) has two symmetric equilibrium points $A_\pm = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ when $bc > 0$. Yu and Zhang [10] investigate the stability of (4.1) and show that the system display Hopf bifurcation under certain conditions. They also drive the conditions of supercritical and subcritical bifurcation. in [7], Mello and Coelho have studied the stability and degenerate Hopf bifurcation which occur at the equilibria $A_\pm$ up to codimension three of system (4.1). Since the system is invariant under the involution $(x, y, z) \to (-x, -y, z)$, so the equilibrium point A_+ and A_- have the same stability.

When $(a, b, c) \in S = \{(a, b, c) : ab > 0, c = \frac{a+b}{3}\}$, $A_\pm$ are Hopf points of system (4.1) because it has two purely imaginary eigenvalues and one real eigenvalue. In this case, the first Liapunov constant of $A_\pm$ is non-zero if and only if $(a - 5b)(2a - b) \neq 0$. In [7], it was shown that when $(a - 5b) = 0$, the second Liapunov constant is different from zero, but when $(2a - b) = 0$, it was shown that the second and third Liapunov constants vanish. Therefore, Mello and Coelho [7], conjectured that the eqilibria $A_\pm$ are centre of (4.1) if the following conditions are held.

$$b = 2c, \quad a = c, \quad \text{and} \quad ab > 0. \tag{4.2}$$

To show that the conjecture concerning the existence of centres on local centre manifold at $A_\pm$ of (4.1) is true, based on Darboux method, Mahdi *et al.* [6] showed that the local centre manifolds are algebraic ruled surface. Buică *et al.* [3] proved that the conjecture is true by finding a global inverse Jacobi multiplier.

Now, we apply the above technique which is described in Section 2 and the main result of this section is the following theorem.

Theorem 4.1. *If the parameters in Lü system* (4.1) *satisfy conditions* (4.2) *, then only one limit cycles can bifurcate from the critical point located at* A_+*.*

Proof. As a first step, we scale the critical point A_+ to the origin by setting $\tilde{x}_1 = x - \sqrt{bc}, \tilde{y} = y - \sqrt{bc}$, $\tilde{z} = z - c$. When system (4.1) satisfies the conditions (4.2), its characteristic polynomial is given by

$$\lambda^3 + \sqrt{2}\omega\lambda^2 + \omega^2\lambda + \omega^3\sqrt{2} = 0,$$

its coefficients satisfy equation (1.3) and the eigenvalues are $\pm i\omega$ and $-\sqrt{2}\omega$, where $\omega = \sqrt{2}c$. The critical point A_+ is centre as we see in Figure 4.1 .

We let $a = \frac{1}{\sqrt{2}}\omega + a_1$ and $b = \sqrt{2}\omega + b_1$ where a_1 and b_1 are parameters after perturbation in the system. Therefore, the unperturbed, $\mathcal{X}_o$, and the perturbed vector field of first order, $\mathcal{X}_1$, are defined by

$$\begin{aligned}
\mathcal{X}_o &= -\frac{\omega}{\sqrt{2}}(\tilde{x}-\tilde{y})\frac{\partial}{\partial\tilde{x}} + \left(\frac{\omega}{\sqrt{2}}(\tilde{y}-\tilde{x}) - \omega\tilde{z} - \tilde{x}\tilde{z}\right)\frac{\partial}{\partial\tilde{y}} + (\omega(\tilde{x}+\tilde{y}-\sqrt{2}\tilde{z}) + \tilde{x}\tilde{y})\frac{\partial}{\partial\tilde{z}},\\
\mathcal{X}_1 &= -a_1(\tilde{x}-\tilde{y})\frac{\partial}{\partial\tilde{x}} - \frac{1}{2\sqrt{2}}b_1\tilde{z}\frac{\partial}{\partial\tilde{y}} + \left(\frac{1}{2\sqrt{2}}b_1(\tilde{x}+\tilde{y}) - b_1\tilde{z}\right)\frac{\partial}{\partial\tilde{z}}.
\end{aligned} \tag{4.3}$$

Using the linear transformation

$$X = PY,\ P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & -\frac{1}{2\sqrt{2}} \\ \frac{3}{\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ 1 & \sqrt{2} & 1 \end{bmatrix}, \tag{4.4}$$

where $X = (\tilde{x}, \tilde{y}, \tilde{z})$, $Y = (y_1, y_2, y_3)$, the linear part of system (4.1) at the origin

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}}\omega & \frac{1}{\sqrt{2}}\omega & 0 \\ -\frac{1}{\sqrt{2}}\omega & \frac{1}{\sqrt{2}}\omega & -\omega \\ \omega & \omega & -\sqrt{2}\omega \end{bmatrix}$$

can be written in the real canonical form as

$$\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & -\sqrt{2}\omega \end{bmatrix},$$

and the new system is given by

$$\begin{aligned}
\dot{y_1} &= -\omega y_2 - \frac{1}{2}y_1^2 - \frac{5}{3\sqrt{2}}y_1y_2 - \frac{1}{9}y_1y_3 - \frac{2}{3}y_2^2 - \frac{7}{18\sqrt{2}}y_2y_3 + \frac{13}{72}y_3^2,\\
\dot{y_2} &= \omega y_1 + \frac{1}{\sqrt{2}}y_1^2 + \frac{4}{3}y_1y_2 - \frac{1}{18\sqrt{2}}y_1y_3 + \frac{\sqrt{2}}{3}y_2^2 + \frac{5}{18}y_2y_3 - \frac{\sqrt{2}}{9}y_3^2,\\
\dot{y_3} &= -\omega\sqrt{2}y_3 + y_1^2 + \sqrt{2}y_1y_2 - \frac{1}{3}y_1y_3 + \frac{1}{3\sqrt{2}}y_2y_3 - \frac{1}{12}y_3^2.
\end{aligned} \tag{4.5}$$

The same transformation in equation (4.4) is used for the perturbed vector field part of system (4.1) and we obtain

$$\begin{aligned}
\dot{y_1} &= \frac{1}{9}\left(2a_1 - \frac{3}{2}b_1\right)y_1 - \frac{1}{9}\left(\sqrt{2}a_1 - \frac{3}{2\sqrt{2}}b_1\right)y2 + \frac{1}{9}\left(a_1 - \frac{1}{2}b_1\right)y_3,\\
\dot{y_2} &= \frac{1}{9}\left(5\sqrt{2}a_1 + \frac{3}{2\sqrt{2}}b_1\right)y_1 - \frac{1}{9}\left(5a_1 + \frac{3}{2}b_1\right)y_2 + \frac{1}{18}\left(5\sqrt{2}a_1 - \frac{5}{\sqrt{2}}b_1\right)y_3,\\
\dot{y_3} &= \frac{-4}{3}a_1y_1 + \frac{1}{3}\left(2\sqrt{2}a_1 - \frac{3}{\sqrt{2}}b_1\right)y_2 - \frac{2}{3}(a_1+b_1)y_3.
\end{aligned} \tag{4.6}$$

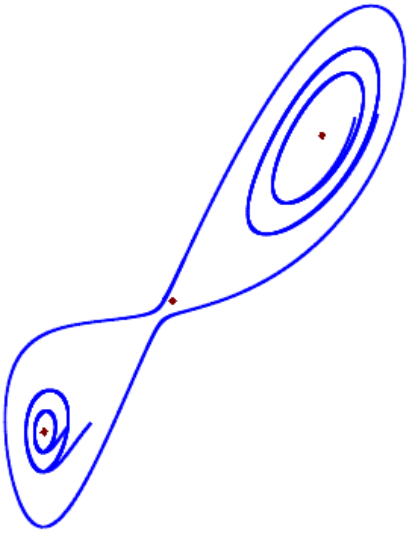

Figure 4.1: Numerical plot of trajectories near critical points of system (4.1) where $a = 1$, $b = 2$, $c = 1$ with initial conditions: $(0, 0.1, 0.2)$, $(1.8, 1.4, 1.5)$, $(-1.3, -1.4, 1.5)$, $(-1.2, -1.4, 1)$ and $(-1, -1.4, 1)$. The red points indicate the critical points.

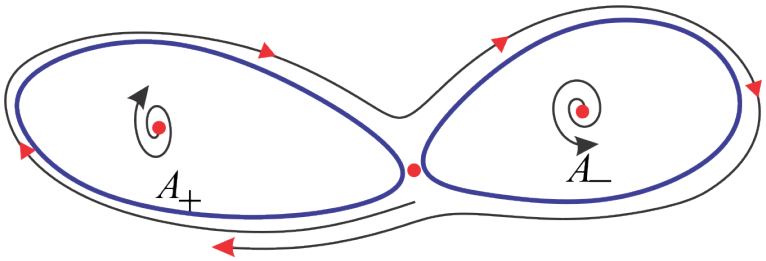

Figure 4.2: Two limit cycles are bifurcated around critical points A_+ and A_-.

Now, we define the unperturbed, F_o, and the perturbed Liapunov function of first order, F_1, by

$$F_o = y_1^2 + y_2^2 + \sum_{k=3}^{N}\sum_{i=0}^{k}\sum_{j=0}^{i} C_{k-i,i-j,j}\, y_1^{k-i} y_2^{i-j} y_3^j,$$

$$F_1 = \sum_{k=3}^{N}\sum_{i=0}^{k}\sum_{j=0}^{i} D_{k-i,i-j,j}\, y_1^{k-i} y_2^{i-j} y_3^j, \tag{4.7}$$

where $N \geq 3$. It is easy to show that the Liapunov function, F_o, of equation (4.5) satisfies $\mathcal{X}_o F_o = 0$. Using computer algebra package MAPLE, equation (2.4) give us the following linear independent terms of Liapunov quantities.

1. $L_1 = \dfrac{-1}{3}(a_1 + b_1)$.

2. $L_2 = -\dfrac{1}{864\omega^2}(91a_1 - 557b_1)$.

The origin of system (4.1) is weak focus of order one if and only if

$$a_1 = -b_1.$$

Since the Jacobian of L_1 and L_2 with respect to a_1 and b_1 is non-zero, then by suitable perturbation of the coefficients of Liapunov quantities, only one limit cycle can be bifurcated from the origin of system (4.1) in the neighborhood of the origin, as we see in Figure 4.2. □

Remark 4.2. By the same way, we can prove that another limit cycle can bifurcate from the critical point located at A_-.

5 Conclusion

In this paper, we presented a simple computational approach to estimate the cyclisity bifurcating from centre. This approach is applied to a special three dimensional system which is introduced in [5] to obtain some bifurcated periodic orbit. In addition, we applied the same approach to Lü system and two bifurcating periodic orbits were obtained.

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