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(ω, c)-periodic functions and mild solutions to abstract fractional integro-differential equations

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Abstract. In this paper we study a new class of functions, which we call (ω, c)-periodic functions. This collection includes periodic, anti-periodic, Bloch and unbounded functions. We prove that the set conformed by these functions is a Banach space with a suitable norm. Furthermore, we show several properties of this class of functions as the convolution invariance. We present some examples and a composition result. As an application, we establish some sufficient conditions for the existence and uniqueness of (ω, c)-periodic mild solutions to a fractional evolution equation.

Keywords: antiperiodic, periodic, (ω, c)-periodic, convolution invariance, fractional integro-differential equations, completeness.

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1 Introduction

The aims of this work are to study the class of (ω, c)-periodic functions and to develop their properties. Also we give applications to fractional integro-differential equations in Banach spaces.

In order to motive the definition of (ω, c)-periodic function, we consider the Mathieu's equation

$$\frac{d^2y}{dt^2} + [a - 2q\cos(2t)]y = 0,$$

which arises as models in many context, as the stability of railroad rails as trains drive over them and seasonally forced population dynamics. This equation is an important special case of the Hill's differential equation.

According to Floquet's theorem, these equations admit a complex valued basis of solutions of the form $y(t) = e^{\mu t}p(t)$, $t \in \mathbb{R}$, where μ is a complex number and p is a complex valued

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function which is ω-periodic (see [7, Ch. 8, Section 4]). We can observe that the solution is not periodic, but

$$y(t+\omega) = cy(t), \qquad c = e^{\mu\omega}, \quad t \in \mathbb{R}. \tag{1.1}$$

On the other hand, Bloch's theorem (which is the analogous to Floquet's theorem in solid-state physics, see [4]) states that the Bloch functions (or wave functions), which satisfy the Schödinger equation, can be written as $\psi_k(r) = e^{ikr}u_k(r)$, where u_k is ω-periodic and (1.1) is satisfied.

Following [12], we say that f is a (ω, c)-periodic function if there is a pair (ω, c), $c \in (\mathbb{C} \setminus \{0\})$, $w > 0$ such that $f(t+\omega) = cf(t)$, for all $t \in \mathbb{R}$ (see [13–15]). This concept is more general than both periodic ($c = 1$) (see [3,5,7,9]) and anti-periodic functions ($c = -1$) (see [1,6,8,10]) and also includes other types of functions such as unbounded (ω, c)-periodic function if $|c| \neq 1$ and *Bloch functions*. We characterize the (ω, c)-periodicity and provide a Banach space structure with a suitable norm. This allows to study unbounded oscillations over $\mathbb{R}$ better than with the direct sup-norm.

Problems as existence and uniqueness of periodic and anti-periodic mild solutions to different abstract equations have been extensively studied due to their several applications in physics, probability, modelling, mechanics and other areas (see [1,6,8,10,11,18]) and the references therein. Particularly, in [2,16] the authors studied the existence and uniqueness of mild solutions to

$$D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)), \tag{1.2}$$

on several subspaces of $BC(\mathbb{R}, X)$, in particular, periodic and antiperiodic mild solutions. In this work we prove the existence and uniqueness of (ω, c)-periodic solutions to (1.2) using a suitable norm.

This paper is organized as follows. In Section 2, we define the space of (ω, c)-periodic functions. Also we present several properties, examples and prove that with a suitable norm the collection of (ω, c)-periodic functions is a Banach space. New convolution and composition theorems are proved. In Section 3, we show an application to (1.2).

2 (ω, c)-periodic functions

Throughout the paper, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, X will denote a complex Banach space with norm $\|\cdot\|$, and the space of continuous functions as

$$C(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is continuous}\}.$$

Definition 2.1. A function $f \in C(\mathbb{R}, X)$ is said to be (ω, c)-periodic if $f(t+\omega) = cf(t)$ for all $t \in \mathbb{R}$. ω is called the c-period of f. The collection of those functions with the same c-period ω will be denoted by $P_{\omega c}(\mathbb{R}, X)$. When $c = 1$ (ω-periodic case) we write $P_\omega(\mathbb{R}, X)$ in spite of $P_{\omega 1}(\mathbb{R}, X)$. Using the principal branch of the complex Logarithm (i.e. the argument in $(-\pi, \pi]$) we define $c^{t/\omega} := \exp((t/\omega)\operatorname{Log}(c))$. Also, we will use the notation $c^\wedge(t) := c^{t/\omega}$ and $|c|^\wedge(t) := |c^\wedge(t)| = e^{(t/\omega)\ln(|c|)}$.

The following proposition gives a characterization of the (ω, c)-periodic functions.

Proposition 2.2. *Let $f \in C(\mathbb{R}, X)$. Then f is (ω, c)-periodic if and only if*

$$f(t) = c^\wedge(t)u(t), \qquad c^\wedge(t) = c^{t/\omega}, \quad u \in P_\omega(\mathbb{R}, X). \tag{2.1}$$

Proof. It is clear that if $f(t) = c^{\wedge}(t)u(t)$ then f is a (ω, c)-periodic function. In order to show the inverse statement, let $f \in P_{\omega c}(\mathbb{R}, X)$. If we write $u(t) := c^{\wedge}(-t)f(t) = c^{-t/\omega}f(t)$, then we have that

$$u(t+\omega) = u(t),$$

hence the function $u(t)$ is an ω-periodic function and $f(t) = c^{\wedge}(t)u(t)$. □

In view of (2.1), for any $f \in P_{\omega c}(\mathbb{R}, X)$ we say that $c^{\wedge}(t)u(t)$ is the c-factorization of f.

Remark 2.3. From Proposition 2.2, we can write all $f \in P_{\omega c}(\mathbb{R}, X)$ as

$$f(t) = c^{\wedge}(t)u(t),$$

where $u(t)$ is ω-periodic on $\mathbb{R}$. We will call $u(t)$ *the periodic part of* f. With this convention, an anti-periodic function f can be written as $f(t) = (-1)^{t/\omega}u(t)$, where its antiperiod is ω. For example, $f(t) = \sin(t)$ can be considered as an anti-periodic function, with $\omega = \pi$. As $\mathrm{Log}(-1) = i\pi$, f has the decomposition $f(t) = c^{\wedge}(t)u(t)$ where

$$c^{\wedge}(t) = (-1)^{t/\omega} = e^{ti} = \cos t + i \sin t,$$

and

$$u(t) = \sin t(\cos t - i \sin t).$$

Let $c = e^{2\pi i/k}$ for some natural number $k \geq 2$ and let f a (ω, c)-periodic function, then f is a periodic function with period $k\omega$ but, in general can be written as $f(t) = e^{2\pi ti/k\omega}u(t)$, where u is a complex periodic function with period ω. In particular if $k = 4$, an $(\omega, e^{\pi i/2})$-periodic function f can be at the same time a Bloch wave: $f(t+\omega) = e^{\pi i/2}f(t)$, an anti-periodic function with antiperiod 2ω: $f(t+2\omega) = -f(t)$ and a 4ω-periodic function: $f(t+4\omega) = f(t)$.

Remark 2.4. From Definition 2.1 we can observe that $P_{\omega c}(\mathbb{R}, X)$ is a translation invariant subspace over $\mathbb{C}$ of $C(\mathbb{R}, X)$. Furthermore, $f \in P_{\omega c}(\mathbb{R}, X)$ derivable implies that $f' \in P_{\omega c}(\mathbb{R}, X)$ and if $|c| = 1$ then $P_{\omega c}(\mathbb{R}, X)$ has only bounded functions, if $|c| < 1$ then any element $f \in P_{\omega c}(\mathbb{R}, X)$ goes to zero when $t \to \infty$, and f is unbounded when $t \to -\infty$, and if $|c| > 1$ then f is unbounded when $t \to \infty$ and f goes to zero when $t \to -\infty$.

Example 2.5. If we consider the linear delayed equation

$$x'(t) = -\rho x(t-r), \qquad t \in \mathbb{R}, \tag{2.2}$$

with $\rho, r > 0$, a solution $\phi(t) = e^{z_0 t}$, with $z_0 = x_0 + iy_0$, $x_0, y_0 \in \mathbb{R}$, $y_0 > 0$, where $z_0 + \rho e^{-z_0 r} = 0$, give us a $(2\pi/y_0, e^{2\pi x_0/y_0})$-periodic solution for (2.2).

Example 2.6. Let $u : \mathbb{R} \to X$ be a X-valued periodic function with period ω. Let $\phi : \mathbb{R} \to \mathbb{C}$ be a function with the semigroup property, that is, $\phi(t+s) = \phi(s)\phi(t)$ for all $t, s \in \mathbb{R}$ and such that $\phi(\omega) \neq 0$. Then

$$v(t) = \phi(t)u(t)$$

is a $(\omega, \phi(\omega))$-periodic function. Taking $\phi(t) = e^{ikt}$ we obtain Bloch functions.

Remark 2.7. In general, if ϕ is a function with the semigroup property such that $\phi(\omega) \neq 0$, and if u is a (ω, c)-periodic function, then $v(t) = \phi(t)u(t)$ is a $(\omega, c\phi(\omega))$-periodic function. Moreover, let $(u_k)_{k \in \mathbb{N}}$ be a sequence of (ω, c)-periodic functions and $(\phi_k)_{k \in \mathbb{N}}$ be a sequence of functions with the semigroup property and such that $\phi_k(\omega) = p \neq 0$ for all $k \in \mathbb{N}$. Assume that

$$\sum_{k=1}^{\infty} \phi_k(t) u_k(t)$$

is a uniformly convergent series on $\mathbb{R}$. Then

$$f(t) = \sum_{k=1}^{\infty} \phi_k(t) u_k(t)$$

is a (ω, cp)-periodic function. As a particular case, if the series

$$\sum_{k=1}^{\infty} \phi_k(t) \frac{\cos[(2k+1)t]}{k^2}$$

is uniformly convergent, then

$$f(t) = \sum_{k=1}^{\infty} \phi_k(t) \frac{\cos[(2k+1)t]}{k^2}$$

is a $(\pi, -p)$-periodic function. In this case, calling $\sigma(p)$ the sign of p, we have

$$c^{\wedge}(t) = (-p)^{t/\omega} = e^{\ln(|p|)+\sigma(p)\pi} = |p|^t(-1 + i\sigma(p)),$$

and hence the π-periodic part of f is

$$u(t) = \sum_{k=1}^{\infty} |p|^{-t}(-1 - i\sigma(p))\phi_k(t) \frac{\cos[(2k+1)t]}{k^2}$$

and the $(-p)$-factorization of f is given by

$$f(t) = c^{\wedge}(t)u(t) = \sum_{k=1}^{\infty} \phi_k(t) \frac{\cos[(2k+1)t]}{k^2}.$$

Next, we show a convolution theorem.

Theorem 2.8. *Let $f \in P_{\omega c}(\mathbb{R}, X)$ with $f(t) = c^{\wedge}(t)p(t)$, $p \in P_{\omega}(\mathbb{R}, X)$. If $k^{\sim}(t) := c^{\wedge}(-t)k(t) \in L^1(\mathbb{R})$, then $(k * f) \in P_{\omega c}(\mathbb{R}, X)$, where*

$$(k * f)(t) = \int_{-\infty}^{+\infty} k(t-s) f(s)\, ds.$$

Proof. The conclusion follows from the fact that $(k * f)(t) = c^{\wedge}(t)(k^{\sim} * p)(t)$. $\square$

Example 2.9. Consider the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = f(x). \end{cases}$$

Let $u(t,x)$ be a regular solution with $u(x,0) = f(x)$. Then it is known that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s) ds.$$

Fix $t_0 > 0$ and assume that $f(x)$ is (ω, c)-periodic. Then, by Theorem 2.8, $u(x+\omega, t_0) = cu(x, t_0)$, hence $u(x,t)$ is (ω, c)-periodic with respect to x.

In order to define a norm over the set $P_{\omega c}(\mathbb{R}, X)$ to give it a Banach structure we need to deal with the following characteristics of their elements: the non-boundedness and its periodicity. The periodicity suggest to use a *sup-norm*, as is possible in $P_{\omega}(\mathbb{R}, X)$, but if $|c| \neq 1$ and we take $\|f\| = \sup_{t \in \mathbb{R}} \|f(t)\|$, then $\|f\| = \infty$, for all $f \in P_{\omega c}(\mathbb{R}, X)$.

The most natural way to avoid the unboundedness of the elements is to restrict the attention to some local bounded case, for example

$$P_{\omega c}^{+} := \{f : \mathbb{R}_{+} \to X : f(t+\omega) = cf(t),\ |c| \leq 1\}$$

with the norm

$$\|f\|_{\infty} := \sup_{t \in \mathbb{R}_{+}} \|f(t)\|, \tag{2.3}$$

but it supposes a strong restriction to the study of the (ω, c)-periodic functions. Moreover, the use of norm (2.3) in the space $P_{\omega c}^{+}$ (bounded case) implies a lost of *periodic structure* in the following sense: if we take $f_1(t) := e^{-t}\cos(t)$ and $f_2(t) := e^{-t}\sin(t)$ in $P_{2\pi\, e^{-2\pi}}^{+}$ with *periodic components* $\cos(t)$ and $\sin(t)$ respectively, which have the same 2π*-period*, and belong to $P_{2\pi}(\mathbb{R}_{+}, \mathbb{R})$, then f_1 and f_2 must have the same norm Nevertheless

$$\|f_1\|_{\infty} = 1, \qquad \|f_2\|_{\infty} = \frac{e^{-\pi/4}}{\sqrt{2}} < 1.$$

Theorem 2.10. *$P_{\omega c}(\mathbb{R}, X)$ is a Banach space with the norm*

$$\|f\|_{\omega c} := \sup_{t \in [0, \omega]} \||c|^{\wedge}(-t) f(t)\|.$$

Proof. Let $\{w_n\}_{n \in \mathbb{N}} \subset P_{\omega c}(\mathbb{R}, X)$ a Cauchy sequence. By Proposition 2.2 we can write $w_n(t) = c^{\wedge}(t) p_n(t)$, where $p_n \in P_{\omega}(\mathbb{R}, X)$. Also $\|p_n - p_m\|_{\omega} = \|w_n - w_m\|_{\omega c}$ implies that $\{p_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $P_{\omega}(\mathbb{R}, X)$, which is a Banach space with respect to the norm $\|\cdot\|_{\omega}$, then there exists a ω-periodic function $p(t)$ such that $p_n \to p$ uniformly in $[0, \omega]$, and in consequence, $w_n(t) \to w(t) := c^{\wedge}(t) p(t)$ with the ωc-norm in $P_{\omega c}(\mathbb{R}, X)$. $\square$

If $F \in C(\mathbb{R} \times X, X)$ and $\varphi \in P_{\omega c}(\mathbb{R}, X)$, we study the invariance on $P_{\omega c}(\mathbb{R}, X)$ for the Nemytskii's operator $\mathcal{N}(\varphi)(\cdot) = F(\cdot, \varphi(\cdot))$.

Theorem 2.11. *Let $F \in C(\mathbb{R} \times X, X)$ and $(\omega, c) \in \mathbb{R}^{+} \times (\mathbb{C} \setminus \{0\})$ given. Then the following are equivalent:*

(1) for every $\varphi \in P_{\omega c}(\mathbb{R}, X)$ we have that $\mathcal{N}(\varphi) \in P_{\omega c}(\mathbb{R}, X)$;

(2) $F(t+\omega, cx) = cF(t, x)$ for all $(t, x) \in \mathbb{R} \times X$.

Proof. It is clear that (1) follows immediately from (2). To prove the reciprocal it is sufficient to consider $\varphi(s) := c^{\wedge}(t-s)\cos\left(\frac{2\pi(t-s)}{\omega}\right)x$, which is in $P_{\omega c}(\mathbb{R}, X)$ and $\varphi(t) = x$. $\square$

Example 2.12. The following functions F satisfy the hypothesis (2) in Theorem 2.11.

1. The function $F(t, u) = f(t)g(u)$ for all $t \in \mathbb{R}$ and for all $u \in X$ where f is a $(\omega, c/g(c))$-periodic function and g is a multiplicative function (i.e. $g(ab) = g(a)g(b)$ for all $a, b \in \mathbb{R}$) with $g(c) \neq 0$.

2. The function $F(t, x) = f(t)g(x)$ for all $t \in \mathbb{R}$ and for all $x \in X$ where f is a (ω, c)-periodic function and $g(cx) = g(x)$ for all $x \in X$. A particular case of this example is obtained taking $g(x) = x^{kn}$ with $c^k = 1$.

3 Existence of an (ω, c)-periodic solution for fractional integro-differential equations in Banach spaces

We consider the problem of existence and uniqueness of (ω, c)-periodic mild solutions for (1.2) where A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on a Banach space X (in the sense of [16]), $a \in L^1_{\text{loc}}(\mathbb{R}_+)$, $\alpha > 0$ and the fractional derivative is understood in the sense of Weyl.

Definition 3.1 ([16])**.** A function $u : \mathbb{R} \to X$ is said to be a mild solution of (1.2) if

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s) f(s, u(s))\, ds \qquad (t \in \mathbb{R})$$

where $\{S_\alpha(t)\}_{t\geq 0}$ is the α-resolvent family generated by A, whenever it exists.

The next theorem is the main result of this section. Note that the norm on $P_{\omega c}(\mathbb{R}, X)$ improve the previous related results.

Theorem 3.2. *Let $f \in C(\mathbb{R} \times X, X)$. Assume the following conditions.*

1. *There exists $(\omega, c) \in \mathbb{R}^+ \times (\mathbb{C} \setminus \{0\})$ such that $f(t+\omega, cx) = cf(t, x)$ for all $t \in \mathbb{R}$ and for all $x \in X$.*

2. *There exists a nonnegative, $(\omega, |c|)$-periodic function $L(t)$ such that $\|f(t, x) - f(t, y)\| \leq L(t)\|x - y\|$ for all $x, y \in X$ and $t \in \mathbb{R}$.*

3. *The operator A generates a uniformly integrable α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that $S_\alpha^\sim(t)$ is integrable and $\sup_{t\in[0,\omega]}(S_\alpha^\sim * L)(t) < 1$ where $S_\alpha^\sim(t) := |c|^{-t/\omega}\|S_\alpha(t)\|$.*

Then equation (1.2) has a unique mild solution in $P_{\omega c}(\mathbb{R}, X)$.

Proof. We define $\mathcal{G} : P_{\omega c}(\mathbb{R}, X) \to P_{\omega c}(\mathbb{R}, X)$ by

$$(\mathcal{G}u)(t) = \int_{-\infty}^{t} S_\alpha(t-s) f(s, u(s))\, ds,$$

for $u \in P_{\omega c}(\mathbb{R}, X)$. By Theorem 2.11 we have that $f(\cdot, u(\cdot)) \in P_{\omega c}(\mathbb{R}, X)$. If $\chi(s)$ is the characteristic function on $(-\infty, s]$, by Theorem 2.8, with $k(t) := \|S_\alpha(t)\| \cdot \chi(t)$ we have $\mathcal{G}u \in P_{\omega c}(\mathbb{R}, X)$. Therefore $\mathcal{G}(P_{\omega c}(\mathbb{R}, X)) \subset P_{\omega c}(\mathbb{R}, X)$. Now, if $u, v \in P_{\omega c}(\mathbb{R}, X)$ we have

$$\begin{aligned}
\|\mathcal{G}(u) - \mathcal{G}(v)\|_{\omega c} &= \sup_{t\in[0,\omega]} \left\| |c|^{-t/\omega} \int_{-\infty}^{t} S_\alpha(t-s)[f(s, u(s)) - f(s, v(s))]\, ds \right\| \\
&\leq \sup_{t\in[0,\omega]} \int_{-\infty}^{t} \|S_\alpha(t-s)|c|^{-(t-s)/\omega}\| \cdot L(s) \cdot |c|^{-s/\omega}\|u(s) - v(s)\|\, ds \\
&\leq \|u - v\|_{\omega c} \sup_{t\in[0,\omega]} \int_{0}^{\infty} S_\alpha^\sim(s) L(t-s)\, ds \\
&= \|S_\alpha^\sim * L\|_\omega \|u - v\|_{\omega c}.
\end{aligned}$$

The conclusion follows from the Banach fixed point theorem. □

Remark 3.3. Considering $a \in P_{\omega c}(\mathbb{R}, X)$ and $b \in P_{\omega \frac{1}{c}}(\mathbb{R}, X)$ we see that the function $f(t, x) = a(t)\cos(b(t)x)$ satisfies the hypotheses of Theorem 3.2, showing an extension of previous results, see [2, Ex. 3.5].

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