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Location of solutions for quasi-linear elliptic equations with general gradient dependence

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Abstract. Existence and location of solutions to a Dirichlet problem driven by (p, q)-Laplacian and containing a (convection) term fully depending on the solution and its gradient are established through the method of subsolution-supersolution. Here we substantially improve the growth condition used in preceding works. The abstract theorem is applied to get a new result for existence of positive solutions with a priori estimates.

Keywords: quasi-linear elliptic equations, gradient dependence, (p, q)-Laplacian, subsolution-supersolution, positive solution.

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1 Introduction

The aim of this paper is to study the following nonlinear elliptic boundary value problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P_μ}$$

by means of the method of subsolution-supersolution on a bounded domain $\Omega \subset \mathbb{R}^N$. For regularity reasons we assume that the boundary $\partial\Omega$ is of class C^2. In order to simplify the presentation we suppose that $N \geq 3$. The lower dimensional cases $N = 1, 2$ are simpler and can be treated by slightly modified arguments.

In the statement of problem (P_μ), there are given real numbers $\mu \geq 0$ and $1 < q < p$. The leading differential operator in (P_μ) is described by the p-Laplacian and q-Laplacian, namely $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$. Hence if $\mu = 0$, problem (P_μ) is governed by the p-Laplacian Δ_p, whereas if $\mu = 1$, it is driven by the (p, q)-Laplacian $\Delta_p + \Delta_q$, which is an essentially different type of nonlinear operator.

The right-hand side of the elliptic equation in (P_μ) is expressed through a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, i.e., $f(\cdot, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$

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is continuous for a.e. $x \in \Omega$. We emphasize that the term $f(x,u,\nabla u)$ (often called convection term) depends not only on the solution u, but also on its gradient ∇u. This fact produces serious difficulties of treatment mainly because the convection term generally prevents to have a variational structure for problem (P_μ), so the variational methods are not applicable.

Existence results for problem (P_μ) or for systems of equations of this form have been obtained in [1,4–7,10–12]. Location of solutions through the method of subsolution-supersolution in the case of systems involving p-Laplacian operators has been investigated in [3]. Here, in the case of an equation possibly involving the (p,q)-Laplacian, we focus on the location of solutions within ordered intervals determined by pairs of subsolution-supersolution of problem (P_μ) under a much more general growth condition on the right-hand side $f(x,u,\nabla u)$ (see hypothesis (H) below). We also provide a new result guaranteeing the existence of positive solutions to (P_μ).

The functional space associated to problem (P_μ) is the Sobolev space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p\,dx\Big)^{\frac{1}{p}}.$$

Its dual space is $W^{-1,p'}(\Omega)$, with $p' = p/(p-1)$, and the corresponding duality pairing is denoted $\langle\cdot,\cdot\rangle$.

A solution of problem (P_μ) is understood in the weak sense, that is any function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \mu\int_\Omega |\nabla u|^{q-2}\nabla u\nabla v\,dx = \int_\Omega f(x,u,\nabla u)v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$.

Our study of problem (P_μ) is based on the method of subsolution-supersolution. We refer to [2,9] for details related to this method. We recall that a function $\overline{u} \in W^{1,p}(\Omega)$ is a *supersolution* for problem (P_μ) if $\overline{u} \geq 0$ on $\partial\Omega$ and

$$\int_\Omega \big(|\nabla\overline{u}|^{p-2}\nabla\overline{u} + \mu|\nabla\overline{u}|^{q-2}\nabla\overline{u}\big)\nabla v\,dx \geq \int_\Omega f(x,\overline{u},\nabla\overline{u})v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geq 0$ a.e. in Ω. A function $\underline{u} \in W^{1,p}(\Omega)$ is a *subsolution* for problem (P_μ) if $\underline{u} \leq 0$ on $\partial\Omega$ and

$$\int_\Omega \big(|\nabla\underline{u}|^{p-2}\nabla\underline{u} + \mu|\nabla\underline{u}|^{q-2}\nabla\underline{u}\big)\nabla v\,dx \leq \int_\Omega f(x,\underline{u},\nabla\underline{u})v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geq 0$ a.e. in Ω.

In the sequel we suppose that $N > p$ (if $N \leq p$ the treatment is easier). Then the critical Sobolev exponent is $p^* = \frac{Np}{N-p}$.

Given a subsolution $\underline{u} \in W^{1,p}(\Omega)$ and a supersolution $\overline{u} \in W^{1,p}(\Omega)$ for problem (P_μ) with $\underline{u} \leq \overline{u}$ a.e. in Ω, we assume that $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfies the growth condition:

(H) There exist a function $\sigma \in L^{\gamma'}(\Omega)$ for $\gamma' = \frac{\gamma}{\gamma-1}$ with $\gamma \in (1,p^*)$ and constants $a > 0$ and $\beta \in \big[0,\frac{p}{(p^*)'}\big)$ such that

$$|f(x,s,\xi)| \leq \sigma(x) + a|\xi|^\beta \quad \text{for a.e. } x\in\Omega,\ \text{all } s\in[\underline{u}(x),\overline{u}(x)],\ \xi\in\mathbb{R}^N.$$

Notice that, under assumption (H), the integrals in the definitions of the subsolution $\underline{u}$ and the supersolution $\overline{u}$ exist.

Our main goal is to obtain a solution $u \in W_0^{1,p}(\Omega)$ of problem (P_μ) with the location property $\underline{u} \le u \le \overline{u}$ a.e. in Ω. This is done through an auxiliary truncated problem termed $(T_{\lambda,\mu})$ depending on a positive parameter λ (for any fixed $\mu \ge 0$). It is shown in Theorem 2.1 that whenever $\lambda > 0$ is sufficiently large, problem $(T_{\lambda,\mu})$ is solvable. The next principal step is performed in Theorem 3.1, where it is proven by adequate comparison that every solution $u \in W_0^{1,p}(\Omega)$ of problem $(T_{\lambda,\mu})$ is within the ordered interval $[\underline{u}, \overline{u}]$ determined by the subsolution-supersolution, that is $\underline{u} \le u \le \overline{u}$ a.e. in Ω. Then the expression of the equation in $(T_{\lambda,\mu})$ enables us to conclude that u is actually a solution of the original problem (P_μ) verifying the location property $\underline{u} \le u \le \overline{u}$ a.e. in Ω. We emphasize that Theorem 2.1 improves all the growth conditions for the convection term $f(x, u, \nabla u)$ considered in the preceding works. Finally, in Theorem 4.1, the procedure to construct solutions located in ordered intervals $[\underline{u}, \overline{u}]$ is conducted to guarantee the existence of a positive solution to problem (P_μ). It is also worth mentioning that this result provides a priori estimates for the obtained solution.

2 Auxiliary truncated problem

This section is devoted to the study of an auxiliary problem related to problem (P_μ). We start with some notation. The Euclidean norm on $\mathbb{R}^N$ is denoted by $|\cdot|$ and the Lebesgue measure on $\mathbb{R}^N$ by $|\cdot|_N$. For every $r \in \mathbb{R}$, we set $r^+ = \max\{r, 0\}$, $r^- = \max\{-r, 0\}$, and if $r > 1$, $r' = \frac{r}{r-1}$.

Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution for problem (P_μ), respectively, with $\underline{u} \le \overline{u}$ a.e. in Ω such that hypothesis (H) is satisfied. We consider the truncation operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ defined by

$$Tu(x) = \begin{cases} \overline{u}(x), & u(x) > \overline{u}(x), \\ u(x), & \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \underline{u}(x), & u(x) < \underline{u}(x), \end{cases} \tag{2.1}$$

which is known to be continuous and bounded.

By means of the constant β in hypothesis (H) we introduce the cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\pi(x,s) = \begin{cases} (s - \overline{u}(x))^{\frac{\beta}{p-\beta}}, & s > \overline{u}(x), \\ 0, & \underline{u}(x) \le s \le \overline{u}(x), \\ -(\underline{u}(x) - s)^{\frac{\beta}{p-\beta}}, & s < \underline{u}(x). \end{cases} \tag{2.2}$$

We observe that π satisfies the growth condition

$$|\pi(x,s)| \le c|s|^{\frac{\beta}{p-\beta}} + \varrho(x) \quad \text{for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \tag{2.3}$$

with a constant $c > 0$ and a function $\varrho \in L^{\frac{p}{\beta}}(\Omega)$. Here it is used that $\underline{u}, \overline{u} \in W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and $\beta < \frac{p}{(p^*)'}$. By (2.3), the fact that $\beta < \frac{p}{(p^*)'}$ and Rellich–Kondrachov compactness embedding theorem, it follows that the Nemytskij operator $\Pi : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ given by $\Pi(u) = \pi(\cdot, u(\cdot))$ is completely continuous. Moreover, (2.2) leads to

$$\int_\Omega \pi(x, u(x))u(x)\,dx \geq r_1 \|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} - r_2 \quad \text{for all } u \in W_0^{1,p}(\Omega), \tag{2.4}$$

with positive constants r_1 and r_2.

Next we consider the Nemytskij operator $N : [\underline{u}, \overline{u}] \to W^{-1,p'}(\Omega)$ determined by the function f in (P_μ), that is

$$N(u)(x) = f(x, u(x), \nabla u(x)),$$

which is well defined by virtue of hypothesis (H).

With the data above, for any $\lambda > 0$ let the auxiliary truncated problem associated to (P_μ) be formulated as follows

$$-\Delta_p u - \mu\Delta_q u + \lambda\Pi(u) = N(Tu). \tag{$T_{\lambda,\mu}$}$$

For problem $(T_{\lambda,\mu})$ we have the following result.

Theorem 2.1. *Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of problem (P_μ), respectively, with $\underline{u} \leq \overline{u}$ a.e. in Ω such that hypothesis (H) is fulfilled. Then there exists $\lambda_0 > 0$ such that whenever $\lambda \geq \lambda_0$ there is a solution $u \in W_0^{1,p}(\Omega)$ of the auxiliary problem $(T_{\lambda,\mu})$.*

Proof. For every $\lambda > 0$ we introduce the nonlinear operator $A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$A_\lambda u = -\Delta_p u - \mu\Delta_q u + \lambda\Pi(u) - N(Tu). \tag{2.5}$$

Due to (2.3) and (H), the operator A_λ is bounded.

We claim that A_λ in (2.5) is a pseudomonotone operator. In order to show this, let a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ satisfy

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \tag{2.6}$$

and

$$\limsup_{n\to\infty} \langle A_\lambda u_n, u_n - u\rangle \leq 0. \tag{2.7}$$

Recalling from (H) that $\sigma \in L^{\gamma'}(\Omega)$ with $\gamma < p^*$, by Hölder's inequality, (2.6) and the Rellich–Kondrachov compact embedding theorem we get

$$\int_\Omega \sigma|u_n - u|\,dx \leq \|\sigma\|_{L^{\gamma'}(\Omega)}\|u_n - u\|_{L^\gamma(\Omega)} \to 0 \quad \text{as } n \to +\infty. \tag{2.8}$$

Let us show that

$$\int_\Omega |\nabla(Tu_n)|^\beta |u_n - u|\,dx \to 0 \quad \text{as } n \to +\infty. \tag{2.9}$$

The definition of the truncation operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ in (2.1) yields

$$\begin{aligned}\int_\Omega |\nabla(Tu_n)|^\beta |u_n - u|\,dx = &\int_{\{u_n<\underline{u}\}} |\nabla\underline{u}|^\beta |u_n - u|\,dx \\ &+ \int_{\{\underline{u}\leq u_n\leq\overline{u}\}} |\nabla u_n|^\beta |u_n - u|\,dx + \int_{\{u_n>\overline{u}\}} |\nabla\overline{u}|^\beta |u_n - u|\,dx.\end{aligned}$$

Using Hölder's inequality, (2.6) and the Rellich–Kondrachov compact embedding theorem, as well as the inequality $\frac{p}{p-\beta} < p^*$, enables us to find that

$$\int_{\{u_n<\underline{u}\}} |\nabla\underline{u}|^\beta |u_n - u|\,dx \leq \|\nabla\underline{u}\|^\beta_{L^p(\Omega)}\|u_n - u\|_{L^{\frac{p}{p-\beta}}(\Omega)} \to 0,$$

$$\int_{\{\underline{u}\leq u_n\leq\overline{u}\}} |\nabla u_n|^\beta |u_n-u|\,dx \leq \|\nabla u_n\|^\beta_{L^p(\Omega)} \|u_n-u\|_{L^{\frac{p}{p-\beta}}(\Omega)} \to 0,$$
$$\int_{\{u_n>\overline{u}\}} |\nabla \overline{u}|^\beta |u_n-u|\,dx \leq \|\nabla \overline{u}\|^\beta_{L^p(\Omega)} \|u_n-u\|_{L^{\frac{p}{p-\beta}}(\Omega)} \to 0.$$

Therefore (2.9) holds true.

Taking into account (2.8), (2.9), hypothesis (H) and the fact that $\underline{u} \leq Tu_n \leq \overline{u}$ a.e. in Ω for every n, it turns out that

$$\lim_{n\to\infty} \int_\Omega f(x, Tu_n, \nabla(Tu_n))(u_n - u)\,dx = 0. \tag{2.10}$$

Using (2.3), (2.6) and the inequality $\frac{p}{p-\beta} < p^*$, the same type of arguments yields

$$\lim_{n\to\infty} \int_\Omega \pi(x, u_n)(u_n - u)\,dx = 0. \tag{2.11}$$

Due to (2.10) and (2.11), inequality (2.7) becomes

$$\limsup_{n\to\infty} \langle -\Delta_p u_n - \mu\Delta_q u_n, u_n - u\rangle \leq 0.$$

Through the $(S)_+$-property of the operator $-\Delta_p - \mu\Delta_q$ (see [9, pp. 39–40]) and (2.6), we obtain the strong convergence $u_n \to u$ in $W^{1,p}_0(\Omega)$, thus

$$-\Delta_p u_n - \mu\Delta_q u_n \to -\Delta_p u - \mu\Delta_q u. \tag{2.12}$$

Taking into account (2.12) and that $u_n \to u$ in $W^{1,p}_0(\Omega)$, we get

$$A_\lambda u_n \rightharpoonup A_\lambda u, \quad \langle A_\lambda u_n, u_n\rangle \to \langle A_\lambda u, u\rangle,$$

which ensures that the operator A_λ is pseudomonotone.

Now we prove that the operator $A_\lambda : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ is coercive meaning that

$$\lim_{\|u\|\to+\infty} \frac{\langle A_\lambda u, u\rangle}{\|u\|} = +\infty.$$

The expression of A_λ in (2.5) allows us to find

$$\langle A_\lambda u, u\rangle \geq \|\nabla u\|^p_{L^p(\Omega)} + \lambda \int_\Omega \pi(x,u)u\,dx - \int_\Omega f(x, Tu, \nabla(Tu))u\,dx. \tag{2.13}$$

Notice that by virtue of (2.1), it holds $\underline{u} \leq Tu \leq \overline{u}$ a.e. in Ω for every $u \in W^{1,p}_0(\Omega)$, so we can use hypothesis (H) with $s = (Tu)(x)$ for a.e. $x \in \Omega$. Then, combining with Young's inequality and Sobolev embedding theorem, we infer for each $\varepsilon > 0$ that

$$\begin{aligned}
&\left|\int_\Omega f(x, Tu, \nabla(Tu))u\,dx\right| \\
&\quad\leq \int_\Omega \left(\sigma|u| + a|\nabla(Tu)|^\beta|u|\right)\,dx \\
&\quad\leq \|\sigma\|_{L^{\gamma'}(\Omega)}\|u\|_{L^\gamma(\Omega)} + \varepsilon\|\nabla u\|^p_{L^p(\Omega)} + c_1(\varepsilon)\|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} + c_2\|u\|_{L^{\frac{p}{p-\beta}}(\Omega)} \\
&\quad\leq \varepsilon\|u\|^p + c_1(\varepsilon)\|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} + d\|u\|,
\end{aligned} \tag{2.14}$$

with positive constants $c_1(\varepsilon)$ (depending on ε), c_2, d .

Inserting (2.4) and (2.14) in (2.13), it turns out that

$$\langle A_\lambda u, u\rangle \geq (1-\varepsilon)\|u\|^p + (\lambda r_1 - c_1(\varepsilon))\|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} - d\|u\| - \lambda r_2. \tag{2.15}$$

Choose $\varepsilon \in (0,1)$ and $\lambda > \frac{c_1(\varepsilon)}{r_1}$. Then (2.15) implies that the operator A_λ is coercive.

Since the operator $A : W_0^{1,p}(\Omega \to W^{-1,p'}(\Omega)$ is bounded, pseudomonotone and coercive, it is surjective (see [2, p. 40]). Therefore we can find $u \in W_0^{1,p}(\Omega)$ that solves equation ($T_{\lambda,\mu}$), which completes the proof. □

3 Main result

We state our main abstract result on problem (P_μ).

Theorem 3.1. *Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of problem (P_μ), respectively, with $\underline{u} \leq \overline{u}$ a.e. in Ω such that hypothesis (H) is fulfilled. Then problem (P_μ) possesses a solution $u \in W_0^{1,p}(\Omega)$ satisfying the location property $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω.*

Proof. Theorem 2.1 guarantees the existence of a solution of the truncated auxiliary problem ($T_{\lambda,\mu}$) provided $\lambda > 0$ is sufficiently large. Fix such a constant λ and let $u \in W_0^{1,p}(\Omega)$ be a solution of ($T_{\lambda,\mu}$).

We prove that $u \leq \overline{u}$ a.e. in Ω. Acting with $(u-\overline{u})^+ \in W_0^{1,p}(\Omega)$ as a test function in the definition of the supersolution $\overline{u}$ of (P_μ) and in the definition of the solution u for the auxiliary truncated problem ($T_{\lambda,\mu}$) results in

$$\langle -\Delta_p \overline{u} - \mu\Delta_q \overline{u}, (u-\overline{u})^+\rangle \geq \int_\Omega f(x,\overline{u},\nabla\overline{u})(u-\overline{u})^+\,dx \tag{3.1}$$

and

$$\langle -\Delta_p u - \mu\Delta_q u, (u-\overline{u})^+\rangle + \lambda\int_\Omega \Pi(u)(u-\overline{u})^+ dx = \int_\Omega f(x,Tu,\nabla(Tu))(u-\overline{u})^+ dx. \tag{3.2}$$

From (3.1), (3.2) and (2.1) we derive

$$\begin{aligned}
&\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u})\nabla(u-\overline{u})^+\,dx \\
&\qquad + \mu\int_\Omega (|\nabla u|^{q-2}\nabla u - |\nabla\overline{u}|^{q-2}\nabla\overline{u})\nabla(u-\overline{u})^+\,dx + \lambda\int_\Omega \pi(x,u)(u-\overline{u})^+\,dx \\
&\leq \int_\Omega \big(f(x,Tu,\nabla(Tu)) - f(x,\overline{u},\nabla\overline{u})\big)(u-\overline{u})^+\,dx \\
&= \int_{\{u>\overline{u}\}} \big(f(x,Tu,\nabla(Tu)) - f(x,\overline{u},\nabla\overline{u})\big)(u-\overline{u})\,dx = 0.
\end{aligned} \tag{3.3}$$

Since

$$\begin{aligned}
&\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u})\nabla(u-\overline{u})^+\,dx \\
&\qquad = \int_{\{u>\overline{u}\}} (|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u})(\nabla u - \nabla\overline{u})\,dx \geq 0
\end{aligned}$$

and

$$\int_\Omega (|\nabla u|^{q-2}\nabla u - |\nabla \overline{u}|^{q-2}\nabla \overline{u})\nabla(u-\overline{u})^+\,dx$$
$$= \int_{\{u>\overline{u}\}} (|\nabla u|^{q-2}\nabla u - |\nabla \overline{u}|^{q-2}\nabla \overline{u})(\nabla u - \nabla \overline{u})\,dx \geq 0,$$

we are able to derive from (2.2) and (3.3) that

$$\int_{\{u>\overline{u}\}} (u-\overline{u})^{\frac{p}{p-\beta}}\,dx = \int_\Omega \pi(x,u)(u-\overline{u})^+\,dx \leq 0.$$

It follows that $u \leq \overline{u}$ a.e in Ω.

In an analogous way, by suitable comparison we can show that $\underline{u} \leq u$ a.e in Ω. Consequently, the solution u of the auxiliary truncated problem $(T_{\lambda,\mu})$ satisfies $Tu = u$ and $\Pi(u) = 0$ (see (2.1) and (2.2)), so it becomes a solution of the original problem (P_μ), which completes the proof. □

4 Existence of positive solutions

In this section we focus on the existence of positive solutions to problem (P_μ). The idea is to construct a subsolution $\underline{u} \in W^{1,p}(\Omega)$ and a supersolution $\overline{u} \in W^{1,p}(\Omega)$ with $0 < \underline{u} \leq \overline{u}$ a.e. in Ω for which Theorem 3.1 can be applied. In this respect, inspired by [6,8], we suppose the following assumptions on the right-hand side f of (P_μ):

$(H1)$ There exist constants $a_0 > 0$, $b > 0$, $\delta > 0$ and $r > 0$, with $r < p-1$ if $\mu = 0$ and $r < q-1$ if $\mu > 0$, such that

$$\left(\frac{a_0}{b}\right)^{\frac{1}{p-r-1}} < \delta \tag{4.1}$$

and

$$f(x,s,\xi) \geq a_0 s^r - b s^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ all } 0 < s < \delta,\ \xi \in \mathbb{R}^N. \tag{4.2}$$

$(H2)$ There exists a constant $s_0 > \delta$, with $\delta > 0$ in $(H1)$, such that

$$f(x,s_0,0) \leq 0 \quad \text{for a.e. } x \in \Omega. \tag{4.3}$$

Our result on the existence of positive solutions for problem (P_μ) is as follows.

Theorem 4.1. *Assume $(H1)$, $(H2)$ and that*

$$|f(x,s,\xi)| \leq \sigma(x) + a|\xi|^\beta \quad \textit{for a.e. } x \in \Omega,\ \textit{all } s \in [0,s_0],\ \xi \in \mathbb{R}^N,$$

with a function $\sigma \in L^{\gamma'}(\Omega)$ for $\gamma \in [1,p^*)$ and constants $a > 0$ and $\beta \in \left[0, \frac{p}{(p^*)'}\right)$. Then, for every $\mu \geq 0$, problem (P_μ) possesses a positive smooth solution $u \in C_0^1(\overline{\Omega})$ satisfying the a priori estimate $u(x) \leq s_0$ for all $x \in \Omega$ (s_0 is the constant in $(H2)$).

Proof. With the notation in hypothesis $(H1)$, consider the following auxiliary problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u + b|u|^{p-2}u = a_0(u^+)^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We are going to show that there exists a solution $\underline{u} \in C_0^1(\overline{\Omega})$ of problem (4.4) satisfying $\underline{u} > 0$ in Ω and

$$b\|\underline{u}\|_{L^\infty(\Omega)}^{p-r-1} \le a_0. \tag{4.5}$$

To this end, we consider the Euler functional associated to (4.5), that is the C^1-function $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + b|u|^p)\,dx + \frac{\mu}{q}\int_\Omega |\nabla u|^q\,dx - \frac{a_0}{r+1}\int_\Omega (u^+)^{r+1}\,dx$$

whenever $u \in W_0^{1,p}(\Omega)$. From the assumption on r in hypothesis $(H1)$ and Sobolev embedding theorem, it is easy to prove that I is coercive. Since I is also sequentially weakly lower semicontinuous, there exists $\underline{u} \in W_0^{1,p}(\Omega)$ such that

$$I(\underline{u}) = \inf_{u \in W_0^{1,p}(\Omega)} I(u).$$

On the basis of the conditions $r < p-1$ if $\mu = 0$ and $r < q-1$ if $\mu > 0$ (see hypothesis $(H1)$), it is seen that for any positive function $v \in W_0^{1,p}(\Omega)$ and with a sufficiently small $t > 0$, there holds $I(tv) < 0$, so $\inf_{u \in W_0^{1,p}(\Omega)} I(u) < 0$. This enables us to deduce that $\underline{u}$ is a nontrivial solution of (4.4). Testing equation (4.4) with $-\underline{u}^-$ yields $\underline{u} \ge 0$. By the nonlinear regularity theory and strong maximum principle we obtain that $\underline{u} \in C_0^1(\overline{\Omega})$ and $\underline{u} > 0$ in Ω.

According to the latter properties, we can utilize $\underline{u}^{\alpha+1}$, with any $\alpha > 0$, as a test function in (4.4). Through Hölder's inequality and because $r+1 < p$, this leads to

$$b\|\underline{u}\|_{L^{p+\alpha}(\Omega)}^{p+\alpha} \le a_0 \int_\Omega \underline{u}^{r+\alpha+1}\,dx \le a_0\|\underline{u}\|_{L^{p+\alpha}(\Omega)}^{r+\alpha+1}|\Omega|_N^{(p-r-1)/(p+\alpha)}.$$

Letting $\alpha \to +\infty$ in the inequality

$$b\|\underline{u}\|_{L^{p+\alpha}(\Omega)}^{p-r-1} \le a_0|\Omega|_N^{(p-r-1)/(p+\alpha)}$$

we arrive at (4.5).

We claim that $\underline{u}$ is a subsolution for problem (P_μ). Specifically, due to (4.1) and (4.5), we can insert $s = \underline{u}(x)$ and $\xi = \nabla\underline{u}(x)$ in (4.2), which in conjunction with (4.4) for $u = \underline{u}$ reads as

$$\begin{aligned}\int_\Omega \left(|\nabla\underline{u}|^{p-2}\nabla\underline{u} + \mu|\nabla\underline{u}|^{q-2}\nabla\underline{u}\nabla v\right)\nabla v\,dx &= \int_\Omega (a_0\underline{u}^r - b\underline{u}^{p-1})v\,dx \\ &\le \int_\Omega f(x,\underline{u},\nabla\underline{u})v\,dx\end{aligned}$$

whenever $v \in W_0^{1,p}(\Omega)$, $v \ge 0$ a.e. in Ω. Thereby the claim is proven.

Now we notice that hypothesis $(H2)$ guarantees that $\overline{u} = s_0$ is a supersolution of problem (P_μ). Indeed, in view of (4.3), we obtain

$$\begin{aligned}\int_\Omega \left(|\nabla\overline{u}|^{p-2}\nabla\overline{u} + \mu|\nabla\overline{u}|^{q-2}\nabla\overline{u}\right)\nabla v\,dx = 0 &\ge \int_\Omega f(x,s_0,0)v\,dx \\ &= \int_\Omega f(x,\overline{u},\nabla\overline{u})v\,dx\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$, $v \ge 0$ a.e. in Ω. We point out from assumption $(H2)$ that $s_0 > \delta$, which in conjunction with (4.1) and (4.5), entails that $\underline{u} < \overline{u}$ in Ω.

We also note that hypothesis (H) holds true for the constructed subsolution-supersolution $(\underline{u}, \overline{u})$ of problem (P_μ). Therefore Theorem 3.1 applies ensuring the existence of a solution $u \in W_0^{1,p}(\Omega)$ to problem (P_μ), which satisfies the enclosure property $\underline{u} \le u \le \overline{u}$ a.e. in Ω. Taking into account that $\underline{u} > 0$, we conclude that the solution u is positive. Moreover, the regularity up to the boundary invoked for problem (P_μ) renders $u \in C_0^1(\overline{\Omega})$, whereas the inequality $u \le \overline{u}$ implies the estimate $u(x) \le s_0$ for all $x \in \Omega$. This completes the proof. $\square$

Remark 4.2. Proceeding symmetrically, a counterpart of Theorem 4.1 for negative solutions can be established.

We illustrate the applicability of Theorem 4.1 by a simple example.

Example 4.3. Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$f(x,s,\xi) = |s|^r - |s|^{p-1} + (2^{\frac{p-r}{p-r-1}} - s)|\xi|^\beta \text{ for all } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

where the constants r, p, β are as in conditions (H) and $(H1)$. For simplicity, we have dropped the dependence with respect to $x \in \Omega$. Hypothesis $(H1)$ is verified by taking for instance $a_0 = b = 1$ and $\delta = 2^{\frac{p-r}{p-r-1}}$ (see (4.1) and (4.2)). Hypothesis $(H2)$ is fulfilled for every $s_0 > \delta = 2^{\frac{p-r}{p-r-1}}$. It is also clear that the growth condition for f on $\Omega \times [0, s_0] \times \mathbb{R}^N$ required in the statement of Theorem 4.1 is satisfied, too. Consequently, Theorem 4.1 applies to problem (P_μ) with the chosen function $f(x,s,\xi)$ giving rise to a positive solution belonging to $C_0^1(\overline{\Omega})$.

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