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Global finite-time stability of differential equation with discontinuous right-hand side

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Abstract. In the paper new sufficient conditions for global finite-time stability of a stationary solution to differential equation with discontinuous right-hand side are given. Time-dependent Lyapunov function which is only continuous is used. Properties of Lyapunov function are described by presubdifferential and contingent derivative.

Keywords: stability, global finite-time stability, ordinary differential equations, Lyapunov function, Dini derivative, contingent derivative, presubdifferential.

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1 Introduction

Many different kinds of stability had been considered since the fundamental works of Lyapunov, e.g. asymptotic, exponential, robust (see e.g. [1, 3, 5–7, 10]) and recently finite-time stability ([2, 7–9, 12, 13]).

The aim of this paper is to present sufficient conditions for global finite-time stability of the origin for the differential equation

$$x' = f(t, x), \tag{1.1}$$

where $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function.

Global finite-time stability was considered in e.g. [2, 7, 8]. In our paper we use only continuous time-varying Lyapunov function. Therefore, in the opposite to the articles cited above, instead of differentiating (even in Dini sense) this function we use presubdifferential and contingent derivative, first considered for this purpose in our previous article [13].

Moreover, we weaken the condition which the Lyapunov function must satisfy in relation to conditions given in [2], [7] and [8]. More precisely, we admit in the crucial inequality (3.1) the presence only of a measurable function which can take zero value even on sets of positive measure. In [2] the authors use only a positive constant. In [7] the function must be greater than a positive constant and in [8] this function is a.e. positive.

In this paper we prove two global finite-time stability theorems. In the first one we do not need to assume uniqueness of solutions to the differential equation. In this theorem some

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crucial condition which should be satisfied by Lyapunov function must occur in the whole space. In the second one we have to assume the uniqueness of solutions to the differential equation but in that case the condition which should satisfy Lyapunov function can occur only in an arbitrary small neighborhood of the origin.

2 Preliminaries

Definitions, assumptions, lemmas, propositions and theorems presented in this section come from [13], where complete necessary proofs are given.

The following assumption holds throughout this section.

Assumption 2.1.

1. *$G \subseteq \mathbb{R}^n$ is an open set containing zero.*
2. *The function $t \mapsto f(t,x)$ is measurable in $[0,\infty)$ for all $x \in G$.*
3. *The function $x \mapsto f(t,x)$ is continuous in G for a.a. $t \in [0,\infty)$.*
4. *$f(t,0) = 0$ for all $t \in [0,\infty)$.*
5. *There exists locally bounded function $m \in L^\infty_{loc}([0,\infty))$ such that $\|f(t,x)\| \le m(t)$ for a.a. $t \ge 0$ and for all $x \in G$.*

Let $V : [0,\infty) \times G \to [0,\infty)$ be a continuous function with respect to both variables. Denote

$$\dot{V}(t,x) = \limsup_{\substack{h \to 0^+ \\ w \to f(t,x)}} \frac{V(t+h, x+hw) - V(t,x)}{h}.$$

Let us see that $\dot{V}(t,x)$ is upper right contingent derivative of V in (t,x) towards $(1, f(t,x))$.

Now we give definition of the presubdifferential from [14].

Definition 2.2. Let $W : \mathbb{R}^j \to \mathbb{R}$, $j \in \mathbb{N}$. The presubdifferential of function W in point v we name the following set

$$\hat{\partial} W(v) = \left\{ v^* \in \mathbb{R}^j : \liminf_{z \to v} \frac{W(z) - W(v) - \langle v^*, z - v \rangle}{\|z - v\|} \ge 0 \right\},$$

for $v \in \mathbb{R}^j$.

Definition 2.3. By $\mathcal{K}_0^\infty$ we name the class of continuous and increasing functions $K : [0,\infty) \to [0,\infty)$ such that $K(0) = 0$ and $K(r) \underset{r\to\infty}{\longrightarrow} \infty$.

Lyapunov function is an important tool which allows investigating stability as well as global finite-time stability of the solution to the differential equation. In the literature there are commonly known conditions for smooth Lyapunov function (see e.g. [3, 8, 9]). In this paper Lyapunov function is only continuous, therefore like in [13] we give other conditions which can be easily checked.

Assumption 2.4. *Let $V : [0,\infty) \times G \to [0,\infty)$ be a continuous function, $K : [0,\infty) \to [0,\infty)$ function of $\mathcal{K}_0^\infty$ class and $\kappa : [0,\infty) \to (-\infty, 0]$ be a continuous and nonpositive function such that*

$$\inf_{t\geq 0} V(t,x) \geq K(\|x\|) > 0 \quad \text{for } x \in G\setminus\{0\}, \tag{2.1}$$

$$V(t,0) = K(0) = 0 \quad \text{for } t \in [0,\infty) \tag{2.2}$$

and $\Gamma \subseteq [0,\infty)$ be a set of measure zero such that

$$\dot{V}(t,x) \leq \kappa(\|x\|) \quad \text{for } t \in [0,\infty)\setminus\Gamma \text{ and } x \in G\setminus\{0\}. \tag{2.3}$$

In addition, there exists at most countable set $C \subseteq [0,\infty)$ such that for all $t \in (0,\infty)\setminus C$ and $x \in G\setminus\{0\}$ there exists $\varepsilon_{tx} \in (0,t)$ and $P_{tx} > 0$ such that for $s \in (t-\varepsilon_{tx}, t+\varepsilon_{tx})$, $z \in B(x,\varepsilon_{tx})$, $\hat{\partial}V(s,z) \neq \varnothing$ and

$$\sup_{\substack{s\in(t-\varepsilon_{tx},t+\varepsilon_{tx})\\ z\in B(x,\varepsilon_{tx})}} \sup_{v^*\in\hat{\partial}V(s,z)} \|v^*\| \leq P_{tx}. \tag{2.4}$$

Let V be a function defined in Assumption 2.4. By ν_φ we denote a function

$$\nu_\varphi : t \mapsto V(t,\varphi(t)), \qquad t \in I. \tag{2.5}$$

The following Lemma was proved in [13, Lemma 2.5].

Lemma 2.5. *Let V and κ satisfy the condition (2.3) from Assumption 2.4 for the differential equation (1.1). Then, for any right-maximally defined solution $\varphi : I \to G$ to the differential equation (1.1), for a.a. $t \in I$, $I \subseteq [0,\infty)$ we have*

$$D^+\nu_\varphi(t) \leq \dot{V}(t,\varphi(t)) \tag{2.6}$$

and hence

$$D^+\nu_\varphi(t) \leq \kappa(\|\varphi(t)\|).$$

An obvious consequence of Lemma 2.5 is the following lemma.

Lemma 2.6. *Under the assumptions of Lemma 2.5 we have that $D^+\nu_\varphi(t) \leq 0$ for a.a. $t \in I$.*

To show that the function $t \mapsto V(t,\varphi(t))$ is nonincreasing (comp. [11, Thm. 7.4.14, p. 174] or [15, Cor. 4]) we need the following lemma, proved in ([13, Lemma 2.7]).

Lemma 2.7. *Let V defined in Assumption 2.4 satisfy (2.4) for the differential equation (1.1). Then, for any right-maximally defined solution $\varphi : I \to G$ to the differential equation (1.1), $I \subseteq [0,\infty)$ and for some at most countable set $\tilde{C} \subseteq [0,\infty)$ we have $D^+\nu_\varphi(t) < \infty$ for $t \in I\setminus\tilde{C}$.*

A direct consequence of Lemma 2.6 and Lemma 2.7 are the following propositions.

Proposition 2.8. *If differential equation (1.1) has continuous Lyapunov function satisfying conditions (2.1)–(2.4), then for any right-maximally defined solution $\varphi : I \to G\setminus\{0\}$ to the differential equation (1.1), $I \subset [0,\infty)$, the function $t \mapsto V(t,\varphi(t))$ is nonincreasing in I.*

Proposition 2.9. *If differential equation (1.1) has a continuous function V satisfying conditions (2.1)–(2.4), then for any right-maximally defined solution $\varphi : I \to G\setminus\{0\}$ to the differential equation (1.1), $I \subseteq [0,\infty)$ and for any $s,\ t \in I$, $s \leq t$ we have $\nu_\varphi(t) \leq \nu_\varphi(s) + \int_s^t \kappa(\|\varphi(\tau)\|)d\tau$, where ν_φ is given by the formula (2.5).*

Proposition 2.10. *If differential equation (1.1) has a continuous function V satisfying (2.1)–(2.4), then for any $t_0 \in [0,\infty)$ and for any solution $\varphi : [t_0, b) \to G$ to the differential equation (1.1), $b \in (t_0,\infty) \cup \{\infty\}$ such that for some $\tau \in [t_0, b)$, $\varphi(\tau) = 0$ we have $\varphi(t) = 0$ for all $t \in [\tau, b)$.*

Definition 2.11. By $\mathcal{S}_{t_0,x_0}$ we mean the set of all right-maximally defined solutions φ to the differential equation (1.1) with initial condition $\varphi(t_0) = x_0$.

The above propositions allow to prove the Lyapunov stability theorem.

Theorem 2.12. *If differential equation (1.1) has continuous function V satisfying (2.1)–(2.4), then the origin for the differential equation is stable.*

To define the settling-time function for the differential equation (1.1) we must start from the following definition.

Definition 2.13. For any $t_0 \geq 0$, $x_0 \in G \setminus \{0\}$ and any $\varphi \in \mathcal{S}_{t_0,x_0}$ denote by $c_\varphi(t_0, x_0)$ finite number (if it exists) belonging to the domain of φ, satisfying the following conditions:

1. $\varphi(t) \in G \setminus \{0\}$ for $t \in (t_0, c_\varphi(t_0, x_0))$
2. $\lim\limits_{t \to c_\varphi(t_0,x_0)^-} \varphi(t) = 0.$

Denote

$$\tau_\varphi(t_0, x_0) = \begin{cases} c_\varphi(t_0, x_0), & \text{if it exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 2.14. As the settling-time function we mean the function $T : [0,\infty) \times G \to \mathbb{R}^+ \cup \{\infty\}$ satisfying the following conditions:

1. $T(t_0, 0) = t_0$ for any $t_0 \geq 0$;
2. for any $t_0 \geq 0$ and $x_0 \in G \setminus \{0\}$ we take $T(t_0, x_0) = \sup\{\tau_\varphi(t_0, x_0), \varphi \in \mathcal{S}_{t_0,x_0}\}$.

Definition 2.15. We tell that the origin is finite-time stable for the differential equation (1.1) if it is stable and for any $t_0 \geq 0$ there exists $\delta = \delta(t_0) > 0$ such that for $x_0 \in G$ satisfying $\|x_0\| < \delta$, the values of $T(t_0, x_0)$ are finite.

Definition 2.16. We denote by $\mathcal{P}$ a class of nonnegative functions $c : [0,\infty) \to [0,\infty)$, which are measurable and upperbounded on each compact subinterval $[0,\infty)$ such that there exists $t_0 \geq 0$ for which $\int_{t_0}^{\infty} c(\tau)d\tau = \infty$.

Let us consider a simple example of a differential equation for which the origin is a finite-time stable equilibrium. In the proof of the global finite-time stability theorem properties of solutions to this differential equation are used.

Let us take any function $c \in \mathcal{P}$, $t \in [0,\infty)$, $z \in \mathbb{R}$, $\alpha \in (0,1)$ and consider Cauchy problem

$$\begin{cases} y' = -c(s)\operatorname{sgn}(y)|y|^\alpha, & (2.7) \\ y(t) = z. & (2.8) \end{cases}$$

Remark 2.17. It is easy to see that for any $c \in \mathcal{P}$ and $t \geq 0$ the function $C_t : s \mapsto \int_t^s c(\tau)d\tau$, $s \in [0,\infty)$ is nondecreasing, absolutely continuous on any compact subset of $[0,\infty)$ and $\int_t^\infty c(\tau)d\tau = \infty$. Hence for any $z \in \mathbb{R}$ and $\alpha \in (0,1)$ there exists $\bar{t} \geq t$ such that

$$C_t(\bar{t}) = \int_t^{\bar{t}} c(\tau)d\tau = \frac{|z|^{1-\alpha}}{1-\alpha}.$$

Let

$$t_{c,z} = \inf\left\{\bar{t} \geq t : \int_t^{\bar{t}} c(\tau)d\tau = \frac{|z|^{1-\alpha}}{1-\alpha}\right\}. \tag{2.9}$$

It is easy to check that the solutions to the Cauchy problem (2.7)–(2.8) are functions

$$\mu_{t,z}(s) = \begin{cases} \operatorname{sgn}(z)\left(|z|^{1-\alpha} - (1-\alpha)\int_t^s c(\tau)d\tau\right)^{\frac{1}{1-\alpha}}, & s \in [t, t_{c,z}),\ z \neq 0, \\ 0, & s \geq t_{c,z},\ z \neq 0, \\ 0, & s \geq t,\ z = 0. \end{cases} \tag{2.10}$$

Then $\mu_{t,z}(s) \neq 0$ for $t \geq 0$, $s \in [t, t_{c,z})$ and $z \neq 0$.

An important tool being used in the proof of finite-time stability theorem of the solution to the differential equation (1.1) is the Comparison Lemma from [15]. The essence of this lemma is assuming only measurability with respect to time and absence of any assumption about monotonicity of the right-hand side of the differential equation and using only Dini derivatives. Therefore it is enough that the inequality (2.3) holds only almost everywhere.

The proof of this lemma will be given in [15] but we include it here for the benefit of readers.

Lemma 2.18 (Comparison Lemma, [15]). *Let $E \subseteq \mathbb{R}$ be an open interval, $\sigma : [0,\infty) \times E \to \mathbb{R}$ a function measurable with respect to t for each $x \in E$ and continuous with respect to x for all $t \in [0,\infty)$. Let $t_0 \in [0,\infty)$, $u_0 \in E$ and $u : [t_0, T) \to E$, where $T \in \mathbb{R} \cup \{\infty\}$, $T > t_0$, means the right-maximal in the set $[0,\infty) \times E$ solution to the equation*

$$u' = \sigma(t, u) \tag{2.11}$$

with boundary condition

$$u(t_0) = u_0 \tag{2.12}$$

as well as:

1. *for each $t_1 \in (0, T)$ and $x \in E$ there exists a neighbourhood $V_{t_1,x} \subseteq E$ of the point x and a constant $L_{t_1,x}$ such that, for all $(t, y) \in [0, t_1] \times V_{t_1,x}$ the following estimation takes place $|\sigma(t, y) - \sigma(t, x)| \leq L_{t_1,x}|y - x|$;*
2. *for each $t_1 \in (0, T)$ and $k \in \mathbb{N}$ such that $E^k = E \cap (-k, k)$ is nonempty, there exists a constant $M^k_{t_1} > 0$ satisfying for all $t \in [0, t_1]$ and $x \in E^k$ the following estimation $|\sigma(t, x)| \leq M^k_{t_1}$.*

If $v : [t_0, T) \to E$ is a continuous function, satisfying $v(t_0) \leq u_0$ and

$$D^+v(t) \leq \sigma(t, v(t)) \quad \text{for a.e. } t \in [t_0, T) \tag{2.13}$$

and there exists at most countable set $C \subseteq [t_0, T)$ such that

$$D^+v(t) < \infty \quad \text{for } t \in [t_0, T) \setminus C, \tag{2.14}$$

then $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

Proof. Choose any $t_1 \in (t_0, T)$. Since the interval $[t_0, t_1]$ is compact (in $\mathbb{R}$), there exists $k_1 \in \mathbb{N}$ such that u restricted to the interval $[t_0, t_1]$ is a solution of the equation $u' = \sigma(t, u)$ in the set $[0,\infty) \times E^{k_1}$. Denote $f(t, x, \mu) = \sigma(t, x) + \mu$, where $(t, x) \in [0,\infty) \times E^{k_1}$ and $\mu \in (-1, 1)$. The function f satisfies in the set $E_0 := [0,\infty) \times E^{k_1} \times (-1, 1)$ the following properties:

- for all $x \in E^{k_1}$ and $\mu \in (-1, 1)$ the function f is Lebesgue measurable with respect to t;
- from the assumption 1, the function f is continuous with respect to x for all $(t, \mu) \in [0, \infty) \times (-1, 1)$ and u is the unique solution to the equation (2.11);
- for all $t \in [0, \infty)$, the function f is continuous with respect to (x, μ);
- from the assumption 2, the constant $M = M^k_{t_1} + 1$ satisfies the inequality $|f(t, x, \mu)| \le M$ for $t \in [t_0, t_1]$, $(t, x, \mu) \in E_0$.

Therefore, using [4, Thm. 4.2, p. 59], for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for $|\mu| < \delta$ each right–maximal solution u_μ of the equation

$$x' = \sigma(t, x) + \mu$$

in the set E_0 with boundary condition (2.12) can be defined at least in the interval $[t_0, t_1]$. Moreover, for $t \in [t_0, t_1]$ the following inequality takes place

$$|u_\mu(t) - u(t)| < \varepsilon. \tag{2.15}$$

To prove the thesis of the theorem we first prove that $v(t) \le u_\mu(t)$ for all $\mu \in (0, \delta)$ and $t \in [t_0, t_1]$. Indeed, in the opposite case, it would exist a point $\beta \in (t_0, t_1]$ for which $v(\beta) > u_\mu(\beta)$. In that case, denote $r_1 = \min\{\min\{v(t), u_\mu(t)\} : t \in [t_0, \beta]\}$ and $r_2 = \max\{\max\{v(t), u_\mu(t)\} : t \in [t_0, \beta]\}$. By assumption 1 of the theorem, there exists $L_\beta > 0$ such that, for all $t \in [t_0, \beta]$ and x, $y \in [r_1, r_2]$ one has $|\sigma(t, y) - \sigma(t, x)| \le L_\beta |y - x|$. Denote $W = \{s > t_0 : v(s) > u_\mu(s)\}$. Continuity of v and u_μ follows, that the set W is open, and hence, the set $Y = [t_0, \beta] \setminus W$ is nonempty (because at least $t_0 \in Y$) and compact. Therefore, there exists some $\alpha \in [t_0, \beta]$ being the maximum of Y. Since $v(\beta) > u_\mu(\beta)$, therefore $\beta \notin Y$ and consequently, $\alpha < \beta$. Continuity of v i u_μ and the Darboux condition follows that α satisfies

$$v(\alpha) = u_\mu(\alpha) \tag{2.16}$$

and

$$v(s) > u_\mu(s) \quad \text{for all } s \in (\alpha, \beta]. \tag{2.17}$$

Define $\rho(t) = \int_\alpha^t \sigma(s, v(s))ds$ for $t \in [\alpha, \beta]$. Then, using inequality (2.13) we immediately get $D^+(v(t) - \rho(t)) = D^+v(t) - \sigma(t, v(t)) \le 0$ for almost all $t \in [\alpha, \beta]$. Since v also satisfies (2.14), we easily conclude from [11, Thm. 7.4.14], that the function $t \to v(t) - \rho(t)$ is nonincreasing in $[\alpha, \beta]$ and therefore $v(t) - \rho(t) \le v(\alpha) - \rho(\alpha)$ for $t \in [\alpha, \beta]$. This means, that

$$v(t) \le v(\alpha) + \int_\alpha^t \sigma(s, v(s))ds \quad \text{for } t \in [\alpha, \beta]. \tag{2.18}$$

Since the functions v and u_μ are continuous, the equality (2.16) implies the existence of $\alpha_1 \in (\alpha, \beta]$ satisfying

$$|v(s) - u_\mu(s)| \le \frac{\mu}{2L_\beta} \quad \text{for } s \in (\alpha, \alpha_1). \tag{2.19}$$

However, taking into account (2.16), (2.17) and (2.19), we get for $t \in (\alpha, \alpha_1)$ the chain of inequalities

$$\begin{aligned} \nu(t) - \nu(\alpha) > u_\mu(t) - u_\mu(\alpha) &= \int_\alpha^t (\sigma(s, u_\mu(s)) + \mu) ds \\ &= \int_\alpha^t (\sigma(s, u_\mu(s)) - \sigma(s, \nu(s)) + \sigma(s, \nu(s)) + \mu) ds \\ &\geq \int_\alpha^t (\sigma(s, \nu(s)) + \mu - L_\beta |\nu(s) - u_\mu(s)|) ds \\ &\geq \int_\alpha^t \left(\sigma(s, \nu(s)) + \frac{\mu}{2}\right) ds > \int_\alpha^t \sigma(s, \nu(s)) ds, \end{aligned}$$

which is in the contradiction with (2.18). This proves, that

$$\nu(t) \leq u_\mu(t) \quad \text{for } t \in [t_0, t_1]. \tag{2.20}$$

Now, it only remains to prove that $\nu(t) \leq u(t)$ for $t \in [t_0, t_1]$. If that was not the case, then it would exist a point $d \in (t_0, t_1]$ such that $\nu(d) > u(d)$. Then, for $\varepsilon = \frac{\nu(d) - u(d)}{2}$, by (2.15) we would get $\nu(d) - u_\mu(d) = \nu(d) - u(d) + u(d) - u_\mu(d) > \varepsilon$, which would contradict the previously proved inequality (2.20).

Therefore, the inequality $\nu(t) \leq u(t)$ holds for all $t \in [t_0, t_1]$. From arbitrariness of $t_1 < T$ it follows, that $\nu(t) \leq u(t)$ for all $t \in [t_0, T)$, which ends the proof of the lemma. $\square$

3 Main results

We shall prove the global finite-time stability theorems basing on the following definition.

Definition 3.1. We call the origin global finite-time stable for the differential equation (1.1) if it is stable and the settling-time function $T : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\}$ has only finite values.

The following assumption, similar to Assumption 2.1 holds throughout this section.

Assumption 3.2.

1. *The function $t \mapsto f(t, x)$ is measurable in $[0, \infty)$ for all $x \in \mathbb{R}^n$.*
2. *The function $x \mapsto f(t, x)$ is continuous in $\mathbb{R}^n$ for a.a. $t \in [0, \infty)$.*
3. *$f(t, 0) = 0$ for all $t \in [0, \infty)$.*
4. *There exists an ascending family of compact sets having nonempty interiors $Q_k \subseteq \mathbb{R}^n$ such that $0 \in \operatorname{int} Q_1$, $\bigcup_{k \in \mathbb{N}} \operatorname{int} Q_k = \mathbb{R}^n$ and there exist locally bounded functions $m_k \in L^\infty_{loc}([0, \infty))$, $k \in \mathbb{N}$ such that $\|f(t, x)\| \leq m_k(t)$ for a.a. $t \geq 0$ and for all $x \in Q_k$.*

Now we prove the first global finite-time stability theorem of the solution to the differential equation (1.1).

Theorem 3.3. *Let $V : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$ specified in Assumption 2.4 with $G = \mathbb{R}^n$ for the differential equation (1.1) be a continuous function satisfying conditions (2.1), (2.2) and (2.4), the function $c : [0, \infty) \to [0, \infty)$ be of class $\mathcal{P}$ and $\alpha \in (0, 1)$ be such that*

$$\dot{V}(t, x) + c(t)(V(t, x))^\alpha \leq 0 \quad \text{for } t \in [0, \infty) \setminus \Gamma \text{ and } x \in \mathbb{R}^n \setminus \{0\}, \tag{3.1}$$

where $\Gamma \subseteq [0, \infty)$ is some set of measure zero. Then the origin for the differential equation (1.1) is globally finite-time stable.

Proof. Let us see that inequality (3.1) implies (2.3) for $t \in [0,\infty) \setminus \Gamma$ and $x \in \mathbb{R}^n \setminus \{0\}$. Let us take any $t_0 \geq 0$, $x_0 \in \mathbb{R}^n \setminus \{0\}$ and $\varphi \in \mathcal{S}_{t_0,x_0}$, $\varphi : [t_0,b) \to \mathbb{R}^n$, where $b \in (t_0,\infty) \cup \{\infty\}$. From Proposition 2.8, for any x_0 and any solution φ we know that the function $t \mapsto V(t,\varphi(t))$ is nonincreasing in $[t_0,b)$. From assumption, for all $t \in [t_0,b)$ we have $0 \leq K(\|\varphi(t)\|) \leq V(t,\varphi(t)) \leq V(t_0,x_0)$. Hence, because the function K is increasing and satisfies $K(R) \underset{R\to\infty}{\longrightarrow} \infty$, then there exists $M > 0$ such that for all $t \in [t_0,b)$ a solution φ to the differential equation (1.1) satisfies condition

$$\|\varphi(t)\| \leq M. \tag{3.2}$$

That means that there exists $k \in \mathbb{N}$ such that $\varphi(t) \in \operatorname{int} Q_k$ for $t \in [t_0,b)$ and using Assumption 3.2, from the a priori estimation theorem we get $b = \infty$.

Based on conditions (2.6) and (3.1) we receive

$$D^+\nu_\varphi(s) \leq -c(s)\big(V(s,\varphi(s))\big)^\alpha \quad \text{for a.a. } s \in [t_0,\infty). \tag{3.3}$$

Let us apply Comparison Lemma (see Theorem 2.18) to inequality (3.3) and the function ν_φ. Then, because of the condition (2.4), from Lemma 2.7, $D^+\nu_\varphi$ takes finite values outside some countable set, then taking into account solutions of the Cauchy problem (2.7)–(2.8) and the fact that the function V is nonnegative, we receive

$$0 \leq \nu_\varphi(s) \leq \mu_{t_0,V(t_0,x_0)}(s) \quad \text{for } s \in [t_0, t_{c,V(t_0,x_0)}),$$

where ν_φ is specified in (2.5), $\mu_{t_0,V(t_0,x_0)}$ given in (2.10) is a solution to the differential equation (2.7) with initial condition (2.8), where $z = V(t_0,x_0)$ and $t_{c,V(t_0,x_0)}$ is given by (2.9). Then $\nu_\varphi(t_0) = V(t_0,\varphi(t_0)) = V(t_0,x_0) = \mu_{t_0,V(t_0,x_0)}(t_0)$ and in consequence $\varphi(s) = 0$ for $s \in [t_{c,V(t_0,x_0)},\infty)$.

Hence, because $\varphi \in \mathcal{S}_{t_0,x_0}$ is arbitrary, we get that $T(t_0,x_0) \leq t_{c,V(t_0,x_0)} < \infty$. $\square$

In this paper, condition (3.1) which is satisfied by Lyapunov function, is weaken then the conditions given in [2], [7] and [8]. In (3.1) we use only measurable functions $c(t)$ which can take zero on sets of positive measure. In [2] instead of function $c(t)$ only a positive constant can be used. In [7] the function $h(t)$ which plays the role of function $c(t)$ in this paper must be greater than a positive constant and in [8] this function must be a.e. positive.

Theorem 3.4. *Let us consider the following differential equation*

$$x' = f(t,x), \tag{3.4}$$

where

$$f(t,x) = \eta(t,x) + \Psi(t,x) \tag{3.5}$$

for $t \in [0,\infty)$, $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $\eta : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\Psi : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ *are measurable in* $t \in [0,\infty)$ *for each fixed* $x \in \mathbb{R}^n$ *and continuous in* $x \in \mathbb{R}^n$ *for each fixed* $t \in [0,\infty)$*. Assume that for some* $t \geq 0$ *and* $l \in \mathbb{N}$ *there exist* $L_l^{1t} > 0$ *and* $L_l^{2t} > 0$ *such that* $\|\eta(s,z)\| \leq L_l^{1t}$ *and* $\|\Psi(s,z)\| \leq L_l^{2t}$ *for* $s \in [0,t]$ *and* $z \in \bar{B}(0,l)$*. In addition assume that* $\eta(t,0) = 0$ *and* $\Psi(t,0) = 0$ *for* $t \in [0,\infty)$*. Let* $\gamma : [0,\infty) \to [0,\infty)$ *be of* $\mathcal{P}$ *class (see Definition 2.16) and denote* $g(t) = \int_0^t \gamma(\tau)d\tau + 1$*. Let* $\beta \in (0,2)$ *and*

$$\delta \geq 1, \qquad \delta + \beta > 2. \tag{3.6}$$

Let $\Gamma \subseteq [0,\infty)$ be at most countable set of Lebesgue measure zero such that for all $t \in [0,\infty) \setminus \Gamma$ and $x \in \mathbb{R}^n \setminus \{0\}$ the inequalities $\langle x, \eta(t,x)\rangle \leq -\gamma(t) g(t)^{\frac{2\delta+\beta-2}{\delta}} \|x\|^{\beta}$ and $\langle x, \Psi(t,x)\rangle \leq -\frac{\gamma(t)}{\delta g(t)}\|x\|^2$ hold and assume the existence of at most countable set $C \subseteq [0,\infty)$ such that for all $t \in (0,\infty) \setminus C$ there exists $\omega_t \in (0,t)$ such that $(t-\omega_t, t+\omega_t) \cap C = \varnothing$ and the function γ is continuous in $(t-\omega_t, t+\omega_t)$.

With the above assumptions the origin for the differential equation (3.4) is globally finite-time stable. More precisely, for any initial conditions $(t,x) \in [0,\infty) \times (\mathbb{R}^n \setminus \{0\})$ the settling-time function can be estimated by the formula

$$T(t,x) \leq \inf\left\{ \bar{t} \geq t : \delta \int_t^{\bar{t}} \gamma(\tau) d\tau = \frac{\left(g(t)\|x\|^{\delta}\right)^{1-\alpha}}{1-\alpha} \right\}, \quad \text{where } \alpha = \frac{\delta+\beta-2}{\delta}.$$

Proof. It is easy to see that the function $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ defined in (3.5) satisfies Assumption 3.2. Indeed, according to the assumptions on functions η and Ψ, the function $t \mapsto f(t,x)$ is measurable in $t \in [0,\infty)$ for each fixed $x \in \mathbb{R}^n$, the function $x \mapsto f(t,x)$ is continuous in $x \in \mathbb{R}^n$ for each fixed $t \in [0,\infty)$, $f(t,0) = 0$ for $t \in [0,\infty)$ and $\|f(t,z)\| \leq m_l(t)$ for $t \in [0,\infty)$ and $z \in \bar{B}(0,l)$, $l \in \mathbb{N}$, where $m_l(t) = L_l^{1t} + L_l^{2t}$.

We will show that the origin for the differential equation (3.4) is globally finite-time stable. For this purpose let us consider Lyapunov function $V(t,x) = g(t)\|x\|^{\delta}$, where $t \in [0,\infty)$, $x \in \mathbb{R}^n$, $g(t) = \int_0^t \gamma(\tau) d\tau + 1$ and δ satisfy (3.6). By the definition of function g and assumptions of function γ we get immediately that

$$g(t) \geq 1 \quad \text{for } t \in [0,\infty) \tag{3.7}$$

and for all $t \in (0,\infty) \setminus C$ there exists $g'(t) = \gamma(t)$. The function $V(t,x)$ satisfies of course conditions (2.1) and (2.2). We will show that V satisfies (2.4) and (3.1) which are also required in Theorem 3.3. Indeed:

- Take any $t \in (0,\infty) \setminus C$ and $x \neq 0$, where C is at most countable set described above. There exists $\omega_t \in (0,t)$ such that $(t-\omega_t, t+\omega_t) \cap C = \varnothing$. Put $\varepsilon_{tx} = \min\{\omega_t, \|x\|\}$. Then for all $z \in B(x, \varepsilon_{tx})$ we get $\|z\| \leq \|x\| + \varepsilon_{tx}$. From [14, Proposition 1.87, p. 90] it follows that

  $$\hat{\partial} V(s,z) = \{\nabla V(s,z)\} = \left\{ \left(\gamma(s)\|z\|^{\delta}, \delta g(s)\|z\|^{\delta-2} z\right) \right\}.$$

  Since $\gamma \in \mathcal{P}$, there exists $L_{tx} > 0$ such that $\gamma(\tau) \in [0, L_{tx}]$ for $\tau \in (t-\varepsilon_{tx}, t+\varepsilon_{tx})$ what immediately follows the inequalities $|g(s)| \leq sL_{tx} + 1 \leq (t+\varepsilon_{tx})L_{tx} + 1$ for $s \in (t-\varepsilon_{tx}, t+\varepsilon_{tx})$. Hence, for $z \in B(x, \varepsilon_{tx})$ and $s \in (t-\varepsilon_{tx}, t+\varepsilon_{tx})$ we get

  $$\begin{aligned} &\|(\gamma(s)\|z\|^{\delta}, \delta g(s)\|z\|^{\delta-2} z)\| \\ &\qquad \leq \sqrt{\left(\left(L_{tx}(\|x\| + \varepsilon_{tx})\right)^2 + \left(\delta((t+\varepsilon_{tx})L_{tx} + 1)\right)^2\right)(\|x\| + \varepsilon_{tx})^{2(\delta-1)}}. \end{aligned}$$

  Therefore the condition (2.4) is satisfied for C and ε_{tx} given above and

  $$P_{tx} = \sqrt{\left(\left(L_{tx}(\|x\| + \varepsilon_{tx})\right)^2 + \left(\delta((t+\varepsilon_{tx})L_{tx} + 1)\right)^2\right)(\|x\| + \varepsilon_{tx})^{2(\delta-1)}}.$$

- Put $\tilde{\Gamma} = \Gamma \cup C \cup \{0\}$. Of course this set is measure zero. For any $t \in [0, \infty) \setminus \tilde{\Gamma}$ and $x \neq 0$ we receive

$$\begin{aligned}
\dot{V}(t,x) &= \limsup_{\substack{h \to 0^+ \\ w \to f(t,x)}} \frac{V(t+h, x+hw) - V(t,x)}{h} \\
&= \limsup_{\substack{h \to 0^+ \\ w \to f(t,x)}} \frac{g(t+h)\|x+hw\|^\delta - g(t)\|x\|^\delta}{h} \\
&= \lim_{\substack{h \to 0^+ \\ w \to f(t,x)}} \frac{(g(t+h) - g(t))\|x+hw\|^\delta}{h} + \lim_{\substack{h \to 0^+ \\ w \to f(t,x)}} \frac{g(t)(\|x+hw\|^\delta - \|x\|^\delta)}{h} \\
&\leq \gamma(t)\|x\|^\delta + \delta g(t)\|x\|^{\delta-2}\left(-\gamma(t)g(t)^{\frac{2\delta+\beta-2}{\delta}}\|x\|^\beta - \frac{\gamma(t)}{\delta g(t)}\|x\|^2\right) \\
&= -\delta\gamma(t)g(t)^{\frac{3\delta+\beta-2}{\delta}}\|x\|^{\delta+\beta-2}.
\end{aligned}$$

From above, for $t \in [0, \infty) \setminus \tilde{\Gamma}$ and $x \in \mathbb{R}^n \setminus \{0\}$ we receive that

$$\dot{V}(t,x) \leq -\delta\gamma(t)g^2(t)\left(g(t)\|x\|^\delta\right)^{\frac{\delta+\beta-2}{\delta}} = -c(t)\left(V(t,x)\right)^\alpha,$$

where

$$c(t) = \delta\gamma(t)g^2(t) \tag{3.8}$$

and $\alpha = \frac{\delta+\beta-2}{\delta} \in (0,1)$.

Since

$$g(\tau) \geq 1, \qquad \tau \in [0, \infty) \tag{3.9}$$

then

$$\int_0^t c(\tau)d\tau = \int_0^t \delta\gamma(\tau)g^2(\tau)d\tau \geq \delta\int_0^t \gamma(\tau)d\tau \xrightarrow[t\to\infty]{} \infty$$

and therefore $c \in \mathcal{P}$.

So, from Theorem 3.3 it follows that the origin for the differential equation (3.4) is globally finite-time stable.

We can estimate the settling-time function. For any $t \geq 0$ and $x \neq 0$ and from inequality (3.9), the settling-time function T satisfies

$$T(t,x) \leq \inf\left\{\bar{t} \geq t : \delta\int_t^{\bar{t}} \gamma(\tau)d\tau = \frac{\left(g(t)\|x\|^\delta\right)^{1-\alpha}}{1-\alpha}\right\}. \qquad \square$$

In the example given below the formula estimating precisely enough of the settling-time function is given.

Example 3.5. Let us consider the following differential equation

$$x' = f(t,x), \tag{3.10}$$

where

$$f(t,x) = \left(f_1(t,x_1), \ldots, f_n(t,x_n)\right),$$

$$f_i(t,x_i) = \begin{cases} 0, & t \in \left[k - \frac{1}{2^{2m}}, k - \frac{1}{2^{2m+1}}\right), \\ -\gamma(t) g(t)^{\frac{7}{4}} \operatorname{sgn}(x_i)|x_i|^{\frac{1}{2}} - \frac{\gamma(t)}{2g(t)} x_i, & t \in \left[k - \frac{1}{2^{2m+1}}, k - \frac{1}{2^{2m+2}}\right) \end{cases}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $i = 1, \ldots, n$, $k = 1, 2, \ldots$, $m = 0, 1, \ldots$, where

$$\gamma(t) = \begin{cases} 0, & t \in \left[k - \frac{1}{2^{2m}}, k - \frac{1}{2^{2m+1}}\right), \\ 1, & t \in \left[k - \frac{1}{2^{2m+1}}, k - \frac{1}{2^{2m+2}}\right) \end{cases} \tag{3.11}$$

for $k = 1, 2, \ldots$, $m = 0, 1, \ldots$ and

$$g(t) = \int_0^t \gamma(\tau) d\tau + 1 \geq 1 \quad \text{for } t \in [0, \infty). \tag{3.12}$$

Let us see that $\gamma \in \mathcal{P}$ and that this function is continuous in every point of the set $(0, \infty) \setminus C$, where $C = \left\{k - \frac{1}{2^m} : k = 1, 2, \ldots,\ m = 0, 1, \ldots\right\}$ is countable and closed set.

Denote

$$\eta(t,x) = (\eta_1(t,x_1), \ldots, \eta_n(t,x_n)),$$

$$\eta_i(t,x_i) = \begin{cases} 0, & t \in \left[k - \frac{1}{2^{2m}}, k - \frac{1}{2^{2m+1}}\right), \\ -\gamma(t) g(t)^{\frac{7}{4}} \operatorname{sgn}(x_i)|x_i|^{\frac{1}{2}}, & t \in \left[k - \frac{1}{2^{2m+1}}, k - \frac{1}{2^{2m+2}}\right) \end{cases}$$

and

$$\Psi(t,x) = (\Psi_1(t,x_1), \ldots, \Psi_n(t,x_n)),$$

$$\Psi_i(t,x_i) = \begin{cases} 0, & t \in \left[k - \frac{1}{2^{2m}}, k - \frac{1}{2^{2m+1}}\right), \\ -\frac{\gamma(t)}{2g(t)} x_i, & t \in \left[k - \frac{1}{2^{2m+1}}, k - \frac{1}{2^{2m+2}}\right) \end{cases}$$

for $k = 1, 2, \ldots$, $m = 0, 1, \ldots$

Let $t_1 > 0$. Then, for any $t \in [0, t_1]$ we receive $|\gamma(t)| \leq 1$ and $|g(t)| \leq \frac{1}{3}\lfloor t_1 + 1 \rfloor + 1$. As a consequence, for any $t \in [0, t_1]$, $z \in \bar{B}(0, \|x\| + l)$ and $l \in \mathbb{N}$ we receive

$$\begin{aligned} \|\eta(t,z)\| &= \left\| \left(-\gamma(t) g(t)^{\frac{7}{4}} \operatorname{sgn}(z_1)|z_1|^{\frac{1}{2}}, \ldots, -\gamma(t) g(t)^{\frac{7}{4}} \operatorname{sgn}(z_n)|z_n|^{\frac{1}{2}}\right) \right\| \\ &\leq \left(\frac{1}{3}\lfloor t_1 + 1 \rfloor + 1\right) \sqrt{ln} \end{aligned}$$

and (because $g(t) \geq 1$)

$$\|\Psi(t,z)\| = \left\| \left(-\frac{\gamma(t)}{2g(t)} z_1, \ldots, -\frac{\gamma(t)}{2g(t)} z_n\right) \right\| \leq l\sqrt{n}.$$

Let us see that for $k = 1, 2, \ldots$, $t \in \left[k - \frac{1}{2^{2m+1}}, k - \frac{1}{2^{2m+2}}\right)$ and $x \in \mathbb{R}^n$ we receive

$$\langle x, \eta(t,x) \rangle = -\gamma(t) g(t)^{\frac{7}{4}} \sum_{i=1}^n (x_i^2)^{\frac{3}{4}} \leq = -\gamma(t) g(t)^{\frac{7}{4}} \|x\|^{\frac{3}{2}} \tag{3.13}$$

and

$$\langle x, \Psi(t,x) \rangle = -\frac{\gamma(t)}{2g(t)} \sum_{i=1}^n x_i^2 = -\frac{\gamma(t)}{2g(t)} \|x\|^2. \tag{3.14}$$

For $k = 1, 2, \ldots$, $t \in \left[k - \frac{1}{2^{2m}}, k - \frac{1}{2^{2m+1}}\right)$, $x \in \mathbb{R}^n$ we have $\gamma(t) = 0$ and as a consequence the following inequalities hold

$$\langle x, \eta(t,x) \rangle = 0 \tag{3.15}$$

and

$$\langle x, \Psi(t,x) \rangle = 0. \tag{3.16}$$

From (3.13) and (3.15) and from (3.14) and (3.16) we receive that the functions η and Ψ satisfy conditions given in Theorem 3.4, with $\beta = \frac{3}{2}$, $\delta = 2$, $C = \{k - \frac{1}{2^m} : k = 1, 2, \ldots, m = 0, 1, \ldots\}$ and $\Gamma = \varnothing$. For these β and δ, from the formula (3.8) in the proof of Theorem 3.4

$$\begin{aligned} c(t) &= \begin{cases} 0, & t \in \left[k - \frac{1}{2^{2m}}, k - \frac{1}{2^{2m+1}}\right), \\ 2\gamma(t) g^2(t), & t \in \left[k - \frac{1}{2^{2m+1}}, k - \frac{1}{2^{2m+2}}\right) \end{cases} \\ &= 2\gamma(t) g^2(t), \end{aligned}$$

for $t \geq 0$, where γ and g are given by the formula (3.11) and (3.12) respectively.

It is easy to see that $c \in \mathcal{P}$. As a consequence, from Theorem 3.4 the origin for the differential equation (3.10) is globally finite-time stable. The settling-time function satisfies $T(t,x) \leq t + 2 + 6\left(\frac{1}{3}t + 1\frac{1}{3}\right)^{\frac{1}{4}} \|x\|^{\frac{1}{2}}$.

Below we prove the second global finite-time stability theorem. In the proof of this theorem global asymptotic stability is used. Therefore we must strengthen the assumptions which Lyapunov function should satisfy. We do this in the following assumption.

Assumption 3.6. *Let us assume that all conditions from Assumption 2.4 are satisfied for $G = \mathbb{R}^n$ and for the continuous and negative function κ. Additionally assume that*

$$\lim_{x \to 0} \sup_{t \geq 0} V(t,x) = 0. \tag{3.17}$$

Definition 3.7. We say that the origin for the differential equation (1.1) is globally asymptotically stable if it is stable and every solution φ to the differential equation (1.1) can be extended to infinity and $\lim_{t\to\infty} \|\varphi(t)\| = 0$.

Now we prove the global asymptotic stability theorem.

Theorem 3.8. *If for the differential equation (1.1) there exists function V satisfying Assumption 3.6, then the origin for this differential equation is globally asymptotically stable.*

Proof. We know from Theorem 2.12 that the origin for the differential equation (1.1) is stable. We will show that the origin for this differential equation is globally asymptotically stable.

Let (t_0, x_0) be any element from $[0, \infty) \times (\mathbb{R}^n \setminus \{0\})$ and let $\varphi \in \mathcal{S}_{t_0,x_0}$, $\varphi : [t_0, b) \to \mathbb{R}^n$, $b \in (t_0, \infty) \cup \{\infty\}$, be any right-maximally defined solution to the differential equation (1.1). From (2.3) we know that the function $\dot{V}(t,x)$ is upperbounded by function κ for $t \in [0,\infty) \setminus \Gamma$ and $x \in \mathbb{R}^n \setminus \{0\}$. As in Theorem 3.3 we receive that $b = \infty$. Moreover, there exists the limit $\beta = \lim_{t\to\infty} V(t, \varphi(t)) \geq 0$. We will show that $\beta = 0$. Let us assume contrary that $\beta > 0$. In this case there exists $\gamma > 0$ such that $\|\varphi(t)\| \geq \gamma$ for $t \geq t_0$. Indeed, otherwise it would exist two possibilities. One of them is existence $\bar{t}$ and a sequence (t_{n_k}) such that $t_{n_k} \to \bar{t}$ and $\varphi(\bar{t}) = 0$ and hence for $t \geq \bar{t}$ the condition $\varphi(t) = 0$ it would be satisfy. Therefore $V(t, \varphi(t)) = 0$ for $t \geq \bar{t}$, so $\beta = \lim_{t\to\infty} V(t,\varphi(t)) = 0$. The second case is existence of a sequence (t_n), $t_n \ge t_0$, $t_n \to \infty$ such that $\varphi(t_n) \to 0$ and $\beta = \lim_{n\to\infty} V(t_n,\varphi(t_n)) \le \lim_{x\to 0}\sup_{t\ge t_0} V(t,x) = 0$, what leads to contradiction in both cases with assumption that $\beta > 0$. Therefore, if $\beta > 0$, then there exists $\gamma > 0$ such that to (3.2) $M \ge \|\varphi(t)\| \ge \gamma$ for all $t \ge t_0$. From the definition of the function κ there exists a constant $L = \max\{\kappa(s) : s \in [\gamma, M]\} < 0$. Let (t_j), $t_j > t_0$, be any sequence such that $t_j \to \infty$. Of course $\|\varphi(t_j)\| \ge \gamma$. Because Assumption 3.6 implies Assumption 2.4, then from Proposition 2.9, for $j \in \mathbb{N}$, we receive $0 < K(\gamma) \le K(\|\varphi(t_j)\|) \le V(t_j,\varphi(t_j)) \le V(t_0,\varphi(t_0)) + \int_{t_0}^{t_j}\kappa(\|\varphi(\tau)\|)d\tau \le V(t_0,\varphi(t_0)) + \int_{t_0}^{t_j} L d\tau = V(t_0,\varphi(t_0)) + L(t_j - t_0) \xrightarrow[j\to\infty]{} -\infty$, what is impossible due to condition $V(t_j,\varphi(t_j)) \ge K(\|\varphi(t_j)\|) > 0$. Hence $\beta = 0$, what means that

$$\lim_{t\to\infty} V(t,\varphi(t)) = 0. \tag{3.18}$$

We will show that from (3.18) follows that $\varphi(t) \xrightarrow[t\to\infty]{} 0$. Let us assume that the function $\varphi(t)$ is not convergent to the origin, when $t \to \infty$. In this case there exist a sequence $t_n \to \infty$ and a constant $\gamma > 0$ such that $\|\varphi(t_n)\| \ge \gamma$ for $n \in \mathbb{N}$. From the fact that $V(t_n,\varphi(t_n)) \ge K(\|\varphi(t_n)\|) > 0$ and the function K is increasing we receive that $V(t_n,\varphi(t_n)) \ge K(\|\varphi(t_n)\|) \ge K(\gamma) > 0$. Hence, $\inf_{n\in\mathbb{N}} V(t_n,\varphi(t_n)) \ge K(\gamma) > 0$, what is impossible due to condition (3.18). Then $\varphi(t) \xrightarrow[t\to\infty]{} 0$. It shows that the origin for the differential equation (1.1) is globally asymptotically stable. □

Let us see that in Theorem 3.3 the condition (3.1) is satisfied for x from the whole space $\mathbb{R}^n$ (except the origin). Below we prove Theorem 3.9, in which the condition (3.1) can occur only in an arbitrarily small neighborhood of the origin $\Omega \setminus \{0\}$, $\Omega \subseteq \mathbb{R}^n$. In this case it is necessary to assume the right-uniqueness of solutions to the differential equation (1.1) and strengthening Assumption 2.4 – see condition (3.17) in Assumption 3.6.

Theorem 3.9. *Let us assume that for the differential equation* (1.1) *there exists a continuous function* $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ *satisfying Assumption* 3.6*, the function* $c : [0,\infty) \to [0,\infty)$ *of* $\mathcal{P}$ *class,* $\alpha \in (0,1)$*, an open neighborhood of the origin* $\Omega \subseteq \mathbb{R}^n$ *and a set* $\Gamma \subseteq [0,\infty)$ *of Lebesgue measure zero such that condition* (3.1) *is satisfied on* $([0,\infty)\setminus\Gamma) \times (\Omega\setminus\{0\})$.

In addition we assume that for any initial conditions from $[0,\infty) \times (\mathbb{R}^n \setminus \{0\})$*, differential equation* (1.1) *has the right-unique solutions in* $[0,\infty) \times (\mathbb{R}^n \setminus \{0\})$.

Then the origin for the differential equation (1.1) *is globally finite-time stable.*

Proof. Let $\eta > 0$ be such that $\bar{B}(0,\eta) \subseteq \Omega$. Let us take any $t_0 \ge 0$ and $x_0 \in \mathbb{R}^n \setminus \{0\}$.

We denote by φ_{t_0,x_0} the right-unique, right-maximally defined Carathéodory solution to the differential equation (1.1) satisfying $\varphi_{t_0,x_0}(t_0) = x_0$. From Theorem 3.8 there exists $t \ge t_0$ such that for a solution φ_{t_0,x_0} to the differential equation (1.1), for all $s \ge t$, $\varphi_{t_0,x_0}(s) \in B(0,\eta) \subseteq \Omega$. Hence, from conditions (2.6) and (3.1) we receive

$$D^+\nu_{\varphi_{t_0,x_0}}(s) \le -c(s)\big(V(s,\varphi_{t_0,x_0}(s))\big)^{\alpha} \quad \text{for a.a. } s \in [t,\infty). \tag{3.19}$$

Apply the Comparison Lemma (see Theorem 2.18) to (3.19) and the function $\nu_{\varphi_{t_0,x_0}}$. Then, by condition (2.4) and Lemma 2.6, $D^+\nu_\varphi$ takes finite values outside some countable set, then having in mind solutions of the Cauchy problem (2.7)–(2.8) and the fact that the function V is nonnegative we receive

$$0 \le \nu_{\varphi_{t_0,x_0}}(s) \le \mu_{t,V(t,\varphi_{t_0,x_0}(t))}(s) \quad \text{for } s \in [t, t_{c,V(t,\varphi_{t_0,x_0}(t))}),$$

where $\nu_{\varphi_{t_0,x_0}}$ is specified in (2.5), $\mu_{t,V(t,\varphi_{t_0,x_0}(t))}$ given by (2.10) is a solution to the differential equation (2.7) with initial condition (2.8), where $z = V(t, \varphi_{t_0,x_0}(t))$ and $t_{c,V(t,\varphi_{t_0,x_0}(t))}$ is given by (2.9). We have $\nu_{\varphi_{t_0,x_0}}(t) = V(t, \varphi_{t_0,x_0}(t)) = \mu_{t,V(t,\varphi_{t_0,x_0}(t))}$.

From (2.1) we get that $\varphi_{t_0,x_0}(s) = 0$ for $s \in [t_{c,V(t,\varphi_{t_0,x_0}(t))}, \infty)$. It means that $T(t_0, x_0) = T(t, \varphi_{t_0,x_0}(t)) < \infty$. □

Theorem 3.10. *Let us consider the following differential equation*

$$x' = f(t, x), \tag{3.20}$$

where

$$f(t, x) = \eta(t, x) + \Psi(t, x) \tag{3.21}$$

for $t \in [0, \infty)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $\eta : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\Psi : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ *are measurable for all* $x \in \mathbb{R}^n$ *with respect to* $t \in [0, \infty)$ *and continuous for all* $t \in [0, \infty)$ *with respect to* $x \in \mathbb{R}^n$. *Assume that for some* $t \geq 0$ *and* $l \in \mathbb{N}$ *there exist* $L_l^{1t} > 0$ *and* $L_l^{2t} > 0$ *such that* $\|\eta(s, z)\| \leq L_l^{1t}$ *and* $\|\Psi(s, z)\| \leq L_l^{2t}$ *for* $s \in [0, t]$ *and* $z \in \bar{B}(0, l)$. *In addition assume that* $\eta(t, 0) = 0$ *and* $\Psi(t, 0) = 0$ *for* $t \in [0, \infty)$. *Let* $\gamma : [0, \infty) \to [0, \infty)$ *be a continuous function satisfying* $\gamma(t) \leq L_1$, $L_1 > 0$, $\gamma(t) > \varepsilon > 0$ *for* $t \in [0, \infty)$. *Let* $\beta \in (0, 2)$ *and*

$$\delta \geq 1, \qquad \delta + \beta > 2. \tag{3.22}$$

Let $C \subseteq [0, \infty)$ *be at most countable set such that for all* $t \in (0, \infty) \setminus C$ *there there exist* $\omega_t \in (0, t)$ *and* $L_{2t} > 0$ *such that* $(t - \omega_t, t + \omega_t) \cap C = \varnothing$ *and for all* $s \in (t - \omega_t, t + \omega_t)$ *there exists* $\gamma'(t)$ *and* $|\gamma'(s)| \leq L_{2t}$. *In addition, for some set* $\Gamma \subseteq [0, \infty)$ *of Lebesgue measure zero, for all* $t \in [0, \infty) \setminus \Gamma$, $\langle x, \Psi(t, x) \rangle \leq -\frac{2\gamma'(t)}{\delta\gamma(t)} \|x\|^2$ *for* $x \in \mathbb{R}^n$ *and*

$$\langle x, \eta(t, x) \rangle \leq \begin{cases} -\rho\gamma(t)\|x\|^\beta, & x \in B(0, 1), \\ -\rho\gamma(t)\|x\|^2, & x \in \mathbb{R}^n \setminus B(0, 1) \end{cases}$$

for some $\rho > 0$.

With the above assumptions the origin for the differential equation (3.20) *is globally finite-time stable.*

Proof. It is easy to see that the function $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ defined in (3.21) satisfies Assumption 3.2.

Let us take Lyapunov function $V(t, x) = \gamma^2(t)\|x\|^\delta$, where $t \in [0, \infty)$, $x \in \mathbb{R}^n$ and δ satisfies (3.22), which satisfies of course conditions (2.1) and (2.2). Take any $t \in [0, \infty) \setminus C$ and $x \neq 0$. By assumption on C there exists $\omega_t \in (0, t)$ such that $(t - \omega_t, t + \omega_t) \cap C = \varnothing$ and put

$$\varepsilon_{tx} = \begin{cases} \min\{\omega_t, 1\}, & t \in [0, \infty),\ x \in B(0, 1), \\ \min\{\omega_t, \|x\|\}, & t \in [0, \infty),\ x \in \mathbb{R}^n \setminus B(0, 1). \end{cases}$$

Hence, similarly as in Theorem 3.4

$$P_{tx} = \sqrt{\left(\left(2L_1 L_{2t}(\|x\| + \varepsilon_{tx})\right)^2 + \delta^2 L_1^4\right)(\|x\| + \varepsilon_{tx})^{2(\delta-1)}}.$$

We only need to calculate $\dot{V}$ and check the additional condition (3.17).

Put $\tilde{\Gamma} = \Gamma \cup C \cup \{0\}$ which is still of course of measure zero. For all $t \in [0,\infty) \setminus \tilde{\Gamma}$ and $x \in B(0,1)$ we receive

$$\begin{aligned}\dot{V}(t,x) &= \limsup_{\substack{h\to 0^+\\ w\to f(t,x)}} \frac{\gamma^2(t+h)\|x+hw\|^\delta - \gamma^2(t)\|x\|^\delta}{h}\\ &\le 2\gamma(t)\gamma'(t)\|x\|^\delta + \delta\gamma^2(t)\|x\|^{\delta-2}\left(-\rho\gamma(t)\|x\|^\beta - \frac{2\gamma'(t)}{\delta\gamma(t)}\|x\|^2\right)\\ &= -\rho\delta\gamma^3(t)\|x\|^{\delta+\beta-2} = -c(t)\left(V(t,x)\right)^\alpha, \end{aligned} \tag{3.23}$$

where $c(t) = \rho\delta\gamma(t)^{\frac{\delta-2\beta+4}{\delta}}$ is certainly of class $\mathcal{P}$ and $\alpha = \frac{\delta+\beta-2}{\delta} \in (0,1)$.

For all $t \in [0,\infty) \setminus \tilde{\Gamma}$ and $x \in \mathbb{R}^n \setminus B(0,1)$ we receive (similarly as above)

$$\begin{aligned}\dot{V}(t,x) &\le 2\gamma(t)\gamma'(t)\|x\|^\delta + \delta\gamma^2(t)\|x\|^{\delta-2}\left(-\rho\gamma(t)\|x\|^2 + \frac{-2\gamma'(t)}{\delta\gamma(t)}\|x\|^2\right)\\ &= -\rho\delta\gamma^3(t)\|x\|^\delta. \end{aligned} \tag{3.24}$$

From (3.23) and (3.24) we receive that condition (2.3) is satisfied for

$$\kappa(r) = \begin{cases} -\rho\delta\varepsilon^3 r^{\delta+\beta-2}, & r \in [0,1),\\ -\rho\delta\varepsilon^3 r^\delta, & r \ge 1. \end{cases}$$

It is obvious that $\kappa : [0,\infty) \to (-\infty, 0)$ is continuous.

From the fact that function γ is bounded we receive

$$\lim_{x\to 0}\sup_{t\ge 0} V(t,x) = \lim_{x\to 0}\sup_{t\ge 0} \gamma^2(t)\|x\|^\delta = 0.$$

From Theorem 3.9 we receive that the origin for the differential equation (3.20) is globally stable in finite-time. □

Example 3.11. Let us consider the following differential equation

$$x' = f(t,x), \tag{3.25}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by the formula

$$f(t,x) = \begin{cases} -(-\sin t + \varepsilon + 1)x + \frac{\cos t}{-\sin t+\varepsilon+1}x, & t \in [0,\infty),\ x \in (-\infty,-1] \cup [1,\infty),\\ -(-\sin t + \varepsilon + 1)\mathrm{sgn}(x)|x|^{\frac{1}{2}} + \frac{\cos t}{-\sin t+\varepsilon+1}x, & t \in [0,\infty),\ x \in (-1,1) \end{cases}$$

for $\varepsilon > 0$. Put $\gamma(t) = -\sin t + \varepsilon + 1 \ge \varepsilon > 0$, $f(t,x) = \eta(t,x) + \Psi(t,x)$, where for $t \in [0,\infty)$ and $x \in \mathbb{R}$, $\Psi(t,x) = \frac{\cos t}{\gamma(t)}x$ and

$$\eta(t,x) = \begin{cases} -\gamma(t)x, & t \in [0,\infty),\ x \in (-\infty,-1] \cup [1,\infty),\\ -\gamma(t)\mathrm{sgn}(x)|x|^{\frac{1}{2}}, & t \in [0,\infty),\ x \in (-1,1). \end{cases}$$

For $t \in [0,\infty)$ and $x \in \mathbb{R}$ it is easy to see that $x\Psi(t,x) = -\frac{\gamma'(t)}{\gamma(t)}\|x\|^2$ and

$$x\eta(t,x) = \begin{cases} -\gamma(t)\|x\|^2, & t \in [0,\infty),\ x \in (-\infty,-1] \cup [1,\infty),\\ -\gamma(t)\|x\|^{\frac{3}{2}}, & t \in [0,\infty),\ x \in (-1,1). \end{cases}$$

Hence the function f defined above satisfies all assumptions of Theorem 3.10 with $\beta = \frac{3}{2}$, $\delta = 2$, $\rho = 1$, $\Gamma = \{0\}$ and $C = \varnothing$.

From Theorem 3.10 the origin for the differential equation (3.25) is globally stable in finite-time.

Remark 3.12. Let us take Lyapunov function for the differential equation (3.25) of the form $V(t,x) = \gamma^2(t)\|x\|^\delta$ like in Theorem 3.10 where $t \in [0,\infty)$, $x \in \mathbb{R}^n$, $\gamma(t) = -\sin t + \varepsilon + 1$ and $\delta = 2$.

For $t \in [0,\infty)$ and $x \in \mathbb{R}^n \setminus B(0,1)$ we have

$$\dot{V}(t,x) = -2(-\sin t + \varepsilon + 1)^3\|x\|^2. \tag{3.26}$$

For $t \in [0,\infty)$ and $x \in B(0,1)$ we receive

$$\dot{V}(t,x) = -2(-\sin t + \varepsilon + 1)^3\|x\|^{\frac{3}{2}} = -c(t)(V(t,x))^\alpha$$

where $c(t) = 2(-\sin t + \varepsilon + 1)^{\frac{3}{2}}$ and $\alpha = \frac{3}{4}$.

From Theorem 3.10 we know that the origin for the differential equation (3.25) is globally finite-time stable. However, let us observe that we can not use Theorem 3.4 because condition (3.1) is not satisfied for $x \in \mathbb{R}^n \setminus B(0,1)$ – see (3.26) (it is true only on some neighborhood of the origin).

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