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Internal exact controllability and uniform decay rates for a model of dynamical elasticity equations for incompressible materials with a pressure term

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Abstract. This paper is concerned with the internal exact controllability of the following model of dynamical elasticity equations for incompressible materials with a pressure term,

$$\phi'' - \Delta\phi = -\nabla p,$$

and it is also devoted to the study of the uniform decay rates of the energy associated with the same model subject to a locally distributed nonlinear damping,

$$\phi'' - \Delta\phi + a(x)g(\phi') = -\nabla p,$$

where Ω is a bounded connected open set of $\mathbb{R}^n$ $(n \geq 2)$ with regular boundary Γ, $\phi = (\phi_1(x,t), \dots, \phi_n(x,t))$, $x = (x_1, \dots, x_n)$ are n-dimensional vectors and p denotes a pressure term. The function $a(x)$ is assumed to be nonnegative and essentially bounded and, in addition, $a(x) \geq a_0 > 0$ a.e. in $\omega \subset \Omega$, where ω satisfies the geometric control condition. The first result is obtained by applying HUM (Hilbert Uniqueness Method) due to J. L. Lions while the second one is obtained by employing ideas first introduced in the literature by Lasiecka and Tataru.

Keywords: internal exact controllability, incompressible materials, non linear damping, uniform decay rates.

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1 Introduction

1.1 Description of the problem.

Let Ω be a bounded connected open set of $\mathbb{R}^n$ $(n \geq 2)$ with regular boundary Γ. Let $Q = \Omega\times]0,T[$ be a cylinder whose lateral boundary is given by $\Sigma = \Gamma\times]0,T[$.

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Consider the following problem

$$\begin{cases} \phi'' - \Delta\phi = -\nabla p \;\text{ in } Q, \\ \operatorname{div}\phi = 0 \;\text{ in } Q, \\ \phi = \psi \;\text{ on } \Sigma, \\ \phi(0) = \phi^0, \;\; \phi'(0) = \phi^1 \;\text{ in } \Omega. \end{cases} \tag{1.1}$$

System (1.1) was studied by J. L. Lions [26], motivated by dynamical elasticity equations for incompressible materials. Assuming that Ω is strictly star-sharped with respect to the origin, that is, there exists $\gamma > 0$ such that

$$m \cdot \nu \geq \gamma > 0 \quad \text{on } \Gamma, \tag{1.2}$$

(where $m(x) = x = (x_1, \dots, x_n)$ and ν is the exterior unitary normal) J.L. Lions [26] proved that the normal derivative of the solution ϕ of the (1.1) belongs to $(L^2(\Sigma))^n$ while A. R. Santos [35] established the boundary exact controllability for problem (1.1). In this direction it is worth mentioning the work due to Cavalcanti et al. [10], in which boundary exact controllability of the viscoelastic equation

$$\phi'' - \Delta\phi - \int_0^t g(t-s)\Delta\phi(s)\,ds = -\nabla p,$$

has been studied.

System (1.1) may be obtained from Newton's second law considering small deflections of Ω, where Ω is a solid body composed of elastic, isotropic, incompressible materials (like some rubber types). For more information on the physical interpretation of this model see A. R. Santos [34] and Cavalcanti et al. [10].

Inspired by the above mentioned works we study, in natural way, in Section 2, the internal exact controllability of the system

$$\begin{cases} \phi'' - \Delta\phi = -\nabla p + h\chi_\omega \;\text{ in } Q, \\ \operatorname{div}\phi = 0 \;\text{ in } Q, \\ \phi = 0 \;\text{ on } \Sigma, \\ \phi(0) = \phi^0, \;\; \phi'(0) = \phi^1 \;\text{ in } \Omega, \end{cases} \tag{1.3}$$

where $\Delta\phi = (\Delta\phi_1, \dots, \Delta\phi_n), \phi'' = (\phi_1'', \dots, \phi_n''), \operatorname{div}\phi = \sum_{i=1}^n \frac{\partial\phi_i}{\partial x_i}$ and $p = p(x,t)$ is the pressure term. In addition, $\omega \subset \Omega$ and χ_ω is the characteristic function of ω where ω is a neighbourhood of the boundary Γ satisfying the well-known geometric control conditions.

The exact controllability problem for system (1.3) is formulated as follows: given $T > 0$ large enough, for every initial date $\{\phi^0, \phi^1\}$ in a suitable space, it is possible to find a control h such that the solution of (1.3) satisfies

$$\phi(x,T) = \phi'(x,T) = 0.$$

Next, in Section 3, we are going to investigate the uniform decay rates of the energy associated with problem (1.1) subject to a locally distributed nonlinear damping as follows

$$\begin{cases} \phi'' - \Delta\phi + a(x)g(\phi') = -\nabla p \;\text{ in } \Omega \times (0,\infty), \\ \operatorname{div}\phi = 0 \;\text{ in } \Omega \times (0,\infty), \\ \phi = 0 \;\text{ on } \Sigma, \\ \phi(0) = \phi^0, \;\; \phi'(0) = \phi^1 \;\text{ in } \Omega, \end{cases} \tag{1.4}$$

where $a \in L^\infty(\Omega)$ is a nonnegative function such that

$$a(x) \geq a_0 > 0 \quad \text{in } \omega \subset \Omega \tag{1.5}$$

and

$$\begin{aligned} g : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ s &\longmapsto g(s) = [g_i(s_i)]_{i=1,\ldots,n} \end{aligned} \tag{1.6}$$

where, for all $i = 1, \ldots, n$, $g_i : \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that

$$\begin{cases} g_i \text{ is continuous monotonic increasing and } g_i(0) = 0, \\ g_i(s_i)s_i > 0 \ \forall \ s_i \neq 0, \\ k|s_i|^2 \leq g_i(s_i)s_i \leq K|s_i|^2 \quad \forall \quad |s_i| \geq 1, \text{ for some positive constants } k \text{ and } K. \end{cases} \tag{1.7}$$

1.2 Main goal, methodology and previous results.

The main goal of this paper is twofold: First, to obtain the exact internal controllability of the system (1.3) and then use this result to prove that the solutions of problem (1.4) decay exponentially to zero.

System (1.1) was first introduced by J. L. Lions [26]. In his work, J. L. Lions proved the hidden regularity property holds for this linear system, under the condition that the domain is star-shaped. Taking advantage of this property and an inverse inequality due to Cavalcanti et al. [10], we are able to obtain the direct and inverse inequalities needed to obtain the internal controllability.

In order to obtain the stabilization result, we use an approach first introduced by Lasiecka and Tataru in [20] which allows us not to impose any growth condition of the function g near the origin and show that the energy decays as fast as the solution of an associated differential equation. Indeed, we are able to establish general decay rates of the energy given by

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right), \qquad \forall t > T_0$$

and driven by the solution $S(t)$ of the nonlinear ODE

$$S_t + q(S(t)) = 0,$$

where q is a strictly increasing function in connection with the damping term $g(u_t)$. Moreover, under some extra conditions on the class of nonlinear dissipation and assuming that the pressure is constant, we give examples of the explicit decay rates.

A different but related approach is provided by Alabau-Boussouira and Ammari in [3], where the authors obtained sharp, simple and quasi-optimal decay rates for nonlinearly damped abstract infinite-dimensional systems. The method employed by the authors relies on an observability inequality for the conservative system and some comparison properties, combining optimal geometric conditions as provided by Bardos et al. [6] with an optimal-weight convexity method of Alabau-Boussouira (see [1] and [2]).

On the other hand, although the bibliography concerning the wave equation subject to a locally distributed damping is truly long, see, for instance, [1,2,6,12–18,21,22,27–29,31,38,40], and a long list of references therein, it seems that there exist just few papers in connection with control or stabilization for the model of dynamical elasticity equations for incompressible materials introduced by J. L. Lions [26]. In [33] Oliveira and Charão also studied the decay properties of the solutions of an incompressible vector wave equation with a locally distributed nonlinear damping. However, in order to obtain the algebraic decay rate to zero of the solution, the authors imposed some extra technical conditions on the function g like the fact that the partial derivative of g is positive definite which will not be necessary in our present study. In order to get this result, the authors used multiplier methods and a lemma due to Nakao. The exponential decay rate of the energy for the case of an incompressible vector wave equation with localized linear dissipation was obtained by Araruna et al. [4]. In a more general framework, a problem that is similar to (1.1), is studied in [19], where Ammari, Feireisl and Nicaise proved exponential and polynomial decay rates for an acoustic system with spatially distributed damping.

2 Internal exact controllability

In what follows, we consider the Hilbert spaces

$$V = \{v \in (H_0^1(\Omega))^n \text{ and } \operatorname{div} v = 0 \text{ in } \Omega\}, \tag{2.1}$$

and

$$H = \{v \in (L^2(\Omega))^n, \operatorname{div} v = 0 \text{ and } v \cdot \nu = 0 \text{ on } \Gamma\}, \tag{2.2}$$

equipped with their respective inner products

$$((u,v)) = \sum_{i=1}^n ((u_i, v_i))_{H_0^1(\Omega)}, \tag{2.3}$$

$$(u,v) = \sum_{i=1}^n (u_i, v_i)_{L^2(\Omega)}. \tag{2.4}$$

We also consider

$$\mathcal{V} = \{\varphi \in (\mathcal{D}(\Omega))^n, \operatorname{div} \varphi = 0\}, \tag{2.5}$$

and

$$W = V \cap (H^2(\Omega))^n. \tag{2.6}$$

We have that $\mathcal{V}$ is dense in V with topology induced by V and

$$H = \overline{\mathcal{V}}^{(L^2(\Omega))^n}. \tag{2.7}$$

We observe that throughout this paper repeated indexes indicate summation from 1 to n.

2.1 Direct and inverse inequalities

Let us consider the following problem

$$\begin{cases} \phi'' - \Delta\phi = -\nabla p \text{ in } Q, \\ \operatorname{div} \phi = 0 \text{ in } Q, \\ \phi = 0 \text{ on } \Sigma, \\ \phi(0) = \phi^0, \ \phi'(0) = \phi^1 \text{ in } \Omega. \end{cases} \tag{2.8}$$

Firstly, observe that following the arguments [37] we can show that for regular initial data the problem (2.8) is equivalent to the problem

$$\begin{cases} \phi'' + A\phi = 0 \ \text{ in } Q, \\ \phi(0) = \phi^0, \ \ \phi'(0) = \phi^1 \ \text{ in } \Omega, \end{cases} \tag{2.9}$$

where A is Stokes operator defined as follows: $A : (H^2(\Omega))^n \cap V \to H$ given by $Au := \mathcal{P}(-\Delta u)$ where $\mathcal{P} : (L^2(\Omega))^n \to H$ is the orthogonal projector in $(L^2(\Omega))^n$ onto H and $\Delta : (H^2(\Omega))^n \cap V \to (L^2(\Omega))^n$ is the Laplace operator with Dirichlet boundary conditions.

In this section we are going to obtain the direct and inverse inequalities to problem (2.8) which is enough to apply HUM (Hilbert Uniqueness Method) in order to obtain the above mentioned exact controllability. For this end we will employ the multiplier technique. The main results of this section are Theorem 2.1 and Theorem 2.4 below.

Theorem 2.1. *Let $\{\phi^0, \phi^1\} \in H \times V'$ and ϕ the ultra weak solution of the problem (2.8). Then, there exists a constant $C_1 > 0$ such that*

$$\int_0^T \int_\omega |\phi|^2 dxdt \leq C_1 \left[|\phi^0|_H^2 + \|\phi^1\|_{V'}^2 \right]. \tag{2.10}$$

Proof. Since ϕ is the ultra weak solution to problem (2.8) then making use of standard properties of ultra weak solutions for linear problems (see Cavalcanti [10, Section 5] or Lions [25, Chap. 1, Section 4]) there exists $C_0 > 0$ such that

$$\|\phi\|^2_{L^\infty(0,T;H)} + \|\phi'\|^2_{L^\infty(0,T;V')} \leq C_0 \left[|\phi^0|_H^2 + \|\phi^1\|_{V'}^2 \right]. \tag{2.11}$$

From (2.11) and $L^\infty(0,T;H) \hookrightarrow L^2(0,T;H)$ we have the desired result. □

Remark 2.2. Arguing as in [7, Chap. 5] or [39, Chap. 2] it is possible to show the existence of a function $p \in H^{-1}(0,T;L_0^2(\Omega))$ such that (1.3) is satisfied in $\mathcal{D}'(Q)$. Moreover, there exists $C > 0$ such that

$$\|p\|_{H^{-1}(0,T;L_0^2(\Omega))} \leq C(|\phi^1|_H + \|\phi^0\|_V + \|h\chi_\omega\|_{L^2(0,T;H)}), \tag{2.12}$$

where $L_0^2(\Omega) \approx L^2(\Omega)/\mathbb{R}$.

Remark 2.3. As it is stated in Lions [25, Chap. I, Lemma 3.7] if $\phi \in (H_0^1(\Omega) \cap H^2(\Omega))^n$, then

$$\frac{\partial \phi_i}{\partial x_k} = \nu_k \frac{\partial \phi_i}{\partial \nu} \quad \text{on } \Gamma; \ \ \forall i,k \in \{1,\dots,n\}. \tag{2.13}$$

Moreover, if $\operatorname{div} \phi = 0$ in Ω, then as in Lions [25, Chap. II, section 5] we have

$$\frac{\partial \phi}{\partial \nu} \cdot \nu = 0 \quad \text{on } \ \Gamma \tag{2.14}$$

and consequently

$$\nu_i \frac{\partial \phi_i}{\partial x_k} = \nu_i \nu_k \frac{\partial \phi_i}{\partial \nu} \quad \text{on } \ \Gamma. \tag{2.15}$$

Let $x^0 \in \mathbb{R}^n$, $m(x) = x - x^0, x \in \mathbb{R}^n$ and $R_0 = \max\{\|m(x)\|; x \in \overline{\Omega}\}$.
We assume that $\omega \subset \Omega$ is a neighborhood of $\overline{\Gamma(x^0)}$ where

$$\begin{aligned} \Gamma_0 &= \Gamma(x^0) = \{x \in \Gamma; m(x) \cdot \nu(x) > 0\}, \\ \Gamma_1 &= \Gamma \setminus \Gamma(x^0) = \{x \in \Gamma; m(x) \cdot \nu(x) \leq 0\}, \\ \Sigma_0 &= \Sigma(x_0) = \Gamma_0(x_0) \times [0, T], \\ \Sigma_1 &= \Sigma \setminus \Sigma_0 = \Gamma_1(x_0) \times [0, T]. \end{aligned} \tag{2.16}$$

As an example of a domain Ω satisfying the above assumption let us consider the Figure 2.1 below:

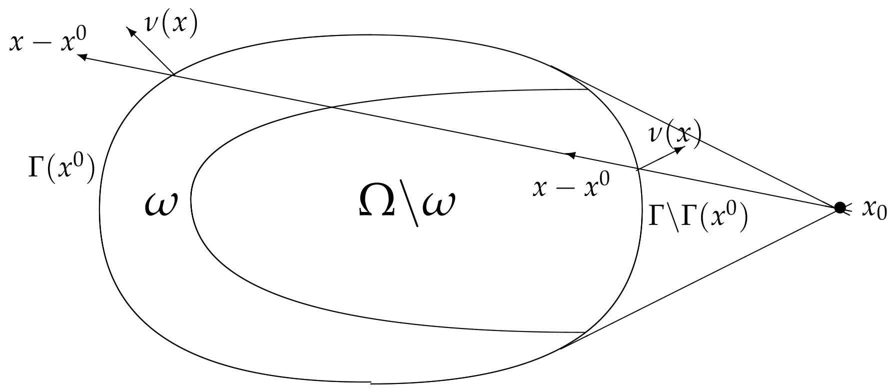

Figure 2.1: Description of a subset ω of a domain Ω which is a neighborhood of $\Gamma(x^0)$.

Theorem 2.4. *Let us consider $T > T_0$, where $T_0 = 2R_0$, as in Theorem 3.3 due to Cavalcanti et al. [10]. Then, there exists a constant $C_2 > 0$ such that*

$$|\phi^0|_H^2 + \|\phi^1\|_{V'}^2 \leq C_2 \int_0^T \int_\omega |\phi|^2 dxdt. \tag{2.17}$$

Proof. Suppose that the following estimate holds

$$\|\theta^0\|_V^2 + |\theta^1|_H^2 \leq C_2 \int_0^T \int_\omega |\theta'|^2 dxdt, \tag{2.18}$$

where θ is solution of the problem (2.8) with initial data $\{\theta^0, \theta^1\} \in V \times H$. Then we have the desired result. Indeed, take $\{\phi^0, \phi^1\} \in H \times V'$. Since $-\phi^1 \in V'$ and the operator $-\Delta : V \longrightarrow V'$ is an isometric isomorphism, there exists $\eta \in V$ such that

$$-\Delta\eta = -\phi^1. \tag{2.19}$$

Let us define

$$\psi(t) = \int_0^t \phi(s)ds + \eta, \tag{2.20}$$

where η satisfies (2.19) and ϕ is the solution to problem

$$\begin{cases} \phi'' - \Delta\phi = -\nabla(\frac{\partial}{\partial t}p) \ \text{ in } Q, \\ \operatorname{div} \phi = 0 \ \text{ in } Q, \\ \phi = 0 \ \text{ on } \Sigma, \\ \phi(0) = \phi^0, \ \ \phi'(0) = \phi^1 \ \text{ in } \Omega. \end{cases} \tag{2.21}$$

Thus we have that ψ is the solution to problem

$$\begin{cases} \psi'' - \Delta\psi = -\nabla p \ \text{ in } Q, \\ \operatorname{div}\psi = 0 \ \text{ in } Q, \\ \psi = 0 \ \text{ on } \Sigma, \\ \psi(0) = \eta, \ \ \psi'(0) = \phi^0 \ \text{ in } \Omega. \end{cases} \tag{2.22}$$

From (2.18) and (2.22) we have

$$|\phi^0|_H^2 + \|\phi^1\|_{V'}^2 \leq C_2 \int_0^T \int_\omega |\phi|^2 dxdt.$$

We will prove (2.18) in several steps.

By Theorem 3.3 due to Cavalcanti et al. [10], for $T > T_0 = 2R_0$ the following inequality holds

$$E_\theta(0) \leq C \int_0^T \int_{\Gamma_0} \left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt, \tag{2.23}$$

where θ is the weak solution of the problem (2.8) with the initial data $\{\theta^0, \theta^1\} \in V \times H$.

Lemma 2.5. *Let $m \in (C^1(\overline{\Omega}))^n$. Then, for all regular solutions of (2.8), the following identity holds*

$$\langle \nabla p, m \cdot \nabla\phi \rangle_{L^2(Q)^n} = \langle \nabla p, \phi \cdot \nabla m \rangle_{L^2(Q)^n} - \langle \nabla p, \phi(\operatorname{div} m) \rangle_{L^2(Q)^n}.$$

Proof. Let us consider

$$X = -\int_0^T \int_\Omega \frac{\partial p}{\partial x_i} m_k \frac{\partial \phi_i}{\partial x_k} dxdt.$$

Integrating by parts with respect to x_k and using the fact that $\phi = 0$ on Σ, we get

$$\begin{aligned} X &= \int_0^T \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial p}{\partial x_i} \cdot m_k\right) \cdot \phi_i dxdt \\ &= \int_0^T \int_\Omega \frac{\partial^2 p}{\partial x_i \partial x_k} m_k \phi_i dxdt + \int_0^T \int_\Omega \frac{\partial p}{\partial x_i}\frac{\partial m_k}{\partial x_k}\phi_i dxdt. \end{aligned} \tag{2.24}$$

Integrating by parts the first integral, with respect to x_i, we obtain

$$\int_0^T \int_\Omega \frac{\partial^2 p}{\partial x_i \partial x_k} m_k \phi_i dxdt = -\int_0^T \int_\Omega \frac{\partial p}{\partial x_k}\frac{\partial m_k}{\partial x_i}\phi_i dxdt - \int_0^T \int_\Omega \frac{\partial p}{\partial x_k} m_k \frac{\partial \phi_i}{\partial x_i} dxdt.$$

Since $\operatorname{div}\phi = 0$ on Q, we conclude that

$$\int_0^T \int_\Omega \frac{\partial^2 p}{\partial x_i \partial x_k} m_k \phi_i dxdt = -\int_0^T \int_\Omega \frac{\partial p}{\partial x_k}\frac{\partial m_k}{\partial x_i}\phi_i dxdt.$$

Therefore,

$$X = -\int_0^T \int_\Omega \frac{\partial p}{\partial x_k}\frac{\partial m_k}{\partial x_i}\phi_i dxdt + \int_0^T \int_\Omega \frac{\partial p}{\partial x_i}\frac{\partial m_k}{\partial x_k}\phi_i dxdt$$

and the proof is finished. □

Lemma 2.6. *Let $T > T_0$ and $\varepsilon > 0$ be such that $T - 2\varepsilon > T_0$. Let θ be the solution of problem (2.8) with initial data $\{\theta^0, \theta^1\} \in V \times H$. Then, there exists $C > 0$ such that*

$$E_\theta(0) \leq C \int_\varepsilon^{T-\varepsilon} \int_{\Gamma_0} \left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt. \tag{2.25}$$

Proof. Since θ is the weak solution of problem (2.8), $\theta \in C([0,T],V)\cap C^1([0,T],H)\cap C^2([0,T],V')$. In particular,

$$\begin{cases} \theta'' - \Delta\theta = -\nabla p \ \text{ in } C([0,T-2\varepsilon],V'), \\ \operatorname{div}\theta = 0 \ \text{ in } \Omega\times(0,T-2\varepsilon), \\ \theta = 0 \ \text{ on } \Gamma\times(0,T-2\varepsilon), \\ \theta(0)=\theta^0, \ \ \theta'(0)=\theta^1 \ \text{ in } \Omega. \end{cases} \tag{2.26}$$

From (2.23) and (2.26) we have

$$E_\theta(0) \le C\int_0^{T-2\varepsilon}\int_{\Gamma_0}\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt. \tag{2.27}$$

Let $0\le t\le T-2\varepsilon$ and define $\eta(t)=\theta(t+\varepsilon)$ and $\nabla q=\nabla p(x,t+\varepsilon)$. Then,

$$\begin{cases} \eta'' - \Delta\eta = -\nabla q \ \text{ in } C([0,T-2\varepsilon],V'), \\ \operatorname{div}\eta = 0 \ \text{ in } \Omega\times(0,T-2\varepsilon), \\ \eta = 0 \ \text{ on } \Gamma\times(0,T-2\varepsilon), \\ \eta(0)=\theta(\varepsilon), \ \ \eta'(0)=\theta'(\varepsilon) \ \text{ in } \Omega, \end{cases} \tag{2.28}$$

analogously to what we have done in (2.27), we obtain,

$$E_\eta(0) \le C\int_0^{T-2\varepsilon}\int_{\Gamma_0}\left|\frac{\partial\eta}{\partial\nu}\right|^2 d\Gamma dt. \tag{2.29}$$

Since $E_\eta(0)=E_\theta(\varepsilon)=E_\theta(0)$ and making the change of variable $s=t+\varepsilon$, we infer

$$E_\theta(0) \le C\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_0}\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt. \qquad \square$$

Fix $T>T_0$ and $\varepsilon>0$ such that $T-2\varepsilon>T_0$. Let us consider $h\in(C^1(\overline{\Omega}))^n$ such that

$$\begin{cases} h\cdot\nu \ge 0 \text{ for every } x\in\Gamma, \\ h=\nu \text{ on } \Gamma_0, \\ h=0 \text{ in } \Omega\setminus\omega. \end{cases}$$

Let $\eta\in C^1(0,T)$ such that $\eta(0)=\eta(T)=0$ and $\eta(t)=1$ in $]\varepsilon,T-\varepsilon[$.

Let us define $r(x,t)=\eta(t)h(x)$ which belongs $[W^{1,\infty}(Q)]^n$ and satisfies

$$\begin{cases} r(x,t)=\nu(x), \text{ for every } (x,t)\in\Gamma_0\times]\varepsilon,T-\varepsilon[, \\ r(x,t)\cdot\nu(x)\ge 0, \text{ for every } (x,t)\in\Gamma\times]0,T[, \\ r(x,0)=r(x,T)=0, \text{ for every } x\in\Omega, \\ r(x,t)=0, \text{ for every } (x,t)\in(\Omega\setminus\omega)\times]0,T[. \end{cases} \tag{2.30}$$

Lemma 2.7. *Let $T>T_0$ and $\varepsilon>0$ be such that $T-2\varepsilon>T_0$. Let θ be the solution of problem (2.8) with initial data $\{\theta^0,\theta^1\}\in V\times H$. Then, there exists $C>0$ such that*

$$E_\theta(0)\le C\int_\varepsilon^{T-\varepsilon}\int_\omega |\theta|^2+|\theta'|^2+|\nabla\theta|^2 dxdt, \tag{2.31}$$

where $\nabla\theta$ *means*

$$\begin{pmatrix} \frac{\partial\theta_1}{\partial x_1} & \cdots & \frac{\partial\theta_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial\theta_1}{\partial x_n} & \cdots & \frac{\partial\theta_n}{\partial x_1} \end{pmatrix}. \tag{2.32}$$

Proof. Initially considering the regular initial data, one obtains the general result using density arguments. In equation $(2.8)_1$, taking the inner product in $(L^2(\Omega))^n$ of $\theta'' - \Delta\theta + \nabla p$ and $r\cdot\nabla\theta$, and integrating in $[0,T]$, we obtain

$$\begin{aligned} \frac{1}{2}\int_0^T\int_\gamma r\cdot\nu\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\gamma dt = \frac{1}{2}\int_0^T\int_\Omega(\operatorname{div} r)(|\theta'|^2 - |\nabla\theta|^2)dxdt + \int_0^T\int_\Omega \frac{\partial\theta_i}{\partial x_j}\frac{\partial r_k}{\partial x_j}\frac{\partial\theta_i}{\partial x_k}dxdt \\ - \int_0^T\int_\Omega \theta_i' r_k'\frac{\partial\theta_i}{\partial x_k}dxdt + \int_0^T\int_\Omega \frac{\partial p}{\partial x_i} r_k \frac{\partial\theta_i}{\partial x_k}dxdt. \end{aligned} \tag{2.33}$$

Indeed, note that

$$\int_0^T(\theta'', r\cdot\nabla\theta)dt - \int_0^T(\Delta\theta, r\cdot\nabla\theta)dt = -\int_0^T(\nabla p, r\cdot\nabla\theta)dt. \tag{2.34}$$

Introduce the notations

$$\begin{aligned} J_1 &:= \int_0^T(\theta'', r\cdot\nabla\theta)dt, \\ J_2 &:= -\int_0^T(\Delta\theta, r\cdot\nabla\theta)dt, \\ J_3 &:= -\int_0^T(\nabla p, r\cdot\nabla\theta)dt. \end{aligned}$$

Next, we are going to estimate these terms.

Using integration by parts and the properties of the function $r(x,t)$ defined in (2.30), we deduce J_1, satisfies

$$J_1 = \int_0^T(\theta'', r\cdot\nabla\theta)dt = -\int_0^T\int_\Omega \theta_i' r_k'\frac{\partial\theta_i}{\partial x_k}dxdt - \underbrace{\int_0^T\int_\Omega \theta_i' r_k \frac{\partial\theta_i'}{\partial x_k}dxdt}_{\widetilde{J}_1}.$$

Then Gauss's Formula yields

$$\int_\Omega \frac{\partial}{\partial x_k}\left(\theta_i'^2 r_k\right)dx = \int_\Gamma \theta_i'^2 r_k \nu_k d\Gamma,$$

and since $\theta_i'(x,t) = 0$ on Σ, we deduce that

$$\widetilde{J}_1 = -\frac{1}{2}\int_0^T\int_\Omega \theta_i'^2\frac{\partial r_k}{\partial x_k}dxdt.$$

Thus

$$\begin{aligned} J_1 &= -\int_0^T\int_\Omega r_k'\frac{\partial\theta_i}{\partial x_k}\theta_i' dxdt + \frac{1}{2}\int_0^T\int_\Omega \theta_i'^2\frac{\partial r_k}{\partial x_k}dxdt \\ &= -\int_0^T(\theta', r'\cdot\nabla\theta)dt + \frac{1}{2}\int_0^T\int_\Omega |\theta'|^2 \operatorname{div} r\, dxdt. \end{aligned} \tag{2.35}$$

Furthermore, we have for the term J_2

$$J_2 := -\int_0^T (\Delta\theta, r\cdot\nabla\theta)dt = -\int_0^T\int_\Omega \frac{\partial^2\theta_i}{\partial x_j^2} r_k \frac{\partial\theta_i}{\partial x_k}dxdt.$$

From Gauss's formula and (2.13), we have

$$J_2 = \underbrace{\int_0^T\int_\Omega \frac{\partial\theta_i}{\partial x_j}\frac{\partial}{\partial x_j}\left(r_k\frac{\partial\theta_i}{\partial x_k}\right)dxdt}_{\widetilde{J_2}} - \int_0^T\int_\Gamma r_k\frac{\partial\theta_i}{\partial x_k}\frac{\partial\theta_i}{\partial \nu}d\Gamma dt. \tag{2.36}$$

Notice that

$$\widetilde{J_2} = \int_0^T\int_\Omega \frac{\partial\theta_i}{\partial x_j}\frac{\partial r_k}{\partial x_j}\frac{\partial\theta_i}{x_k}dxdt + \underbrace{\frac{1}{2}\int_0^T\int_\Omega r_k\frac{\partial}{\partial x_k}\left[\left(\frac{\partial\theta_i}{\partial x_j}\right)^2\right]dxdt}_{\widetilde{\widetilde{J_2}}}. \tag{2.37}$$

From Gauss's formula and (2.13), we deduce that

$$\widetilde{\widetilde{J_2}} = \frac{1}{2}\int_0^T\int_\Gamma r_k\nu_k\left(\frac{\partial\theta_i}{\partial\nu}\right)^2 d\Gamma dt - \frac{1}{2}\int_0^T\int_\Omega \frac{\partial r_k}{\partial x_k}\left(\frac{\partial\theta_i}{\partial x_j}\right)^2 dxdt. \tag{2.38}$$

Thus from (2.36), (2.37), (2.38) and (2.13), we have

$$J_2 = \int_0^T\int_\Omega \frac{\partial\theta_i}{\partial x_j}\frac{\partial r_k}{\partial x_j}\frac{\partial\theta_i}{x_k}dxdt - \frac{1}{2}\int_0^T\int_\Omega \operatorname{div} r(|\nabla\theta|^2)dxdt - \frac{1}{2}\int_0^T\int_\Gamma (r\cdot\nu)\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt. \tag{2.39}$$

For the term J_3 we have

$$J_3 = -\int_0^T (\nabla p, r\cdot\nabla\theta)dt = -\int_0^T\int_\Omega \frac{\partial p}{\partial x_i} r_k\frac{\partial\theta_i}{\partial x_k}dxdt. \tag{2.40}$$

Therefore, from (2.34), (2.35), (2.39) and (2.40) we infer

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Gamma (r\cdot\nu)\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt &= \frac{1}{2}\int_0^T\int_\Omega (\operatorname{div} r)(|\theta'|^2 - |\nabla\theta|^2)dxdt - \int_0^T (\theta', r'\cdot\nabla\theta)dt \\ &\quad + \int_0^T\int_\Omega \frac{\partial\theta_i}{\partial x_j}\frac{\partial r_k}{\partial x_j}\frac{\partial\theta_i}{\partial x_k}dxdt + \int_0^T\int_\Omega \frac{\partial p}{\partial x_i} r_k\frac{\partial\theta_i}{\partial x_k}dxdt.\end{aligned} \tag{2.41}$$

Note that by Lemma 2.5 we have that

$$\int_0^T\int_\Omega \frac{\partial p}{\partial x_i} r_k\frac{\partial\theta_i}{\partial x_k}dxdt = \langle \nabla p, -\theta\cdot\nabla r + \theta(\operatorname{div} r)\rangle_{H^{-1}(Q)^n, H_0^1(Q)^n}. \tag{2.42}$$

Therefore

$$\left|\int_0^T\int_\Omega \frac{\partial p}{\partial x_i} r_k\frac{\partial\theta_i}{\partial x_k}dxdt\right| \le \delta\|\nabla p\|^2_{H^{-1}(Q)^n} + C_\delta\int_0^T\int_\omega |\theta|^2 + |\theta'|^2 + |\nabla\theta|^2 dxdt.$$

Using the properties of $r(x,t)$ and estimating others term on the right side of equality (2.41), we obtain that

$$\frac{1}{2}\int_0^T\int_\Gamma (r\cdot\nu)\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt \le \delta\|\nabla p\|^2_{H^{-1}(Q)^n} + C\int_0^T\int_\omega |\theta|^2 + |\theta'|^2 + |\nabla\theta|^2 dxdt. \tag{2.43}$$

Therefore, by Lemma 2.6 and (2.43) we have

$$\begin{aligned} E_\theta(0) \le \frac{1}{2}\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_0}\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt &= \frac{1}{2}\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_0}(r\cdot\nu)\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt \\ &\le \frac{1}{2}\int_0^T\int_\Gamma (r\cdot\nu)\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\Gamma dt \\ &\le \delta\|\nabla p\|^2_{H^{-1}(Q)^n} + C\int_0^T\int_\omega |\theta|^2+|\theta'|^2+|\nabla\theta|^2 dxdt. \end{aligned} \tag{2.44}$$

Using the fact $\|\nabla p\|^2_{H^{-1}(Q)^n} \le CE_\theta(0)$, by (2.44) and choosing δ small enough we have

$$E_\theta(0) \le C\int_0^T\int_\omega |\theta|^2+|\theta'|^2+|\nabla\theta|^2 dxdt.$$

Since the inequality above is valid for all $T > T_0$, in particular for $T-2\varepsilon$, proceeding as in the demonstration of Lemma 2.6 we have the desired result. □

Remark 2.8. According to the proof of Lemma 2.3 in J. L. Lions [25] we can construct a neighbourhood $\hat\omega$ of $\overline{\Gamma_0}$ such that $\Omega\cap\overline{\hat\omega}\subset\omega$ and we can to build a vector field $\hat r$ for $\hat w$. Then, we get analogously, that

$$E_\theta(0) \le C\int_\varepsilon^{T-\varepsilon}\int_{\hat\omega} |\theta|^2+|\theta'|^2+|\nabla\theta|^2 dxdt.$$

Now, let us consider a function $r = r(x,t) \in W^{1,\infty}(Q)$ that satisfies

$$\begin{cases} r(x,t) \ge 0, \text{ for every } (x,t)\in\Omega\times]0,T[, \\ r(x,t) = 1, \text{ for every } (x,t)\in\hat\omega\times]\varepsilon,T-\varepsilon[, \\ r(x,t) = 0, \text{ for every } (x,t)\in(\Omega\setminus\omega)\times]0,T[, \\ 0<r(x,t)<1, \text{for every } (x,t)\in(\omega\setminus\hat\omega)\times]0,T[\text{ and } \hat\omega\times(]0,\varepsilon]\cup[T-\varepsilon,T[), \\ r(x,0)=r(x,T)=0, \text{for every } x\in\Omega, \\ \frac{|\nabla r|^2}{r}\in L^\infty(\omega\times]0,T[). \end{cases} \tag{2.45}$$

The function r can be chosen as follows $r(x,t) = \rho^2(x)\eta(t)$ where $\eta\in C^1(0,T)$ and it satisfies

$$\begin{cases} \eta(0)=\eta(T)=0, \\ \eta(t)=1, \text{ in }]\varepsilon,T-\varepsilon[, \\ 0<\eta(t)<1, \text{ in }]0,\varepsilon[\ \cup \ ]T-\varepsilon,T[, \end{cases}$$

and $\rho\in C^1(\overline{\Omega})$ satisfies

$$\begin{cases} \rho(x)=1, \text{ for every } x\in\hat\omega, \\ \rho(x)=0, \text{ for every } x\in\Omega\setminus\omega, \\ 0<\rho(x)<1, \text{ for every } x\in\omega\setminus\hat\omega. \end{cases}$$

Proposition 2.9. *Let us consider $T > T_0$ and $\varepsilon > 0$ such that $T-2\varepsilon > T_0$ and θ the solution of problem (2.8) with initial data $\{\theta^0,\theta^1\}\in V\times H$. Then, there exists a constant $C>0$ such that*

$$E_\theta(0) \le C\int_0^T\int_\omega |\theta|^2+|\theta'|^2 dxdt. \tag{2.46}$$

Proof. Initially we consider regular initial data and one obtains the general result using density arguments. In equation $(2.8)_1$, taking the inner product in $(L^2(\Omega))^n$ of $\theta'' - \Delta\theta + \nabla p$ and $r\theta$ and integrating in $[0, T]$, we obtain

$$\int_0^T \int_\Omega \theta_i'' r\theta_i dxdt - \int_0^T \int_\Omega \Delta\theta_i r\theta_i dxdt = -\int_0^T \int_\Omega \frac{\partial p}{\partial x_i} r\theta_i dxdt. \tag{2.47}$$

Next, we are going to estimate the terms in (2.47).

Denote by

$$I_1 := \int_0^T \int_\Omega \theta_i'' r\theta_i dxdt.$$
$$I_2 := -\int_0^T \int_\Omega \Delta\theta_i r\theta_i dxdt,$$
$$I_3 := -\int_0^T \int_\Omega \frac{\partial p}{\partial x_i} r\theta_i dxdt.$$

Using the properties of the function $r(x,t)$ defined in (2.45) and making use of the equality $(\theta_i', r\theta_i)' = (\theta_i'', r\theta_i) + (\theta_i', r'\theta_i) + (\theta_i', r\theta_i')$, we have that I_1 can be estimated as follows

$$I_1 = \int_0^T \int_\Omega \theta_i'' r\theta_i dxdt = -\int_0^T \int_\Omega \theta_i' r'\theta_i dxdt - \int_0^T \int_\Omega \theta_i' r\theta_i' dxdt. \tag{2.48}$$

Notice that using Gauss's formula and taking $\theta = 0$ on Σ into account, we infer that I_2 verifies

$$I_2 = -\int_0^T \int_\Omega \Delta\theta_i r\theta_i dxdt = \int_0^T \int_\Omega \nabla\theta_i (\nabla r)\theta_i + r|\nabla\theta_i|^2 dxdt. \tag{2.49}$$

Replacing (2.48), (2.49) in (2.47), we obtain

$$\begin{aligned}\int_0^T \int_\Omega r|\nabla\theta_i|^2 dxdt = &\int_0^T \int_\Omega r|\theta_i'|^2 dxdt + \int_0^T \int_\Omega r'\theta_i'\theta_i dxdt \\ &- \int_0^T \int_\Omega \nabla\theta_i(\nabla r)\theta_i dxdt - \int_0^T \int_\Omega \frac{\partial p}{\partial x_i} r\theta_i dxdt.\end{aligned} \tag{2.50}$$

By Young's inequality we obtain that

$$\int_0^T \int_\Omega r'\theta_i'\theta_i dxdt \le C \int_0^T \int_\Omega (|\theta_i'|^2 + |\theta_i|^2) dxdt. \tag{2.51}$$

Making use of the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, we can write

$$\begin{aligned}\left| \int_0^T \int_\Omega \nabla\theta_i(\nabla r)\theta_i dxdt \right| &= \left| \int_0^T \int_\Omega r^{\frac{1}{2}} \nabla\theta_i \frac{\nabla r}{r^{\frac{1}{2}}} \theta_i dxdt \right| \\ &\le \frac{1}{2} \int_0^T \int_\Omega r|\nabla\theta_i|^2 + \frac{1}{2} \int_0^T \int_\Omega \frac{|\nabla r|^2}{r} |\theta_i|^2 dxdt.\end{aligned} \tag{2.52}$$

By the Cauchy–Schwarz inequality and by Young's inequality we have that

$$\int_0^T \int_\Omega \frac{\partial p}{\partial x_i} r\theta_i dxdt \le \delta \|p\|^2_{H^{-1}(0,T;L^2(\Omega)^n)} + C_\delta \|r\theta\|_{H_0^1(0,T;L^2(\Omega)^n)} \tag{2.53}$$

for any $\delta > 0$.

Combining (2.51), (2.52) and (2.53) we obtain

$$\int_0^T \int_\omega r|\nabla\theta|^2 dxdt \le C\int_0^T \int_\omega |\theta'|^2 + |\theta|^2 dxdt + \delta\|p\|^2_{H^{-1}(0,T;L^2(\Omega)^n)}. \tag{2.54}$$

However,

$$\int_\varepsilon^{T-\varepsilon} \int_{\hat\omega} |\nabla\theta_i|^2 dxdt = \int_\varepsilon^{T-\varepsilon} \int_{\hat\omega} r|\nabla\theta_i|^2 dxdt \le \int_0^T \int_\omega r|\nabla\theta_i|^2 dxdt. \tag{2.55}$$

From (2.54) and 2.55, we obtain

$$\int_\varepsilon^{T-\varepsilon} \int_{\hat\omega} |\nabla\theta_i|^2 dxdt \le C\int_0^T \int_\omega |\theta_i'|^2 + |\theta_i|^2 dxdt + \delta\|p\|^2_{H^{-1}(0,T;L^2(\Omega)^n)}. \tag{2.56}$$

From Remark 2.8 and (2.56) we obtain

$$E_\theta(0) \le C\int_0^T \int_\omega |\theta'|^2 + |\theta|^2 dxdt + \delta\|p\|^2_{H^{-1}(0,T;L^2(\Omega)^n)}. \tag{2.57}$$

By (2.57) and choosing δ small enough we have

$$E_\theta(0) \le C\int_0^T \int_\omega |\theta'|^2 + |\theta|^2 dxdt. \tag{2.58}$$

□

Proposition 2.10. *Let $T > T_0$ and $\varepsilon > 0$ be such that $T - 2\varepsilon > T_0$, ω a neighborhood of $\overline{\Gamma_0}$ previously cited and θ the solution of the (2.8) with $\{\theta^0, \theta^1\} \in V \times H$. Then, there exists a constant $C > 0$ such that*

$$\int_0^T \int_\omega |\theta|^2 dxdt \le C\int_0^T \int_\omega |\theta'|^2 dxdt. \tag{2.59}$$

Proof. We argue by contradiction. Let us suppose that (2.59) is not verified, then for every $m \in \mathbb{N}$ let $\{\theta^0_m, \theta^1_m\} \in V \times H$ be a sequence of initial data where the corresponding solutions $\{\theta_m\}$ of the problem (2.8) verify

$$\lim_{m\to\infty} \frac{\int_0^T \int_\omega |\theta_m|^2 dxdt}{\int_0^T \int_\omega |\theta'_m|^2 dxdt} = +\infty, \tag{2.60}$$

or, equivalently,

$$\lim_{m\to\infty} \frac{\int_0^T \int_\omega |\theta'_m|^2 dxdt}{\int_0^T \int_\omega |\theta_m|^2 dxdt} = 0. \tag{2.61}$$

Defining

$$\alpha_m = \sqrt{\|\theta_m\|_{L^2(0,T,H_\omega)}} \quad \text{and} \quad \psi_m = \frac{\theta_m}{\alpha_m}, \tag{2.62}$$

where $H_\omega = \{u;\ \ u \in (L^2(\omega))^n;\ \operatorname{div} u = 0 \ \text{ and } \ u\cdot\nu = 0 \ \text{ on } \ \Gamma\}$, then we obtain

$$\|\psi_m\|_{L^2(0,T,H_\omega)} = 1. \tag{2.63}$$

It is easy to see that

$$E_{\psi_m} = \frac{1}{\alpha_m^2} E_{\theta_m} \tag{2.64}$$

From Proposition 2.9 we have

$$E_{\psi_m}(0) \leq \int_0^T \int_\omega |\psi_m'|^2 + |\psi_m|^2 dx dt.$$

then, by (2.61), (2.62) and (2.63)

$$E_{\psi_m}(0) \leq L. \tag{2.65}$$

Therefore, there exist subsequences of the $\{\psi_m^0\}, \{\psi_m^1\}$, denoted in the same way, such that

$$\psi_m^0 \rightharpoonup \psi^0 \ \text{ in } V,$$
$$\psi_m^1 \rightharpoonup \psi^1 \ \text{ in } H.$$

Since $E_{\psi_m}(t) = E_{\psi_m}(0) \leq L$ we conclude that there exist subsequences of $\{\psi_m\}$, denoted in the same way, such that

$$\{\psi_m\} \ \text{ is bounded in } \ L^\infty(0,T;V), \tag{2.66}$$
$$\{\psi_m'\} \ \text{ is bounded in } \ L^\infty(0,T;H). \tag{2.67}$$

From (2.66) and (2.67), we infer that there exist subsequences, denoted in the same way, such that

$$\psi_m \overset{*}{\rightharpoonup} \psi \ \text{ (weak star) in } \ L^\infty(0,T;V), \tag{2.68}$$
$$\psi_m' \overset{*}{\rightharpoonup} \psi' \ \text{ (weak star) in } \ L^\infty(0,T;H). \tag{2.69}$$

Since $V \hookrightarrow H$ is compact, then from Theorem 5.1 due to J. L. Lions [24], we have

$$\psi_m \to \psi \quad \text{in } L^2(0,T,H). \tag{2.70}$$

From (2.61) and (2.69) we have that

$$\psi'(x,t) = 0 \quad \text{in } \omega \times (0,T) \tag{2.71}$$

and ψ is independent of t in ω.

From the above convergence results, we have that ψ is solution of problem (2.9) with initial data $\{\psi^0, \psi^1\} \in V \times H$. In order to achieve a contradiction it is enough to prove that $E_\psi(0) = 0$. Indeed, since $E_\psi(t) = E_\psi(0)$, we have that $|\psi'|_H^2 + \|\psi\|_V^2 = 0$ hence $\psi = 0$ in Q contradicting (2.63) and (2.70). In order to prove this, we consider the following system

$$\begin{cases} \xi'' + A\xi = 0 \ \text{ in } Q, \\ \xi(0) = -\psi^1, \ \ \xi'(0) = A\psi^0 \ \text{ in } \Omega, \end{cases} \tag{2.72}$$

with $\{-\psi^1, A\psi^0\} \in H \times V'$, where A is the Stokes operator.

Taking $v(x,t) = \psi^0(x) - \int_0^T \xi(x,s)ds$, then v solves (2.9) with initial dates $\{\psi^0, \psi^1\} \in V \times H$.

Therefore, from the uniqueness of solutions of (2.9) , we have that

$$v = \psi \tag{2.73}$$

and by (2.71) we have that

$$\xi \equiv 0 \quad \text{in } \omega \times (0,T). \tag{2.74}$$

Let us show that $\xi \equiv 0$ in $\Omega \times (0,T)$. Indeed, applying the curl operator in (2.72), we that $u = \operatorname{curl} \xi$ satisfies

$$\begin{cases} u'' - \Delta u = 0 \ \text{ in } Q, \\ u = 0 \ \text{ in } \omega \times (0,T). \end{cases} \tag{2.75}$$

Then by Holmgren's uniqueness theorem, see [25, p.62] we deduce that $u = 0$ in $\Omega \times (0,T)$, that is, $\operatorname{curl} \xi = 0$, therefore there exists a functional $\varphi = \varphi(x,t)$ such that

$$\xi = \nabla \varphi \quad \text{in } Q. \tag{2.76}$$

Hence by $(2.72)_2$, we have

$$\Delta \varphi = 0 \quad \text{in } Q. \tag{2.77}$$

By (2.74) and (2.76) we have that

$$\varphi = f(t) \quad \text{in } \omega \times (0,T). \tag{2.78}$$

Then, by (2.77), (2.78) and unique continuation for the Laplace equation we infer that

$$\varphi = f(t) \quad \text{in } Q.$$

Therefore

$$\xi = \nabla \varphi = 0 \quad \text{in } Q.$$

Hence, we have that $\psi^0 = \psi^1 = 0$. Then, $E_\psi(0) = 0$ as we desired to prove. □

Remark 2.11. From Proposition 2.9 and Proposition 2.10 we obtain the inequality

$$E_\theta(0) \leq C \int_0^T \int_\omega |\theta'|^2 dx dt$$

which proves (2.18) and finishes the proof of Theorem 2.4. □

3 Uniform decay rate

3.1 Wellposedness

Let us consider the following problem

$$\begin{cases} u'' - \Delta u + a(x) g(u') = -\nabla p \ \text{ in } \Omega \times (0,\infty), \\ \operatorname{div} u = 0 \ \text{ in } \Omega \times (0,\infty), \\ u = 0 \ \text{ on } \Sigma, \\ u(0) = u^0, \ \ u'(0) = u^1 \ \text{ in } \Omega. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $u^0 \in W = V \cap (H^2(\Omega))^n$, $u^1 \in V$ and assume that hypotheses (1.5), (1.6) and (1.7) hold. Then, there exists a unique function $u : Q \to \mathbb{R}^n$ such that*

$$u \in L^\infty(0,\infty; W), \quad u' \in L^\infty(0,\infty; V) \quad \textit{and} \quad u'' \in L^\infty(0,\infty; H);$$

$$u'' - \Delta u + a(x) g(u') = -\nabla p \quad \textit{in } (\mathcal{D}'(Q))^n;$$

$$\operatorname{div} u = 0;$$

$$u(0) = u^0, \quad u'(0) = u^1 \quad \textit{in } \Omega.$$

Proof. Let $\mathcal{P}$ be the orthogonal projector in $(L^2(\Omega))^n$ onto H and consider the problem

$$\begin{cases} u'' + Au + Bu' = 0, \\ u(0) = u^0, \ \ u'(0) = u^1, \end{cases} \tag{3.2}$$

where the operator A is the Stokes operator, that is,

$$\begin{aligned} A : D(A) \subset V &\longrightarrow V' \\ u &\longmapsto Au = \mathcal{P}(-\Delta u) \end{aligned}$$

is defined by

$$\langle Au, v\rangle_{V',V} = \int_\Omega \nabla u : \nabla v dx, \qquad \forall\, v \in V$$

with $D(A) = \{v \in V; Av \in H\}$.

We have by [7, Proposition IV.5.9] that $D(A) = V \cap (H^2(\Omega))^n$.

Observe that since $v \in V \subset H$ then by integration by parts

$$\int_\Omega \nabla u : \nabla v dx = \sum_{i=1}^n \int_\Omega \nabla u_i \nabla v_i dx = \sum_{i=1}^n \left[\int_\Omega -\Delta u_i v_i dx + \int_\Gamma \frac{\partial u_i}{\partial \nu_i} v_i d\Gamma \right],$$

however $\int_\Gamma \frac{\partial u_i}{\partial \nu_i} v_i d\Gamma = 0$ because $v_i \in H_0^1(\Omega)$.

Therefore

$$\langle Au, v\rangle_{V',V} = \int_\Omega \nabla u : \nabla v dx = \sum_{i=1}^n \langle -\Delta u_i, v_i\rangle_{H^{-1},H_0^1}. \tag{3.3}$$

Moreover the operator A mentioned above is linear and maximal monotone in $V \times V'$.

To show that A is maximal, define

$$b(u,v) = \int_\Omega \nabla u : \nabla v + u \cdot v dx \quad \text{for all } u, v \in V.$$

Thus b is a form bilinear, continuous and coercive in $V \times V$, so we can define an operator $I + A$ from V to V' by

$$\langle (I + A)u, v\rangle_{V',V} = \int_\Omega \nabla u : \nabla v + u \cdot v dx$$

where I is the identity operator. From the Lax–Milgram theorem this operator is an isomorphism from V onto V'. Therefore A is maximal monotone.

Since I is continuous and monotone then by [5, Corollary 1.3], we conclude that $I + A$ is maximal monotone in $V \times V'$.

The operator B is defined by

$$\begin{aligned} B : V &\longrightarrow V' \\ u &\longmapsto Bu = \mathcal{P}(a(x)g(u(x))). \end{aligned}$$

Note that if $v \in V \subset H$, then $v \in (L^2(\Omega))^n$. Thus $ag(v) \in (L^2(\Omega))^n$, so $\mathcal{P}(ag(v)) \in H$ and therefore $D(B) = \{v \in V; Bv \in H\}$.

Also note that since the projection is self-adjoint and $v \in V \subset H$, we have

$$\langle Bu, v\rangle_{V',V} = (Bu, v) = (\mathcal{P}(ag(u)), v) = (ag(u), \mathcal{P}(v)) = (ag(u), v). \tag{3.4}$$

The operator B is monotone. Indeed, since $\mathcal{P}$ is linear and self-adjoint we have

$$\begin{aligned}\langle Bu - Bv, u - v\rangle_{V',V} &= (Bu - Bv, u - v)\\ &= (\mathcal{P}(ag(u)) - \mathcal{P}(ag(v)), u - v)\\ &= (\mathcal{P}(ag(u) - ag(v)), u - v)\\ &= (ag(u) - ag(v), \mathcal{P}(u - v))\\ &= (ag(u) - ag(v), u - v)\\ &= \sum_{i=1}^{n}\int_{\Omega} a(x)(g_i(u_i(x)) - g_i(v_i(x)))(u_i(x) - v_i(x))dx \geq 0\end{aligned} \tag{3.5}$$

because a is nonnegative and g_i is monotonic by hypothesis. Then B is monotone.

We claim that B is hemicontinuous. In fact,

Take $u, v \in D(B)$, $t_m \in \mathbb{R}$ such that $t_m \to 0$ when $m \to \infty$. Note that for all $w \in V$

$$\begin{aligned}\lim_{m\to\infty}\langle B(u + t_m v), w\rangle_{V',V} &= \lim_{m\to\infty}(\mathcal{P}(ag(u + t_m v)), w)\\ &= \lim_{m\to\infty}(ag(u + t_m v)), w)\\ &= \sum_{i=1}^{n}\lim_{m\to\infty}\int_{\Omega} a(x)(g_i(u_i(x) + t_m v_i(x))w_i(x)dx.\end{aligned} \tag{3.6}$$

Let $f_{m_i} = a(x)g_i(u_i(x) + t_m v_i(x))w_i(x)$. Thus if $|u_i(x) + t_m v_i(x)| \geq 1$ we have by (1.7)

$$\begin{aligned}|f_{m_i}(x)| &= |a(x)g_i(u_i(x) + t_m v_i(x))||w_i(x)|\\ &\leq K\|a\|_\infty |u_i(x) + t_m v_i(x)||w_i(x)|\\ &\leq K\|a\|_\infty [|u_i(x)||w_i(x)| + c_1|v_i(x)||w_i(x)|]\\ &\leq c_2|u_i(x)||w_i(x)| + c_3|v_i(x)||w_i(x)|\end{aligned}$$

almost everywhere in Ω, where c_1 is such that $|t_m| \leq c_1$.

Since $u_i(x), v_i(x), w_i(x) \in L^2(\Omega)$, it results that $f_{m_i} \in L^1(\Omega)$, for all $m \in \mathbb{N}$.

In the same way we obtain $f_{m_i} \in L^1(\Omega)$, for all $m \in \mathbb{N}$, if $|u_i(x) + t_m v_i(x)| \leq 1$ because of the continuity of g_i.

We observe that, since g_i is continuous we have that

$$\lim_{m\to\infty} g_i(u_i(x) + t_m v_i(x))w_i(x) = g_i(u_i(x))w_i(x).$$

Thus, from Lebesgue's dominated convergence theorem we conclude that that

$$\lim_{m\to\infty}\int_{\Omega} a(x)g_i(u_i(x) + t_m v_i(x))w_i(x)dx = \int_{\Omega} a(x)g_i(u_i(x))w_i(x)dx.$$

Therefore since $w \in V \subset H$, we have

$$\begin{aligned}\sum_{i=1}^{n}\lim_{m\to\infty}\int_{\Omega} a(x)g_i(u_i(x) + t_m v_i(x))w_i(x)dx &= \sum_{i=1}^{n}\int_{\Omega} a(x)g_i(u_i(x))w_i(x)dx\\ &= \sum_{i=1}^{n}(a(g_i(u_i)), w_i)_{L^2(\Omega)}\\ &= (a(g(u)), w)\\ &= (a(g(u)), \mathcal{P}(w))\\ &= (\mathcal{P}(a(g(u))), w)\\ &= \langle Bu, w\rangle_{V',V}.\end{aligned} \tag{3.7}$$

Thus by (3.6) and (3.7) we obtain

$$\lim_{m\to\infty} \langle B(u+t_m v), w\rangle_{V',V} = \langle Bu, w\rangle_{V',V}$$

and therefore B is hemicontinuous.

Moreover, we have that $I+A+B$ is coercive. In fact, using condition (1.7), then

$$\begin{aligned}\langle u_m + Au_m + Bu_m, u_m\rangle_{V',V} &= \langle u_m, u_m\rangle_{V',V} + \langle Au_m, u_m\rangle_{V',V} + \langle Bu_m, u_m\rangle_{V',V} \\ &\geq (u_m, u_m) + \sum_{i=1}^{n}\int_\Omega |\nabla u_{m_i}|^2 dx \\ &= |u_m|^2 + |\nabla u_m|^2 \\ &= |u_m|^2 + \|u_m\|^2 \\ &\geq \|u_m\|^2.\end{aligned} \tag{3.8}$$

We can reformulate the problem (3.2) to obtain

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t\end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -I \\ A & B\end{bmatrix}}_{\mathbb{A}}\begin{bmatrix} u \\ u_t\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix} \quad \text{in } V\times H. \tag{3.9}$$

Thus we have a matrix operator $\mathbb{A} : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H} = V \times H$, defined by

$$\mathbb{A}\begin{pmatrix} v \\ h\end{pmatrix} = \begin{pmatrix} -h \\ Av + Bh\end{pmatrix}$$

whose domain is $D(\mathbb{A}) = \{(v,h) \in \mathcal{H};\ \ h \in V \text{ and } Av + Bh \in H\}$. We shall prove that $\mathbb{A}$ is maximal monotone in $\mathcal{H}$. Indeed, $\mathbb{A}$ is monotone, since

$$\begin{aligned}&\left(\mathbb{A}\begin{bmatrix} v_1 \\ h_1\end{bmatrix} - \mathbb{A}\begin{bmatrix} v_2 \\ h_2\end{bmatrix}, \begin{bmatrix} v_1 \\ h_1\end{bmatrix} - \begin{bmatrix} v_2 \\ h_2\end{bmatrix}\right)_{\mathcal{H}} \\ &= \left(\begin{bmatrix} -h_1 + h_2 \\ A(v_1 - v_2) + Bh_1 - Bh_2\end{bmatrix}, \begin{bmatrix} v_1 - v_2 \\ h_1 - h_2\end{bmatrix}\right)_{\mathcal{H}} \\ &= -(h_1 - h_2, v_1 - v_2)_V + (A(v_1 - v_2), h_1 - h_2)_H + (Bh_1 - Bh_2, h_1 - h_2)_H \\ &= (Bh_1 - Bh_2, h_1 - h_2)_H \geq 0\end{aligned}$$

because B is monotone.

In order to prove the maximality of $\mathbb{A}$, it is sufficient to prove that $R(I+\mathbb{A}) = \mathcal{H}$, that is, given $(v_0, h_0) \in \mathcal{H}$, we have to show that there exists $(v,h) \in D(\mathbb{A})$ such that

$$(I+\mathbb{A})\begin{bmatrix} v \\ h\end{bmatrix} = \begin{bmatrix} v_0 \\ h_0\end{bmatrix}, \tag{3.10}$$

that is,

$$\begin{cases} v - h = v_0 \\ h + Av + Bh = h_0.\end{cases}$$

Combining the above identities we deduce

$$h + Ah + Bh = h_0 - Av_0. \tag{3.11}$$

Therefore, it is sufficient to prove that $I + A + B$ is maximal monotone in $V \times V'$, that is, we have to show that $R(I + A + B) = V'$. Since B is monotone and hemicontinuous, $I + A$ is maximal monotone em $V \times V'$ and $I + A + B$ is coercive, we conclude by [5, Corollary 1.3] that $(I + A) + B$ is maximal monotone. Then, (3.11) possesses a unique solution $h \in V$. Since $v = v_0 + h$ and $Av + Bh = h_0 - h$, we conclude that $v \in V$ and $Av + Bh \in H$.

Consequently, the system (3.10) has a unique solution $(v, h) \in D(\mathbb{A})$, and, therefore, $\mathbb{A}$ is maximal monotone in $\mathcal{H}$.

Finally, from the above and making use of Theorem 3.1 due to Brezis [8] and given $\{u^0, u^1\} \in D(A) \times V$ there exists a unique $u(t)$ regular solution of problem (3.2) in the class

$$u \in L^\infty(0, \infty; V \cap (H^2(\Omega))^n), \qquad u' \in L^\infty(0, \infty; V), \qquad u'' \in L^\infty(0, \infty; H). \tag{3.12}$$

Now we are going to recover the pressure term of the regular solution. Let $v \in V$ be any time-independent test function. For any $t > 0$ we thus have

$$0 = (u''(t) + Au(t) + Bu'(t), v) = (u''(t), v) + \langle Au(t), v\rangle_{V',V} + \langle Bu'(t), v\rangle_{V',V}.$$

Since $v \in V \subset H$, we obtain as in (3.3)

$$\langle Au(t), v\rangle_{V',V} = \sum_{i=1}^{n} \langle -\Delta u_i, v_i\rangle_{H^{-1},H_0^1}$$

and by (3.4) we have

$$\langle Bu'(t), v\rangle_{V',V} = \sum_{i=1}^{n} \langle a g_i(u_i'), v_i\rangle_{H^{-1},H_0^1}.$$

It follows

$$0 = \sum_{i=1}^{n} \Big\langle \underbrace{u_i''(t) - \Delta u_i(t) + a g_i(u_i'(t))}_{L_i}, v_i \Big\rangle_{H^{-1},H_0^1}.$$

Let

$$L(v) = \sum_{i=1}^{n} \langle L_i, v_i\rangle_{H^{-1},H_0^1},$$

then $L \in (H^{-1}(\Omega))^n$ and in addition $L(v) = 0 \ \ \forall \ \ v \in V$ a.e. in $(0, T)$.

This being true for any $v \in V$, by [7, Theorem IV.2.3] we obtain that there exists a unique $p(t) \in L_0^2(\Omega)$ such that

$$u''(t) - \Delta u(t) + a g(u'(t)) = -\nabla p,$$

that is,

$$L_i = -\frac{\partial p}{\partial x_i} \ \text{ a.e. in } (0, T).$$

Therefore

$$L_i(v_i) = \Big\langle -\frac{\partial p}{\partial x_i}, v_i \Big\rangle \quad \text{for all } v_i \in H_0^1(\Omega) \text{ a.e. in } (0, T)$$

thus

$$\langle L_i(v_i), \theta_i\rangle = \Big\langle \Big\langle -\frac{\partial p}{\partial x_i}, v_i \Big\rangle, \theta_i \Big\rangle \ \text{ for all } \ \theta_i \in \mathcal{D}(0, T), \ \text{ for all } \ v_i \in H_0^1(\Omega).$$

In particular,

$$\langle\langle u_i''(t) - \Delta u_i(t) + a(x) g_i(u_i'(t)), v_i\rangle, \theta_i\rangle = \Big\langle \Big\langle -\frac{\partial p}{\partial x_i}, v_i \Big\rangle, \theta_i \Big\rangle \quad \text{for all } \theta_i \in \mathcal{D}(0, T), \quad \text{for all } v_i \in \mathcal{D}(\Omega).$$

Since $\psi_i = v_i\theta_i \in \mathcal{D}(\Omega);\ \theta_i \in \mathcal{D}(0,T)$ is dense in $\mathcal{D}(Q)$ we have

$$u_i'' - \Delta u_i + a(x)g(u_i') = -\frac{\partial p}{\partial x_i} \quad \text{in } \mathcal{D}'(Q),$$

where $p(t) \in L^2(\Omega)$ a.e. in $(0,T)$, therefore

$$u'' - \Delta u + a(x)g(u') = -\nabla p \quad \text{in } (\mathcal{D}'(Q))^n.$$

In addition, since $u(t) \in V \cap (H^2(\Omega))^n$ then $\Delta u(t) \in H \subset (L^2(\Omega))^n, ag(u') \in (L^2(\Omega))^n$ and $u''(t) \in H \subset (L^2(\Omega))^n$ we have from Cattabriga's lemma in [9] that $p(t) \in H^1(\Omega)$. □

Theorem 3.2. *Let $u^0 \in V$, $u^1 \in H$ and assume that hypothesis (1.5), (1.6) and (1.7) hold. Then, there exists a unique weak solution u of problem (3.1) such that*

$$u \in C([0,\infty);V) \cap C^1([0,\infty);H).$$

Proof. Proceeding analogously to Theorem 3.1, we prove that $\mathbb{A}$ is maximal monotone in $\mathcal{H} = V \times H$. Thus given $\{u^0,u^1\} \in V \times H$, by [8, Theorem 3.4] there exists an unique $u(t)$ weak solution of problem (3.2) in the class

$$u \in C([0,\infty);V) \cap C^1([0,\infty);H). \tag{3.13}$$

To determine the pressure term of the weak solution, we note that $u'' \in L^1(0,T;V')$. Then

$$|\langle u''(t), v\rangle| \leq \|u''(t)\|_{V'}\|v\|.$$

Let us consider $\{\varphi_m\}_{m\in\mathbb{N}}$ a sequence of functions in $\mathcal{V}$ such that $\varphi_m \to 0$ in $(\mathcal{D}(\Omega))^n$. Then we have $|\langle u''(t), \varphi_m\rangle| \to 0$.

Thus, $u''(t)$ is a linear and continuous form in $\mathcal{V}$ with the norm of $(\mathcal{D}(\Omega))^n$. Then by the Hahn–Banach theorem $u''(t)$ can be continuously extended to $(\mathcal{D}(\Omega))^n$, which will still be denoted by $u''(t)$. In this sense, $L = u''(t) - \Delta u(t) + a(x)g(u'(t))$ is a linear and continuous form in $\mathcal{D}(\Omega)$ a.e. in $[0,T)$. Similar to the statement previously made, we have $L \in (\mathcal{D}'(Q))^n$ and $L(\varphi) = 0$ in $(\mathcal{D}'(0,T))^n$ $\forall \varphi \in \mathcal{V}$. From above and Rham's theorem (cf. J. L. Lions [24] and R. Teman [39]), we have

$$L = -\nabla p, \qquad p \in \mathcal{D}'(Q)$$

that is,

$$u'' - \Delta u + a(x)g(u') = -\nabla p, \qquad p \in \mathcal{D}'(Q).$$

□

3.2 Stability result

The energy related to problem (3.1) is given by

$$E(t) = \frac{1}{2}\|u(t)\|_V^2 + \frac{1}{2}|u'(t)|_H^2. \tag{3.14}$$

In addition, multiplying the regular solution of problem (3.1) by u', performing integrations by parts having in mind that $\operatorname{div} u' = 0$ in $\Omega \times (0,T)$ we deduce the following identity of the energy:

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_\Omega a(x)g(u')\cdot u'\,dxdt, \qquad 0 \leq t_1 \leq t_2 < \infty. \tag{3.15}$$

By standard density arguments the above identity (3.15) remains valid for weak solutions of (3.1) as well.

Before presenting our stability result, we will define some needed functions. For this purpose, we are following ideas first introduced in the literature by Lasiecka and Tataru [20]. For the reader's comprehension, we will repeat them briefly. Let $h : \mathbb{R} \to \mathbb{R}$ be a concave, strictly increasing function with $h(0) = 0$, and such that

$$h(s_i g_i(s_i)) \geq |s_i|^2 + |g_i(s_i)|^2 \quad \text{for } |s_i| < 1,\ i = 1, \ldots, n, \tag{3.16}$$

from which we deduce having in mind that $h(s \cdot g(s)) \geq h(s_i g_i(s_i))$, since h is increasing,

$$n\, h(s \cdot g(s)) \geq |s|^2 + |g(s)|^2 \quad \text{for } |s| < 1. \tag{3.17}$$

Note that such a function can be straightforwardly constructed, given the hypotheses of g given in (1.6) and (1.7). With this function, we define

$$r(\cdot) = h\left(\frac{\cdot}{\text{meas}(Q)}\right). \tag{3.18}$$

As r is monotone increasing, $cI + r$ is invertible for all $c \geq 0$. For L a positive constant, we then set, respectively,

$$z(x) = (cI + r)^{-1}(Lx), \qquad L := (C\,\text{meas}(Q))^{-1}, \tag{3.19}$$

where C ia a positive constant that will be determined later.

The function z is easily seen to be positive, continuous, and strictly increasing with $z(0) = 0$. Finally, let

$$q(x) = x - (I + z)^{-1}(x). \tag{3.20}$$

We can now proceed to state our stability result.

Theorem 3.3 (Uniform decay rates)**.** *Assume that hypotheses* (1.5)–(1.7) *and* (3.17) *hold. Let u be the weak solution of problem* (3.1)*. With the energy $E(t)$ as defined in* (3.14)*, there exists a $T_0 > 0$ such that*

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right) \qquad \forall\, t > T_0 \tag{3.21}$$

with $\lim_{t\to\infty} S(t) = 0$, where the contraction semigroup $S(t)$ is the solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \qquad S(0) = E(0), \tag{3.22}$$

where q is given in (3.20)*. Here the constant c (from definition* (3.19)*) is $c \equiv \frac{k^{-1}+K}{\text{meas}(Q)(1+\|a\|_\infty)}$.*

Proof. Writing $u = v + w$ we have that (3.1) is equivalent to

$$\begin{cases} v'' - \Delta v = -\nabla p \ \text{ in } Q, \\ \text{div}\, v = 0 \ \text{ in } Q, \\ v = 0 \ \text{ on } \Sigma, \\ v(0) = u^0, \ \ v'(0) = u^1 \ \text{ in } \Omega, \end{cases} \quad \text{and} \quad \begin{cases} w'' - \Delta w = -a(x)g(u') \ \text{ in } Q, \\ \text{div}\, w = 0 \ \text{ in } Q, \\ w = 0 \ \text{ on } \Sigma, \\ w(0) = 0, \ \ v'(0) = 0 \ \text{ in } \Omega. \end{cases} \tag{3.23}$$

From (1.5), Remark 2.11 and from Lemma 2.2 due to Cavalcanti et al. [10], we have

$$\begin{aligned} E_u(T) \le E_u(0) = E_v(0) &\le c_1 \int_0^T \int_\omega |v'|^2 dxdt \\ &\le c_2 \int_0^T \int_\omega |u'|^2 dxdt + c_3 \int_0^T \int_\omega |w'|^2 dxdt \\ &\le \tilde{c} \int_0^T \int_\Omega a(x) \left(|u'|^2 + |g(u')|^2 \right) dxdt \end{aligned} \tag{3.24}$$

where $\tilde{c}$ depends on T.

Let

$$\Sigma_\alpha = \{(t,x) \in Q; |u_i'| > 1 \text{ a.e.}\},$$
$$\Sigma_\beta = Q \setminus \Sigma_\alpha.$$

Then using hypothesis (1.7), we obtain

$$\begin{aligned} \int_{\Sigma_\alpha} a(x)\left(|g(u')|^2 + |u'|^2\right) d\Sigma_\alpha &= \sum_{i=1}^n \int_{\Sigma_\alpha} a(x)\left(|g_i(u_i')|^2 + |u_i'|^2\right) d\Sigma_\alpha \\ &\le (k^{-1} + K) \sum_{i=1}^n \int_{\Sigma_\alpha} a(x) g_i(u_i') u_i' d\Sigma_\alpha \\ &= (k^{-1} + K) \int_{\Sigma_\alpha} a(x) g(u') \cdot u' d\Sigma_\alpha. \end{aligned} \tag{3.25}$$

Moreover, from (3.17) and from the fact that h is concave and increasing, having in mind that $a(x) \le \|a\|_\infty + 1$ and $\frac{a(x)}{1+\|a\|_\infty} < a(x)$ we deduce that

$$\begin{aligned} \int_{\Sigma_\beta} a(x)\left(|g(u')|^2 + |u'|^2\right) d\Sigma_\beta &\le n \int_{\Sigma_\beta} a(x) h(g(u') \cdot u') d\Sigma_\beta \\ &= n \int_{\Sigma_\beta} (1 + \|a\|_\infty) \frac{a(x)}{1 + \|a\|_\infty} h(g(u') \cdot u') d\Sigma_\beta \\ &\le n \int_{\Sigma_\beta} (1 + \|a\|_\infty) h\left(\frac{a(x)}{1 + \|a\|_\infty}(g(u') \cdot u')\right) d\Sigma_\beta \\ &\le n \int_{\Sigma_\beta} (1 + \|a\|_\infty) h(a(x) g(u') \cdot u') d\Sigma_\beta. \end{aligned} \tag{3.26}$$

By Jensen's inequality

$$\begin{aligned} & n(1 + \|a\|_\infty) \int_{\Sigma_\beta} h(a(x) g(u') \cdot u') d\Sigma_\beta \\ & \qquad \le n(1 + \|a\|_\infty) \operatorname{meas}(Q) h\left(\frac{1}{\operatorname{meas}(Q)} \int_Q a(x) g(u') \cdot u' dQ\right) \\ & \qquad = n(1 + \|a\|_\infty) \operatorname{meas}(Q) r\left(\int_Q a(x) g(u') \cdot u' dQ\right), \end{aligned}$$

where $r(s) = h(\frac{s}{\operatorname{meas}(Q)})$ is defined in (3.18). Thus

$$\begin{aligned} \int_Q a(x)\left(|g(u')|^2 + |u'|^2\right) dQ \le (k^{-1} + K) &\int_{\Sigma_\alpha} a(x) g(u') \cdot u' d\Sigma_\alpha \\ &+ n(1 + \|a\|_\infty) \operatorname{meas}(Q) r\left(\int_Q a(x) g(u') \cdot u' dQ\right). \end{aligned} \tag{3.27}$$

Splicing together (3.24) and (3.27), we have

$$E(T) \le (1+\|a\|_\infty)C\left[\frac{K_0}{(1+\|a\|_\infty)}\int_Q a(x)g(u')\cdot u'dQ + \text{meas}(Q)r\left(\int_Q a(x)g(u')\cdot u'dQ\right)\right] \quad (3.28)$$

where $K_0 = K^{-1} + K$ and $C = C(\tilde{c}, n)$. Setting

$$L = \frac{1}{C\,\text{meas}(Q)(1+\|a\|_\infty)},$$
$$c = \frac{K_0}{\text{meas}(Q)(1+\|a\|_\infty)},$$

we obtain

$$z[E(T)] \le \int_Q a(x)g(u')\cdot u'dQ = E(0) - E(T), \quad (3.29)$$

where the function z is as defined in (3.19). We recall that the above inequality holds for a fixed T large enough, say, for all $T > T_0$, for some $T_0 > 0$. To finish the proof of Theorem 3.3, we invoke the following result due to Lasiecka and Tataru [20].

Lemma A. Let z be a positive, increasing function such that $z(0) = 0$. Since z is increasing, we can define an increasing function $q, q(x) = x - (I+z)^{-1}(x)$. Consider a sequence s_m of positive numbers which satisfies

$$s_{m+1} + z(s_{m+1}) \le s_m.$$

Then $s_m \le S(m)$, where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \qquad S(0) = s_0.$$

Moreover, if $z(x) > 0$ for $x > 0$, then $\lim_{t\to\infty} S(t) = 0$.

With this result in mind, we replace T (resp. 0) in (3.29) with $m(T+1)$ (resp. mT) to obtain

$$E(m(T+1)) + z(E(m(T+1))) \le E(mT) \quad \text{for } m = 0, 1, \ldots \quad (3.30)$$

Applying Lemma A with $s_m = E(mT)$, thus results in

$$E(mT) \le S(m), \quad \text{for } m = 0, 1, \ldots \quad (3.31)$$

Finally, using the dissipativity of $E(t)$, we have for $t = mT + \tau, 0 \le \tau \le T$,

$$E(t) \le E(mT) \le S(m) = S\left(\frac{t-\tau}{T}\right) \le S\left(\frac{t}{T} - 1\right) \quad \text{for } t > T, \quad (3.32)$$

where we have used above the fact that $S(\cdot)$ is dissipative. The proof of Theorem 3.3 is now complete. □

We shall give, for illustration, several examples of explicit decay rate estimates.

Example 3.4. Initially assuming that $k|s_i| \le |g_i(s_i)| \le K|s_i|$ for all $s_i \in \mathbb{R}$ and some positive constants k and K or if $g_i(s_i) = c\,s_i$ the exponential decay holds. Indeed, from (3.24) we deduce that

$$E(T) \le c\int_0^T \int_\Omega a(x)|u'|^2\,dxdt, \quad \text{for all } T > T_0,$$

with jointly with the identity of the energy

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2} \int_\Omega a(x)|u'|^2 \, dxdt,$$

allows to conclude that $E(T) \leq \left(\frac{c}{1+c}\right) E(0)$ for all $T > T_0$, and, therefore, since we are dealing with an autonomous system, $E(2T) \leq \left(\frac{c}{1+c}\right) E(T) \leq \left(\frac{c}{1+c}\right)^2 E(0)$, and, by iteration, $E(nT) \leq \left(\frac{c}{1+c}\right)^n E(0)$ for all $n \in \mathbb{N}$ and $T > T_0$ which allows to conclude the desired exponential decay.

If p is constant everywhere, we can give a wide assortment of examples which we borrowed from [11, Section 8, Corollary 8.1 and Corollary 8.2] by following ideas firstly introduced in [1,2]. Indeed, in this specific case denoting

$$E_i(t) := \frac{1}{2}\left(\|u_i(t)\|^2_{H_0^1(\Omega)} + \|u_i'(t)\|^2_{L^2(\Omega)}\right), \qquad i = 1, \dots, n$$

each portion of the full energy verifies the identity of the energy

$$E_i(t_2) - E_i(t_1) = -\int_{t_1}^{t_2} \int_\Omega a(x) g_i(u_i') u_i' \, dxdt, \qquad i = 1, \dots, n,$$

and, furthermore, for each $i = 1, \dots, n$ the wave equation is in place, namely

$$\begin{cases} u_i'' - \Delta u_i + a(x) g_i(u_i') = \nabla p = 0, \text{ in } \Omega \times (0, \infty), \\ u_i = 0 \text{ on } \Gamma \times (0, \infty), \\ u_i(0) = u_i^0, \ u_i'(0) = u_i^1. \end{cases} \tag{3.33}$$

As a consequence, we deduce, for the purely wave equation, as before that

$$E_i(T) \leq c \int_0^T \int_\Omega (|u_i'|^2 + |g_i(u_i')|^2) \, dxdt,$$

which implies that decay of each $E_i(T)$ is driven by the Corollary 8.1 and Corollary 8.2 due to [11] and the decay associate with the full energy is given by

$$E(t) := \sum_{i=1}^n E_i(t) \leq \sum_{i=1}^n S_i \left(\frac{t}{T_0} - 1\right), \qquad \forall t > T_0. \tag{3.34}$$

Let us see some examples:

Example 3.5. We consider $g_i(s_i) = s_i^p$, $p > 1$ at the origin. Since the function $s_i^{\frac{p+1}{2}}$ is convex for $p \geq 1$ we will be solving

$$S_{i,t} + S_i^{\frac{p+1}{2}} = 0. \tag{3.35}$$

This equation can be integrated directly, of course. However, for sake of illustration of the general formula we find

$$G(s, S_0) = \int_{\sqrt{S_0}}^{\sqrt{s}} u^{-p} du = \frac{1}{1-p}\left[s^{\frac{-p+1}{2}} - S_0^{\frac{-p+1}{2}}\right].$$

From here $G^{-1}(t) = \left[S_0^{\frac{-p+1}{2}} - t(1-p)\right]^{\frac{2}{-p+1}}$. Thus

$$E_i(t) \leq C(E_i(0)) \left[E_i(0)^{\frac{-p+1}{2}} + t(p-1)\right]^{\frac{2}{-p+1}}.$$

If $g_i(s_i) = s_i^p$ for all $i = 1, \dots, n$, the decay of $E(t)$ would be the sum of decays of the same type. However, if $g_i(s_i) = s_i^{p_i}$ the decay would be the worst one given by the largest p_i.

Example 3.6. We take $g_i(s_i) = s_i^3 e^{-\frac{1}{s_i^2}}$ for s_i at the origin. Since the function $s_i^2 e^{-\frac{1}{s_i}}$ is convex in the neighbourhood of the origin we solve

$$S_{i,t} + S_i^2 e^{-\frac{1}{S_i}} = 0. \tag{3.36}$$

In this case $G(S, S_0) = -1/2\big[e^{-\frac{1}{S}} - e^{-\frac{1}{S_0}}\big]$ and $G^{-1}(t, S_0)) = \big[\ln(e^{\frac{1}{S_0}} - 2t)\big]^{-1}$. Hence

$$E_i(t) \le C(E_i(0)) \left[\ln\left(e^{\frac{1}{E_i(0)}} + t\right)\right]^{-1}.$$

Example 3.7. We consider $g_i(s_i) = s_i|s_i|e^{-\frac{1}{|s_i|}}$ for s_i near zero . Since the function $s_i^{3/2} e^{-\frac{1}{\sqrt{s_i}}}$ is convex on $[0, s_0]$ for some small s_0 we are led to differential equation

$$S_{i,t} + S_i^{3/2} e^{-\frac{1}{\sqrt{S_i}}} = 0. \tag{3.37}$$

Function $G(S, S_0)$ is given by $G(S, S_0) = -\big[e^{\frac{1}{\sqrt{S}}} - e^{\frac{1}{\sqrt{S_0}}}\big]$. Hence

$$G^{-1}(t, S_0) = \frac{1}{\ln^2\left[e^{\frac{1}{\sqrt{S_0}}} - t\right]}$$

and

$$E_i(t) \le C(E_i(0)) \frac{1}{\ln^2\left[e^{\frac{1}{\sqrt{E_i(0)}}} + \frac{1}{2}t\right]}.$$

Example 3.8. We take $g_i(s_i) = |s_i|^{\theta-1}s_i, 0 < \theta < 1$. In this case the analysis is identical to the case of example 1 since $g^{-1}(s_i) = s_i^{\frac{1}{\theta}}, s_i > 0$ and $\frac{1}{\theta} > 1$. Thus the decay rates in that case become

$$E_i(t) \le C(E_i(0)) \left[E_i(0)^{\frac{-1+\theta}{2\theta}} + t\frac{1-\theta}{\theta}\right]^{\frac{2\theta}{\theta-1}}.$$

Summarizing we can choose $g_i(s_i)$ as the above examples so that for each one we have a decay $E_i(t)$ but the total one $E(t) := \sum_{i=1}^n E_i(t)$ will be driven by the worst one.

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