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Periodic solutions for a delay model of plankton allelopathy on time scales†

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Abstract

In this paper, a delay model of plankton allelopathy is investigated. By using the coincidence degree theory, sufficient conditions for existence of periodic solutions are obtained. The presented criteria improve and extend previous results in the literature.
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1 Introduction

Recently, Song and Chen proposed a nonautonomous system that arises in plankton allelopathy involving discrete time delays and periodic environmental factors in [1] as follows,

$$\begin{cases} \dot{N}_1(t) = N_1[k_1(t) - \alpha_1(t)N_1(t) - \beta_{12}(t)N_2(t) - \gamma_1(t)N_1(t)N_2(t-\tau_2(t))], \\ \dot{N}_2(t) = N_2[k_2(t) - \alpha_2(t)N_2(t) - \beta_{21}(t)N_1(t) - \gamma_2(t)N_2(t)N_1(t-\tau_1(t))], \end{cases} \tag{1}$$

where $N_1(t)$ and $N_2(t)$ stand for the population density of two competing species, γ_1 and γ_2 are the rates of toxic inhibition of the first species by the second and vice versa, respectively. All the coefficients and time delays are positive $\omega-$periodic functions.

However, the following discrete time model is more appropriate when the populations have non–overlapping generations [2],

$$\begin{cases} N_1(n+1) = N_1(n)\exp\{k_1(n) - \alpha_1(n)N_1(n) \\ \qquad\qquad -\beta_{12}(n)N_2(n) - \gamma_1(n)N_1(n)N_2(n-\tau_2(n))\}, \\ N_2(n+1) = N_2(n)\exp\{k_2(n) - \alpha_2(n)N_2(n) \\ \qquad\qquad -\beta_{21}(n)N_1(n) - \gamma_2(n)N_2(n)N_1(n-\tau_1(n))\}, \end{cases} \tag{2}$$

By using the coincidence degree theory, existences of periodic solutions for system (1) and (2) were studied in [1–2]. It is obvious that the results and approaches are

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astonishingly similar. To unify these two models, we consider the dynamic equations on time scales motivated by the new idea of Stefan Hilger in [3–4],

$$\begin{cases} x_1^{\Delta}(t) = r_1(t) - \alpha_1(t)e^{x_1(t)} - \beta_{12}(t)e^{x_2(t)} - \gamma_1(t)e^{x_1(t)+x_2(t-\tau_2(t))}, \\ x_2^{\Delta}(t) = r_2(t) - \alpha_2(t)e^{x_2(t)} - \beta_{21}(t)e^{x_1(t)} - \gamma_2(t)e^{x_2(t)+x_1(t-\tau_1(t))}, \end{cases} \tag{3}$$

where $r_i(t), \alpha_i(t), \beta_{ij}(t), \gamma_i$ and $\tau_i(t)$ $(i, j = 1, 2; i \neq j)$ are $rd-$continuous positive $\omega-$periodic functions on time scale $\mathbb{T}$. Set $N_i(t) = e^{x_i(t)}$, $i = 1, 2$, then system (3) can be reduced to (1) and (2) when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively.

The main purpose of this paper is to explore the periodic solutions of system (3) by using coincidence degree theory and we refer the reader to [5–6]. Moreover, with the help of new inequality on time scales [7], we can find the sharp priori bounds and improve existence criteria for periodic solutions. In next section, some preliminary results are presented. In Section 3, existence of periodic solutions is established.

2 Preliminaries

For convenience, we first present some basic definitions and lemmas about time scales and the continuation theorem of the coincidence degree theory; more details can be found in [3, 8]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. Throughout this paper, we assume that the time scale $\mathbb{T}$ is unbounded above and below, such as $\mathbb{R}$, $\mathbb{Z}$ and $\bigcup_{k\in\mathbb{Z}}[2k, 2k+1]$. The following definitions and lemmas about time scales are from [3].

Definition 2.1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, +\infty)$ are defined, respectively, by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \mu(t) = \sigma(t) - t$. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

Definition 2.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, $f^{\Delta}(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta or Hilger differentiable on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for } r, s \in \mathbb{T}.$$

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T})$.

Lemma 2.4. Every rd-continuous function has an antiderivative.

Lemma 2.5. If $a, b \in \mathbb{T}$, α, $\beta \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T})$,then
(a) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;
(b) if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$;
(c) if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.
Lemma 2.6.([7]) Let t_1, $t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R} \in C_{rd}(\mathbb{T})$ is $\omega-$periodic, then

$$g(t) \leq g(t_1) + \frac{1}{2}\int_k^{k+\omega} |g^\Delta(s)|\Delta s$$

and

$$g(t) \geq g(t_2) - \frac{1}{2}\int_k^{k+\omega} |g^\Delta(s)|\Delta s,$$

the constant factor $\frac{1}{2}$ is the best possible.

For simplicity, we use the following notations throughout this paper. Let $\mathbb{T}$ be ω-periodic, that is $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$,

$$k = \min\{\mathbb{R}^+ \cap \mathbb{T}\}, \quad I_\omega = [k, k+\omega] \cap \mathbb{T}, \quad g^L = \inf_{t\in\mathbb{T}} g(t),$$
$$g^M = \sup_{t\in\mathbb{T}} g(t), \quad \bar{g} = \frac{1}{\omega}\int_{I_\omega} g(s)\Delta s = \frac{1}{\omega}\int_k^{k+\omega} g(s)\Delta s,$$

where $g \in C_{rd}(\mathbb{T})$ is an ω-periodic real function, i.e., $g(t + \omega) = g(t)$ for all $t \in \mathbb{T}$.

Now, we introduce some concepts and a useful result from [8].

Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$, then it follows that $L| \operatorname{Dom} L \cap \ker P : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

Next, we state the Mawhin's continuation theorem, which is a main tool in the proof of our theorem.
Lemma 2.7. Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose

(a) for each $\lambda \in (0, 1)$, every solution u of $Lu = \lambda Nu$ is such that $u \notin \partial\Omega$;

(b) $QNu \neq 0$ for each $u \in \partial\Omega \cap \ker L$ and the Brouwer degree $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.

Then the operator equation $Lu = Nu$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

3 Main Results

Theorem 3.1. If

$$\frac{\bar{\alpha}_i}{\bar{\beta}_{ji}} > \max\left(\frac{\bar{\gamma}_i}{\bar{\gamma}_j}, \frac{\bar{r}_i}{\bar{r}_j} e^{\bar{r}_i \omega}\right), \quad (i, j = 1, 2; i \neq j)$$

then system (3) has at least one ω−periodic solution.

Let $X = Z = \{(u_1, u_2)^T \in C(\mathbb{T}, \mathbb{R}^2) : u_i(t + \omega) = u_i(t), \ i = 1, 2, \forall t \in \mathbb{T}\}, \|(u_1, u_2)^T\| = \sum_{i=1}^{2} \max_{t \in I_\omega} |u_i(t)|, \quad (u_1, u_2)^T \in X(Z)$.

Then X and Z are both Banach spaces when they are endowed with the above norm $\| \cdot \|$.

Let

$$N \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} r_1(t) - \alpha_1 e^{x_1(t)} - \beta_{12}(t) e^{x_2(t)} \\ \qquad -\gamma_1 e^{x_1(t) + x_2(t - \tau_2(t))} \\ r_2(t) - \alpha_2 e^{x_2(t)} - \beta_{21}(t) e^{x_1(t)} \\ \qquad -\gamma_2 e^{x_2(t) + x_1(t - \tau_1(t))} \end{bmatrix},$$

$$L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^\Delta \\ x_2^\Delta \end{bmatrix},$$

$$P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_\kappa^{\kappa+\omega} x_1(t) \Delta t \\ \frac{1}{\omega} \int_\kappa^{\kappa+\omega} x_2(t) \Delta t \end{bmatrix}.$$

Obviously, $\ker L = \{(x_1, x_2)^T \in X : (x_1(t), x_2(t))^T = (h_1, h_2)^T \in \mathbb{R}^2, t \in \mathbb{T}\}, \operatorname{Im} L = \{(x_1, x_2)^T \in Z : \bar{x}_1 = \bar{x}_2 = 0, t \in \mathbb{T}\}, \dim \ker L = 2 = \operatorname{codim} \operatorname{Im} L$. Since $\operatorname{Im} L$ is closed in Z, then L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projections such that $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$. Furthermore, the generalized inverse (of L) $K_P : \operatorname{Im} L \to \ker P \cap \operatorname{Dom} L$ exists and is given by

$$K_P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \int_\kappa^t x_1(s) \Delta s - \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \int_\kappa^t x_1(s) \Delta s \Delta t \\ \int_\kappa^t x_2(s) \Delta s - \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \int_\kappa^t x_2(s) \Delta s \Delta t \end{bmatrix}.$$

Thus,

$$QN \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \big(r_1(t) - \alpha_1 e^{x_1(t)} - \beta_{12}(t) e^{x_2(t)} \\ \qquad -\gamma_1 e^{x_1(t) + x_2(t - \tau_2(t))} \big) \Delta t \\ \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \big(r_2(t) - \alpha_2 e^{x_2(t)} - \beta_{21}(t) e^{x_1(t)} \\ \qquad -\gamma_2 e^{x_2(t) + x_1(t - \tau_1(t))} \big) \Delta t \end{bmatrix},$$

and

$$K_P(I - Q) N \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \int_\kappa^t N_1(s) \Delta s - \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \int_\kappa^t N_1(s) \Delta s \Delta t \\ \qquad - \left(t - \kappa - \frac{1}{\omega} \int_\kappa^{\kappa+\omega} (t - \kappa) \Delta t \right) \bar{N}_1 \\ \int_\kappa^t N_2(s) \Delta s - \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \int_\kappa^t N_2(s) \Delta s \Delta t \\ \qquad - \left(t - \kappa - \frac{1}{\omega} \int_\kappa^{\kappa+\omega} (t - \kappa) \Delta t \right) \bar{N}_2 \end{bmatrix}.$$

Clearly, QN and $K_P(I-Q)N$ are continuous. According to Arzela-Ascoli theorem, it is not difficulty to show that $K_P(I-Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$ and $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on $\bar{\Omega}$.

Now, we shall search an appropriate open bounded subset Ω for the application of the continuation theorem, Lemma 2.7. For the operator equation $Lu = \lambda Nu$, where $\lambda \in (0,1)$, we have

$$\begin{cases} x_1^{\Delta}(t) = \lambda \big(r_1(t) - \alpha_1 e^{x_1(t)} - \beta_{12}(t) e^{x_2(t)} \\ \qquad\qquad -\gamma_1 e^{x_1(t)+x_2(t-\tau_2(t))} \big), \\ x_2^{\Delta}(t) = \lambda \big(r_2(t) - \alpha_2 e^{x_2(t)} - \beta_{21}(t) e^{x_1(t)} \\ \qquad\qquad -\gamma_2 e^{x_2(t)+x_1(t-\tau_1(t))} \big). \end{cases} \tag{4}$$

Assume that $(u_1, u_2)^T \in X$ is a solution of (4) for a certain $\lambda \in (0,1)$. Integrating (4) on both sides from k to $k+\omega$, we obtain

$$\begin{cases} \bar{r}_1 \omega = \int_{\kappa}^{\kappa+\omega} \alpha_1 e^{x_1(t)} \Delta t + \int_{\kappa}^{\kappa+\omega} \beta_{12}(t) e^{x_2(t)} \Delta t \\ \qquad + \int_{\kappa}^{\kappa+\omega} \gamma_1 e^{x_1(t)+x_2(t-\tau_2(t))} \Delta t, \\ \bar{r}_2 \omega = \int_{\kappa}^{\kappa+\omega} \alpha_2 e^{x_2(t)} \Delta t + \int_{\kappa}^{\kappa+\omega} \beta_{21}(t) e^{x_1(t)} \Delta t \\ \qquad + \int_{\kappa}^{\kappa+\omega} \gamma_2 e^{x_2(t)+x_1(t-\tau_1(t))} \Delta t. \end{cases} \tag{5}$$

Since $(x_1, x_2)^T \in X$, there exist $\xi_i, \eta_i \in [k, k+\omega]$, $i = 1,2$, such that

$$x_i(\xi_i) = \min_{t\in[\kappa,\kappa+\omega]} \{x_i(t)\}, \quad x_i(\eta_i) = \max_{t\in[\kappa,\kappa+\omega]} \{x_i(t)\}. \tag{6}$$

From (4) and (5), we have

$$\int_{\kappa}^{\kappa+\omega} \left| x_1^{\Delta}(t) \right| \Delta t < 2\bar{r}_1 \omega$$

and

$$\int_{\kappa}^{\kappa+\omega} \left| x_2^{\Delta}(t) \right| \Delta t < 2\bar{r}_2 \omega.$$

From the first equation of (5) and (6), we have

$$\bar{r}_1 \omega > \bar{\alpha}_1 \omega e^{x_1(\xi_1)},$$

and

$$x_1(\xi_1) < \ln \frac{\bar{r}_1}{\bar{\alpha}_1} := l_1,$$

thus,

$$x_1(t) \le x_1(\xi_1) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |x_1^{\Delta}(t)| \Delta t < \ln \frac{\bar{r}_1}{\bar{\alpha}_1} + \bar{r}_1 \omega := M_1.$$

Similarly, we have

$$x_2(\xi_2) < \ln \frac{\bar{r}_2}{\bar{\alpha}_2} := l_2,$$

so,

$$x_2(t) \le x_2(\xi_2) + \frac{1}{2}\int_\kappa^{\kappa+\omega} |x_2^\Delta(t)|\Delta t \le \ln\frac{\bar{r}_2}{\bar{\alpha}_2} + \bar{r}_2\omega := M_2.$$

By (5) and (6),

$$\bar{r}_i\omega \le \omega(\bar{\alpha}_i e^{x_i(\eta_i)} + \bar{\beta}_{ij}e^{x_j(\eta_j)} + \gamma_i e^{x_i(\eta_i)+x_j(\eta_j)}),$$

where $i, j = 1, 2$; $i \ne j$. Hence,

$$\bar{r}_i \le (\bar{\alpha}_i + \gamma_i e^{M_j})e^{x_i(\eta_i)} + \bar{\beta}_{ij}e^{M_j},$$

and

$$x_i(\eta_i) \ge \ln\frac{\bar{r}_i - \bar{\beta}_{ij}e^{M_j}}{\bar{\alpha}_i + \gamma_i e^{M_j}} := L_i, \quad i = 1, 2.$$

Thus,

$$x_i(t) \ge x_i(\eta_i) - \frac{1}{2}\int_\kappa^{\kappa+\omega} |x_i^\Delta(t)|\Delta t \ge L_i - \bar{r}_1\omega := M_{i+2}.$$

So, we have

$$\max_{t\in I_\omega} |x_1(t)| \le \max\{|M_1|, |M_3|\} := R_1,$$
$$\max_{t\in I_\omega} |x_2(t)| \le \max\{|M_2|, |M_4|\} := R_2.$$

Clearly, R_1 and R_2 are independent of λ. Let $R = R_1 + R_2 + R_0$, where R_0 is taken sufficiently large such that $R_0 \ge |l_1| + |l_2| + |L_1| + |L_2|$. Now, we consider the algebraic equations:

$$\begin{cases} \bar{r}_1 - \bar{\alpha}_1 e^x - \bar{\beta}_{12}e^y - \bar{\gamma}_1 e^{x+y} = 0, \\ \bar{r}_2 - \bar{\alpha}_2 e^x - \bar{\beta}_{21}e^y - \bar{\gamma}_2 e^{x+y} = 0, \end{cases} \tag{7}$$

every solution $(x^*, y^*)^T$ of (7) satisfies $\|(x^*, y^*)^T\| < R$. Now, we define $\Omega = \{(u_1(t), u_2(t))^T \in X, \|(u_1(t), u_2(t))^T\| < R\}$. Then it is clear that Ω verifies the requirement (a) of Lemma 2.7. If $(x_1, x_2)^T \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}^2$, then $(x_1, x_2)^T$ is a constant vector in $\mathbb{R}^2$ with $\|(x_1, x_2)^T\| = |x_1| + |x_2| = R$, so we have

$$QN\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By direct computation, we can obtain $\deg(JQN, \Omega \cap \ker L, 0) = 1 \ne 0$. By now, we have verified that Ω fulfills all requirements of Lemma 2.7; therefore, (3) has at least one ω-periodic solution in $\operatorname{Dom} L \cap \bar{\Omega}$. The proof is complete.

4 Conclusion

We investigated a time–delay plankton allelopathy model on time scales. By using the analytical approach, we show that the time delays have no influence on the periodicity of both species. If $\mathbb{T} = \mathbb{R}$, then system (1) is the special case of (3) and our results are more general than those in [1]. We can also obtain the existence theorem of periodic solutions for difference equations (2) when $\mathbb{T} = \mathbb{Z}$. Furthermore, the conditions in Theorem 3.1 are easier then the corresponding conditions in [1–2] with the help of sharp inequality.

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