



# A note on dissipativity and permanence of delay difference equations

*Dedicated to Professor László Hatvani on the occasion of his 75th birthday*

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**Abstract.** We give sufficient conditions on the uniform boundedness and permanence of non-autonomous multiple delay difference equations of the form

$$x_{k+1} = x_k f_k(x_{k-d}, \dots, x_{k-1}, x_k),$$

where  $f_k: D \subseteq (0, \infty)^{d+1} \rightarrow (0, \infty)$ . Moreover, we construct a positively invariant absorbing set of the phase space, which implies also the existence of the global (pullback) attractor if the right-hand side is continuous. The results are applicable for a wide range of single species discrete time population dynamical models, such as (non-autonomous) models by Ricker, Pielou or Clark.

**Keywords:** delay difference equation, higher order difference equation, absorbing set, global pullback attractor, permanence, positive invariance, population dynamics.

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## 1 Introduction


We consider difference equations with multiple delays of the form

$$x_{k+1} = x_k f_k(x_{k-d}, \dots, x_{k-1}, x_k), \tag{1.1}$$

with positive integer maximum delay  $d$  and positive growth functions  $f_k: I^{d+1} \rightarrow (0, \infty)$ ,  $k \in \mathbb{Z}$ , where  $I \subseteq (0, \infty)$  is a real (possibly infinite) interval, and  $\zeta(d) f_k(\zeta) \in I$  for all  $\zeta \in I^{d+1}$  and  $k \in \mathbb{Z}$  (where  $\zeta = (\zeta(0), \dots, \zeta(d)) \in I^{d+1}$ ). These equations are highly motivated by population dynamical models with non-overlapping generations.

In the study of such equations, uniform boundedness is a relevant and important question both from the biological and from the mathematical point of view. On the one hand, the population size should not be arbitrarily big in a realistic model, and on the other hand,

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uniform boundedness implies – in case of continuous right-hand side – the existence of the global (pullback) attractor, and allows to restrict the analysis to a bounded absorbing set. The latter is also crucial for the numerical exploration of the asymptotic behavior. Whether one merely wants to do simulations, or aims to apply validated numerics (e.g. interval arithmetics) to rigorously prove some behavior regarding the dynamics, it is essential to have a bounded subset of the state-space at hand, where all the relevant dynamics happen.

Uniform persistence means that solutions are eventually uniformly bounded away from zero. Uniform persistence together with uniform boundedness is usually called permanence. The biological benefit of uniform persistence is obvious: it ensures that the population does not go extinct. However, it can also be decisive for the mathematical analysis. To mention only one of our motivations, in many applications, such as in Ricker's or Pielou's equation, there exists a unique, nontrivial equilibrium state, whose attractivity in the positive orthant is conjectured under certain conditions (for more details on the topic see e.g. [6]). A possible way to tackle this problem is via the application of graph representations of maps and interval arithmetics (see e.g. [1, 2]), where the vertices of the graph are given usually by small enough  $d + 1$  dimensional cubes in the state space and the edges mean possible transitions determined by the map. For such a proof one needs to exclude the origin and the vertex (box) enclosing it from the graph representation, and uniform persistence allows us to do so.

In this note we construct a positively invariant, bounded absorbing set – which obviously implies uniform boundedness – in Theorem 3.1 under mild conditions on the right-hand side, and give sufficient conditions for uniform persistence in Theorem 3.2 for equation (1.1). Relevant notions and notations are defined in Section 2, whereas applicability of the results is demonstrated in Section 4 via three examples.

For studying (1.1) it is more convenient to consider the equivalent, first order,  $d + 1$  dimensional equation:

$$y_{k+1} = \begin{pmatrix} y_{k+1}(0) \\ y_{k+1}(1) \\ \vdots \\ y_{k+1}(d) \end{pmatrix} = F_k(y_k) := \begin{pmatrix} y_k(1) \\ y_k(2) \\ \vdots \\ y_k(d) f_k(y_k) \end{pmatrix}, \quad (1.2)$$

with  $F_k: I^{d+1} \rightarrow I^{d+1}$  for all  $k \in \mathbb{Z}$ .

Our general hypothesis on the growth functions  $f_k$  is

(H<sub>1</sub>) there exists  $M \in (1, \infty)$ , such that  $f_k(y) \leq M$  holds for all  $k \in \mathbb{Z}$  and all  $y \in I^{d+1}$ .

For uniform boundedness we further assume

(H<sub>2</sub>) there exist  $D > 0$ ,  $j \in \{0, 1, \dots, d\}$ , and  $\delta \in (0, 1)$  such that for all  $k \in \mathbb{Z}$  and  $y \in I^{d+1}$ ,

$$f_k(y) \leq \frac{1}{M^j} \quad \text{holds if} \quad y(d-j) > D, \quad (H_{2a})$$

and moreover,

$$f_k(y) \leq \frac{\delta}{M^j} \quad \text{holds if} \quad y(d-j) > DM. \quad (H_{2b})$$

The following remark offers a practical alternative to check whether hypothesis (H<sub>2</sub>) is fulfilled.

**Remark 1.1.** It is worth noticing that if there exist  $j \in \{0, 1, \dots, d\}$  and a strictly monotone decreasing function  $h: I \rightarrow I$ , such that  $f_k(y) \leq h(y(d-j))$  holds for all  $k \in \mathbb{Z}$  and for all  $y \in I^{d+1}$ , and moreover, there exists  $D_h \in I$  such that  $h(D_h) \leq M^{-j}$  and  $D_h M \in I$  hold, then (H<sub>2</sub>) is fulfilled with  $j, D := D_h$  and  $\delta := \frac{h(D_h M)}{h(D_h)}$ , where  $\delta \in (0, 1)$  follows from the strict monotonicity of  $h$ .

To see this, observe first that if  $y \in I^{d+1}$  with  $y(d-j) > D_h$ , then  $f_k(y) \leq h(y(d-j)) < h(D_h) \leq M^{-j}$  holds, yielding (H<sub>2a</sub>). If in addition  $y(d-j) > D_h M$  also holds, then one obtains that  $f_k(y) \leq h(y(d-j)) < h(D_h M) = \delta h(D_h) \leq \delta M^{-j}$ , so (H<sub>2b</sub>) is also satisfied.

For uniform persistence we will make use of hypothesis (H<sub>3</sub>), which assumes the existence of an absorbing set, i.e. that there exists a bounded set  $A \subseteq I^{d+1}$  that all bounded sets eventually enter under iteration of the map (1.2) (see Definition 2.2 for the exact formulation).

(H<sub>3</sub>) Equation (1.2) possesses an absorbing set  $A$ , and there exist positive numbers  $a, \varepsilon$ , and  $m \in (0, 1)$ , such that

$$m \leq f_k(y) \quad \text{holds for all } y \in A \text{ and } k \in \mathbb{Z}, \quad (\text{H}_{3a})$$

$$1 + \varepsilon \leq f_k(y) \quad \text{holds for all } y \in (0, a]^{d+1} \cap A \text{ and } k \in \mathbb{Z}. \quad (\text{H}_{3b})$$

Our results are stated in Theorems 3.1 and 3.2. For comparison, we mention the most relevant results from the literature: Kocić and Ladas [4, Theorem 2.2.1] give sufficient conditions for the permanence of an autonomous version of equation (1.1). Their basic assumptions are that the right-hand side is continuous, it has a unique positive fixed point and  $f$  is non-increasing in each variable, except possibly the first one, and moreover,  $xf(x, \cdot)$  converges to a finite, positive limit, as the first variable  $x$  converges to 0, while all the others are fixed. Although their setting is obviously different from ours, some of their main ideas were applicable to prove permanence for an autonomous Ricker type equation,

$$x_{k+1} = x_k \exp\left(1 - \sum_{i=0}^d \alpha_i x_{k-i}\right), \quad (1.3)$$

where constants  $\alpha_i \geq 0, i \in \{0, \dots, d\}$  are parameters, such that not all of them vanish, [8, Theorem 3.1]. Clearly, equation (1.3) fulfills assumptions (H<sub>1</sub>)–(H<sub>3</sub>) – see also Example 4.1. Our assumptions are more general in many ways: first of all, equation (1.1) is non-autonomous, moreover, we need neither continuity, nor monotonicity of the growth functions. Therefore our equation allows more realistic models: on the one hand, nature does not behave autonomously, and on the other hand, dropping the continuity and monotonicity conditions on the growth functions makes more elaborate (self-)control of the population possible.

Pötzsche [7, Proposition 3.2(a)] gives uniform boundedness for Clark type equations of the form

$$x_{k+1} = \lambda x_k + g_k(x_{k-d}, \dots, x_{k-1}, x_k) =: \tilde{g}_k(x_{k-d}, \dots, x_{k-1}, x_k), \quad (1.4)$$

where  $\lambda \in (0, 1)$  is a constant parameter,  $I \subseteq [0, \infty)$  is an interval, functions  $\tilde{g}_k$  map  $I^{d+1}$  into  $I$  for all  $k \in \mathbb{Z}$ , and there exists  $K > 0$  such that it is an upper bound of  $g_k: I^{d+1} \rightarrow [0, \infty)$  for all  $k \in \mathbb{Z}$ . Moreover, he shows that  $[0, \frac{R^+}{1-\lambda}]^{d+1} \cap I^{d+1}$  is a pullback absorbing set for all  $R^+ > K$ . In Example 4.3 we demonstrate that our result applies for (1.4) with slightly more general growth-functions  $g_k$ .

Finally we mention a paper by Li, Sun and Yan [5] in which they prove permanence for the single delay difference equation

$$x_{k+1} = (1 - p)x_k + \frac{qx_k}{1 + x_{k-d}^m},$$

where  $p \in (0, 1]$ ,  $q \in (p, \infty)$ ,  $m \in (0, \infty)$  and  $d \in \mathbb{N}$ . To this generalization of Pielou's equation  $x_{k+1} = \lambda x_k (1 + \alpha_d x_{k-d})^{-1}$  our theory does not directly apply, however, we can apply our results for a different generalization of it in Example 4.2.

## 2 Preliminaries

In the following, we introduce some notations and give definitions of some relevant notions.

For an arbitrary  $n \in \mathbb{Z}$ , let us use notation  $\mathbb{Z}_{\geq n} = \mathbb{Z} \cap [n, \infty)$  – note that  $n$  can be negative. Given an initial time  $k_0 \in \mathbb{Z}$ , a *solution* of equation (1.2) is a sequence  $\phi: \mathbb{Z}_{\geq k_0} \rightarrow I^{d+1}$  that satisfies (1.2), i.e.  $\phi_{k+1} = F_k(\phi_k)$  holds for all  $k \in \mathbb{Z}_{\geq k_0}$ . For a given initial state  $\xi \in I^{d+1}$  at time  $k_0 \in \mathbb{Z}$ , let the sequence  $\varphi(\cdot; k_0, \xi): \mathbb{Z}_{\geq k_0} \rightarrow I^{d+1}$  denote the unique solution to the initial value problem (1.2) with  $y_{k_0} = \xi$ , i.e.  $\varphi(k+1; k_0, \xi) = F_k(\varphi(k; k_0, \xi))$  holds for all  $k \in \mathbb{Z}_{\geq k_0}$ , and in addition  $\varphi(k_0; k_0, \xi) = \xi$  is fulfilled. We call this map  $\varphi$  the *general solution* to (1.2).

**Definition 2.1.** Let us call  $\mathbb{I}$  a *discrete interval* if it is the intersection of a real interval with the integers. For a discrete interval  $\mathbb{I}$ ,  $\mathcal{Y} \subseteq \mathbb{I} \times \mathbb{R}^{d+1}$  is called a *non-autonomous set* with *k-fiber*

$$\mathcal{Y}(k) := \{y \in \mathbb{R}^{d+1} : (k, y) \in \mathcal{Y}\}$$

for all  $k \in \mathbb{I}$ .

**Definition 2.2** ([7]). Equation (1.2) is *pullback dissipative* if there exists a bounded set  $A \subseteq I^{d+1}$ , such that for all bounded  $B \subseteq I^{d+1}$  there exists  $N = N(B) \in \mathbb{Z}_{\geq 0}$ , such that

$$\varphi(k; k-n, \xi) \in A, \quad \text{for all } k \in \mathbb{Z}, n \geq N, \xi \in B.$$

The set  $A$  is then called a *pullback absorbing set* of (1.2).

**Definition 2.3** ([7]). A *global pullback attractor* of (1.2) is a non-autonomous set  $\mathcal{A} \subseteq \mathbb{Z} \times I^{d+1}$ , having the following properties:

- (a)  $\mathcal{A}(k)$  is compact for all  $k \in \mathbb{Z}$ ,
- (b)  $\mathcal{A}(k) = \varphi(k; k_0, \mathcal{A}(k_0))$  for all  $k, k_0 \in \mathbb{Z}$  with  $k \geq k_0$  (*invariance*), and
- (c)  $\lim_{n \rightarrow \infty} \text{dist}(\varphi(k_0; k_0 - n, \xi), \mathcal{A}(k_0 - n)) = 0$  for all  $k_0 \in \mathbb{Z}$ ,  $\xi \in I^{d+1}$  (*attractivity*).

**Remark 2.4.** Provided (1.2) possesses a pullback absorbing set, and the functions  $f_k$  are continuous for all  $k \in \mathbb{Z}$ , then (1.2) has a global pullback attractor  $\mathcal{A} \subseteq \mathbb{Z} \times A$  [3, Theorem 3.6]. This implies that  $\mathcal{A}$  is uniformly bounded (i.e. there exists  $R > 0$  so that  $\mathcal{A}(k) \subseteq (0, R]^{d+1}$  for all  $k \in \mathbb{Z}$ ). Thus  $\mathcal{A}$  is uniquely defined and has the dynamical characterization, that it consists of all pairs  $(k_0, \xi) \in \mathbb{Z} \times A$ , such that there exists a bounded solution  $\phi: \mathbb{Z} \rightarrow I^{d+1}$  with  $\phi(k_0) = \xi$ . Consequently,  $\mathcal{A}$  contains all equilibria, as well as periodic and homo- and heteroclinic solutions of (1.2).

Let us also note that the above notions are natural generalizations of the corresponding ones for autonomous equations, that is, if (1.2) is autonomous, then Definitions 2.2 and 2.3 reduce to the “usual” dissipativity, absorbing set and global attractor notions, respectively.

### 3 Results

In our first theorem we construct a positively invariant pullback absorbing set under conditions  $(H_1)$  and  $(H_2)$ .

**Theorem 3.1.** *Assume that  $(H_1)$  and  $(H_2)$  are fulfilled with some fixed  $D > 0$ ,  $M > 1$ ,  $\delta \in (0, 1)$ , and  $j \in \{0, 1, \dots, d\}$ . Then the set*

$$A = \bigcup_{i=0}^j A_i \cup \tilde{A}_0,$$

defined by

$$\begin{aligned} \tilde{A}_0 &= \{y \in (0, DM^{j+1}]^{d+1} \cap I^{d+1} : y(d) \leq D\}, \\ A_i &= \left\{ y \in (0, DM^{j+1}]^{d+1} \cap I^{d+1} : D < y(d-i) \leq DM, \text{ and} \right. \\ &\quad \left. y(d-\ell) \leq DM^{i+1-\ell}, \text{ for } 0 \leq \ell \leq i \right\}, \quad \forall i \in \{0, \dots, j\}, \end{aligned}$$

is positively invariant w.r.t. all maps  $F_k$ ,  $k \in \mathbb{Z}$ , in (1.2). Moreover,  $A$  is a pullback absorbing set for equation (1.2).

Consequently the set  $(0, C]^{d+1} \cap I^{d+1}$  is also pullback absorbing with  $C := DM^{j+1}$ .

If in addition the functions  $f_k$  in (1.2) are continuous for all  $k \in \mathbb{Z}$ , then a global pullback attractor  $\mathcal{A}$  exists and is a subset of  $\mathbb{Z} \times A$ .

*Proof. Positive invariance of  $A$ .* Fix arbitrary  $\eta \in A$  and  $k \in \mathbb{Z}$ , and for brevity let  $\bar{\eta} := F_k(\eta)$ . We need to show that  $\eta \in A$  implies  $\bar{\eta} \in A$ .

Note that  $\eta \in A$  implies  $\eta(l) \in (0, DM^{j+1}] \cap I$  for all  $l \in \{0, \dots, d\}$ . Therefore, as (1.2) gives  $\bar{\eta}(l) = \eta(l+1)$  for all  $l \in \{0, \dots, d-1\}$ ,  $\bar{\eta}(l) \in (0, DM^{j+1}] \cap I$  holds for all  $l \in \{0, \dots, d-1\}$ , and certainly, also  $\bar{\eta}(d) \in I$ .

Further, if  $\eta \in \tilde{A}_0$ , then  $\eta(d) \leq D$  and thus, by  $(H_1)$ ,  $\bar{\eta}(d) \leq DM$  holds, so  $\bar{\eta} \in \tilde{A}_0 \cup A_0$ .

Next we prove that  $\eta \in A_i$  implies  $\bar{\eta} \in A_{i+1}$  for all  $i \in \{0, \dots, j-1\}$ , in case  $j \geq 1$ . To see this, observe that on the one hand,  $\bar{\eta}(d) \leq \eta(d)M \leq DM^{i+2}$  holds as desired, and on the other hand, the other bounds given in  $A_{i+1}$  follow directly from the bounds in  $A_i$  and from  $\bar{\eta}(d-\ell) = \eta(d-\ell+1)$  for  $\ell \in \{1, \dots, i+1\}$ .

Finally, if  $\eta \in A_j$ , then by definition of  $A_j$ ,  $D < \eta(d-j) \leq DM$  and  $\eta(d) \leq DM^{j+1}$  hold. Then, by  $(H_{2a})$ ,  $f_k(\eta) \leq M^{-j}$  holds, from which one infers that  $\bar{\eta}(d) = \eta(d)f_k(\eta) \leq DM$ . This means that  $\bar{\eta} \in \tilde{A}_0 \cup A_0$  holds true, which completes the proof of positive invariance of  $A$ .

**Attractivity of  $A$ .** Suppose that  $B$  is a fixed, arbitrary bounded subset of  $I^{d+1}$ , i.e. there exists  $K > 0$ , such that  $B \subseteq (0, K]^{d+1}$ . Let  $k \in \mathbb{Z}$  be arbitrary and consider  $\varphi(k; k-n, B)$ . As we have already shown positive invariance of  $A$ , it remains to prove that there exists  $N = N(B) \in \mathbb{Z}_{\geq 0}$ , such that for all  $\zeta \in B$  there exists a positive integer  $n = n(\zeta) \leq N$ , such that  $\varphi(k; k-n, \zeta) \in A$  holds.

For the moment, fix  $k_0$  and  $\zeta \in B$ , and for brevity let us introduce the notation  $y_k := \varphi(k; k_0, \zeta)$  for all  $k \in \mathbb{Z}_{\geq k_0}$ .

First we claim that there exists  $k_1 \in \mathbb{Z}_{\geq k_0}$ , such that  $y_{k_1}(d) \leq DM$ . There are two cases: either  $y_{k_0}(d) \leq DM$ , and then the choice  $k_1 = k_0$  is appropriate, or  $y_{k_0}(d) > DM$  holds.

In the latter case we have  $DM < y_{k_0}(d) = y_{k_0+j}(d-j) \leq K$ , and on the other hand,  $y_{k_0+j}(d) \leq y_{k_0}(d)M^j$  holds by  $(H_1)$ . Combining these with (1.2) and hypothesis  $(H_{2b})$  for  $k = k_0 + j$  gives

$$y_{k_0+j+1}(d) \leq y_{k_0+j}(d) \frac{\delta}{M^j} \leq y_{k_0}(d) M^j \frac{\delta}{M^j} = \delta y_{k_0}(d).$$

By induction one infers immediately that there exists a smallest  $l \in \mathbb{N}$ , such that  $k_1 := k_0 + l(j+1)$  fulfills  $y_{k_1}(d) \leq DM$ . Since  $y_{k_0}(d) \leq K$ ,

$$l = l(\xi, k_0) \leq \left\lceil \log_{\delta} \frac{DM}{K} \right\rceil =: l_1$$

holds for arbitrary  $\xi \in B$  and  $k_0 \in \mathbb{Z}$ , where  $\lceil \cdot \rceil$  denotes the ceiling function, and thus

$$k_1 = k_1(\xi, k_0) \leq k_0 + l_1(j+1) \quad (3.1)$$

holds independently of the choice of  $\xi \in B$  and  $k_0 \in \mathbb{Z}$ .

Next, we fix  $l, l_1$  and  $k_1$  from above, and show that  $y_{k_1+d+j+1} \in \tilde{A}_0 \cup A_0$ .

From  $y_{k_1}(d) \leq DM$ , and after applying (1.2) and assumption  $(H_1)$   $j$  times one obtains that  $y_{k_1+j}(d-j+i) \leq DM^{i+1}$  holds for all  $i \in \{0, 1, \dots, j\}$ , and in particular

$$y_{k_1+j}(d-i) \leq DM^{i+1}, \quad \text{for all } i \in \{0, 1, \dots, j\}. \quad (3.2)$$

We claim that provided  $y_k(d-i) \leq DM^{i+1}$  holds for some  $k \geq k_0 + j$ , and for all  $i \in \{0, \dots, j\}$  (note that this is the case for  $k = k_1 + j$ ), then  $y_{k+1}(d) \leq DM^{j+1}$ . Assume to the contrary that  $y_{k+1}(d) > DM^{j+1}$ . Then, by  $(H_1)$  and equation (1.2),  $y_{k-j}(d) > D$  must hold, and thus  $y_k(d-j) = y_{k-j}(d) > D$ . Using this, hypothesis  $(H_{2a})$ , and that  $y_k(d) \leq DM^{j+1}$  is fulfilled by assumption, we infer

$$y_{k+1}(d) \leq y_k(d)M^{-j} \leq y_k(d) \leq DM^{j+1},$$

a contradiction to  $y_{k+1}(d) > DM^{j+1}$ . This proves the claim.

Combining the above claim with (3.2) and (1.2) we infer by induction that

$$y_k \in (0, DM^{j+1}]^{d+1} \cap I^{d+1} \quad (3.3)$$

holds for all  $k \geq k_1 + d$ . Keeping this in mind, either  $y_{k_1+d}(d) \leq D$ , and then  $y_{k_1+d} \in \tilde{A}_0 \subseteq A$ , or  $y_{k_1+d+j}(d-j) = y_{k_1+d}(d) > D$  holds true.

In the latter case, as we have  $y_{k_1+d+j}(d) \leq DM^{j+1}$  by (3.3), the application of assumption  $(H_{2a})$  to equation (1.2) (with  $k = k_1 + d + j$ ) yields that

$$y_{k_1+d+j+1}(d) \leq \frac{y_{k_1+d+j}(d)}{M^j} \leq \frac{DM^{j+1}}{M^j} = DM.$$

This together with (3.3) means that  $y_{k_1+d+j+1} \in \tilde{A}_0 \cup A_0 \subseteq A$ .

All in all, by (3.1) we have

$$\begin{aligned} k_1 + d + j + 1 - k_0 &\leq k_0 + l_1(j+1) + d + (j+1) - k_0 \\ &= (l_1 + 1)(j+1) + d, \end{aligned}$$

thus the choice

$$N := N(B) := (l_1 + 1)(j + 1) + d$$

is appropriate, i.e. the above argument proves that  $\varphi(k; k - n, \xi) \in A$  holds for all  $k \in \mathbb{Z}$ ,  $\xi \in B$  and  $n \geq N$ , and thus  $A$  is pullback absorbing.

Consequently, as a superset of  $A$ ,  $(0, C]^{d+1} \cap I^{d+1}$  (with  $C := DM^{j+1}$ ) is also pullback absorbing.

Finally, the last statement of the theorem follows directly from [3, Theorem 3.6] and the fact that  $A$  is pullback absorbing.  $\square$

The next theorem gives sufficient conditions on the uniform persistence of equation (1.2).

**Theorem 3.2.** *Assume that hypotheses (H<sub>1</sub>) and (H<sub>3</sub>) are fulfilled with absorbing set  $A$  and positive numbers  $a, \varepsilon, M > 1$  and  $m \in (0, 1)$ . Then for all  $\xi \in I^{d+1}$ , there exists  $n_0 = n_0(\xi) \in \mathbb{Z}_{\geq 0}$ , such that  $\varphi(k; k - n, \xi) \in [c, \infty)^{d+1} \cap A$  holds for all  $k \in \mathbb{Z}$  and  $n \in \mathbb{Z}_{\geq n_0}$ , where  $\varphi$  denotes the general solution to (1.2) and*

$$c := a \min \left\{ \frac{1}{M^{d+1}}, m^{d+1} \right\}. \quad (3.4)$$

*Proof.* Let us again temporarily fix arbitrary  $k_0 \in \mathbb{Z}$  and  $\xi \in A$ , and in order to shorten notations, introduce  $y_k := \varphi(k; k_0, \xi)$  for  $k \in \mathbb{Z}_{\geq k_0}$ . Note also that  $c < a$  trivially holds by definition (3.4).

As  $A$  is pullback absorbing by assumption (H<sub>3</sub>), we may assume that  $y_k \in A$  for all  $k \in \mathbb{Z}_{\geq k_0}$ , and moreover, it is sufficient to show, that there exists  $n_0 \in \mathbb{Z}_{\geq 0}$ , independent of  $k_0$ , such that  $y_k \in [c, \infty)^{d+1} \cap A$  holds for all  $k \in \mathbb{Z}_{\geq k_0 + n_0}$ . We do this in two steps: first we show that the sequence  $y_k$  meets the set  $[c, a]^{d+1} \subseteq [c, \infty)^{d+1}$ , then in Step 2 we prove that from that point, the solution will not leave the set  $[c, \infty)^{d+1}$ .

**Step 1.** Suppose that there exists  $k' \in \mathbb{Z}_{\geq k_0 + 2d}$ , such that  $y_{k'} \notin [c, \infty)^{d+1}$ , i.e. there exists an  $l \in \{0, \dots, d\}$ , such that  $y_{k'}(d - l) < c$ . Note that if such a  $k'$  does not exist, then the statement of the theorem holds here with  $n_0 = 2d$ .

Since  $y_{k'-l}(d) = y_{k'}(d - l)$ , hence  $y_{k''}(d) < c$  holds for  $k'' := k' - l \geq k_0 + d$ . Then equation (1.2) and  $i$  times application of (H<sub>3a</sub>) yield that

$$y_{k''}(d) \geq m^i y_{k''-i}(d) = m^i y_{k''}(d - i)$$

holds for all  $i \in \{0, \dots, d\}$ . Using this,  $y_{k''}(d) < c$ , definition (3.4), and  $m < 1$ , respectively, we obtain that the inequalities

$$y_{k''}(d - i) \leq \frac{y_{k''}(d)}{m^i} < \frac{c}{m^i} \leq am^{d-i+1} < a$$

hold true for  $i \in \{0, \dots, d\}$ , i.e.  $y_{k''} \in (0, a]^{d+1}$  and  $y_{k''}(d) < c$  hold.

Observe that from equations (1.2) and (H<sub>3b</sub>) it is clear, that from now on (i.e. for  $k \geq k''$ ),  $y_{k+1}(d) \geq (1 + \varepsilon)y_k(d)$  and  $y_k \in (0, a]^{d+1}$  will be fulfilled as long as  $y_k(d) \leq a$ .

Consequently, there exists a smallest  $k_1 > k''$  such that  $y_{k_1}(d) \geq c$ . As  $y_{k_1-1}(d) < c$ , hence  $c \leq y_{k_1}(d) < cM$  holds by (H<sub>1</sub>). This implies in particular that  $y_{k_1} \in (0, a]^{d+1}$  is satisfied, as  $cM \leq aM^{-d} \leq a$  holds by (3.4).

Now, applying  $(d - i)$  times hypothesis (H<sub>1</sub>), one gets from  $y_{k_1}(d) < cM$ , that

$$y_{k_1+d-i}(d) \leq y_{k_1}(d)M^{d-i} < cM^{d-i+1} \leq a \quad (3.5)$$

holds for all  $i \in \{0, \dots, d\}$ . Thus, in the light of the above observation,  $(H_{3b})$  can also be applied  $(d - i)$  times to  $c \leq y_{k_1}(d)$  to obtain that

$$y_{k_1+d-i}(d) \geq y_{k_1}(d)(1 + \varepsilon)^{d-i} \geq c(1 + \varepsilon)^{d-i} \quad (3.6)$$

holds for all  $i \in \{0, \dots, d\}$ .

Inequalities (3.5) and (3.6) combined with  $y_{k_1+d}(d - i) = y_{k_1+d-i}(d)$  yield that

$$y_{k_1+d} \in [c, a]^{d+1} \cap A \subseteq [c, \infty)^{d+1} \cap A.$$

**Step 2.** We claim that  $y_k$  stays in  $[c, \infty)^{d+1} \cap A$  for all  $k \geq k_1 + d$ . Otherwise there is a smallest  $k_2 > k_1 + d$ , such that  $y_{k_2} \notin [c, \infty)^{d+1} \cap A$ , for which  $y_{k_2}(d) < c$  must hold. But then, applying  $(i + 1)$  times inequality  $(H_{3a})$ , one gets that

$$y_{k_2}(d) \geq m^{i+1} y_{k_2-i-1}(d)$$

holds for all  $i \in \{0, \dots, d\}$ . From this, and making use of inequality  $y_{k_2}(d) < c$  and (3.4), respectively, we deduce that

$$y_{k_2-1}(d - i) = y_{k_2-i-1}(d) \leq \frac{y_{k_2}(d)}{m^{i+1}} < \frac{c}{m^{i+1}} \leq am^{d-i}$$

holds for all  $i \in \{0, \dots, d\}$ , which implies in particular that  $y_{k_2-1} \in (0, a]^{d+1} \cap A$ .

On the other hand, the definition of  $k_2$  guarantees that  $y_{k_2-1} \in [c, \infty)^{d+1} \cap A$ , which in turn yields  $y_{k_2-1} \in [c, a]^{d+1} \cap A$ . However, this contradicts to  $y_{k_2}(d) < c$  combined with  $(H_{3b})$ , which concludes the proof of the claim, and ensures that  $y_k \in [c, \infty)^{d+1} \cap A$  holds for all  $k \geq k_1 + d$ .

This proves that  $n_0 = k_1 + d - k_0$  is appropriate, i.e.  $\varphi(k; k - n, \zeta) \in [c, \infty)^{d+1} \cap A$  holds for all  $k \in \mathbb{Z}$  and  $n \in \mathbb{Z}_{\geq n_0}$ .  $\square$

## 4 Examples

For the first two examples let us define the coefficient functions  $\alpha_{k,i}: I^{d+1} \rightarrow [a_i, b_i]$  for all  $k \in \mathbb{Z}$  and indices  $i \in \{0, 1, \dots, d\}$ , where  $0 \leq a_i \leq b_i < \infty$  are constant bounds, such that  $\sum_{i=0}^d a_i \neq 0$ . We emphasize that the coefficient functions can be both discontinuous and non-monotone. To simplify notations, let  $\mathbb{J}$  denote the set of indices  $j \in \{0, 1, \dots, d\}$ , such that  $a_j > 0$ .

**Example 4.1** (Generalized Ricker equation). Let us consider positive solutions of the equation

$$x_{k+1} = x_k \exp\left(1 - \sum_{i=0}^d \alpha_{k,i}(x_{k-d}, \dots, x_{k-1}, x_k) x_{k-i}\right), \quad k \in \mathbb{Z}.$$

Then the corresponding  $d + 1$  dimensional equation (1.2) is defined by  $f_k: (0, \infty)^{d+1} \rightarrow (0, \infty)$ ,

$$f_k(y) = \exp\left(1 - \sum_{i=0}^d \alpha_{k,i}(y) y(d - i)\right) \quad \text{for all } k \in \mathbb{Z}.$$

Then

$$f_k(y) \leq \exp(1 - a_j y(d - j)) \quad (4.1)$$



holds on  $(0, \infty)^{d+1}$  for all  $k \in \mathbb{Z}$  and  $j \in \mathbb{J}$ , so  $(H_1)$  is satisfied with  $M = e$ . We also need to find  $D$  with which  $(H_2)$  is fulfilled, too. As the right-hand side of (4.1) is strictly monotone decreasing in  $y(d-j)$ , then by virtue of Remark 1.1,  $D := D(j)$  defined by  $\exp(1 - a_j D) = M^{-j} = e^{-j}$  gives an appropriate choice. This yields

$$D = D(j) = \frac{1+j}{a_j}.$$

Therefore, Theorem 3.1 can be applied to obtain that the corresponding set  $A$ , defined in the theorem, is positive invariant and pullback absorbing for equation (1.2) for any  $j \in \mathbb{J}$  and  $D = D(j)$ , and in particular the set  $(0, C]^{d+1}$  is pullback absorbing with

$$C := \min_{j \in \mathbb{J}} \left\{ \frac{(1+j)e^{j+1}}{a_j} \right\}.$$

Moreover, if the coefficient functions are continuous, then there exists a global pullback attractor  $\mathcal{A} \subseteq \mathbb{Z} \times (0, C]^{d+1}$ .

Now set  $m := \exp(1 - C \sum_{i=0}^d b_i)$  and fix  $a < (\sum_{i=0}^d b_i)^{-1}$ . Then it is easy to see that  $m < 1$  and  $(H_{3a})$  hold, and moreover, there exists  $\varepsilon = \varepsilon(a) > 0$ , such that  $(H_{3b})$  is also fulfilled. Thus Theorem 3.2 can be applied to obtain that with

$$c := a \min\{e^{-d-1}, m^{d+1}\},$$

for all  $\xi \in (0, \infty)^{d+1}$  there exist  $n_0 = n_0(\xi) \in \mathbb{Z}_{\geq 0}$ , such that  $\varphi(k; k-n, \xi) \in [c, C]^{d+1}$  holds for all  $k \in \mathbb{Z}$  and  $n \in \mathbb{Z}_{\geq n_0}$ .

As the size of the absorbing set can be important in certain applications, we note that even for the constant-coefficient and autonomous case (1.3), our theorem may provide better bounds, than the one given by Sun and Li [8, Theorem 3.1]. Their proof shows that eventually every solution will be below  $e^{d+1} / (\sum_{i=0}^d \alpha_i)$ , which gives better bounds in case  $\alpha_d > 0$  and every other  $\alpha_i$  vanishes, but it can be worse than ours otherwise. Both our and their lower bound depend on the upper bound  $C$ , and similarly, it depends on the concrete parameters, which result gives a better lower bound.

**Example 4.2** (Generalized Pielou equation). Let us consider now positive solutions of the equation

$$x_{k+1} = \frac{x_k \lambda_k(x_{k-d}, \dots, x_{k-1}, x_k)}{1 + \sum_{i=0}^d \alpha_{k,i}(x_{k-d}, \dots, x_{k-1}, x_k) x_{k-i}}, \quad k \in \mathbb{Z}.$$

Then the corresponding  $d+1$  dimensional equation (1.2) is defined by  $f_k: (0, \infty)^{d+1} \rightarrow (0, \infty)$ ,

$$f_k(y) = \frac{\lambda_k(y)}{1 + \sum_{i=0}^d \alpha_{k,i}(y) y(d-i)} \quad \text{for all } k \in \mathbb{Z},$$

where coefficient functions  $\lambda_k$  map also  $(0, \infty)^{d+1}$  to some common interval  $[\lambda_-, \lambda_+] \subset (1, \infty)$  for all  $k \in \mathbb{Z}$ .

Then the estimate

$$f_k(y) \leq \frac{\lambda_+}{1 + a_j y(d-j)} \tag{4.2}$$

holds for all  $y \in (0, \infty)^{d+1}$  and  $j \in \mathbb{J}$ . One can already see that with  $M := \lambda_+$ ,  $(H_1)$  is fulfilled. Moreover, as the right-hand side of (4.2) is strictly monotone decreasing, we obtain

from Remark 1.1, that  $(H_2)$  is also satisfied with

$$D := D(j) := \frac{\lambda_+^{j+1} - 1}{a_j},$$

which is deduced from  $\frac{\lambda_+}{1+a_j D} = M^{-j} = \lambda_+^{-j}$ .

Thus the assumptions of Theorem 3.1 are satisfied for any  $j \in \mathbb{J}$ , and in particular  $(0, C]^{d+1}$  is a pullback absorbing set for the equation (1.2) for

$$C = \min_{j \in \mathbb{J}} \left\{ \frac{\lambda_+^{j+1} (\lambda_+^{j+1} - 1)}{a_j} \right\}. \quad (4.3)$$

Moreover, if the coefficient functions are continuous, then there exists a global pullback attractor  $\mathcal{A} \subseteq \mathbb{Z} \times (0, C]^{d+1}$ .

Furthermore, by setting

$$m := \frac{\lambda_-}{1 + C \sum_{i=0}^d b_i} \quad \text{and fixing} \quad a < \frac{\lambda_- - 1}{\sum_{i=0}^d b_i},$$

it is not hard to see that  $m < 1$  and  $(H_{3a})$  hold, and if  $\varepsilon > 0$  is small enough, then  $(H_{3b})$  is also satisfied. Therefore, Theorem 3.2 yields that with  $c := a \min\{\lambda_+^{-d-1}, m^{d+1}\}$ , for all  $\xi \in (0, \infty)^{d+1}$  there exists  $n_0 = n_0(\xi) \in \mathbb{Z}_{\geq 0}$ , such that  $\varphi(k; k - n, \xi) \in [c, C]^{d+1}$  holds for all  $k \in \mathbb{Z}$  and  $n \in \mathbb{Z}_{\geq n_0}$ .

Our last example concerns a general Clark type equation, which covers some famous population dynamical models, e.g. non-autonomous, discrete time versions of Lasota–Ważewski or Mackey–Glass equations.

**Example 4.3** (Generalized Clark type equation). Consider equation

$$z_{k+1} = \lambda z_k + g_k(z_{k-d}, \dots, z_{k-1}, z_k), \quad (4.4)$$

where  $\lambda \in (0, 1)$ , and  $g_k$  maps  $[0, \infty)^{d+1}$  into  $[0, \infty)$  for all  $k \in \mathbb{Z}$ , and there exist  $K > 0$  and  $\beta_0 \in [0, 1 - \lambda)$ , such that

$$g_k(y) < \beta_0 y(d) + K \quad (4.5)$$

holds for all  $y \in [0, \infty)^{d+1}$  and for all  $k \in \mathbb{Z}$ . Without further assumptions on the feedback functions  $g_k$ , persistence clearly does not hold, so our aim here is to make use of Theorem 3.1 in order to obtain an absorbing set.

Applying the transformation  $x_k := \exp(z_k)$ , we get an equation of the form (1.1) with  $f_k: [1, \infty)^{d+1} \rightarrow [1, \infty)$ ,  $f_k(y) = y(d)^{-(1-\lambda)} \exp(g_k(\ln(y(0)), \dots, \ln(y(d))))$ , for all  $k \in \mathbb{Z}$  and  $y \in [1, \infty)^{d+1}$ . By assumption (4.5) one infers that

$$f_k(y) \leq e^K y(d)^{\lambda + \beta_0 - 1}$$

holds on  $[1, \infty)^{d+1}$  for all  $k \in \mathbb{Z}$ .

Since  $\lambda + \beta_0 - 1 < 0$ , on the one hand,  $f_k(y) \leq e^K$  holds for all  $y \in [1, \infty)^{d+1}$ , and on the other hand,  $f_k(y)$  is estimated from above by a strictly monotone decreasing function of  $y(d)$ . Hence,  $(H_1)$  is satisfied with  $M = e^K$ , and due to Remark 1.1,  $(H_2)$  is also fulfilled with  $j = 0$  and

$$D := \exp\left(\frac{K}{1 - \lambda - \beta_0}\right),$$

where  $D$  is deduced from  $e^K D^{\lambda+\beta_0-1} = M^{-j} = 1$ .

Thus Theorem 3.1 yields the pullback absorbing set  $[1, C]^{d+1}$  for

$$C = \exp\left(\frac{K(2 - \lambda - \beta_0)}{1 - \lambda - \beta_0}\right).$$

For the original equation (4.4) – more precisely, for the corresponding  $d + 1$  dimensional equation – this means that  $A := [0, C']^{d+1}$  is a pullback absorbing set with

$$C' := \frac{K(2 - \lambda - \beta_0)}{1 - \lambda - \beta_0},$$

and if the right-hand side is continuous, then the global pullback attractor  $\mathcal{A}$  exists and is contained in  $\mathbb{Z} \times A$ .

We note that for the frequently studied special case, when functions  $g_k$  in (4.4) are uniformly bounded by some  $K > 0$ , i.e.  $\beta_0 = 0$ , then Proposition 3.2 (a) of [7] provides a smaller upper bound for the size of the absorbing set, namely,  $[0, R^+]^{d+1}$  is pullback absorbing for any  $R^+ > \frac{K}{1-\lambda}$ .

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