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Positive solutions for a Kirchhoff type problem with fast increasing weight and critical nonlinearity

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Abstract. In this paper, we study the following Kirchhoff type problem

$$-\left(a+b\int_{\mathbb{R}^3}K(x)|\nabla u|^2dx\right)\operatorname{div}(K(x)\nabla u)=\lambda K(x)|x|^\beta|u|^{q-2}u+K(x)|u|^4u,\quad x\in\mathbb{R}^3,$$

where $K(x)=\exp(|x|^\alpha/4)$ with $\alpha\geq 2$, $\beta=(\alpha-2)(6-q)/4$ and the parameters $a,b,\lambda>0$. When $6-\frac{4}{\alpha}<q<6$, we obtain a positive ground state solution for any $\lambda>0$. When $2<q<4$, we obtain a positive solution for $\lambda>0$ small enough. In the proof we use variational methods.

Keywords: Kirchhoff type equation, critical nonlinearity, variational methods.

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1 Introduction

In this paper, we consider the existence of positive solutions for the following Kirchhoff type problem

$$-\left(a+b\int_{\mathbb{R}^3}K(x)|\nabla u|^2dx\right)\operatorname{div}(K(x)\nabla u)=\lambda K(x)|x|^\beta|u|^{q-2}u+K(x)|u|^4u,\quad x\in\mathbb{R}^3,\tag{1.1}$$

where $K(x)=\exp(|x|^\alpha/4)$ with $\alpha\geq 2$, $\beta=(\alpha-2)(6-q)/4$ and the parameters $a,b,\lambda>0$.

It is well known that Kirchhoff type problems are presented by Kirchhoff in [9] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. When $K(x)\equiv 1$, the general Kirchhoff type problem involving critical exponent

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u=f(x,u)+u^5,\quad x\in\mathbb{R}^3,\tag{1.2}$$

has been studied by many researchers. Under different assumptions on $f(x,u)$, some interesting studies for (1.2) can be found in [12–14, 23, 25]. There are also several existence results

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for (1.2) on a bounded domain $\Omega \subset \mathbb{R}^3$. For this case, we refer the interested readers to [4,10,11,16,22].

On the other hand, as pointed out in [3,8], one of the motivations for studying problem (1.1) due to the fact that, for $\alpha = q = 2, a = 1, b = 0$ and $\lambda = 1/5$, (1.1) arises naturally when one seeks self-similar solutions of the form

$$w(t,x) = t^{-1/5}u(xt^{-1/2})$$

to the evolution equation

$$w_t - \Delta w = |w|^4 w \quad \text{on } (0,\infty) \times \mathbb{R}^3.$$

For more detailed description, see [3,8].

In [5], Furtado et al. concerned the following equation

$$-\operatorname{div}(K(x)\nabla u) = \lambda K(x)|x|^\gamma |u|^{q-2}u + K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, $\gamma = (\alpha-2)\frac{(2^*-q)}{(2^*-2)}$ and $2 < q < 2^*$. In that article, the authors obtained the existence of a positive solution for (1.3) by using Mountain Pass Theorem. In particular, when $N = 3$, they proved that there is a positive solution for large value of λ if $2 < q \leq 6 - \frac{4}{\alpha}$, and no restriction on λ if $6 - \frac{4}{\alpha} < q < 6$.

Subsequently, Furtado et al. [6] studied the number of solutions for the following problem

$$-\operatorname{div}\big(K(x)\nabla u\big) = K(x)f(u) + \lambda K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $f(u)$ is superlinear and subcritical. More precisely, for any given $k \in \mathbb{N}$, the authors shown that there exists $\lambda^* = \lambda^*(k) > 0$ such that (1.4) has at least k pairs of solutions for $\lambda \in \big(0, \lambda^*(k)\big)$. But they can not give any information about the sign of these solutions.

Recently, we investigated the following Kirchhoff type of problem with concave-convex nonlinearities and critical exponent (see [21])

$$-\left(a + \epsilon \int_{\mathbb{R}^3} K(x)|\nabla u|^2 dx\right) \operatorname{div}(K(x)\nabla u) = \lambda K(x) f(x)|u|^{q-2}u + K(x)|u|^4 u, \quad x \in \mathbb{R}^3,$$

where $1 < q < 2$, and $\epsilon > 0$ is small enough. Under some conditions on $f(x)$, we gave the existence of two positive solutions and obtained uniform lower estimates for extremal values for the problem. For more results of related problem, please see [2,7,17–20] and the references therein.

From these results above, we do not see any existence of positive solutions for problem (1.1) in the case of $2 < q < 6$, the term $\int K(x)|\nabla u|^2 dx$ and critical nonlinearity, hence it is natural to ask what the case would be. Our aim of this paper is to show how variational methods can be employed to establish some existence of positive solutions for the Kirchhoff type problem (1.1).

In order to state our main results, let H denote the Hilbert space obtained as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} K(x)|\nabla u|^2 dx\right)^{1/2}.$$

Define the weighted Lebesgue spaces for each $q \in [2,6]$

$$L_K^q(\mathbb{R}^3) = \left\{u \text{ measurable in } \mathbb{R}^3 \ : \ \int_{\mathbb{R}^3} K(x)|x|^\beta |u|^q dx < \infty\right\}$$

with the norm

$$\|u\|_q = \left(\int_{\mathbb{R}^3} K(x)|x|^\beta |u|^q dx \right)^{1/q}.$$

By [5, Proposition 2.1], we have that the embedding $H \hookrightarrow L^q_K(\mathbb{R}^3)$ is continuous for $2 \le q \le 6$, and compact for $2 \le q < 6$. This enables us to define for each $q \in [2,6]$

$$S_q = \inf \left\{ \int_{\mathbb{R}^3} K(x)|\nabla u|^2 dx \quad : \quad u \in H, \int_{\mathbb{R}^3} K(x)|x|^\beta |u|^q dx = 1 \right\}. \tag{1.5}$$

In particular, when $q = 6$, we put $S = S_6$ for simplicity. It is worth mentioning that this constant is equal to the best constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, see [2].

By the above embedding, it is easy to see the following functional associated to (1.1)

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q}\int_{\mathbb{R}^3} K(x)|x|^\beta |u|^q dx - \frac{1}{6}\int_{\mathbb{R}^3} K(x)|u|^6 dx$$

is well defined on H and $I \in C^1(H, \mathbb{R})$. It is commonly known that there exists a one to one correspondence between the critical points of I and the weak solutions of (1.1). Here, we say $u \in H$ is a weak solution of (1.1), if for any $\phi \in H$, there holds

$$\left(a + b\|u\|^2\right) \int_{\mathbb{R}^3} K(x)\nabla u \nabla \phi dx - \lambda \int_{\mathbb{R}^3} K(x)|x|^\beta |u|^{q-2} u\phi dx - \int_{\mathbb{R}^3} K(x)|u|^4 u\phi dx = 0.$$

Additionally, we say a nontrivial solution $u \in H$ to (1.1) is a ground state solution, if $I(u) \le I(v)$ for any nontrivial solution $v \in H$ to (1.1).

Our main results for (1.1) are the following theorems.

Theorem 1.1. *Assume that $a, b > 0$, $\alpha \ge 2$ and $6 - \frac{4}{\alpha} < q < 6$. Then for any $\lambda > 0$, problem* (1.1) *has at least a positive ground state solution.*

Theorem 1.2. *Assume that $a, b > 0$, $\alpha \ge 2$ and $2 < q < 4$. Then there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$, problem* (1.1) *has at least a positive solution.*

Kirchhoff type problems are often treated as nonlocal in view of the presence of the term $\int K(x)|\nabla u|^2 dx$ which implies that equation (1.1) is no longer a pointwise identity. And so, the methods employed in [5] cannot be used here. For Theorem 1.1, motivated by [23] (see also [15]), we shall use Nehari Manifold method to prove the existence of a positive ground state solution for problem (1.1). For Theorem 1.2, we cannot proceed as in proof of Theorem 1.1 since $2 < q < 4$. We also remark that the method used in [5] by letting λ sufficiently large do not apply here, due to the appearance of the term $\int K(x)|\nabla u|^2 dx$. On the contrary, we overcome this difficulty by letting λ small enough, which is inspired by [12].

This paper is organized as follows. In the next section, we give some notations and preliminaries. Then we prove Theorem 1.1 in Section 3, and Theorem 1.2 in Section 4.

2 Notations and preliminaries

Throughout this paper, we write $\int u$ instead of $\int_{\mathbb{R}^3} u(x)dx$. $B_r(x)$ denotes a ball centered at x with radius $r > 0$. Let $\to$ denote strong convergence. Let $\rightharpoonup$ denote weak convergence. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \le C$ as $\varepsilon \to 0$, and $o(\varepsilon^t)$ denotes $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$. All limitations hold as $n \to \infty$ unless otherwise stated. C and C_i denote various positive constants whose values may vary from line to line.

Lemma 2.1. *Let $a, b > 0$ and $2 < q < 6$, then the functional I satisfies the mountain-pass geometry:*

(i) There exist $\rho, \theta > 0$ such that $I(u) \geq \theta > 0$ for any $\|u\| = \rho$.

(ii) There exists $e \in H$ with $\|e\| > \rho$ such that $I(e) < 0$.

Proof. (i) By (1.5), we have that

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q}\int K(x)|x|^\beta|u|^q - \frac{1}{6}\int K(x)|u|^6 \\ &\geq \frac{a}{2}\|u\|^2 - \frac{\lambda}{q}S_q^{-q/2}\|u\|^q - \frac{1}{6}S^{-3}\|u\|^6. \end{aligned}$$

Therefore, since $2 < q < 6$, it follows that there are $\rho, \theta > 0$ such that $I(u) \geq \theta > 0$ for any $\|u\| = \rho$.

(ii) Let $u \in H \setminus \{0\}$. Thus, we have for $2 < q < 6$

$$\lim_{t\to+\infty} I(tu) = \lim_{t\to+\infty}\left[\frac{a}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|u\|^4 - \frac{\lambda}{q}t^q\int K(x)|x|^\beta|u|^q - \frac{1}{6}t^6\int K(x)|u|^6\right] = -\infty.$$

Thus, there exists $e := tu$ such that $\|e\| > \rho$ and $I(e) < 0$. □

3 Positive ground state solution for $6 - \frac{4}{\alpha} < q < 6$

In this section, we will employ Nehari method to prove the existence of a positive ground state solution of the considered problem for $6 - \frac{4}{\alpha} < q < 6$. And, suppose that the assumptions of Theorem 1.1 hold throughout this section.

Define the Nehari manifold

$$\Lambda = \{u \in H \setminus \{0\} : G(u) = 0\},$$

where

$$G(u) = \langle I'(u), u\rangle = a\|u\|^2 + b\|u\|^4 - \lambda\int K(x)|x|^\beta|u|^q - \int K(x)|u|^6.$$

Let

$$c_* = \inf_{u\in\Lambda} I(u) \tag{3.1}$$

be the infimum of I on the Nehari manifold.

Lemma 3.1. *For any $u \in \Lambda$, there are $\delta, \sigma > 0$ such that $\|u\| \geq \delta$ and $\langle G'(u), u\rangle \leq -\sigma$.*

Proof. For any $u \in \Lambda$,

$$\begin{aligned} 0 &= \langle I'(u), u\rangle \\ &= a\|u\|^2 + b\|u\|^4 - \lambda\int K(x)|x|^\beta|u|^q - \int K(x)|u|^6 \\ &\geq a\|u\|^2 - \lambda S_q^{-q/2}\|u\|^q - S^{-3}\|u\|^6. \end{aligned}$$

From $q > 6 - \frac{4}{\alpha}$ and $\alpha \geq 2$, we have $q > 4$ and hence, there exists some $\delta > 0$ such that $\|u\| \geq \delta$. Furthermore,

$$\begin{aligned}\langle G'(u), u\rangle &= 2a\|u\|^2 + 4b\|u\|^4 - q\lambda \int K(x)|x|^\beta |u|^q - 6\int K(x)|u|^6 \\ &= (2a - qa)\|u\|^2 + (4b - qb)\|u\|^4 - (6-q)\int K(x)|u|^6 \\ &< (2a - qa)\|u\|^2 \\ &< -(qa - 2a)\delta^2 < 0.\end{aligned}$$

Set $\sigma = (qa - 2a)\delta^2$, this finishes the proof. □

Lemma 3.2. *The functional I is coercive and bounded from below on Λ.*

Proof. For $u \in \Lambda$, it follows from Lemma 3.1 and $6 - \frac{4}{\alpha} < q < 6$ that

$$\begin{aligned}I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q}\int K(x)|x|^\beta|u|^q - \frac{1}{6}\int K(x)|u|^6 \\ &= \left(\frac{a}{2} - \frac{a}{q}\right)\|u\|^2 + \left(\frac{b}{4} - \frac{b}{q}\right)\|u\|^4 + \left(\frac{1}{q} - \frac{1}{6}\right)\int K(x)|u|^6 \\ &\geq \left(\frac{a}{2} - \frac{a}{q}\right)\|u\|^2 \\ &\geq \left(\frac{a}{2} - \frac{a}{q}\right)\delta^2 > 0.\end{aligned}$$

Thus, the coercivity and lower boundedness of I hold. The proof of Lemma 3.2 is completed. □

Lemma 3.3. *Given $u \in \Lambda$, there exist $\rho_u > 0$ and a continuous function $g_{\rho_u} : B_{\rho_u}(0) \to \mathbb{R}^+$ defined for $w \in H$, $w \in B_{\rho_u}(0)$ such that*

$$g_{\rho_u}(0) = 1, \quad g_{\rho_u}(w)(u-w) \in \Lambda$$

and

$$\langle g'_{\rho_u}(0), \phi\rangle = \frac{(2a + 4b\|u\|^2)\int K(x)\nabla u \nabla\phi - q\lambda\int K(x)|x|^\beta|u|^{q-2}u\phi - 6\int K(x)|u|^4 u\phi}{a\|u\|^2 + 3b\|u\|^4 - \lambda(q-1)\int K(x)|x|^\beta|u|^q - 5\int K(x)|u|^6}.$$

Proof. Fix $u \in \Lambda$ and define $F : \mathbb{R}^+ \times H \to \mathbb{R}$ as below

$$F(t,w) = at\|u-w\|^2 + bt^3\|u-w\|^4 - \lambda t^{q-1}\int K(x)|x|^\beta|u-w|^q - t^5\int K(x)|u-w|^6.$$

Since $u \in \Lambda$, we have $F(1,0) = 0$. Moreover, using Lemma 3.1, we also have for $6 - \frac{4}{\alpha} < q < 6$

$$\begin{aligned}F_t(1,0) &= a\|u\|^2 + 3b\|u\|^4 - \lambda(q-1)\int K(x)|x|^\beta|u|^q - 5\int K(x)|u|^6 \\ &= a(2-q)\|u\|^2 + b(4-q)\|u\|^4 - (6-q)\int K(x)|u|^6 \\ &< a(2-q)\|u\|^2 \\ &\leq a(2-q)\delta^2 < 0.\end{aligned}$$

Using the implicit function theorem for F at the point $(1,0)$, we can conclude that there exists $\rho_u > 0$ satisfying for $w \in H$, $\|w\| < \rho_u$, the equation $F(t,w) = 0$ has a unique continuous solution $t = g_{\rho_u}(w) > 0$ with $g_{\rho_u}(0) = 1$. Since $F(g_{\rho_u}(w), w) = 0$ for $w \in H$, $\|w\| < \rho_u$, we get

$$\begin{aligned} & a g_{\rho_u}(w)\|u-w\|^2 + b g_{\rho_u}^3(w)\|u-w\|^4 - \lambda g_{\rho_u}^{q-1}(w)\|u-w\|_q^q - g_{\rho_u}^5(w)\|u-w\|_6^6 \\ &\quad = \frac{a\|g_{\rho_u}(w)(u-w)\|^2 + b\|g_{\rho_u}(w)(u-w)\|^4 - \lambda\|g_{\rho_u}(w)(u-w)\|_q^q - \|g_{\rho_u}(w)(u-w)\|_6^6}{g_{\rho_u}(w)} \\ &\quad = 0, \end{aligned}$$

that is,

$$g_{\rho_u}(w)(u-w) \in \Lambda, \quad \text{for all } w \in H,\ \|w\| < \rho_u.$$

Furthermore, we have for all $\phi \in H$, $r > 0$

$$\begin{aligned} & F(1, 0 + r\phi) - F(1,0) \\ &\quad = a\|u - r\phi\|^2 + b\|u - r\phi\|^4 - \lambda \int K(x)|x|^\beta |u - r\phi|^q - \int K(x)|u - r\phi|^6 \\ &\qquad - a\|u\|^2 - b\|u\|^4 + \lambda \int K(x)|x|^\beta |u|^q + \int K(x)|u|^6 \\ &\quad = -a \int K(x)\left(2r\nabla u \nabla\phi - r^2|\nabla\phi|^2\right) \\ &\qquad - b\left[2\int K(x)|\nabla u|^2 \int K(x)\left(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2\right) - \left(\int K(x)\left(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2\right)\right)^2\right] \\ &\qquad - \lambda \int K(x)|x|^\beta \Big(|u - r\phi|^q - |u|^q\Big) - \int K(x)\Big(|u - r\phi|^6 - |u|^6\Big) \end{aligned}$$

and consequently

$$\begin{aligned} & \langle F_w, \phi\rangle|_{t=1,w=0} \\ &\quad = \lim_{r\to 0} \frac{F(1, 0 + r\phi) - F(1,0)}{r} \\ &\quad = -(2a + 4b\|u\|^2)\int K(x)\nabla u\nabla\phi + q\lambda\int K(x)|x|^\beta|u|^{q-2}u\phi + 6\int K(x)|u|^4 u\phi. \end{aligned}$$

Thus,

$$\begin{aligned} \langle g'_{\rho_u}(0), \phi\rangle &= -\frac{\langle F_w, \phi\rangle}{F_t}\Big|_{t=1,w=0} \\ &= \frac{(2a + 4b\|u\|^2)\int K(x)\nabla u\nabla\phi - q\lambda\int K(x)|x|^\beta|u|^{q-2}u\phi - 6\int K(x)|u|^4u\phi}{a\|u\|^2 + 3b\|u\|^4 - \lambda(q-1)\int K(x)|x|^\beta|u|^q - 5\int K(x)|u|^6}. \end{aligned}$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *For any $u \in H \setminus \{0\}$, there exists a unique $t(u) > 0$ satisfying $t(u)u \in \Lambda$ and $I(t(u)u) = \max_{t>0} I(tu)$.*

Proof. The proof is similar to [24, Lemma 4.1], and is omitted here. □

Lemma 3.5. $c_* < c_1 := \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{3/2}}{24}$.

Proof. Let $\varphi(x) \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function satisfying $\varphi(x) \equiv 1$ in $B_\eta(0)$, $\varphi(x) \equiv 0$ outside $B_{2\eta}(0)$ and $0 \le \varphi \le 1$. Define

$$u_\varepsilon(x) = K^{-1/2}\varphi(x)\left(\frac{1}{\varepsilon + |x|^2}\right)^{1/2},$$

and set

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_6}.$$

According to [2], we have that

$$\|v_\varepsilon\|^2 = \int K(x)|\nabla v_\varepsilon|^2 = S + O(\varepsilon^{1/2}) + O(\varepsilon^{\alpha/2}) \tag{3.2}$$

and

$$\|u_\varepsilon\|_6^6 = \int K(x)|u_\varepsilon|^6 = \varepsilon^{-3/2}A_0 + O(1), \quad \text{with } A_0 = \int \frac{1}{(1+|x|^2)^3}. \tag{3.3}$$

Then, we obtain the following estimate

$$\|u_\varepsilon\|_6^q = \left(\varepsilon^{-3/2}A_0 + O(1)\right)^{q/6} = A_0^{q/6}\varepsilon^{-q/4} + O\left(\varepsilon^{-(q-6)/4}\right). \tag{3.4}$$

In addition, we also have

$$\begin{aligned}
\int K(x)|x|^\beta |u_\varepsilon|^q &= \int_{B_{2\eta}(0)} \frac{K(x)|x|^\beta K(x)^{-q/2}\varphi^q(x)}{(\varepsilon + |x|^2)^{q/2}} \\
&\ge C\int_{B_{2\eta}(0)} \frac{|x|^\beta \varphi^q(x)}{(\varepsilon + |x|^2)^{q/2}} \\
&= C\left(\int_{B_{2\eta}(0)} \frac{|x|^\beta}{(\varepsilon + |x|^2)^{q/2}} + \int_{B_{2\eta}(0)} \frac{|x|^\beta\left(\varphi^q(x) - 1\right)}{(\varepsilon + |x|^2)^{q/2}}\right) \\
&= C\left(\varepsilon^{\frac{\beta}{2} - \frac{q}{2} + \frac{3}{2}}\int_{B_{2\eta/\sqrt{\varepsilon}}(0)} \frac{|x|^\beta}{(1 + |x|^2)^{q/2}} + \int_{B_{2\eta}(0)} \frac{|x|^\beta\left(\varphi^q(x) - 1\right)}{(\varepsilon + |x|^2)^{q/2}}\right) \\
&= O\left(\varepsilon^{\frac{\beta}{2} - \frac{q}{2} + \frac{3}{2}}\right) + O(1),
\end{aligned}$$

whenever $\frac{6\alpha}{2+\alpha} < q < 6$. This and (3.4) imply that for $\frac{6\alpha}{2+\alpha} < q < 6$ and ε small enough, we have

$$\begin{aligned}
\int K(x)|x|^\beta |v_\varepsilon|^q &= \frac{\int K(x)|x|^\beta u_\varepsilon^q}{\|u_\varepsilon\|_6^q} \\
&\ge \frac{O\left(\varepsilon^{\frac{\beta}{2} - \frac{q}{2} + \frac{3}{2}}\right) + O(1)}{A_0^{q/6}\varepsilon^{-q/4} + O\left(\varepsilon^{-(q-6)/4}\right)} \\
&= O\left(\varepsilon^{\frac{\beta}{2} - \frac{q}{4} + \frac{3}{2}}\right) + O\left(\varepsilon^{q/4}\right).
\end{aligned} \tag{3.5}$$

By Lemma 3.4 and the definition of c_*, it is easy to see that Lemma 3.5 follows if we can show that

$$\sup_{t>0} I(tv_\varepsilon) < \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{\left(b^2S^4 + 4aS\right)^{3/2}}{24}.$$

To this goal, let

$$g(t) = I(tv_\varepsilon) = \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{\lambda}{q}t^q\int K(x)|x|^\beta v_\varepsilon^q - \frac{t^6}{6}, \quad t \geq 0$$

and

$$g_1(t) = \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{t^6}{6}, \quad t \geq 0.$$

Note that $\|v_\varepsilon\|_6 = 1$. Thus, by Lemma 3.4, we know that $g(t)$ has a unique maximum point $t_\varepsilon := t(v_\varepsilon) > 0$. We claim that $t_\varepsilon \geq C_0 > 0$ for some positive constant C_0 and any $\varepsilon > 0$. Otherwise, there is some sequence $\varepsilon_n \to 0$ satisfying $t_{\varepsilon_n} \to 0$ and $g(t_{\varepsilon_n}) = \sup_{t>0} I(tv_{\varepsilon_n})$. Then, by Lemma 2.1 and the continuity of I, we conclude that

$$0 < \theta \leq c_* \leq \lim_{n\to\infty} I(t_{\varepsilon_n}v_{\varepsilon_n}) = 0$$

which is a contradiction. Hence, the claim holds.

By (3.2), we also have that

$$\begin{aligned}\sup_{t>0} g_1(t) &= \frac{ab\|v_\varepsilon\|^6}{4} + \frac{b^3\|v_\varepsilon\|^{12}}{24} + \frac{\left(b^2\|v_\varepsilon\|^8 + 4a\|v_\varepsilon\|^2\right)^{3/2}}{24} \\ &= \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{3/2}}{24} + O(\varepsilon^{1/2}).\end{aligned} \tag{3.6}$$

Obviously, we have $6 - \frac{4}{\alpha} > \frac{6\alpha}{2+\alpha}$ provided $\alpha \geq 2$. Furthermore, by using (3.5) and (3.6), we obtain for $6 - \frac{4}{\alpha} < q < 6$ and ε small enough

$$\begin{aligned}\sup_{t>0} I(tv_\varepsilon) &= I(t_\varepsilon v_\varepsilon) \\ &= g_1(t_\varepsilon) - \frac{\lambda}{q}t_\varepsilon^q\int K(x)|x|^\beta v_\varepsilon^q \\ &\leq \sup_{t>0} g_1(t) - \frac{\lambda}{q}C_0^q\int K(x)|x|^\beta v_\varepsilon^q \\ &\leq \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{3/2}}{24} + O(\varepsilon^{1/2}) - O\left(\varepsilon^{\frac{\beta}{2} - \frac{q}{4} + \frac{3}{2}}\right) + O\left(\varepsilon^{\frac{q}{4}}\right) \\ &< \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{3/2}}{24}.\end{aligned}$$

This completes the proof. □

Lemma 3.6. *Let $\{u_n\} \subset \Lambda$ be a $(PS)_{c_*}$ sequence for I with $c_* < c_1$, where c_1 is given in Lemma 3.5. Then $\{u_n\}$ has a convergent subsequence.*

Proof. Let $\{u_n\} \subset \Lambda$ be a $(PS)_{c_*}$ sequence for I, namely

$$\lim_{n\to\infty} I(u_n) = c_* \quad \text{and} \quad \lim_{n\to\infty} I'(u_n) = 0. \tag{3.7}$$

Firstly, we show that $\|u_n\|$ is bounded. By (3.7), we have that for $q > 4$

$$\begin{aligned}c_* + 1 + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{a}{4}\|u_n\|^2 - \left(\frac{\lambda}{q} - \frac{\lambda}{4}\right)\int K(x)|x|^\beta|u_n|^q - \left(\frac{1}{6} - \frac{1}{4}\right)\int K(x)|u_n|^6 \\ &\geq \frac{a}{4}\|u_n\|^2\end{aligned}$$

which implies that $\|u_n\|$ is bounded. Up to a subsequence (still denoted by $\{u_n\}$), we may assume that

$$\begin{aligned} u_n &\rightharpoonup u_* \quad \text{in } H, \\ u_n &\to u_* \quad \text{in } L^r_K(\mathbb{R}^3),\ 2 \le r < 6, \\ u_n &\to u_* \quad \text{a.e. on } \mathbb{R}^3 \end{aligned}$$

and

$$\|u_n\|^2 \to \iota^2.$$

Since $u_n \in \Lambda$, it then follows from Lemma 3.1 that $\iota^2 > 0$.

Secondly, we prove that $u_* \not\equiv 0$. If not, we have $u_* \equiv 0$ and so $\int K(x)|x|^\beta |u_n|^q = o(1)$. On the other hand, by (3.7) and the boundedness of $\{u_n\}$, we have

$$o(1) = \langle I'(u_n), u_n \rangle = a\|u_n\|^2 + b\|u_n\|^4 - \lambda \int K(x)|x|^\beta |u_n|^q - \int K(x)|u_n|^6$$

and thus from (1.5),

$$a\|u_n\|^2 + b\|u_n\|^4 = \int K(x)|u_n|^6 + o(1) \le S^{-3}\|u_n\|^6 + o(1). \tag{3.8}$$

Letting $n \to \infty$ in (3.8), we have

$$\iota^2 \ge \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2}.$$

Consequently,

$$\begin{aligned} c_* &= \lim_{n\to\infty} I(u_n) \\ &= \lim_{n\to\infty} \left[\frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \frac{\lambda}{q}\int K(x)|x|^\beta |u_n|^q - \frac{1}{6}\int K(x)|u_n|^6 \right] \\ &= \lim_{n\to\infty} \left[\frac{a}{3}\|u_n\|^2 + \frac{b}{12}\|u_n\|^4 - \lambda \frac{6-q}{6q} \int K(x)|x|^\beta |u_n|^q \right] \\ &= \frac{a}{3}\iota^2 + \frac{b}{12}\iota^4 \\ &\ge c_1 \end{aligned}$$

in contradiction to the assumption $c_* < c_1$.

Finally, we claim that $\|u_n\|^2 \to \|u_*\|^2$. Indeed, if to the contrary, it follows from Fatou Lemma that

$$\iota^2 > \|u_*\|^2. \tag{3.9}$$

Since $I'(u_n) \to 0$, we also have for all $v \in H$

$$o(1) = \left(a + b\|u_n\|^2\right) \int K(x)\nabla u_n \nabla v - \lambda \int K(x)|x|^\beta |u_n|^{q-2} u_n v - \int K(x)|u_n|^4 u_n v.$$

Then, passing to the limit as $n \to \infty$, we get

$$0 = \left(a + b\iota^2\right) \int K(x)\nabla u_* \nabla v - \lambda \int K(x)|x|^\beta |u_*|^{q-2} u_* v - \int K(x)|u_*|^4 u_* v.$$

Taking $v = u_*$ in the above equation, we obtain

$$0 = \left(a + b\iota^2\right) \|u_*\|^2 - \lambda \int K(x)|x|^\beta |u_*|^q - \int K(x)|u_*|^6.$$

This together with (3.9) imply that $\langle I'(u_*), u_*\rangle < 0$. By Lemma 3.4, then it is easy to see that there exists $t_0 \in (0,1)$ such that $\langle I'(t_0 u_*), t_0 u_*\rangle = 0$. Therefore,

$$\begin{aligned}
c_* &\le \sup_{t>0} I(tu_*) \\
&= I(t_0 u_*) \\
&= I(t_0 u_*) - \frac{1}{4}\langle I'(t_0 u_*), t_0 u_*\rangle \\
&= \frac{a}{4} t_0^2 \|u_*\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right) t_0^q \int K(x)|x|^\beta |u_*|^q + \frac{1}{12} t_0^6 \int K(x)|u_*|^6 \\
&< \frac{a}{4}\|u_*\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right) \int K(x)|x|^\beta |u_*|^q + \frac{1}{12}\int K(x)|u_*|^6 \\
&\le \liminf_{n\to\infty} \left[\frac{a}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right) \int K(x)|x|^\beta |u_n|^q + \frac{1}{12}\int K(x)|u_n|^6\right] \\
&= \liminf_{n\to\infty} \left[I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle\right] \\
&= c_*
\end{aligned}$$

a contradiction. Hence, $\|u_n\| \to \|u_*\|$. This and the weak convergence of $\{u_n\}$ in H implies that $u_n \to u_*$ in H, and Lemma 3.6 is proved. □

Lemma 3.7. *For any $\lambda > 0$, there exists a sequence $\{u_n\} \subset \Lambda$ such that:*

$$u_n \ge 0, \quad I(u_n) \to c_* \quad \textit{and} \quad I'(u_n) \to 0.$$

Proof. In view of Lemma 3.2, we can apply Ekeland variational principle to construct a minimizing sequence $\{u_n\} \subset \Lambda$ satisfying the following properties:

(i) $I(u_n) \to c_*$,

(ii) $I(z) \ge I(u_n) - \frac{1}{n}\|u_n - z\|$ for all $z \in \Lambda$.

Since $I(|u|) = I(u)$, we can assume that $u_n \ge 0$ on $\mathbb{R}^3$. Let $0 < \rho < \rho_n \equiv \rho_{u_n}$, $g_n \equiv g_{u_n}$, where ρ_{u_n} and g_{u_n} are defined according to Lemma 3.3. Let $v_\rho = \rho u$ with $\|u\| = 1$. Fix n and let $z_\rho = g_n(v_\rho)(u_n - v_\rho)$. Since $z_\rho \in \Lambda$, by the property (ii), one gets

$$I(z_\rho) - I(u_n) \ge -\frac{1}{n}\|z_\rho - u_n\|.$$

It then follows from the definition of Fréchet derivative that

$$\langle I'(u_n), z_\rho - u_n\rangle + o(\|z_\rho - u_n\|) \ge -\frac{1}{n}\|z_\rho - u_n\|.$$

Thus,

$$\langle I'(u_n), -v_\rho + \left(g_n(v_\rho) - 1\right)(u_n - v_\rho)\rangle \ge -\frac{1}{n}\|z_\rho - u_n\| + o(\|z_\rho - u_n\|)$$

which implies

$$-\rho\langle I'(u_n), u\rangle + \big(g_n(v_\rho) - 1\big)\langle I'(u_n), u_n - v_\rho\rangle \geq -\frac{1}{n}\|z_\rho - u_n\| + o(\|z_\rho - u_n\|).$$

Therefore,

$$\langle I'(u_n), u\rangle \leq \frac{1}{n}\frac{\|z_\rho - u_n\|}{\rho} + \frac{o(\|z_\rho - u_n\|)}{\rho} + \frac{g_n(v_\rho) - 1}{\rho}\langle I'(u_n), u_n - v_\rho\rangle. \tag{3.10}$$

From $\|u\| = 1$, Lemma 3.3 and the boundedness of $\{u_n\}$, it follows that

$$\begin{aligned}\lim_{\rho\to 0}\frac{|g_n(v_\rho) - 1|}{\rho} &= \lim_{\rho\to 0}\frac{|g_n(0+\rho u) - g_n(0)|}{\rho}\\ &= \langle g_n'(0), u\rangle\\ &\leq \|g_n'(0)\|\\ &\leq C_1.\end{aligned}$$

Note that

$$\begin{aligned}\|z_\rho - u_n\| &= \|g_n(v_\rho)(u_n - v_\rho) - (u_n - v_\rho) - v_\rho\|\\ &= \|\,\big(g_n(v_\rho) - 1\big)\,(u_n - v_\rho) - v_\rho\|\\ &\leq |g_n(v_\rho) - 1|\cdot\|u_n - v_\rho\| + \|v_\rho\|\\ &= |g_n(v_\rho) - 1|C_2 + \rho.\end{aligned}$$

Furthermore, for fixed n, since $\langle I'(u_n), u_n\rangle = 0$ and $(u_n - v_\rho) \to u_n$ as $\rho \to 0$, by letting $\rho \to 0$ in (3.10) we can deduce that

$$\langle I'(u_n), u\rangle \leq \frac{C}{n},$$

which shows that $I'(u_n) \to 0$. This finishes the proof of Lemma 3.7. □

With the previous preparations, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.7, we see that there exists a minimizing sequence $\{u_n\} \subset \Lambda$ satisfying $u_n \geq 0$, $I(u_n) \to c_*$ and $I'(u_n) \to 0$ for any $\lambda > 0$. It then follows from Lemma 3.6 that $u_n \to u_*$, $I(u_*) = c_*$ and $u_* \geq 0$ is a weak solution of (1.1). By standard elliptic regularity argument and the strong maximum principle we have that $u_* > 0$. This and the definition of c_* imply that u_* is a positive ground state solution of (1.1) and the proof is complete. □

4 Positive solution for $2 < q < 4$

In this section, we apply Mountain Pass Theorem to obtain the existence of a positive solution for problem (1.1) when $2 < q < 4$.

Define

$$\tilde{c}_* := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I\left(\gamma(t)\right) \tag{4.1}$$

where

$$\Gamma := \{\gamma \in C\left([0,1], H\right):\ \gamma(0) = 0 \text{ and } I\left(\gamma(1)\right) < 0\}.$$

Lemma 4.1. *Assume* $2 < q < 4$. *Then there exists* $\lambda_* > 0$ *satisfying for all* $\lambda \in (0, \lambda_*)$

$$\tilde{c}_* < c_1 - D_0 \lambda^{\frac{4}{4-q}}$$

where c_1 *given in Lemma 3.5 and* $D_0 = \frac{4-q}{4}\left(\frac{3q}{b}\right)^{\frac{q}{4-q}}\left[\frac{(6-q)}{6q} S_q^{-q/2}\right]^{\frac{4}{4-q}}$.

Proof. We first recall that

$$u_\varepsilon(x) = K^{-1/2}\varphi(x)\left(\frac{1}{\varepsilon + |x|^2}\right)^{1/2}. \tag{4.2}$$

According to [5], we have that

$$\|u_\varepsilon\|^2 = \int K(x)|\nabla u_\varepsilon|^2 = \varepsilon^{-1/2} A_1 + O(1), \quad \text{with } A_1 = \int \frac{|x|^2}{(1+|x|^2)^3}. \tag{4.3}$$

Let

$$h(t) = \frac{a}{2} t^2 \|u_\varepsilon\|^2 + \frac{b}{4} t^4 \|u_\varepsilon\|^4 - \frac{1}{6} t^6 \int K(x)|u_\varepsilon|^6, \quad t \geq 0$$

By (3.3), (4.3) and the fact $A_1 A_0^{-1/3} = S$ (see [1]), we have that for $\varepsilon > 0$ small enough

$$\begin{aligned}
\sup_{t>0} h(t) &= \frac{ab\|u_\varepsilon\|^6}{4\|u_\varepsilon\|_6^6} + \frac{b^3\|u_\varepsilon\|^{12}}{24\|u_\varepsilon\|_6^{12}} + \frac{\left(b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2\|u_\varepsilon\|_6^6\right)^{3/2}}{24\|u_\varepsilon\|_6^{12}} \\
&= \frac{ab}{4}\frac{\left(A_1 + O(\varepsilon^{1/2})\right)^3}{A_0 + O(\varepsilon^{3/2})} + \frac{b^3}{24}\frac{\left(A_1 + O(\varepsilon^{1/2})\right)^6}{\left(A_0 + O(\varepsilon^{3/2})\right)^2} \\
&\quad + \frac{1}{24}\frac{\left[b^2\left(A_1 + O(\varepsilon^{1/2})\right)^4 + 4a\left(A_1 + O(\varepsilon^{1/2})\right)\left(A_0 + O(\varepsilon^{3/2})\right)\right]^{3/2}}{\left(A_0 + O(\varepsilon^{3/2})\right)^2} \\
&= \frac{abA_1^3A_0^{-1}}{4} + \frac{b^3A_1^6A_0^{-2}}{24} + \frac{\left(b^2A_1^4A_0^{-4/3} + 4aA_1A_0^{-1/3}\right)^{3/2}}{24} + O(\varepsilon^{1/2}) + O(\varepsilon^{3/2}) \\
&= \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{3/2}}{24} + O(\varepsilon^{1/2}).
\end{aligned} \tag{4.4}$$

By the definition of c_1 and D_0, we may choose small λ_1 such that for all $\lambda \in (0, \lambda_1)$,

$$c_1 - D_0\lambda^{\frac{4}{4-q}} > 0. \tag{4.5}$$

Clearly, $\lim_{t\to 0^+} h(t) = 0$ and hence, there exists $t_1 > 0$ satisfying for all $\lambda \in (0, \lambda_1)$

$$\sup_{0<t\leq t_1} I(tu_\varepsilon) < c_1 - D_0\lambda^{\frac{4}{4-q}}. \tag{4.6}$$

We consider the case $t > t_1$ next. Let $\varepsilon < \eta^2$, then

$$\begin{aligned}
\int K(x)|x|^\beta|u_\varepsilon|^q &\geq \int_{B_\eta(0)} \frac{K(x)|x|^\beta K(x)^{-q/2}\varphi^q(x)}{(\varepsilon + |x|^2)^{q/2}} \\
&\geq \exp\left(\left(1 - \frac{q}{2}\right)\eta^\alpha/4\right) \int_{B_\eta(0)} \frac{|x|^\beta}{(\varepsilon + |x|^2)^{q/2}} \\
&\geq \exp\left(\left(1 - \frac{q}{2}\right)\eta^\alpha/4\right) \frac{1}{(2\eta^2)^{q/2}} \int_{B_\eta(0)} |x|^\beta \\
&= \exp\left(\left(1 - \frac{q}{2}\right)\eta^\alpha/4\right) \frac{4\pi}{(2\eta^2)^{q/2}} \int_0^\eta r^{2+\beta} dr \\
&= \frac{4\pi\eta^{3+\beta-q}}{(3+\beta)2^{q/2}} \exp\left(\left(1 - \frac{q}{2}\right)\eta^\alpha/4\right).
\end{aligned} \tag{4.7}$$

By using (4.4) and (4.7), we have for all $\varepsilon = \lambda^{\frac{8}{4-q}} \in (0, \eta)$ and $t > t_1$

$$\begin{aligned} I(tu_\varepsilon) &= h(t) - \lambda \frac{t^q}{q} \int K(x)|x|^\beta u_\varepsilon^q \\ &\leq \sup_{t>0} h(t) - \lambda \frac{t_1^q}{q} \int K(x)|x|^\beta u_\varepsilon^q \\ &\leq \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{3/2}}{24} + O(\varepsilon^{1/2}) \\ &\quad - \lambda \frac{t_1^q}{q} \frac{4\pi\eta^{3+\beta-q}}{(3+\beta)2^{q/2}} \exp\left(\left(1 - \frac{q}{2}\right)\eta^\alpha/4\right) \\ &= c_1 + O(\lambda^{\frac{4}{4-q}}) - \lambda \frac{t_1^q}{q} \frac{4\pi\eta^{3+\beta-q}}{(3+\beta)2^{q/2}} \exp\left(\left(1 - \frac{q}{2}\right)\eta^\alpha/4\right). \end{aligned} \tag{4.8}$$

From $q \in (2, 4)$, we have $\frac{4}{4-q} > 2$. Thus, there exists $\lambda_2 > 0$ sufficient small such that for all $\lambda \in (0, \lambda_2)$

$$O(\lambda^{\frac{4}{4-q}}) - \lambda \frac{t_1^q}{q} \frac{4\pi\eta^{3+\beta-q}}{(3+\beta)2^{q/2}} \exp\left(\left(1 - \frac{q}{2}\right)\eta^\alpha/4\right) < -D_0\lambda^{\frac{4}{4-q}}. \tag{4.9}$$

Let $\lambda_* = \min\{\lambda_1, \lambda_2, \eta^{(4-q)/8}\}$ and $\varepsilon = \lambda^{\frac{8}{4-q}}$, then by (4.6), (4.8) and (4.9), we have that for all $\lambda \in (0, \lambda_*)$

$$\sup_{t>0} I(tu_\varepsilon) < c_1 - D_0\lambda^{\frac{4}{4-q}}.$$

This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $\{\tilde{u}_n\} \subset H$ be a $(PS)_c$ sequence for I with $c < c_1 - D_0\lambda^{\frac{4}{4-q}}$, where c_1 is defined as in Lemma 3.5. Then $\{\tilde{u}_n\}$ has a convergent subsequence.*

Proof. Let $\{\tilde{u}_n\} \subset H$ be a $(PS)_c$ sequence for I, that is

$$\lim_{n\to\infty} I(\tilde{u}_n) = c \quad \text{and} \quad \lim_{n\to\infty} I'(\tilde{u}_n) = 0. \tag{4.10}$$

Firstly, we show that $\|\tilde{u}_n\|$ is bounded. By (4.10), we have that

$$\begin{aligned} c + 1 + o(1)\|\tilde{u}_n\| &\geq I(\tilde{u}_n) - \frac{1}{6}\langle I'(\tilde{u}_n), \tilde{u}_n\rangle \\ &= \frac{a}{3}\|\tilde{u}_n\|^2 + \frac{b}{12}\|\tilde{u}_n\|^4 - \lambda\frac{6-q}{6q}\int K(x)|x|^\beta|\tilde{u}_n|^q \\ &\geq \frac{a}{3}\|\tilde{u}_n\|^2 + \frac{b}{12}\|\tilde{u}_n\|^4 - \lambda\frac{6-q}{6q}S_q^{-q/2}\|\tilde{u}_n\|^q \end{aligned}$$

which means that $\|\tilde{u}_n\|$ is bounded as $2 < q < 4$. After passing to a subsequence, we may assume that

$$\begin{aligned} &\tilde{u}_n \rightharpoonup \tilde{u}_* \quad \text{in } H, \\ &\tilde{u}_n \to \tilde{u}_* \quad \text{in } L_K^r(\mathbb{R}^3),\ 2 \leq r < 6, \\ &\tilde{u}_n \to \tilde{u}_* \quad \text{a.e. on } \mathbb{R}^3. \end{aligned}$$

Write $v_n = \tilde{u}_n - \tilde{u}_*$ and we claim that $\|v_n\| \to 0$. Otherwise, up to a subsequence (still denoted by $\{v_n\}$), we may suppose $\|v_n\| \to l$ with $l > 0$. From (4.10), we have that $\langle I'(\tilde{u}_n), \tilde{u}_* \rangle = o(1)$ and hence

$$0 = a\|\tilde{u}_*\|^2 + b(l^2 + \|\tilde{u}_*\|^2)\|\tilde{u}_*\|^2 - \lambda \int K(x)|x|^\beta |\tilde{u}_*|^q - \int K(x)|\tilde{u}_*|^6. \tag{4.11}$$

On the other hand, by $\langle I'(\tilde{u}_n), \tilde{u}_n \rangle = o(1)$, we can use Brézis–Lieb lemma to obtain

$$\begin{aligned} 0 &= a(\|v_n\|^2 + \|\tilde{u}_*\|^2) + b(\|v_n\|^4 + 2\|v_n\|^2\|\tilde{u}_*\|^2 + \|\tilde{u}_*\|^4) \\ &\quad - \lambda \int K(x)|x|^\beta |\tilde{u}_*|^q - \int K(x)|v_n|^6 - \int K(x)|\tilde{u}_*|^6 + o(1). \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we obtain

$$o(1) = a\|v_n\|^2 + b\|v_n\|^4 + b\|v_n\|^2\|\tilde{u}_*\|^2 - \int K(x)|v_n|^6 \tag{4.13}$$

and so, from (1.5) it follows that

$$a\|v_n\|^2 + b\|v_n\|^4 + b\|v_n\|^2\|\tilde{u}_*\|^2 = \int K(x)|v_n|^6 + o(1) \le S^{-3}\|v_n\|^6 + o(1).$$

Taking the limit as $n \to \infty$, we obtain that

$$l^2 \ge \frac{bS^3 + \sqrt{b^2S^6 + 4(a + b\|u_*\|^2)S^3}}{2} \ge \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2}. \tag{4.14}$$

By (4.11) and Hölder's inequality, we obtain

$$\begin{aligned} I(\tilde{u}_*) &= \frac{a}{2}\|\tilde{u}_*\|^2 + \frac{b}{4}\|\tilde{u}_*\|^4 - \frac{\lambda}{q}\int K(x)|x|^\beta|\tilde{u}_*|^q - \frac{1}{6}\int K(x)|\tilde{u}_*|^6 \\ &= \frac{a}{3}\|\tilde{u}_*\|^2 + \frac{b}{12}\|\tilde{u}_*\|^4 - \lambda\frac{6-q}{6q}\int K(x)|x|^\beta|\tilde{u}_*|^q - \frac{b}{6}l^2\|\tilde{u}_*\|^2 \\ &\ge \frac{b}{12}\|\tilde{u}_*\|^4 - \lambda\frac{6-q}{6q}S_q^{-q/2}\|\tilde{u}_*\|^q - \frac{b}{6}l^2\|\tilde{u}_*\|^2. \end{aligned}$$

For $A := \lambda\frac{6-q}{6q}S_q^{-q/2}$, define

$$\psi(t) = \frac{b}{12}t^4 - At^q.$$

By easy calculation, it follows that $\psi(t)$ achieves its minimum value at $t_{min} = \left(\frac{3pA}{b}\right)^{1/(4-q)}$ and

$$\psi(t_{min}) = -\left(\frac{3q}{b}\right)^{\frac{q}{4-q}}\frac{4-q}{4}A^{\frac{4}{4-q}}.$$

Thus, we have

$$\begin{aligned} I(\tilde{u}_*) &\ge -\frac{4-q}{4}\left(\frac{3q}{b}\right)^{\frac{q}{4-q}}\left[\frac{(6-q)}{6q}S_q^{-q/2}\right]^{\frac{4}{4-q}}\lambda^{\frac{4}{4-q}} - \frac{b}{6}l^2\|\tilde{u}_*\|^2 \\ &= -D_0\lambda^{\frac{4}{4-q}} - \frac{b}{6}l^2\|\tilde{u}_*\|^2. \end{aligned} \tag{4.15}$$

Then, we can use (4.13)–(4.15) to obtain

$$\begin{aligned}
c + o(1) &= I(\tilde{u}_n) \\
&= \frac{a}{2}\|\tilde{u}_n\|^2 + \frac{b}{4}\|\tilde{u}_n\|^4 - \frac{\lambda}{q}\int K(x)|x|^\beta|\tilde{u}_n|^q - \frac{1}{6}\int K(x)|\tilde{u}_n|^6 \\
&= \frac{a}{2}\|\tilde{u}_*\|^2 + \frac{b}{4}\|\tilde{u}_*\|^4 - \frac{\lambda}{q}\int K(x)|x|^\beta|\tilde{u}_*|^q - \frac{1}{6}\int K(x)|\tilde{u}_*|^6 \\
&\quad + \frac{a}{2}\|v_n\|^2 + \frac{b}{4}\|v_n\|^4 + \frac{b}{2}\|v_n\|^2\|\tilde{u}_*\|^2 - \frac{1}{6}\int K(x)|v_n|^6 + o(1) \\
&= I(\tilde{u}_*) + \frac{a}{2}\|v_n\|^2 + \frac{b}{4}\|v_n\|^4 + \frac{b}{2}\|v_n\|^2\|\tilde{u}_*\|^2 - \frac{1}{6}\int K(x)|v_n|^6 + o(1) \\
&= I(\tilde{u}_*) + \frac{a}{3}\|v_n\|^2 + \frac{b}{12}\|v_n\|^4 + \frac{b}{3}\|v_n\|^2\|\tilde{u}_*\|^2 + o(1) \\
&\geq I(\tilde{u}_*) + \frac{a}{3}l^2 + \frac{b}{12}l^4 + \frac{b}{6}l^2\|\tilde{u}_*\|^2 + o(1) \\
&\geq I(\tilde{u}_*) + c_1 + \frac{b}{6}l^2\|\tilde{u}_*\|^2 + o(1) \\
&\geq c_1 - D_0\lambda^{\frac{4}{4-q}}
\end{aligned}$$

which is a contradiction with our assumption $c < c_1 - D_0\lambda^{\frac{4}{4-q}}$. Thus, the claim follows, that is, $\tilde{u}_n \to \tilde{u}_*$ in H. This finishes the proof of Lemma 4.2. □

Now, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Using Lemma 2.1, we can apply Mountain Pass Theorem to obtain a sequence $\{\tilde{u}_n\} \subset \Lambda$ satisfying $I(\tilde{u}_n) \to \tilde{c}_*$ and $I'(\tilde{u}_n) \to 0$. It then follows from Lemmas 4.1 and 4.2 that there is $\lambda_* > 0$ such that $\tilde{u}_n \to \tilde{u}_*$ for all $\lambda \in (0, \lambda_*)$, and $\tilde{u}_*$ is a weak solution of (1.1). Furthermore, if we replace I by the following functional

$$\tilde{I}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q}\int_{\mathbb{R}^3} K(x)|x|^\beta(u^+)^q - \frac{1}{6}\int_{\mathbb{R}^3} K(x)(u^+)^6.$$

It is easy to see that all the above calculations can be repeated word for word. Then, we can infer from $\langle \tilde{I}'(\tilde{u}_*), \tilde{u}_*^- \rangle = 0$ that $\tilde{u}_*^- = 0$. In turn, we obtain $\tilde{u}_* \geq 0$. By standard elliptic regularity argument and the strong maximum principle, we also have that $\tilde{u}_* > 0$, that is, $\tilde{u}_*$ is a positive solution of (1.1). This completes the proof. □

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