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Regularity in generalized Morrey spaces of solutions to higher order nondivergence elliptic equations with *VMO* coefficients

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Abstract. We study the boundedness of the sublinear integral operators generated by Calderón–Zygmund operator and their commutators with *BMO* functions on generalized Morrey spaces. These obtained estimates are used to get regularity of the solution of Dirichlet problem for higher order linear elliptic operators.

Keywords: higher order elliptic equations, generalized Morrey spaces, Calderón–Zygmund integrals, commutators, *VMO*.

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1 Introduction

In recent years studying local and global regularity of the solutions of elliptic and parabolic differential equations with discontinuous coefficients is of great interest. In the case of smooth coefficients higher order elliptic equations studying in [1, 2, 13, 20, 34, 36]. They received the solvability of the Dirichlet problem, boundary estimates of the solutions and regularity of solutions. For parabolic operators these questions are studied in [4, 15, 19, 35].

However, the task is complicated by discontinuous coefficients. In general, with arbitrary discontinuous coefficients as L_p theory so strong solvability not true (see, [9–11]).

In particular, if we consider nondivergent elliptic equations of second order at $a_{ij}(x) \in W^1_n(\Omega)$ and the differences between the largest and lowest eigenvalues $\{a_{ij}\}$ are small enough, that is the condition of Cordes is satisfied, then $Lu \in L_2(\Omega)$ and $u \in W^2_2(\Omega)$. This result is extended to $W^2_p(\Omega)$ for $p \in (2-\varepsilon, 2+\varepsilon)$ with small enough ε.

In recent years Sarason introduced the *VMO* class of functions of vanishing mean oscillation, as tending to zero mean oscillation allowed to study local and global properties of second order elliptic equations. Chiarenza, Franciosi, Frasca and Longo [10, 11] show that if

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$a_{ij} \in VMO \cap L_\infty(\Omega)$ and $Lu \in L_p(\Omega)$, then $u \in \overset{\circ}{W}{}_2^2(\Omega)$, for $p \in (1,\infty)$. They also proved the solvability of Dirichlet problem in $W_p^2(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$. This result is extended to quasilinear equations with VMO coefficients in [18].

As a consequence, Hölder property of the solutions and their gradients for sufficiently small p are obtained. On the other hand, for small p with $Lu \in L_{p,\lambda}(\Omega)$ also takes place Hölder properties of solutions. There is a question of studying the properties of regularity of an operator L in Morrey spaces with VMO coefficients. In [6] Caffarelli proved that the solution from $W_p^2(\Omega)$ belongs to $C_{\rm loc}^{1+\alpha}(\Omega)$ if the function f is in Morrey $L_{n,n\alpha}^{\rm loc}(\Omega)$ with $\alpha \in (0,1)$. These conditions may be relaxed at $f \in L_{p,\lambda}^{\rm loc}(\Omega)$, $p < n$, $\lambda > 0$. In [17] inner regularity of second order derivatives from $W_p^2(\Omega)$ is proved. Moreover $D^2u \in L_{p,\lambda}^{\rm loc}(\Omega)$ at $f \in L_{p,\lambda}^{\rm loc}(\Omega)$ is shown if $a_{ij} \in VMO \cap L_\infty(\Omega)$.

Guliyev and Softova studied the global regularity of solution to nondivergence elliptic equations with VMO coefficients [27] in generalized Morrey space. These authors also considered parabolic operators with discontinuous coefficients [28]. Guliyev and Gadjiev [26] considered the second order elliptic equations in generalized Morrey spaces.

In fact, the better inclusion between the Morrey and the Hölder spaces permits to obtain regularity of the solutions to different elliptic and parabolic boundary problems. For the properties and applications of the classical Morrey spaces, we refer the readers to [6,17,22,23,33] and references therein.

The boundedness of the Hardy–Littlewood maximal operator in the Morrey spaces that allows us to prove continuity of fractional and classical Calderón–Zygmund operators in these spaces [7,8]. Recall that the integral operators of that kind appear in the representation formulas of the solutions of elliptic, parabolic equations and systems. Thus the continuity of the Calderón–Zygmund integrals implies regularity of the solutions in the corresponding spaces.

For more recent results on boundedness and continuity of singular integral operators in generalized Morrey and new function spaces and their application in the differential equations theory see [5,9,14,16,18,21,26,38–40] and the references therein.

Guliyev and Gadjiev considered higher order elliptic equations in generalized Morrey spaces in [29]. The solvability of Dirichlet boundary value problems for the higher order uniformly elliptic equations in generalized Morrey spaces is proved, see also [32], and the references in [29].

Our goal in these paper is to show the continuity of sublinear integral operators generated by Calderón–Zygmund operator and their commutators with BMO functions in generalized Morrey spaces. These obtained estimates are used to study regularity of the solution of Dirichlet problem for higher order linear uniformly elliptic operators.

2 Definition and statement of the problem

In this paper the following notations will be used: $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x = (x', x_n), \ \ x' \in \mathbb{R}^{n-1}, x_n > 0\}$, S^{n-1} is the unit sphere in $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a domain and $\Omega_r = \Omega \cap B_r(x)$, $x \in \Omega$, where $B_r = B(x_0,r) = \{x \in \mathbb{R}^n : |x-x_0| < r\}$, $B_r^c = \mathbb{R}^n \setminus B_r$, $B_r^+ = B^+(x_0,r) = B(x_0,r) \cap \{x_n > 0\}$. $D_i u = \frac{\partial u}{\partial x_i}$, $Du = (D_1 u, \ldots, D_n u)$ means the gradient of u, $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, where $|\alpha| = \sum_{k=1}^n \alpha_k$. The letter C are used for various positive constants and may change from one occurrence to another.

The domain $\Omega \subset \mathbb{R}^n$ supposed to be bounded with $\partial\Omega \in C^{1,1}$. Although this condition can be relaxed and task to consider in nonsmooth domains.

Definition 2.1. Let $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $1 \le p < \infty$. The generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ consists of all $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi^{-1}(x,r) \left(r^{-n} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

For any bounded domain Ω we define $M_{p,\varphi}(\Omega)$ taking $f \in L_p(\Omega)$ and Ω_r instead of $B(x,r)$ in the norm above.

The generalized Sobolev–Morrey space $W^{2m}_{p,\varphi}(\Omega)$ consists of all Sobolev functions $u \in W_p^{2m}(\Omega)$ with distributional derivatives $D^\alpha u \in M_{p,\varphi}(\Omega)$, endowed with the norm

$$\|u\|_{W^{2m}_{p,\varphi}(\Omega)} = \sum_{0\le|\alpha|\le 2m} \|D^\alpha u\|_{M_{p,\varphi}(\Omega)}.$$

The space $W^{2m}_{p,\varphi}(\Omega) \cap \overset{\circ}{W}{}^m_p$ consists of all functions $u \in W^{2m}_p(\Omega) \cap \overset{\circ}{W}{}^m_p$ with $D^\alpha u \in M_{p,\varphi}(\Omega)$ and is endowed by the same norm. Recall that $\overset{\circ}{W}{}^m_p$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in W^m_p.

Let a be a locally integrable function on R^n, then we shall define the commutators generated by an operator T and a as follows

$$T_a f(x) = [a, T] f(x) = T(af)(x) - a(x)T(f)(x).$$

Definition 2.2. Let Ω be an open set in $\mathbb{R}^n$ and $a(\cdot) \in L^1_{\text{loc}}(\Omega)$. We say that $a(\cdot) \in BMO$ (bounded mean oscillation) if

$$\|a\|_* = \sup_{x\in\Omega, \rho>0} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |a(y) - a_{\Omega(x,\rho)}| dy < \infty,$$

where $a_Q = \frac{1}{|Q|}\int_Q a(y)dy$ is the mean integral of $a(\cdot)$. The quantity $\|a\|_*$ is a norm in BMO of function $a(\cdot)$ and BMO is a Banach space.

We say that $a(\cdot) \in VMO(\Omega)$ (vanishing mean oscillation) if $a \in BMO(\Omega)$ and $r > 0$ define

$$\eta(r) = \sup_{x\in\Omega, \rho\le r} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |a(y) - a_{\Omega(x,\rho)}| dy < \infty,$$

and

$$\lim_{r\to 0} \eta(r) = \lim_{r\to 0} \sup_{x\in\Omega, \rho\le r} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |a(y) - a_{\Omega(x,\rho)}| dy = 0.$$

The quantity $\eta(r)$ is called VMO-modulus of a.

We consider the boundary value Dirichlet problem for higher order nondivergence uniformly elliptic equations with VMO coefficients in generalized Morrey spaces as follows

$$\begin{aligned} Lu(x) &:= \sum_{|\alpha|,|\beta|\le 2m} a_{\alpha\beta}(x) D^\alpha D^\beta u(x) = f(x) \quad \text{in } \Omega, \\ \frac{\partial^j u(x)}{\partial n^j} &= g(x), \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

$j = 0, \ldots, m-1$. The conditions for coefficients $a_{\alpha\beta}(\cdot)$ and right hand we give later.

3 Auxiliary results and interior estimate

In this section we present some results concerning continuity of sublinear operators generated by Calderón–Zygmund singular integrals. We also give continuity of commutators generated by sublinear operators and *BMO* functions in $M_{p,\varphi}(\mathbb{R}^n)$.

Lemma 3.1. *Let $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable function and $1 < p < \infty$. There exists a constant C such that for any $x \in \mathbb{R}^n$ and for all $t > 0$*

$$\int_r^\infty \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\,\varphi(x,r). \tag{3.1}$$

If T is a Calderón-Zygmund operator, then T is bounded in $M_{p,\varphi}(\mathbb{R}^n)$ for any $f \in M_{p,\varphi}(\mathbb{R}^n)$:

$$\|Tf\|_{M_{p,\varphi}(\mathbb{R}^n)} \le C\,\|f\|_{M_{p,\varphi}(\mathbb{R}^n)} \tag{3.2}$$

with constant C is independent of f.

This result is obtained in [3]. The following Corollary is obtained from this lemma and its proof is similar to the proof in Theorem 2.11 in [27].

Corollary 3.2. *Let Ω be an open set in $\mathbb{R}^n$ and C be a constant. Then for any $x \in \Omega$ and for all $t > 0$ we have*

$$\int_r^\infty \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\,\varphi(x,r), \qquad 1 < p < \infty.$$

If T is a Calderón-Zygmund operator, then T is bounded in $M_{p,\varphi}(\Omega)$ for any $f \in M_{p,\varphi}(\Omega)$, i.e.,

$$\|Tf\|_{M_{p,\varphi}(\Omega)} \le C\,\|f\|_{M_{p,\varphi}(\Omega)} \tag{3.3}$$

with constant C is independent of f.

Lemma 3.3. *Let $a \in BMO(\mathbb{R}^n)$ and the function φ satisfy the condition*

$$\int_r^\infty \left(1 + \log\frac{t}{r}\right) \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\,\varphi(x,r), \qquad 1 < p < \infty. \tag{3.4}$$

where C is independent of x and r. If the linear operator T satisfies the condition

$$|Tf(x)| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \qquad x \in \operatorname{supp} f \tag{3.5}$$

for any $f \in L_1(\mathbb{R}^n)$ with compact support and $[a, T]$ is bounded on $L_p(\mathbb{R}^n)$, then the operator $[a, T]$ is bounded on $M_{p,\varphi}(\mathbb{R}^n)$.

This result is obtained in [3, 24, 25]. From these lemmas and [18] we have the following.

Corollary 3.4. *Let the function $\varphi(\cdot)$ satisfy the condition (3.4) and $a \in BMO(\mathbb{R}^n)$. If T is a Calderón–Zygmund operator, then there exist a constant $C = C(n, p, \varphi)$, such that for any $f \in M_{p,\varphi}(\mathbb{R}^n)$ and $1 < p < \infty$,*

$$\|[a, T]\|_{M_{p,\varphi}(\mathbb{R}^n)} \le C\,\|a\|_* \|f\|_{M_{p,\varphi}(\mathbb{R}^n)}. \tag{3.6}$$

As in [6] we have the local version of Corollary 3.4.

Corollary 3.5. *Let the function $\varphi(\cdot)$ satisfy the condition (3.4). Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and $a(\cdot) \in VMO(\Omega)$. If T is a Calderón–Zygmund operator, then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon, \eta)$ such that for any ball $B_r(0)$ with radius $r \in (0, \rho_0)$, $\Omega(0,r) \neq \emptyset$ and for any $f \in M_{p,\varphi}(\Omega(0,r))$*

$$\|[a,T]\|_{M_{p,\varphi}(\Omega(0,r))} \leq C\,\varepsilon\,\|f\|_{M_{p,\varphi}(\Omega(0,r))}, \tag{3.7}$$

where $C = C(n,p,\varphi)$ is independent of ε, f, r.

These type of results are also valid for different generalized Morrey spaces $M_{p,\varphi_1}(\Omega)$ and $M_{p,\varphi_2}(\Omega)$. If $p = 1$, then the operator T is bounded from $M_{1,\varphi_1}(\mathbb{R}^n)$ to $WM_{1,\varphi_1}(\mathbb{R}^n)$. For example, we give the following results.

Lemma 3.6. *Let $a \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \leq C\,\varphi_2(x,r), \qquad 1 < p < \infty, \tag{3.8}$$

where C does not depend on x and r. Suppose T_a is a sublinear operator satisfying (3.5) and bounded on $L_p(\mathbb{R}^n)$. Then the operator $T_a = [a,T]$ is bounded from M_{p,φ_1} to M_{p,φ_2}, i.e.,

$$\|T_a f\|_{M_{p,\varphi_2}(\mathbb{R}^n)} \leq C\,\|a\|_*\|f\|_{M_{p,\varphi_1}(\mathbb{R}^n)}$$

with constant C is independent of f.

Besides that, *BMO* and *VMO* classes contain also discontinuous functions and the following example shows the inclusion $W_n^1(\mathbb{R}^n) \subset VMO \subset BMO$.

Example 3.7. $f_\alpha(x) = |\log|x||^\alpha \in VMO$ for any $\alpha \in (0,1)$; $f_\alpha \in W_n^1(\mathbb{R}^n)$ for $\alpha \in (0, 1-\frac{1}{n})$, $f_\alpha \notin W_n^1(\mathbb{R}^n)$ for $\alpha \in [1-\frac{1}{n}, 1)$; $f(x) = |\log|x|| \in BMO \setminus VMO$; $\sin f_\alpha(x) \in VMO \cap L_\infty(\mathbb{R}^n)$.

Now using boundedness of Calderón–Zygmund integral operators in generalized Morrey spaces we will get internal estimates for solutions of the problem (2.1) with coefficients from *VMO* spaces.

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 3$. We suppose that non-smooth boundary of Ω is Reifenberg flat (see Reifenberg [37]). It means that $\partial\Omega$ is well approximated by hyperplanes at each point and at each scale. This kind of regularity of the boundary mean also that the boundary has no inner or outer cusps.

Let coefficients $a_{\alpha\beta}$, $|\alpha|, |\beta| \leq m$ be symmetric and satisfy the conditions uniform ellipticity, essential boundedness of the coefficients $a_{\alpha\beta} \in L_\infty(\Omega)$ and regularity $a_{\alpha\beta} \in VMO(\Omega)$.

Let $f \in M_{p,\varphi}(\Omega)$, $1 < p < \infty$ and $\varphi(\cdot) : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable, and satisfy the condition

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \leq C\,\varphi(x,r), \tag{3.9}$$

where C does not depend on x,r.

From [2,10,17,30] we have interior representation, such that if $u \in \overset{\circ}{W}{}_p^{2m}$

$$\begin{aligned} D^\alpha u(x) = P.V. \int_B D^\alpha\Gamma(x, x-y) \left[\sum_{|\alpha|,|\beta|\leq m} (a_{\alpha\beta}(x) - a_{\alpha\beta}(y))D^\alpha u(y) + Lu(y) \right] dy \\ + Lu(x) \int_{|y|=1} D^\beta\Gamma(x,y)y_j d\delta_y \end{aligned} \tag{3.10}$$

for a.e. $x \in B \subset \Omega$, where B is a ball, $|\alpha| = |\beta| = m$, and $\Gamma(x,t)$ is the fundamental solution of L. Note that, $\Gamma(x,t)$ can be repsentated in the form

$$\Gamma(x,t) = \frac{1}{(n-2)\omega_n (\det a_{\alpha\beta})^{\frac{1}{2}}} \left(\sum_{i,j=1}^{n} A_{\alpha\beta}(x) t_i t_j \right)^{\frac{2-n}{2}},$$

for a.e. $x \in B$ and $\forall t \in \mathbb{R}^n \backslash \{0\}$, where $(A_{\alpha\beta})_{n\times n}$ is inverse matrix for $\{a_{\alpha\beta}\}_{n\times n}$.

Theorem 3.8 (Interior estimate). *Let Ω be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$ and the function $\varphi(\cdot)$ satisfy (3.9), $a_{\alpha\beta} \in VMO(\Omega)$, $|\alpha|, |\beta| \le m$, and*

$$M = \max_{i,j=1,n} \sup_{t\in\mathbb{R}^n} \|\Gamma(\cdot,t)\|_{L_\infty(\Omega)} < \infty.$$

Then there exists a positive constant $C(n,p,\varphi,M)$ such that for any $\Omega' \subset \Omega'' \subset \Omega$ and $u \in \overset{\circ}{W}{}_p^{2m}(\Omega)$ we have $D^\alpha D^\beta u \in M_{p,\varphi}(\Omega')$, $|\alpha|, |\beta| \le m$ and

$$\|D^\alpha D^\beta u\|_{M_{p,\varphi}(\Omega')} \le C \left(\|Lu\|_{M_{p,\varphi}(\Omega')} + \|u\|_{M_{p,\varphi}(\Omega')} \right). \tag{3.11}$$

Proof. We take an arbitrary point $x \in \operatorname{supp} u$ and a ball $B_r(x) \subset \Omega'$, and choose a point $x_0 \in B_r(x)$. Fix the coefficients of L in x_0. Consider the operator $L_0 = a_{\alpha\beta}(x_0) D^\alpha$. These operator have the constant coefficients. We know that a solution $\vartheta \in C_0^\infty(B_r(x_{x_0}))$ of $L_0\vartheta = (L_0 - L)\vartheta + L\vartheta$ can be presented as Newtonian type potential

$$\vartheta(x) = \int_{B_r} \Gamma^0(x-y) \left[(L_0 - L)\vartheta(y) + L\vartheta(y) \right] dy,$$

where $\Gamma^0(x-y) = \Gamma(x_0, x-y)$ is the fundamental solution of L_0. Taking $D^\alpha D^\beta \vartheta$ and unfreezing the coefficients we get for all $|\alpha|, |\beta| \le m$ by (3.10)

$$\begin{aligned} D^\alpha D^\beta \vartheta(x) &= P.V. \int_{B_r} D^\alpha D^\beta \Gamma(x, x-y) \left[(a_{\alpha\beta}(x) - a_{\alpha\beta}(y)) D^\alpha D^\beta u(y) + L\vartheta(y) \right] \\ &\quad + L\vartheta(x) \int_{S^n} D^\beta \Gamma(x,y) y_i d\sigma_y \\ &= R(L\vartheta)(x) + [a_{\alpha\beta}, R] D^\alpha D^\beta \vartheta(x) + L\vartheta(x) \int_{S^{n-1}} D^\beta \Gamma(x,y) y_i d\delta_y. \end{aligned} \tag{3.12}$$

The known properties of the fundamental solution imply that $D^\alpha D^\beta \Gamma(x,\xi)$ are variable Calderón–Zygmund kernels. The formula (3.12) holds for any $\vartheta \in W_p^{2m}(B_r) \cap \overset{\circ}{W}{}_p^m(B_r)$ because of the approximation properties of the Sobolev functions with C_0^∞ functions. For each $\varepsilon > 0$ there exists $r_0(\varepsilon)$ such that for any $r < r_0(\varepsilon)$

$$\|D^\alpha D^\beta \vartheta\|_{M_{p,\varphi}(B_r^+)} \le C \left(\varepsilon \|D^\alpha D^\beta \vartheta\|_{M_{p,\varphi}(B_r^+)} + \|L\vartheta\|_{M_{p,\varphi}(B_r^+)} \right).$$

Choosing ε small enough we can move the norm of $D^\alpha D^\beta \vartheta$ on the left-hand side that gives

$$\|D^\alpha D^\beta \vartheta\|_{M_{p,\varphi}(B_r^+)} \le C \, \|L\vartheta\|_{M_{p,\varphi}(B_r^+)} \tag{3.13}$$

with constant independent of ϑ.

Define a cut-off function $\eta(x)$ such that for $\theta \in (0,1)$, $\theta' = \frac{\theta(3-\theta)}{2} > 0$ and $|\alpha| \le m$ we have

$$\eta(x) = \begin{cases} 1, & x \in B_{\theta r}, \\ 0, & x \notin B_{\theta r}, \end{cases}$$

$\eta(x) \in C_0^\infty(B_r)$, $|D^\alpha \eta| \le C\,[\theta(1-\theta)r]^{-\alpha}$.

Applying (3.13) to $\vartheta(x) = \eta(x)u(x) \in W_p^{2m}(B_r) \cap \overset{\circ}{W}{}_p^m(B_r)$ we get

$$\begin{aligned}\|D^\alpha D^\beta \vartheta\|_{M_{p,\varphi}(B_{\theta r})} &\le C\,\|L\vartheta\|_{M_{p,\varphi}(B_{\theta' r})}\\ &\le C\left(\|L\vartheta\|_{M_{p,\varphi}(B'_{\theta r})} + \frac{\|Du\|_{M_{p,\varphi}(B_{\theta' r})}}{\theta(1-\theta)r} + \frac{\|u\|_{M_{p,\varphi}(B_{\theta' r})}}{[\theta(1-\theta)r]^2}\right)\end{aligned}$$

with constant independent of ϑ.

Define the weighted semi-norm

$$\Theta_\alpha = \sup_{0<\theta<1} [\theta(1-\theta)r]^{-\alpha} \|D^\alpha u\|_{M_{p,\varphi}(B_{\theta r})}, \qquad |\alpha| \le 2m.$$

Because of the choice of θ' we have $\theta(1-\theta) \le 2\theta'(1-\theta')$. Thus, after standard transformations and taking the supremum with respect to $\theta \in (0,1)$ the last inequality can be rewritten as

$$\Theta_{2m} \le C\,(r^2 \|Lu\|_{M_{p,\varphi}(B_r)} + \Theta_m + \Theta_0). \tag{3.14}$$

Now we use following interpolation inequality

$$\Theta_m \le \varepsilon \Theta_{2m} + \frac{C}{\varepsilon}\Theta_0 \quad \text{for any } \varepsilon \in (0, 2m).$$

Indeed, by simple scaling arguments we get in $M_{p,\varphi}(\mathbb{R}^n)$ an interpolation inequality analogous to [12, Theorem 7.28]

$$\|D^\alpha u\|_{M_{p,\varphi}(B_r)} \le \delta \|D^\alpha D^\beta \vartheta\|_{M_{p,\varphi}(B_r)} + \frac{C}{\delta}\|u\|_{M_{p,\varphi}}, \qquad \delta \in (0, r).$$

We can always find some $\varepsilon_0 \in (0,1)$ such that

$$\begin{aligned}\Theta_m &\le 2[\Theta_0(1-\Theta_0)r]\,\|D^\alpha u\|_{M_{p,\varphi}(B_{\Theta_0 r})}\\ &\le 2[\Theta_0(1-\Theta_0)r]\left(\delta\|D^\alpha D^\beta \vartheta\|_{M_{p,\varphi}(B_{\varepsilon_0 r})} + \frac{C}{\delta}\|u\|_{M_{p,\varphi}(B_{\varepsilon_0 r})}\right).\end{aligned}$$

The assertion follows choosing $\delta = \frac{\varepsilon}{2}[\varepsilon_0(1-\varepsilon_0)r] < \varepsilon_0 r$ for any $\varepsilon \in (0, 2m)$. Interpolating Θ_1 in (3.14) we obtain

$$\frac{r^2}{4}\|D^\alpha D^\beta u\|_{M_{p,\varphi}(B_{\frac{r}{2}})} \le \Theta_2 \le C\,(r^2\|Lu\|_{M_{p,\varphi}(B_r)} + \|u\|_{M_{p,\varphi}(B_r)})$$

and hence the Caccioppoli type estimate

$$\|D^\alpha D^\beta u\|_{M_{p,\varphi}(B_{\frac{r}{2}})} \le C\left(\|Lu\|_{M_{p,\varphi}(B_r)} + \frac{1}{r^2}\|u\|_{M_{p,\varphi}(B_r)}\right). \tag{3.15}$$

Let $\vartheta = \{\vartheta_{ij}\}_{i,j=1}^n \in [M_{p,\omega}(B_r)]^{n^2}$ be arbitrary function matrix. Define the operators

$$S_{ij\alpha\beta}(\vartheta_{ij})(x) = [a_{\alpha\beta}, R]\vartheta_{ij}(x), \qquad i,j = \overline{1,n},\ |\alpha|, |\beta \le m.$$

Because of the *VMO* properties of $a_{\alpha\beta}$'s we can choose r so small that

$$\sum_{i,j=1}^n \sum_{|\alpha|,|\beta|\le m} \|S_{ij\alpha\beta}\| < 1. \tag{3.16}$$

Now for a given $u \in W_p^{2m}(B_r) \cap \overset{\circ}{W}{}_p^m(B_r)$ with $Lu \in M_{p,\varphi}(B_r)$ we define

$$H(x) = RLu(x) + Lu(x) \int_{S^{n-1}} D^\beta \Gamma(x,y) y_i d\sigma_y.$$

Corollary 3.5 implies that $H \in M_{p,\varphi}(B_r)$. Define the operator W as

$$W\vartheta = \left\{ \sum_{|\alpha|,|\beta| \le m} \left(S_{ij\alpha\beta}\vartheta + H(x) \right) \right\}_{i,j=1}^{n} : [M_{p,\varphi}(B_r)]^{n^2} \to [M_{p,\varphi}(B_r)]^{n^2}.$$

By virtue (3.16) the operator W is a contraction mapping and there exists a unique fixed point $\tilde{\vartheta} = \{\tilde{\vartheta}_{ij}\}_{i,j=1}^n \in [M_{p,\varphi}(B_r)]^{n^2}$ of W such that $W\tilde{\vartheta} = \tilde{\vartheta}$. On the other hand it follows from the representation formula (3.12) that also $D^\alpha D^\beta u$ $|\alpha|, |\beta| \le m$ is a fixed point of W. Hence $D^\alpha D^\beta u = \tilde{\vartheta}$, that is $D^\alpha D^\beta u \in M_{p,\omega}(B_r)$ and in addition (3.15) holds. The interior estimate (3.11) follows from (3.15) by a finite covering of Ω' with balls $B_{\frac{r}{2}}$, $r < \text{dis}(\Omega', \partial\Omega'')$. □

4 Sublinear operators generated by nonsingular integral operators

We are passing to boundary estimates. Firstly we give some results by sublinear operators generated on nonsingular integral operators in the space $M_{p,\varphi}(\mathbb{R}^n_+)$.

In the beginning we consider a known result concerning the Hardy operator

$$Hg(r) = \frac{1}{r} \int_0^r g(t)dt, \qquad 0 < r < \infty.$$

Lemma 4.1 ([27]). *If*

$$A = C \sup_{r>0} \frac{\omega(r)}{r} \int_0^r \frac{dt}{\operatorname*{ess\,sup}_{0<s<t} \vartheta(s)} < \infty, \tag{4.1}$$

then the inequality

$$\operatorname*{ess\,sup}_{r>0} \omega(r) Hg(r) \le A \operatorname*{ess\,sup}_{r>0} \vartheta(r) g(r) \tag{4.2}$$

holds for all non-negative and non-increasing g on $(0,\infty)$.

For any $x \in \mathbb{R}^n_+$ define $\widetilde{x} = (x', -x_n)$ and recall that $x^0 = (x', 0)$. Let $\widetilde{T}$ be a sublinear operator such that for any function $f \in L_1(\mathbb{R}^n_+)$ with a compact support the inequality

$$|\widetilde{T} f(x)| \le C \int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\widetilde{x} - y|^n} dy, \tag{4.3}$$

holds, where constant C is independent of f.

Lemma 4.2. *Suppose that $f \in L_p^{loc}(\mathbb{R}^n_+)$ and $1 \le p < \infty$. Let*

$$\int_1^\infty t^{-\frac{n}{p}-1} \|f\|_{L_p(B^+(x^0,t))} dt < \infty \tag{4.4}$$

and $\widetilde{T}$ be a sublinear operator satisfying (4.3).

1. *If $p > 1$ and $\widetilde{T}$ is bounded on $L_p(\mathbb{R}^n_+)$, then*

$$\|\widetilde{T} f\|_{L_p(B^+(x^0,t))} \le C r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}-1} \|f\|_{L_p(B^+(x^0,t))} dt. \tag{4.5}$$

2. *If $p > 1$ and $\widetilde{T}$ is bounded from $L_1(\mathbb{R}^n_+)$ on $WL_1(\mathbb{R}^n_+)$, then*

$$\|\widetilde{T}f\|_{WL_1(B^+(x^0,t))} \le C \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B^+(x^0,t))} dt, \tag{4.6}$$

where the constant C is independent of x^0, r and f.

This lemma is proved in [27].

Lemma 4.3. *Let $1 < p < \infty$, $\varphi_1, \varphi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying for any $x \in \mathbb{R}^n$ and for any $t > 0$*

$$\int_r^{\infty} \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x,r) \tag{4.7}$$

and $\widetilde{T}$ be a sublinear operator satisfying (4.3).

1. *If $p > 1$ and $\widetilde{T}$ is bounded in $L_p(\mathbb{R}^n_+)$, then it is bounded from $M_{p,\varphi_1}(\mathbb{R}^n_+)$ to $M_{p,\varphi_2}(\mathbb{R}^n_+)$ and*

$$\|\widetilde{T}f\|_{M_{p,\varphi_2}(\mathbb{R}^n_+)} \le C\, \|f\|_{M_{p,\varphi_1}(\mathbb{R}^n_+)}. \tag{4.8}$$

2. *If $p = 1$ and $\widetilde{T}$ is bounded in $L_1(\mathbb{R}^n_+)$ to $WL_1(\mathbb{R}^n_+)$, then it is bounded from $M_{1,\varphi_1}(\mathbb{R}^n_+)$ to $WM_{1,\varphi_2}(\mathbb{R}^n_+)$ and*

$$\|\widetilde{T}f\|_{M_{1,\varphi_2}(\mathbb{R}^n_+)} \le C\, \|f\|_{WM_{1,\varphi_1}(\mathbb{R}^n_+)}$$

with constant C is independent of f.

This lemma is proved in [27].

5 Commutators of sublinear operators generated by nonsingular integrals

Now we consider commutators of sublinear operators generated by nonsingular integrals in the space $M_{p,\varphi}(\mathbb{R}^n_+)$.

For a function $a \in BMO$ and sublinear operator $\widetilde{T}$ satisfying (4.3) we define the commutator as $\widetilde{T}_a f = \widetilde{T}[a,f] = a\widetilde{T}f - \widetilde{T}(af)$. Suppose that for any $f \in L_1(\mathbb{R}^n_+)$ with compact support and $x \notin \operatorname{supp} f$ the following inequality is valid

$$|\widetilde{T}_a f(x)| \le C \int_{\mathbb{R}^n_+} |a(x) - a(y)| \frac{|f(y)|}{|x-y|^n} dy, \tag{5.1}$$

where the constant a is independent of f and x. Suppose also that $\widetilde{T}_a$ is bounded in $L_p(\mathbb{R}^n_+)$, $p \in (1,\infty)$, and satisfy the following inequality

$$\|\widetilde{T}_a f\|_{L_p(\mathbb{R}^n_+)} \le C\, \|a\|_* \|f\|_{L_p(\mathbb{R}^n_+)},$$

where the constant C is independent of f. Our aim is to show boundedness of $\widetilde{T}_a$ in $M_{p,\varphi}(\mathbb{R}^n_+)$. We recall properties of the BMO functions. The following lemma is proved by John–Nirenberg in [31].

Lemma 5.1. *Let $a \in BMO(\mathbb{R}^n)$ and $p \in (1,\infty)$. Then for any ball B the following inequality holds*

$$\left(\frac{1}{|B|}\int_B |a(y)-a_B|^p dy\right)^{\frac{1}{p}} \le C\,(p)\|a\|_*.$$

As a consequence of Lemma 5.1 we get the following corollary.

Corollary 5.2. *If $a \in BMO$, then for all $0 < 2r < t$ the following inequality holds*

$$|a_{B_r} - a_{B_t}| \le C\,\|a\|_* \ln\frac{t}{r}, \tag{5.2}$$

where the constant C is independent of a.

For the estimate of the commutator we use the following lemma in the proof of Theorem 5.4.

Lemma 5.3 ([27])**.** *Let $\widetilde{T}_a$ be a bounded operator in $L_p(\mathbb{R}^n_+)$ satisfying (5.1) and $1 < p < \infty$, $a \in BMO$. Suppose that for $f \in L_p^{\rm loc}(\mathbb{R}^n_+)$ and $r > 0$ the following holds*

$$\int_t^\infty \left(1+\ln\frac{t}{r}\right) t^{-\frac{n}{p}-1} \|f\|_{L_p(B_t^+(x^0,t))} dt < \infty. \tag{5.3}$$

Then we have

$$\|\widetilde{T}_a f\|_{L_p(B_r^+)} \le C\,\|a\|_* r^{\frac{n}{p}} \int_{2r}^\infty \left(1+\ln\frac{t}{r}\right) \|f\|_{L_p(B_t^+(x^0,t))} \frac{dt}{t^{\frac{n}{p}+1}},$$

where the constant C is independent of f.

Theorem 5.4. *Let $\varphi_1, \varphi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying (4.7) and $1 < p < \infty$, $a \in BMO$. Suppose $\widetilde{T}_a$ is a sublinear operator bounded on $L_p(\mathbb{R}^n_+)$ and satisfying (5.1). Then $\widetilde{T}_a$ is bounded from $M_{p,\varphi_1}(\mathbb{R}^n_+)$ to $M_{p,\varphi_2}(\mathbb{R}^n_+)$ and*

$$\|\widetilde{T}_a f\|_{M_{p,\varphi_2}(\mathbb{R}^n_+)} \le C\,\|a\|_* \|f\|_{M_{p,\varphi_1}(\mathbb{R}^n_+)}, \tag{5.4}$$

where the constant C is independent of f.

The proof of the Theorem 5.4 follows from Lemmas 4.2, 5.1 and 5.3.

6 Singular and nonsingular integral operators

Now we consider singular and nonsingular integral operators in the spaces $M_{p,\varphi}$. We deal with Calderón–Zygmund type integrals and their commutators with BMO functions.

A measurable function $K(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \backslash\{0\} \to \mathbb{R}$ is called a variable Calderón–Zygmund kernel if

1. $K(x,\xi)$ is a Calderón–Zygmund kernel for all $x \in \mathbb{R}^n$:

   1_a $K(x,\cdot) \in C^\infty(\mathbb{R}^n\backslash\{0\})$;

   1_b $K(x,\mu\xi) = \mu^{-n}K(x,\xi), \quad \forall \mu > 0$;

   1_c $\int_{S^{n-1}} K(x,\xi)d\sigma_\xi = 0$, $\int_{S^{n-1}} |K(x,\xi)|d\sigma_\xi < +\infty$.

2. $\max\limits_{|\alpha|,|\beta|\le m} \|D_x^\alpha D_\xi^\beta K(x,\xi)\|_{L_\infty(\mathbb{R}^n\times S^{n-1})} = M < \infty$

and M is independent of x.

The singular integral

$$Rf(x) = P.V. \int_{\mathbb{R}^n} K(x, x-y) f(y) dy$$

and its commutators

$$[a, R]f(x) := P.V. \int_{\mathbb{R}^n} K(x, x-y) f(y)[a(x) - a(y)] dy = a(x)Rf(x) - R(af)(x)$$

are bounded in $L_p(\mathbb{R}^n)$ (see [9]). Moreover

$$|K(x, \xi)| \le |\xi|^{-n} |K(x, \frac{\xi}{|\xi|})| \le M|\xi|^{-n}.$$

Then we have

$$|Rf(x)| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy,$$
$$|[a, R]f(x)| \le C \int_{\mathbb{R}^n} \frac{|a(x) - a(y)||f(y)|}{|x-y|^n} dy$$

where the constants C are independent of f.

Lemma 6.1. *Let the function $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the condition (3.9) and $1 < p < \infty$. Then for any $f \in M_{p,\varphi}(\mathbb{R}^n)$ and $a \in BMO$ there exist constants depending on n, p, φ and the Kernel such that*

$$\|Rf\|_{M_{p,\varphi}(\mathbb{R}^n)} \le C \, \|f\|_{M_{p,\varphi}(\mathbb{R}^n)},$$
$$\|[a, R]f\|_{M_{p,\varphi}(\mathbb{R}^n)} \le C \, \|a\|_* \|f\|_{M_{p,\varphi}(\mathbb{R}^n)}$$

where constants are independent of f.

The assertion of this lemma follows by (4.8) and (3.6).

For studying regularity properties of the solution of Dirichlet problem (2.1) we need some additional local results.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $a \in BMO(\Omega)$. Suppose the function $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the condition (3.9) and $f \in M_{p,\varphi}(\Omega)$ with $1 < p < \infty$. Then*

$$\|Rf\|_{M_{p,\varphi}(\Omega)} \le C \, \|f\|_{M_{p,\varphi}(\Omega)},$$
$$\|[a, R]f\|_{M_{p,\varphi}(\Omega)} \le C \, \|a\|_* \|f\|_{M_{p,\varphi}(\Omega)}, \tag{6.1}$$

where $C = C(n, p, \varphi, \Omega, K)$ is independent of f.

Lemma 6.3. *Let the conditions of Lemma 6.1 be satisfied and $a \in VMO(\mathbb{R}^n_+)$ with VMO-modulus γ_a. Then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ such that for any ball B_r with a radius $r \in (0, \rho_0)$ and all $f \in M_{p,\varphi}(B_r)$ the following inequality holds*

$$\|[a, R]f\|_{M_{p,\varphi}(B_r^+)} \le C \, \varepsilon \|f\|_{M_{p,\varphi}(B_r^+)} \tag{6.2}$$

with $C = C(n, p, \varphi, \Omega, K)$ being independent of f.

To obtain above estimates it is sufficient to extend $K(x,\cdot)$ and $f(\cdot)$ as zero outside Ω. This extension keeps its *BMO* norm or *VMO* modulus according to [10].

For any $x,y\in\mathbb{R}^n_+$, $\widetilde{x}=(x',-x_n)$ define the generalized reflection $\mathcal{T}(x,y)$ as

$$\mathcal{T}(x,y)=x-2x_n\frac{a^n_{\alpha\beta}(y)}{a^{nn}_{\alpha\beta}(y)},$$
$$\mathcal{T}(x)=\mathcal{T}(x,x):\mathbb{R}^n_+\to\mathbb{R}^n_-,$$

where $a^n_{\alpha\beta}$ is the last row of the coefficients matrix $(a_{\alpha\beta})_{\alpha,\beta}$. Then there exists a positive constant C depending on n and Λ, such that

$$C^{-1}\,|\widetilde{x}-y|\le|\mathcal{T}(x)|\le C\,|\widetilde{x}-y|,\qquad\forall\,x,y\in\mathbb{R}^n_+.$$

For any $f\in M_{p,\varphi}(\mathbb{R}^n_+)$ and $a\in BMO(\mathbb{R}^n_+)$ consider the nonsingular integral operators

$$\widetilde{R}f(x)=\int_{\mathbb{R}^n_+}K(x,\mathcal{T}(x)-y)f(y)dy,$$
$$[a,\widetilde{R}]f(x)=a(x)\widetilde{R}f(x)-\widetilde{R}(af)(x).$$

The kernel $K(x,\mathcal{T}(x)-y):\mathbb{R}^n\times\mathbb{R}^n_+\to R$ is not singular and verifies the conditions 1_b and 2 from Calderón–Zygmund kernel. Moreover

$$|K(x,\mathcal{T}(x)-y)|\le M|\mathcal{T}(x)-y|^{-n}\le C\,|\widetilde{x}-y|^{-n}$$

implies

$$|\widetilde{R}f(x)|\le C\int_{\mathbb{R}^n_+}\frac{|f(y)|}{|\widetilde{x}-y|^n}dy,$$
$$|[a,\widetilde{R}]f(x)|\le C\int_{\mathbb{R}^n_+}\frac{|a(x)-a(y)||f(y)|}{|\widetilde{x}-y|^n}dy,$$

where constant C is independent of f.

The following estimates are simple consequence of the previous results.

Lemma 6.4. *Let φ be measurable function satisfying condition (6.1) and $a\in BMO(\Omega)$, $p\in(1,\infty)$. Then the operator $\widetilde{R}f$ and $[a,\widetilde{R}]f$ are continuous in $M_{p,\varphi}(\mathbb{R}^n_+)$ and for all $f\in M_{p,\varphi}(\mathbb{R}^n_+)$ the following holds*

$$\|\widetilde{R}f\|_{M_{p,\varphi}(\mathbb{R}^n_+)}\le C\,\|f\|_{M_{p,\varphi}(\mathbb{R}^n_+)},$$
$$\|[a,\widetilde{R}]f\|_{M_{p,\varphi}(\mathbb{R}^n_+)}\le C\,\|a\|_*\|f\|_{M_{p,\varphi}(\mathbb{R}^n_+)},\tag{6.3}$$

where constants C are dependent on known quantities only.

Lemma 6.5. *Let φ be measurable function satisfying condition (6.1), $a\in VMO(\mathbb{R}^n_+)$ with VMO-modulus γ_a and $p\in(1,\infty)$. Then for any $\varepsilon>0$ there exists a positive number $\rho_0=\rho_0(\varepsilon,\gamma_a)$ such that for any ball B^+_r with a radius $r\in(0,\rho_0)$ and all $f\in M_{p,\varphi}(B^+_r)$ the following holds*

$$\|[a,\widetilde{R}]f\|_{M_{p,\varphi}(B^+_r)}\le C\,\varepsilon\,\|f\|_{M_{p,\varphi}(B^+_r)}\tag{6.4}$$

where C is independent of ε, f and r.

The proof is as in [9].

7 Boundary estimates of solutions

We formulate the problem (2.1) again. We consider the Dirichlet problem for linear nondivergent equation of order $2m$

$$Lu(x) = \sum_{|\alpha|,|\beta|\le m} a_{\alpha\beta}(x) D^\alpha D^\beta u(x) = f(x), \qquad x \in \Omega,$$
$$u \in W^{2m}_{p,\varphi}(\Omega) \cap \mathring{W}^m_p(\Omega), \qquad p \in (1,\infty) \tag{7.1}$$

subject to the following conditions: there exists a constant $\lambda > 0$ such that

$$\begin{aligned} \lambda^{-1}|\xi|^{2m} \le \sum_{|\alpha|,|\beta|\le m} a_{\alpha\beta}\xi_\alpha\xi_\beta \le \lambda|\xi|^{2m} \\ a_{\alpha\beta}(x) = a_{\beta\alpha}(x), \qquad |\alpha|,|\beta| \le m, \end{aligned} \tag{7.2}$$

i.e. the operator L has uniform ellipticity. The last assumption implies immediately essential boundedness of the coefficients $a_{\alpha\beta}(x) \in L_\infty(\Omega)$ and $a_{\alpha\beta}(x) \in VMO(\Omega)$, $f \in M_{p,\varphi}(\Omega)$ with $1 < p < \infty$, $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is measurable.

To prove a local boundary estimate for the norm $D^\alpha D^\beta u$ we define the space $W^{2m,\gamma_0}_p(B^+_r)$ as a closure of $C_{\gamma_0} = \{u \in C^\infty_0(B(x^0,r)) : D^\alpha u(x) = 0 \text{ for } x_n \le 0\}$ with respect to the norm of W^{2m}_p.

Theorem 7.1 (Boundary estimate)**.** *Suppose that $u \in W^{2m,\gamma_0}_p(B^+_r)$ and $Lu \in M_{p,\varphi}(B^+_r)$ with $1 < p < \infty$ and φ satisfies (6.1). Then $D^\alpha D^\beta u(x) \in M_{p,\varphi}(B^+_r)$, $|\alpha|,|\beta| \le m$ and for each $\varepsilon > 0$ there exists $r_0(\varepsilon)$ such that*

$$\|D^\alpha D^\beta u\|_{M_{p.\varphi}(B^+_r)} \le C\,\|Lu\|_{M_{p.\varphi}(B^+_r)} \tag{7.3}$$

for any $r \in (0, r_0)$.

Proof. For $u \in W^{2m,\gamma_0}_p(B^+_r)$ the boundary representation formula holds (see [29])

$$\begin{aligned} D^\alpha D^\beta u(x) = P.V. \int_{B^+_r} D^\alpha D^\beta \Gamma(x, x-y) Lu(y) dy \\ + P.V. \int_{B^+_r} D^\alpha D^\beta \Gamma(x, x-y)[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)] D^\alpha D^\beta u(y) dy \\ + Lu(x) \int_{S^{n-1}} D^\alpha \Gamma(x,y) y_i d\sigma_y + I_{\alpha,\beta}(x), \end{aligned} \tag{7.4}$$

$\forall\, i = \overline{1,n}$, $|\alpha|,|\beta| \le m$, where we have set

$$\begin{aligned} I_{\alpha,\beta}(x) = \int_{B^+_r} D^\alpha D^\beta (x, \mathcal{T}(x) - y) Lu(y) dy \\ + \int_{B^+_r} D^\alpha D^\beta (x, \mathcal{T}(x) - y)[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)] D^\alpha D^\beta u(y) dy, \end{aligned}$$

$|\alpha|,|\beta| \le m-1$,

$$\begin{aligned} I_{\alpha,m}(x) &= I_{m,\alpha}(x) \\ &= \int_{B^+_r} D^\alpha D^\beta (x, \mathcal{T}(x) - y)(D^m \mathcal{T}(x))^\ell \{[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)] D^\alpha D^\beta u(y) + Lu(y)\} dy, \\ I_{mm}(x) &= \int_{B^+_r} D^\alpha D^\beta (x, \mathcal{T}(x) - y)(D^m \mathcal{T}(x))^\ell (D^m \mathcal{T}(x))^s \\ &\quad \times \{[a_{\alpha\beta}(x) - a_{\alpha\beta}(y)] D^\alpha D^\beta u(y) + Lu(y)\} dy, \end{aligned}$$

where $D^m\mathcal{T}(x) = ((D_m\mathcal{T}(x))^1, \dots, (D_m\mathcal{T}(x))^n) = \mathcal{T}(\ell_n, x)$. Applying estimates (6.3), (6.4) and taking into account the *VMO* properties of the coefficients $a_{\alpha\beta}$'s, it is possible to choose r_0 so small that

$$\|D^\alpha D^\beta u\|_{M_{p,\varphi}(B_r^+)} \le C\,\|Lu\|_{M_{p,\varphi}(B_r^+)}$$

for each $r < r_0$. For an arbitrary matrix function $\mathrm{w} = \{\mathrm{w}_{ij}\}_{i,j=1}^n \in [M_{p,\varphi}(B_r^+)]^{n^2}$ define

$$\begin{aligned}
S_{ij\alpha\beta}(\mathrm{w}_{\alpha\beta})(x) &= [a_{\alpha\beta}, B_{ij}]\mathrm{w}_{\alpha\beta}(x), \qquad i,j=\overline{1,n}, |\alpha|\le m, |\beta|\le m,\\
\widetilde{S}_{ij\alpha\beta}(\mathrm{w}_{\alpha\beta})(x) &= [a_{\alpha\beta}, \widetilde{B}_{ij}]\mathrm{w}_{\alpha\beta}(x), \qquad i,j=\overline{1,n-1}, |\alpha|\le m, |\beta|\le m,\\
\widetilde{S}_{in\alpha\beta}(\mathrm{w}_{\alpha\beta})(x) &= [a_{\alpha\beta}, \widetilde{B}_{ij}]\mathrm{w}_{\alpha\beta}(D_n\mathcal{T}(x))^\ell, \qquad i,j=\overline{1,n}, |\alpha|\le m, |\beta|\le m,\\
\widetilde{S}_{nn\alpha\beta}(\mathrm{w}_{\alpha\beta})(x) &= [a_{\alpha\beta}, \widetilde{B}_{\ell s}]\mathrm{w}_{\alpha\beta}(D_n\mathcal{T}(x))^\ell(D_n\mathcal{T}(x))^s, \qquad |\alpha|\le m, |\beta|\le m.
\end{aligned}$$

From (6.2) and (6.4) we can take r so small that

$$\sum_{i,j=1}^n \sum_{|\alpha|,|\beta|\le m} \|S_{ij\alpha\beta} + \widetilde{S}_{ij\alpha\beta}\| < 1. \tag{7.5}$$

Now given $u \in W_p^{2m,\gamma_0}(B_r^+)$ with $Lu \in M_{p,\varphi}(B_r^+)$ we set

$$\begin{aligned}
\widetilde{H}(x) &= RLu(x) + \widetilde{R}Lu(x) + \widetilde{R}Lu(x)(D_n\mathcal{T}(x))^\ell\\
&\quad + \widetilde{R}_{\ell s}Lu(x)(D_n\mathcal{T}(x))^\ell(D_n\mathcal{T}(x))^s + Lu(x)\int_{S^{n-1}} D^\alpha\Gamma(x,y)y_i d\sigma_y.
\end{aligned}$$

Then estimates (6.1) and (6.3) imply $\widetilde{H} \in M_{p,\varphi}(B_r^+)$. Define the operator

$$U\mathrm{w} = \left\{ \sum_{|\alpha|,|\beta|\le m} \Big(S_{ij\alpha\beta}(\mathrm{w}_{\alpha\beta}) + \widetilde{S}_{ij\alpha\beta}(\mathrm{w}_{\alpha\beta}) + \widetilde{H}_{ij}(x) \Big) \right\}_{i,j=1}^n .$$

By virtue of (7.5) it is a contraction mapping in $[M_{p,\varphi}(B_r^+)]^{n^2}$ and there is a unique fixed point $\widetilde{\mathrm{w}} = \{\widetilde{\mathrm{w}}_{\alpha\beta}\}_{|\alpha|,|\beta|\le m}^n$ such that $U\widetilde{\mathrm{w}} = \widetilde{\mathrm{w}}$. On the other hand, it follows from the representation formula (7.4) that also $D^\alpha D^\beta u = \{D^\alpha D^\beta u\}_{|\alpha|,|\beta|\le m}$ is a fixed point of U. Hence $D^\alpha D^\beta u = \widetilde{\mathrm{w}}$, $D^\alpha D^\beta u \in M_{p,\omega}(B_r^+)$ and estimate (7.3) holds. Thus the theorem is proved. □

Theorem 7.2. *Let operator L in problem (7.1) be uniformly elliptic and $a_{\alpha\beta} \in VMO(\Omega)$. Then for any function $f \in M_{p,\varphi}(\Omega)$ the unique solution of the problem (7.1) has $2m$ derivatives in $M_{p,\varphi}(\Omega)$. Moreover,*

$$\Big\| \sum_{|\alpha|,|\beta|\le m} D^\alpha D^\beta u \Big\|_{M_{p,\varphi}(\Omega)} \le C\left(\|u\|_{M_{p,\varphi}(\Omega)} + \|f\|_{M_{p,\varphi}(\Omega)} \right) \tag{7.6}$$

with the constant C depends on known quantities.

Proof. Since $M_{p,\varphi}(\Omega) \subset L_p(\Omega)$ the problem (7.1) is uniquely solvable in the Sobolev space $W_p^{2m}(\Omega) \cap \overset{\circ}{W}{}_p^m(\Omega)$ according to [2] and [11]. By local flattering of the boundary, covering with semi-balls, taking a partition of unity subordinated to that covering and applying of estimate (7.3) we get a boundary a priori estimate that unified with (3.11) ensures validity of (7.6). □

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