Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **41**, 1–15; https://doi.org/10.14232/ejqtde.2019.1.41 www.math.u-szeged.hu/ejqtde/

Nonhomogeneous fractional p-Kirchhoff problems involving a critical nonlinearity

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Received 15 March 2019, appeared 26 June 2019
Communicated by Patrizia Pucci

Abstract. This paper is concerned with the existence of solutions for a kind of nonhomogeneous critical p-Kirchhoff type problem driven by an integro-differential operator $\mathcal{L}_K^p$. In particular, we investigate the equation:

$$\mathcal{M}\left(\iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}dxdy\right)\mathcal{L}_K^p v(x) = \mu g(x)|v|^{q-2}v + |v|^{p_s^*-2}v + \mu f(x) \quad \text{in } \mathbb{R}^n,$$

where $g(x) > 0$, and $f(x)$ may change sign, $\mu > 0$ is a real parameter, $0 < s < 1 < p < \infty$, dimension $n > ps$, $1 < q < p < p_s^*$, $p_s^* = \frac{np}{n-ps}$ is the critical exponent of the fractional Sobolev space $W_K^{s,p}(\mathbb{R}^n)$. By exploiting Ekeland's variational principle, we show the existence of non-trivial solutions. The main feature and difficulty of this paper is the fact that $\mathcal{M}$ may be zero and lack of compactness at critical level $L^{p_s^*}(\mathbb{R}^n)$. Our conclusions improve the related results on this topic.

Keywords: fractional p-Kirchhoff problems, non-homogeneous, critical nonlinearity, Ekeland's variational principle.

2010 Mathematics Subject Classification: 35R11, 35J60, 35A15, 47G20.

1 Introduction and main results

In this paper, we study the existence of solutions of the following the fractional p-Kirchhoff type equation involving a critical nonlinearity:

$$\mathcal{M}\left(\iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}dxdy\right)\mathcal{L}_K^p v(x) = \mu g(x)|v|^{q-2}v + |v|^{p_s^*-2}v + \mu f(x) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n > ps$, $p > 1$ and $s \in (0,1)$, $p_s^* = \frac{np}{n-ps}$, μ is a positive parameter, $\mathcal{M} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function, where $g(x) > 0$, and $f(x)$ may change sign on $\mathbb{R}^n$, and $\mathcal{L}_K^p$ is the

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non-local p-fractional type operator defined as follows:

$$\mathcal{L}_K^p \phi(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} |\phi(x) - \phi(y)|^{p-2}(\phi(x) - \phi(y)) K(x-y) dy \quad \text{for all } x \in \mathbb{R}^n,$$

along any $\phi \in C_0^\infty(\mathbb{R}^n)$, where $B_\varepsilon(x)$ denotes the open ball in $\mathbb{R}^n$ of radius $\varepsilon > 0$ at the centre $x \in \mathbb{R}^n$ and $K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ is a measurable function which satisfies the following properties:

$$\begin{cases} \zeta K \in L^1(\mathbb{R}^n), & \text{where } \zeta(x) = \min\{|x|^p, 1\}; \\ \text{there exists } \kappa_0 > 0 \text{ such that } K(x) \geq \kappa_0 |x|^{-(n+ps)} & \text{for any } x \in \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{1.2}$$

When K is a standard type (i.e. $K(x) = |x|^{-(n+ps)}$). In this case, the problem (1.1) becomes

$$\mathcal{M}\left(\iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^p}{|x-y|^{n+ps}} dxdy \right) (-\triangle)_p^s v = \mu g(x)|v|^{q-2}v + |v|^{p_s^*-2}v + \mu f(x) \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $(-\triangle)_p^s$ is a fractional p-Laplace operator defined by

$$(-\triangle)_p^s v(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{n+ps}} dy$$

for $x \in \mathbb{R}^n$. For more details about the fractional p-Laplacian, we refer to [6,7,12,21,25,26] and the references therein. Moreover, if $f(x) = 0$, then the problem (1.3) reduces to

$$\mathcal{M}\left(\iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^p}{|x-y|^{n+ps}} dxdy \right) (-\triangle)_p^s v = \mu g(x, v) + |v|^{2_s^*-2}v \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

In [27], Zhang *et al*. obtained infinitely many solutions for the problem (1.4) by using Kajikiya's new version of the symmetric mountain pass lemma.

In recent years, fractional and nonlocal problems have received extensive attention, especially involving critical nonlinear terms. For instance, in bounded domains, we refer to [11,18]; in the whole space, see [13]. It is worth pointing out that the interest in nonlocal fractional problems is beyond the curiosity of mathematics. Indeed, there is much literature on nonlocal operators and their applications, here we list only a few, see for example [14,16,19,28,29] and the references therein. For the basic nature of Sobolev spaces, we recommend readers to read the literature [15,17].

Recently, in [10], Fiscella *et al*. proposed a stationary Kirchhoff variational equation and investigated a model given by the following formulation:

$$\begin{cases} \mathcal{M}\left(\iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} dxdy \right) (-\triangle)^s v = \mu g(x, v) + |v|^{2_s^*-2}v & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.5}$$

where $\mathcal{M} = a + bt$ for all $t \in \mathbb{R}_0^+$, here $a > 0$, $b \geq 0$. Kirchhoff problems like (1.5) are said to be non-degenerate if $a > 0$ and $b \geq 0$, while it is named degenerate if $a = 0$ and $b > 0$. For the two separate cases, we also refer to [3,20] about non-degenerate Kirchhoff type problems and to [9,22] about degenerate Kirchhoff type problems for the recent advances in this direction.

For non-homogeneous cases, in [23] Xiang *et al.* showed the multiplicity of solutions for the nonhomogeneous fractional p-Kirchhoff equations involving concave-convex nonlinearities by using the mountain pass theorem and Ekeland's variational principle. Taking the same approach as in [23], Chen *et al.* [4] studied the multiplicity of solutions for a kind of Dirichlet boundary value problem in bounded domain. Furthermore, Pucci *et al.* in [21] investigated the existence of multiple solutions for the nonhomogeneous fractional p-Laplacian equations of Schrödinger–Kirchhoff type in $\mathbb{R}^n$.

Inspired by the above papers, we will study the existence of solutions for nonhomogeneous fractional p -Kirchhoff problems involving critical nonlinearities and weight terms. As far as we know, there are no works on the the problem (1.1). Thus, our conclusions are new on this topic. To this purpose, we first suppose that

(M_1) There exists $\theta \in \big(1, \frac{n}{n-ps}\big)$, such that

$$\theta\widetilde{\mathcal{M}}(t) = \theta \int_0^t \mathcal{M}(s)ds \geq \mathcal{M}(t)t, \qquad \forall\, t \in \mathbb{R}_0^+;$$

(M_2) for every $\sigma > 0$ there exists $m = m(\sigma) > 0$ such that $\mathcal{M}(t) \geq m$ for all $t \geq \sigma$;

(M_3) there exists a_0 such that $\mathcal{M}(t) \geq a_0 t^{\theta-1}$ for all $t \in [0,1]$.

At present, the assumptions about Kirchhoff functions $\mathcal{M}$ are diverse; we refer the interested reader to [5, 10] and references therein. Here we do not require Kirchhoff functions to satisfy any monotonicity, and our paper covers the degenerate case.

Next, concerning the positive weight $g : \mathbb{R}^n \to \mathbb{R}$, we assume that

(Z_1) $g \in L^{q_1}(\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n)$, $f \in L^\xi(\mathbb{R}^n)$, with $q_1 = \frac{p_s^*}{p_s^*-q}$, $\xi = \frac{p_s^*}{p_s^*-1}$.

(Z_2) $1 < q < p < \theta p < p_s^*$.

Before we present the main results, we give some notations. The function space $W_K^{s,p}(\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\mathbb{R}^n)$, and $W_K^{s,p}(\mathbb{R}^n)$ is a Banach space which can be endowed with the norm, defined as

$$\|\phi\|_{s,p} = \left(\iint_{\mathbb{R}^{2n}} |\phi(x) - \phi(y)|^p K(x-y)dxdy\right)^{\frac{1}{p}},$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$.

Now we define weak solutions for the problem (1.1):

Definition 1.1. We say that $u \in W_K^{s,p}(\mathbb{R}^n)$ is a weak solution to the problem (1.1), if

$$\begin{aligned}\mathcal{M}(\|v(x)\|_{s,p}^p) &\iint_{\mathbb{R}^{2n}} |v(x) - v(y)|^{p-2}(v(x) - v(y))(\phi(x) - \phi(y)K(x-y)dxdy \\ &= \mu \int_{\mathbb{R}^n} g(x)|v(x)|^{q-2}v(x)\phi(x)dx + \int_{\mathbb{R}^n} |v(x)|^{p_s^*-2}v(x)\phi(x)dx + \mu \int_{\mathbb{R}^n} f(x)\phi(x)dx,\end{aligned}$$

for any $\phi \in X_0$.

The main results of this paper are as follows.

Theorem 1.2. *Set $K : \mathbb{R}^n \setminus \{0\} \to (0,\infty)$ be a function fulfilling (1.2) and if (M_1)–(M_3) and (Z_1)–(Z_2) hold. Then, there exist μ_0, $\tau_0 > 0$ such that for any $\mu \in (0,\mu_0)$, the problem (1.1) has at least one non-trivial solution with negative energy in $W_K^{s,p}(\mathbb{R}^n)$ when $\|f\|_\xi \leq \tau_0$.*

Theorem 1.3. *Assume that all conditions in Theorem 1.2 are fulfilled. Then, there exist $\mu_0^{'}$, $\tau_0^{'} > 0$ such that for any $\mu \in (0, \mu_0^{'})$, the problem (1.1) has at least one non-trivial non-negative solution with negative energy in $W_K^{s,p}(\mathbb{R}^n)$, provided that $f \geq 0$ a.e. in $\mathbb{R}^n$ and $\|f\|_\xi \leq \tau_0^{'}$.*

Remark 1.4. The main novelty of our paper is that we discuss the problem (1.1) containing a critical nonlinearity, which is not considered in previous references, such as [23]. To overcome this difficulty about lack of compactness at critical level $L^{p_s^*}(\mathbb{R}^n)$, we fix parameter μ under a suitable threshold strongly depending on conditions (M_2) and (M_3).

Finally, we give a simple example to show a direct application of our main results.

Example 1.5. Let $n > 1$ and $\theta > 1$. We consider the following the problem

$$\left(a + b\left(\iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x-y|^{n+1}} dxdy\right)^{\theta-1}\right)(-\triangle)^{\frac{1}{2}} v(x) = \mu g(x) v^{q-1} + v^{\frac{n+1}{n-1}} + \mu f(x) \quad \text{in } \mathbb{R}^n,$$

where a, b are non-negative constants with $a + b > 0$, g and f satisfy

$$g(x) = (1 + |x|^2)^{-\frac{2_s^* - q}{2_s^*}}, \quad \text{for all } x \in \mathbb{R}^n,$$

and

$$f(x) = (1 + |x|^2)^{-\frac{1}{\xi}}, \quad \text{for all } x \in \mathbb{R}^n.$$

Obviously, $g \in L^{\frac{2_s^*}{2_s^* - q}}(\mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}^n)$ and $f \in L^\xi(\mathbb{R}^n)$. Thus, they satisfy conditions (Z_1) and (Z_2). It is clearly that (M_3) hold, and for each $\sigma > 0$

$$\mathcal{M}(t) = a + bt^{\theta-1} \geq a + b\sigma^{\theta-1} =: m(\sigma) > 0 \quad \text{for all } t \geq \sigma,$$

and

$$\widetilde{\mathcal{M}}(t) = \int_0^t \mathcal{M}(s)ds \geq \frac{1}{\theta}\mathcal{M}(t)t \quad \text{for all } t \geq 0.$$

Therefore, (M_1) and (M_2) hold. Then, Theorem 1.3 implies that the above problem admits one non-trivial non-negative solutions in $W_K^{s,p}(\mathbb{R}^n)$.

The framework of this paper is as follows. Section 2 introduces the necessary definitions and properties of space $W_K^{s,p}(\mathbb{R}^n)$. In Section 3, we give the proofs of the main results.

2 Variational framework

In this section, we first recall the basic variational frameworks and main lemmas for the the problem (1.1). Let $L^q(\mathbb{R}^n, g)$ be the weighted Lebesgue space, endowed with the norm

$$\|v\|_{q,g}^q = \int_{\mathbb{R}^n} g(x)|v(x)|^q dx.$$

The Banach space $L^q(\mathbb{R}^n, g) = (L^q(\mathbb{R}^n, g), \|\cdot\|_{q,g})$ is uniformly convex according to Proposition A.6 of [1]. Moreover, it follows from Lemma 2.1 of [3] that the embedding $W_K^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n, g)$ is compact, so we have

$$\|v\|_{q,g} \leq C_g \|v\|_{s,p} \quad \text{for all } v \in W_K^{s,p}(\mathbb{R}^n), \tag{2.1}$$

and $C_g = S^{-\frac{1}{p}}\|g\|_{q_1}^{\frac{1}{q}} > 0$, where $S = S(n, p, s)$ is the best fractional critical Sobolev constant, given by

$$S = \inf_{v \in W_K^{s,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|v\|_{s,p}^p}{\|v\|_{L^{p_s^*}(\mathbb{R}^n)}^p}. \tag{2.2}$$

Obviously, from the above expression of S and (1.2), we can get the fractional Sobolev embedding inequality that we know well:

$$\|v\|_{L^{p_s^*}(\mathbb{R}^n)} \leq S^{-\frac{1}{p}}\|v\|_{s,p} \quad \text{for all } v \in W_K^{s,p}(\mathbb{R}^n). \tag{2.3}$$

Similar to [24], we can also get that $(W_K^{s,p}(\mathbb{R}^n), \|v\|_{s,p})$ is a uniformly convex Banach space. Hence, it is a reflexive Banach space. For $v \in W_K^{s,p}(\mathbb{R}^n)$, we define

$$\begin{aligned}\mathcal{I}_\mu(v) = \frac{1}{p}\widetilde{\mathcal{M}}\left(\iint_{\mathbb{R}^{2n}} |v(x)-v(y)|^p K(x-y)dxdy\right) - \frac{\mu}{q}\int_{\mathbb{R}^n} g(x)|v|^q dx \\ - \frac{1}{p_s^*}\int_{\mathbb{R}^n} |v|^{p_s^*}dx - \mu\int_{\mathbb{R}^n} f(x)vdx.\end{aligned}$$

Clearly, by assumptions (1.2) and (Z_1)–(Z_2), the energy functional $\mathcal{I}_\mu : W_K^{s,p}(\mathbb{R}^n) \to \mathbb{R}$ associated with the problem (1.1) is well defined. Obviously, the functional $\mathcal{I}_\mu$ is of class $C^1(W_K^{s,p}(\mathbb{R}^n))$ and

$$\begin{aligned}\langle \mathcal{I}_\mu'(v), u\rangle = \mathcal{M}(\|v\|_{s,p}^p)\iint_{\mathbb{R}^{2n}} |v(x)-v(y)|^{p-2}(v(x)-v(y))(u(x)-u(y)K(x-y)dxdy \\ - \mu\int_{\mathbb{R}^n} g(x)|v|^{q-2}vudx - \int_{\mathbb{R}^n}|v|^{p_s^*-2}vudx - \mu\int_{\mathbb{R}^n} f(x)udx,\end{aligned}$$

for all $v, u \in W_K^{s,p}(\mathbb{R}^n)$, see for example ([3]:lemmas 4.2) with slight changes. Thus, the critical points of functional $\mathcal{I}_\mu$ are weak solutions of the problem (1.1).

3 Proof of main results

In this section, we prove the main results of this article. For convenience, we use $\|\cdot\|_q$ to represent the norm of Lebesgue space $L^q(\mathbb{R}^n)$.

Lemma 3.1. *If (M_1)–(M_2) and (Z_1)–(Z_2) hold. Then, there exist $\rho, \alpha = \alpha(\rho), \tau_0, \mu_1 > 0$, such that $\mathcal{I}_\mu(v) \geq \alpha$ for any $\mu \in (0, \mu_0)$ with $\|v\|_{s,p} = \rho$ and $\|f\| \leq \tau_0$, for all $\mu \in (0, \mu_1)$.*

Proof. From (M_1), we get

$$\widetilde{\mathcal{M}}(t) \geq \widetilde{\mathcal{M}}(1)t^\theta \quad \text{for all } t \in [0,1]. \tag{3.1}$$

According to the Hölder inequality and (2.3), we have

$$\left|\int_{\mathbb{R}^n} g(x)|v|^q dx\right| \leq \|g\|_{q_1}\|v\|_{p_s^*}^q \leq S^{-\frac{q}{p}}\|g\|_{q_1}\|v\|_{s,p}^q \quad \text{for all } v \in W_K^{s,p}(\mathbb{R}^n), \tag{3.2}$$

where $q_1 = \frac{p_s^*}{p_s^*-q}$. In the same way,

$$\left|\int_{\mathbb{R}^n} |v|^{p_s^*}dx\right| = \|v\|_{p_s^*}^{p_s^*} \leq S^{-\frac{p_s^*}{p}}\|v\|_{s,p}^{p_s^*} \tag{3.3}$$

and

$$\left|\int_{\mathbb{R}^n} f(x)vdx\right| \le \|f\|_\xi \|v\|_{p_s^*} \le S^{-\frac{1}{p}}\|f\|_\xi\|v\|_{s,p} \le \varepsilon\|v\|_{s,p}^{p\theta} + C_\varepsilon \|f\|_\xi^{(p\theta)'} \tag{3.4}$$

for all $v \in W_K^{s,p}(\mathbb{R}^n)$, where $\xi = \frac{p_s^*}{p_s^*-1}$. $\varepsilon > 0$, $C_\varepsilon > 0$. Hence, from (3.1)–(3.4), we get for all $v \in W_K^{s,p}(\mathbb{R}^n)$ with $\|v\|_{s,p} \le 1$

$$\begin{aligned}\mathcal{I}_\mu(v) &\ge \frac{\widetilde{\mathcal{M}}(1)}{p}\|v\|_{s,p}^{p\theta} - \mu S^{-\frac{q}{p}}\|g\|_{q_1}\|v\|_{s,p}^q - S^{-\frac{p_s^*}{p}}\|v\|_{s,p}^{p_s^*} - \mu\varepsilon\|v\|_{s,p}^{p\theta} - \mu C_\varepsilon\|f\|_\xi^{(p\theta)'} \\ &\ge \frac{\widetilde{\mathcal{M}}(1)}{2p}\|v\|_{s,p}^{p\theta} - \mu S^{-\frac{q}{p}}\|g\|_{q_1}\|v\|_{s,p}^q - S^{-\frac{p_s^*}{p}}\|v\|_{s,p}^{p_s^*} - \mu C_\varepsilon\|f\|_\xi^{(p\theta)'},\end{aligned} \tag{3.5}$$

with $0 < \varepsilon\mu < \frac{\widetilde{\mathcal{M}}(1)}{2p}$, $(p\theta)' = \frac{p\theta}{p\theta-1}$. Set

$$h(z) = \mu S^{-\frac{q}{p}}\|g\|_{q_1} z^{q-p\theta} + S^{-\frac{p_s^*}{p}} z^{p_s^*-p\theta} \quad \text{for all } z > 0. \tag{3.6}$$

It suffices to prove that $h(z_0) < \frac{\widetilde{\mathcal{M}}(1)}{2p}$ for some $z_0 = \|v\|_{s,p} > 0$. Obverse that $h(z) \to \infty$ as $z \to 0^+$. Then, h has a minimum at $z_0 > 0$. To obtain z_0, we get

$$h'(z) = \mu S^{-\frac{q}{p}}(q-p\theta)\|g\|_{q_1} z^{q-p\theta-1} + S^{-\frac{p_s^*}{p}}(p_s^*-p\theta)z^{p_s^*-p\theta-1}.$$

Let $h'(z_0) = 0$, we have

$$z_0 = \left(\frac{\mu S^{-\frac{q}{p}}(p\theta-q)\|g\|_{q_1}}{S^{-\frac{p_s^*}{p}}(p_s^*-p\theta)}\right)^{\frac{1}{p_s^*-q}} = \mu^{\frac{1}{p_s^*-q}} S_0^{\frac{1}{p_s^*-q}} \in (0,1],$$

where $S_0 = S^{-\frac{q}{p}}(p\theta-q)\|g\|_{q_1}/S^{-\frac{p_s^*}{p}}(p_s^*-p\theta)$ and $\mu \le S_0^{-1}$. Furthermore, $h(z_0) < \frac{\widetilde{\mathcal{M}}(1)}{2p}$ means that

$$h(z_0) = S^{-\frac{q}{p}}(p\theta-q)\|g\|_{q_1}(p_s^*-q)(p_s^*-p\theta)^{-1}\mu^{\frac{p_s^*-p\theta}{p_s^*-q}} S_0^{\frac{q-p\theta}{p_s^*-q}} < \frac{\widetilde{\mathcal{M}}(1)}{2p}. \tag{3.7}$$

Then, by (3.5) and (3.7), there exist $\mu_1, \tau_0, \alpha > 0$ such that $\mathcal{I}_\mu(v) \ge \alpha$, with $\mu \in (0, \mu_1)$, $\rho = z_0 = \|v\|_{s,p}$ and $\|f\|_\xi \le t_0$ for every $f \in L^\xi(\mathbb{R}^n)$. Thus, the Lemma 3.1 is complete. □

Next, we verify the compactness condition. For the convenience of the reader, we give the following definition.

Definition 3.2. Let $\mathcal{I}_\mu \in C^1(X, \mathbb{R})$, we say that $\mathcal{I}_\mu$ satisfies the $(PS)_c$ condition at the level $c \in \mathbb{R}$, if any sequence $\{v_j\} \subset X$ such that

$$\mathcal{I}_\mu(v_j) \to c, \;\; \mathcal{I}_\mu'(v_j) \to 0 \quad \text{as } j \to \infty, \tag{3.8}$$

possesses a convergent subsequence in X.

Lemma 3.3. *If (M_1)–(M_2) and (Z_1)–(Z_2) hold. Set $c < 0$. Then, there exists μ_2 such that for any $\mu \in (0, \mu_2)$, the functional $\mathcal{I}_\mu$ fulfills $(PS)_c$ condition.*

Proof. Considering $\mu_2 > 0$ small enough such that

$$\left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{q}{p_s^*-q}} \left[\left(\frac{1}{q} - \frac{1}{p\theta}\right)\mu_2\|g\|_{q_1}\right]^{\frac{p_s^*}{p_s^*-q}} - \left(1 - \frac{1}{p\theta}\right)\mu_2\|f\|_\xi\|v\|_{p_s^*}$$
$$< \min\left[\left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(a_0^{\frac{1}{\theta}}S\right)^{\frac{p_s^*\theta}{p_s^*-p\theta}}, \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)(mS)^{\frac{p_s^*}{p_s^*-p}}\right], \tag{3.9}$$

where $q < p < p\theta < p_s^*$, a_0 comes from (M_3), $m = m(1)$ is defined in (M_2) with $\sigma = 1$, while S is given in (2.2). Set $\mu \in (0, \mu_2)$ and suppose that $\{v_j\}_j$ be a $(PS)_c$ sequence in $W_K^{s,p}(\mathbb{R}^n)$. Due to the degenerate nature of the equation (1.1), we will discuss it in two cases.

Case 1. $\inf_{j\in\mathbb{N}} \|v_j\|_{s,p} = d_\mu > 0$. First, we prove that the sequence $\{v_j\}_j$ is bounded in $W_K^{s,p}(\mathbb{R}^n)$. From (M_2), with $\sigma = d_\mu^p$, there exists $m = m(d_\mu^p) > 0$ such that

$$\mathcal{M}(\|v_j\|_{s,p}^p) \geq m \quad \text{for all } j \in \mathbb{N}. \tag{3.10}$$

Moreover, it follows from (M_1), (3.10) and (2.1) that

$$\begin{aligned}\mathcal{I}_\mu(v_j) - \frac{1}{p_s^*}\langle \mathcal{I}'_\mu(v_j), v_j\rangle &= \frac{1}{p}\widetilde{\mathcal{M}}(\|v_j\|_{s,p}^p) - \frac{1}{p_s^*}\mathcal{M}(\|v_j\|_{s,p}^p)\|v_j\|_{s,p}^p - \left(\frac{1}{q} - \frac{1}{p_s^*}\right)\mu\|v_j\|_{q,g}^q \\ &\quad - \left(1 - \frac{1}{p_s^*}\right)\mu\int_{\mathbb{R}^n} f(x)v_j dx \\ &\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)m\|v_j\|_{s,p}^p - \left(\frac{1}{q} - \frac{1}{p_s^*}\right)\mu C_g^q\|v_j\|_{s,p}^q \\ &\quad - \left(1 - \frac{1}{p_s^*}\right)\mu S^{-\frac{1}{p}}\|f\|_\xi\|v_j\|_{s,p}.\end{aligned} \tag{3.11}$$

Therefore, from (3.8) there exists $\zeta > 0$ and $\eta > 0$ such that as $j \to \infty$

$$c + \zeta\|v_j\|_{s,p}^q + \eta\|v_j\|_{s,p} + o(1) \geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)m\|v_j\|_{s,p}^p, \tag{3.12}$$

with $q < p < p\theta < p_s^*$. It means that $\{v_j\}_j$ is bounded in $W_K^{s,p}(\mathbb{R}^n)$.

Taking into account the above fact and Lemmas 2.1, 2.4 of [3], there exist $v \in W_K^{s,p}(\mathbb{R}^n)$ and $\beta_\mu \geq 0$ such that, up to a subsequence still relabeled $\{v_j\}_j$, it follows that

$$\begin{aligned} &v_j \rightharpoonup v \quad \text{in } W_K^{s,p}(\mathbb{R}^n), && \|v_j\|_{s,p} \to \beta_\mu, \\ &v_j \rightharpoonup v \quad \text{in } L^{p_s^*}(\mathbb{R}^n), && \|v_j - v\|_{p_s^*} \to \omega, \\ &v_j \to v \quad \text{in } L^q(\mathbb{R}^n, g), && v_j \to v \quad a.e. \text{ in } \mathbb{R}^n. \end{aligned} \tag{3.13}$$

Further, by the above formula and Proposition A.8 of [1], we have

$$|v_j|^{p_s^*-2}v_j \rightharpoonup |v|^{p_s^*-2}v \quad \text{in } L^{(p_s^*)'}(\mathbb{R}^n), \qquad |v_j|^{q-2}v_j \rightharpoonup |v|^{q-2}v \quad \text{in } L^{q'}(\mathbb{R}^n, g). \tag{3.14}$$

Discussion similar to Lemma 3.1 in reference [27], we can easily obtain that v is a critical point of the $C^1(W_K^{s,p}(\mathbb{R}^n))$ functional

$$\widetilde{\mathcal{I}}_\mu(v) = \frac{1}{p}\mathcal{M}(\beta_\mu^p)\|v\|_{s,p}^p - \frac{\mu}{q}\|v\|_{q,g}^q - \frac{1}{p_s^*}\|v\|_{p_s^*}^{p_s^*} - \mu\int_{\mathbb{R}^n} f(x)v dx. \tag{3.15}$$

By (3.13), we get

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} g(x)(|v_j(x)|^{q-2}v_j(x)-|v(x)|^{q-2}v(x))(v_j(x)-v(x))dx=0. \tag{3.16}$$

Moreover, by again (3.13) and the well-known Brézis and Lieb lemma of [2], we have

$$\|v_j\|_{s,p}^p=\|v_j-v\|_{s,p}^p+\|v\|_{s,p}^p+o(1),\quad \|v_j\|_{p_s^*}^{p_s^*}=\|v_j-v\|_{p_s^*}^{p_s^*}+\|v\|_{p_s^*}^{p_s^*}+o(1) \tag{3.17}$$

as $j\to\infty$. In particular, (3.8), (3.13), (3.15), (3.16) and (3.17) imply that as $j\to\infty$

$$\begin{aligned}
o(1)&=\langle \mathcal{I}_\mu'(v_j)-\widetilde{\mathcal{I}}_\mu'(v),v_j-v\rangle\\
&=\mathcal{M}(\|v_j\|_{s,p}^p)\|v_j\|_{s,p}^p+\mathcal{M}(\beta_\mu^p)\|v\|_{s,p}^p-\langle v_j,v\rangle_{s,p}[\mathcal{M}(\|v_j\|_{s,p}^p)+\mathcal{M}(\beta_\mu^p)]\\
&\quad-\mu\int_{\mathbb{R}^n} g(x)(|v_j(x)|^{q-2}v_j(x)-|v(x)|^{q-2}v(x))(v_j(x)-v(x))dx\\
&\quad-\int_{\mathbb{R}^n}(|v_j(x)|^{p_s^*-2}v_j(x)-|v(x)|^{p_s^*-2}v(x))(v_j(x)-v(x))dx\\
&=\mathcal{M}(\beta_\mu^p)(\beta_\mu^p-\|v\|_{s,p}^p)-\|v_j\|_{p_s^*}^{p_s^*}+\|v\|_{p_s^*}^{p_s^*}+o(1)\\
&=\mathcal{M}(\beta_\mu^p)\|v_j-v\|_{s,p}^p-\|v_j-v\|_{p_s^*}^{p_s^*}+o(1),
\end{aligned}$$

where $\langle v_j,v\rangle_{s,p}=\iint_{\mathbb{R}^{2n}}|v_j(x)-v_j(y)|^{p-2}(v_j(x)-v_j(y))(v(x)-v(y)K(x-y)dxdy$. Thus, we obtain the crucial formula

$$\mathcal{M}(\beta_\mu^p)\lim_{j\to\infty}\|v_j-v\|_{s,p}^p=\lim_{j\to\infty}\|v_j-v\|_{p_s^*}^{p_s^*}. \tag{3.18}$$

Combining (2.2), (3.13) and (3.18), we have

$$\omega^{p_s^*}\geq S\mathcal{M}(\beta_\mu^p)\omega^p. \tag{3.19}$$

If $\omega=0$, thanks to $\beta_\mu>0$ and $\mathcal{M}$ admits a unique zero at 0, then (3.18) yields at once that $v_j\to v$ in $W_K^{s,p}(\mathbb{R}^n)$, concluding the proof. Instead, suppose that $\omega>0$. Observing that (3.17), we can get

$$\mathcal{M}(\beta_\mu^p)(\beta_\mu^p-\|v\|_{s,p}^p)=\omega^{p_s^*},$$

By (3.19), we obtain that

$$(\omega^{p_s^*})^{\frac{ps}{n}}=\mathcal{M}(\beta_\mu^p)^{\frac{ps}{n}}(\beta_\mu^p-\|v\|_{s,p}^p)^{\frac{ps}{n}}\geq S\mathcal{M}(\beta_\mu^p). \tag{3.20}$$

Because we do not know the exact behavior of $\mathcal{M}$, we have to think about both of these scenarios: either $0<\beta_\mu<1$ or $\beta_\mu\geq 1$. To do this, we separate the certificate the two subcases in the first case.

Subcase 1. $0<\beta_\mu<1$. It follows from (3.20) and (M_3) that

$$\beta_\mu^{\frac{p^2s}{n}}\geq(\beta_\mu^p-\|v\|_{s,p}^p)^{\frac{ps}{n}}\geq S\mathcal{M}(\beta_\mu^p)^{\frac{n-ps}{n}}\geq a_0^{\frac{n-ps}{n}}S\beta_\mu^{\frac{p(\theta-1)(n-ps)}{n}}$$

and considering $n<ps\frac{\theta}{\theta-1}=ps\theta'$, it can be seen that

$$\beta_\mu^p\geq\left(a_0^{\frac{n-ps}{n}}S\right)^{\frac{n}{ps\theta-n(\theta-1)}}. \tag{3.21}$$

Indeed, this limitation $\frac{n}{p\theta'} < s$ can be derived directly from this fact $1 < \theta < \frac{p_s^*}{p} = \frac{n}{n-ps}$. Making use of (M_3), (3.20) and (3.21), we get

$$\omega^{p_s^*} \geq \left(S\mathcal{M}(\beta_\mu^p)\right)^{\frac{n}{ps}} \geq \left(Sa_0\beta_\mu^{p(\theta-1)}\right)^{\frac{n}{ps}} \geq \left(a_0^{\frac{1}{\theta}}S\right)^{\frac{n\theta}{ps\theta-n(\theta-1)}}. \tag{3.22}$$

Now, using (M_1) for any $j \in \mathbb{N}$ we get

$$\begin{aligned}
&\mathcal{I}_\mu(v_j) - \frac{1}{p\theta}\langle \mathcal{I}'_\mu(v_j), v_j\rangle \\
&= \frac{1}{p}\widetilde{\mathcal{M}}(\|v_j\|_{s,p}^p) - \frac{1}{p\theta}\mathcal{M}(\|v_j\|_{s,p}^p)\|v_j\|_{s,p}^p - \left(\frac{1}{q} - \frac{1}{p\theta}\right)\mu\|v_j\|_{q,g}^q - \left(\frac{1}{p_s^*} - \frac{1}{p\theta}\right)\|v_j\|_{p_s^*}^{p_s^*} \\
&\quad - \left(1 - \frac{1}{p\theta}\right)\mu\int_{\mathbb{R}^n} f(x)v_j dx \\
&\geq \left(\frac{1}{p\theta} - \frac{1}{q}\right)\mu\|v_j\|_{q,g}^q + \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\|v_j\|_{p_s^*}^{p_s^*} - \left(1 - \frac{1}{p\theta}\right)\mu\int_{\mathbb{R}^n} f(x)v_j dx.
\end{aligned}$$

For this, as $j \to \infty$, it follows from (3.8), (3.13), (3.17), (Z_1), the Hölder inequality and Young inequality that

$$\begin{aligned}
c &\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(\omega^{p_s^*} + \|v\|_{p_s^*}^{p_s^*}\right) - \left(\frac{1}{q} - \frac{1}{p\theta}\right)\mu\|v\|_{q,g}^q - \left(1 - \frac{1}{p\theta}\right)\mu\int_{\mathbb{R}^n} f(x)v dx \\
&\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(\omega^{p_s^*} + \|v\|_{p_s^*}^{p_s^*}\right) - \left(\frac{1}{q} - \frac{1}{p\theta}\right)\mu\|g\|_{q_1}\|v\|_{p_s^*}^q - \left(1 - \frac{1}{p\theta}\right)\mu\|f\|_\xi\|v\|_{p_s^*} \\
&\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(\omega^{p_s^*} + \|v\|_{p_s^*}^{p_s^*}\right) - \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\|v\|_{p_s^*}^{p_s^*} \\
&\quad - \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{q}{p_s^*-q}}\left[\left(\frac{1}{q} - \frac{1}{p\theta}\right)\mu\|g\|_{q_1}\right]^{\frac{p_s^*}{p_s^*-q}} - \left(1 - \frac{1}{p\theta}\right)\mu\|f\|_\xi\|v\|_{p_s^*}.
\end{aligned} \tag{3.23}$$

Finally, according to (3.22) we have

$$\begin{aligned}
0 > c &\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(a_0^{\frac{1}{\theta}}S\right)^{\frac{p_s^*\theta}{p_s^*-p\theta}} - \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{q}{p_s^*-q}}\left[\left(\frac{1}{q} - \frac{1}{p\theta}\right)\mu\|g\|_{q_1}\right]^{\frac{p_s^*}{p_s^*-q}} \\
&\quad - \left(1 - \frac{1}{p\theta}\right)\mu\|f\|_\xi\|v\|_{p_s^*} > 0,
\end{aligned}$$

where the above inequality by (3.9). In this subcase, we get our contradiction concluding the proof.

Subcase 2. $\beta_\mu \geq 1$. Here, it follows from (3.20) and (M_2) with $\sigma = 1$, we get

$$\omega^{p_s^*} \geq (mS)^{\frac{n}{ps}},$$

with $m = m(1) > 0$. Hence, by (3.23), we get

$$\begin{aligned}
0 > c &\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)(mS)^{\frac{p_s^*}{p_s^*-p}} - \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{q}{p_s^*-q}}\left[\left(\frac{1}{q} - \frac{1}{p\theta}\right)\mu\|g\|_{q_1}\right]^{\frac{p_s^*}{p_s^*-q}} \\
&\quad - \left(1 - \frac{1}{p\theta}\right)\mu\|f\|_\xi\|v\|_{p_s^*} > 0,
\end{aligned}$$

where again the above inequality by (3.9). We get a contradiction which completes for the proof of the fist case.

Case 2. $\inf_{j\in\mathbb{N}}\|v_j\|_{s,p}=0$. Thinking about two cases at zero. When 0 is an accumulation point of the real sequence $\{\|v_j\|_{s,p}\}_j$, and so there is a subsequence of $\{v_j\}_j$ strongly converging to $v=0$. This case can not happen thanks to it means that the trivial solution is a critical point at level c. When 0 is an isolated point of $\{v_j\}_j$, this case can happen, so that there is a subsequence still denoted by $\{\|v_{j_k}\|_{s,p}\}_k$, such that $\inf_{k\in\mathbb{N}}\|v_{j_k}\|_{s,p}=d_\mu>0$ and we can prove as before. The proof of the second case is complete. □

Proof of Theorem 1.2. Throughout this paper, considering $\mu_0=\min\{\mu_1,\mu_2\}$. Now, we look for the solution v_∞.

Case 1. $f\not\equiv 0$. We first show that there exists a $\phi_2\in C_0^\infty(\mathbb{R}^n)$, with $\|\phi_2\|_{s,p}=1$ such that $\int_{\mathbb{R}^n}f(x)\phi_2(x)dx>0$. Indeed, since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{\xi'}(\mathbb{R}^n)$ and $|f|^{\xi-2}f\in L^{\xi'}(\mathbb{R}^n)$. Then, there exists $j_0>0$ such that

$$\|f_{j_0}-|f|^{\xi-2}f\|_{L^{\xi'}(\mathbb{R}^n)}\le\frac{1}{2}\|f\|^{\xi-1}_{L^{\xi}(\mathbb{R}^n)}.$$

Hence, we obtain

$$\int_{\mathbb{R}^n}f(x)f_{j_0}(x)dx\ge-\|f_{j_0}-|f|^{\xi-2}f\|_{L^{\xi'}(\mathbb{R}^n)}\|f\|_{L^{\xi}(\mathbb{R}^n)}+\int_{\mathbb{R}^n}|f(x)|^{\xi}dx\ge\frac{1}{2}\int_{\mathbb{R}^n}|f(x)|^{\xi}dx>0.$$

Obviously, $f_{j_0}\in W_K^{s,p}(\mathbb{R}^n)$. Let $\phi_2=\frac{f_{j_0}}{\|f_{j_0}\|_{s,p}}$, we have that $\int_{\mathbb{R}^n}f(x)\phi_2(x)dx>0$. Like that

$$\begin{aligned}\mathcal{I}_\mu(t\phi_2)&=\frac{1}{p}\widetilde{\mathcal{M}}(\|t\phi_1\|^p_{s,p})-\frac{\mu t^q}{q}\int_{\mathbb{R}^n}g(x)|\phi_2|^q dx-\frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^n}|\phi_2|^{p_s^*}dx-\mu t\int_{\mathbb{R}^n}f(x)\phi_2 dx\\&\le\frac{\max\limits_{0\le\varrho\le1}\mathcal{M}(\varrho)}{p}t^p-\frac{\mu t^q}{q}\int_{\mathbb{R}^n}g(x)|\phi_2|^q dx-\frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^n}|\phi_2|^{p_s^*}dx-\mu t\int_{\mathbb{R}^n}f(x)\phi_2 dx\\&<0\end{aligned}\tag{3.24}$$

for small $t\in(0,1)$, on account of $p,q,p_s^*>1$.

Case 2. $f\equiv 0$. It is clearly that (3.24) still holds with $\|\phi_2\|_{s,p}=1$ for small enough $t\in(0,1)$, because of $1<q<p<p_s^*$. Therefore, for any open ball $B_{\mathbb{R}}\subset W_K^{s,p}(\mathbb{R}^n)$, we know

$$-\infty<c_r=\inf_{v\in\widetilde{B}_{\mathbb{R}}}\mathcal{I}_\mu(v)<0.\tag{3.25}$$

Hence,

$$c=\inf_{v\in\widetilde{B}_\rho}\mathcal{I}_\mu(v)<0\quad\text{and}\quad\inf_{v\in\partial B_\rho}\mathcal{I}_\mu(v)>0,\tag{3.26}$$

where $\rho>0$ is given lemma 3.1. Set $\varepsilon_j\downarrow 0$ such that

$$0<\varepsilon_j<\inf_{v\in\partial B_\rho}\mathcal{I}_\mu(v)-\inf_{v\in B_\rho}\mathcal{I}_\mu(v).\tag{3.27}$$

So, by Ekeland's variational principle in [8], there exists $\{v_j\}_j\subset\widetilde{B}_\rho$ such that

$$c\le\mathcal{I}_\mu(v_j)\le c+\varepsilon_j\tag{3.28}$$

and

$$\mathcal{I}_\mu(v_j) \le \mathcal{I}_\mu(v) + \varepsilon_j \|v_j - v\|_{s,p} \quad \text{for all } v \in \widetilde{B}_\rho, v \ne v_j. \tag{3.29}$$

In that way, by (3.27)–(3.29), we have that

$$\mathcal{I}_\mu(v_j) \le c + \varepsilon_j \le \inf_{v \in B_\rho} \mathcal{I}_\mu(v) + \varepsilon_j < \inf_{v \in \partial B_\rho} \mathcal{I}_\mu(v), \tag{3.30}$$

so that $v_j \in B_\rho$. At present, considering the functional $T : \widetilde{B}_\rho \to \mathbb{R}$ expressed by

$$T(v) = \mathcal{I}_\mu(v) + \varepsilon_j \|v_j - v\|_{s,p} \quad \text{for all } v \in \widetilde{B}_\rho.$$

Like that, (3.29) implies that $T(v_j) < T(v)$ for all $v \in \widetilde{B}_\rho$, with $v \ne v_j$ and thus v_j is a strict local minimum of T. Furthermore,

$$\frac{T(v_j + tu) - T(v_j)}{t} \ge 0 \quad \text{for small } t > 0 \text{ and for all } u \in B_1.$$

Hence,

$$\frac{\mathcal{I}_\mu(v_j + tu) - \mathcal{I}_\mu(v_j)}{t} + \varepsilon_j \|v\|_{s,p} \ge 0.$$

Passing to the limit as $t \to 0^+$, we get

$$\langle \mathcal{I}_\mu'(v_j), u\rangle + \varepsilon_j \|v\|_{s,p} \ge 0 \quad \text{for all } u \in B_1.$$

In the above inequality, replacing u with $-u$, we have

$$-\langle \mathcal{I}_\mu'(v_j), u\rangle + \varepsilon_j \|v\|_{s,p} \ge 0 \quad \text{for all } u \in B_1.$$

Thus, $\|\mathcal{I}_\mu'(v_j)\|_{(W_K^{s,p}(\mathbb{R}^n))'} \le \varepsilon_j$.

To sum up, there exist a sequence $\{v_j\}_j \subset B_\rho$ such that $\mathcal{I}_\mu(v_j) \to c < 0$ and $\mathcal{I}_\mu'(v_j) \to 0$ in $(W_K^{s,p}(\mathbb{R}^n))'$ as $j \to \infty$. According to lemma 3.3, $\{v_j\}_j$ has a convergent subsequence in $W_K^{s,p}(\mathbb{R}^n)$, still denoted by $\{v_j\}_j$, such that $v_j \to v_\infty$ in $W_K^{s,p}(\mathbb{R}^n)$. Thus, v_∞ is a solution of (1.1), with $\mathcal{I}_\mu(v_\infty) < 0$. The proof of Theorem 1.2 is completed. □

Next, we explore the existence of the non-trivial non-negative solution. For this purpose, first, some notations need to be introduced. Set $\mathcal{I}_\mu^+ : W_K^{s,p}(\mathbb{R}^n) \to \mathbb{R}$ be defined by

$$\mathcal{I}_\mu^+(v) = \frac{1}{p}\widetilde{\mathcal{M}}\left(\|v\|_{s,p}^p\right) - \frac{\mu}{q}\int_{\mathbb{R}^n} g(x)|v^+|^q dx - \frac{1}{p_s^*}\int_{\mathbb{R}^n} |v^+|^{p_s^*} dx - \mu\int_{\mathbb{R}^n} f(x) v dx,$$

for all $v \in W_K^{s,p}(\mathbb{R}^n)$, where $v^+ = \max\{v, 0\}$. We claim that for $u \in W_K^{s,p}(\mathbb{R}^n)$, we have $u^+ \in W_K^{s,p}(\mathbb{R}^n)$, where $u^+ = \max\{u, 0\} = \frac{|u|+u}{2}$. Indeed, we know

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} |u^+(x) - u^+(y)|^p K(x-y) dxdy &= \iint_{\mathbb{R}^{2n}} \left|\frac{|u(x)| - |u(y)| + u(x) - u(y)}{2}\right|^p K(x-y) dxdy \\ &\le \iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^p K(x-y) dxdy.\end{aligned}$$

Thus, $u^+ \in W_K^{s,p}(\mathbb{R}^n)$. Similarly, $u^- = \max\{-u, 0\}$ is also in $W_K^{s,p}(\mathbb{R}^n)$.

Like that, $\mathcal{I}_\mu^+$ is well defined on $W_K^{s,p}(\mathbb{R}^n)$ and of class $C^1(W_K^{s,p}(\mathbb{R}^n))$ and

$$\begin{aligned}\langle(\mathcal{I}_\mu^+(v))',\phi\rangle &= \mathcal{M}(\|v\|_{s,p}^p)\iint_{\mathbb{R}^{2n}}|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y)K(x-y)dxdy\\&\quad-\mu\int_{\mathbb{R}^n}g(x)|v^+(x)|^{q-2}v^+(x)\phi(x)dx-\int_{\mathbb{R}^n}|v^+(x)|^{p_s^*-2}v^+(x)\phi(x)dx\\&\quad-\mu\int_{\mathbb{R}^n}f(x)\phi(x)dx.\end{aligned}\tag{3.31}$$

Observed that all critical points of $\mathcal{I}_\mu^+$ are non-negative. Indeed, if v is a critical point of $\mathcal{I}_\mu^+$, thus from (3.31), we obtain

$$\begin{aligned}\langle(\mathcal{I}_\mu^+(v))',-v^-\rangle &= \mathcal{M}(\|v\|_{s,p}^p)\iint_{\mathbb{R}^{2n}}|v(x)-v(y)|^{p-2}(v(x)-v(y))(-v^-(x)+v^-(y)K(x-y)dxdy\\&\quad+\mu\int_{\mathbb{R}^n}g(x)|v^+(x)|^{q-2}v^+(x)v^-(x)dx+\int_{\mathbb{R}^n}|v^+(x)|^{p_s^*-2}v^+(x)v^-(x)dx\\&\quad+\mu\int_{\mathbb{R}^n}f(x)v^-(x)dx=0.\end{aligned}\tag{3.32}$$

We note that

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}}|v^-(x)-v^-(y)|^pK(x-y)dxdy\\&\qquad\le\iint_{\mathbb{R}^{2n}}|v(x)-v(y)|^{p-2}(v(x)-v(y))(-v^-(x)+v^-(y)K(x-y)dxdy.\end{aligned}$$

By (3.32), $\mu>0$ and $f\ge 0$ a.e. in $\mathbb{R}^n$, we get

$$\iint_{\mathbb{R}^{2n}}|v^-(x)-v^-(y)|^pK(x-y)dxdy\le 0.$$

It means that $v^-=0$ a.e. in $\mathbb{R}^n$. Therefore, $v\ge 0$ a.e. in $\mathbb{R}^n$.

Proof of Theorem 1.3. We replace $\mathcal{I}_\mu$ with $\mathcal{I}_\mu^+$ in the Theorem 1.2 and make use of the similar arguments as in Theorem 1.2, we get that the problem (1.1) admits one non-trivial non-negative solutions in $W_K^{s,p}(\mathbb{R}^n)$. □

Acknowledgements

The authors thank the anonymous referees for invaluable comments and insightful suggestions which improved the presentation of this manuscript.

Funding

The work is supported by the Fundamental Research Funds for Central Universities (2017B19714, 2017B07414, 2013/B19020449, 2019B44914), National Key Research and Development Program of China (2018YFC1508106), Natural Science Foundation of Jiangsu Province (BK20180500) and Scientific Research Fund of Jilin Engineering Normal University (XYB201814 and XYB201812). The work is also supported by Program for Innovative Research Team of Jilin Engineering Normal University and Applied mathematics research center of Jilin Engineering Normal University. This paper is also supported by China Scholarship Council.

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