Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **15**, 1–16; https://doi.org/10.14232/ejqtde.2020.1.15 www.math.u-szeged.hu/ejqtde/

Monotone solutions for singular fractional boundary value problems

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Received 10 August 2019, appeared 24 February 2020
Communicated by Jeff R. L. Webb

Abstract. In this paper, we investigate a boundary value problem of fractional differential equation. The nonlinear term includes fractional derivatives and is singular with respect to space variables. By means of Schaefer's fixed point theorem and Vitali convergence theorem, an existence result of monotone solutions is obtained. The proofs are based on regularization and sequential techniques. An example is also given to illustrate our main result.

Keywords: Caputo fractional derivative, monotone solution, boundary value problem, singularity.

2020 Mathematics Subject Classification: 34B16, 26A33.

1 Introduction

In this work, we consider the following boundary value problem (BVP for short)

$$\begin{cases} {}^C D^{\alpha}_{0^+} u(t) = f(t, u(t), u'(t), {}^C D^{\beta}_{0^+} u(t)), \\ u(0) + u(1) = 0, \ u'(0) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $0 < \beta < 1$, ${}^C D^{\alpha}_{0^+}$ and ${}^C D^{\beta}_{0^+}$ are Caputo fractional derivatives, $f(t, x, y, z)$ is singular at the value 0 of its space variables x, y and z. We establish an existence result of monotone increasing and continuous solution $u(t)$ of BVP (1.1). Since $\lim_{x \to 0} f(t, x, y, z) = \infty$, it follows from the condition $u(0) + u(1) = 0$ that there exists $\xi \in (0, 1)$ such that $u(\xi) = 0$ and thus ξ is a singular point of f.

Throughout the paper, $AC[0, 1]$ and $AC^k[0, 1]$ are the set of absolutely continuous functions on $[0, 1]$ and the set of functions having absolutely continuous kth derivatives on $[0, 1]$ respectively, $AC^0[0, 1] = AC[0, 1]$ for $k = 0$. $\|x\|_p = \big(\int_0^1 |x(t)|^p dt \big)^{\frac{1}{p}}$ is the norm in $L^p[0, 1]$, $1 \le p < \infty$. The basic space used in this paper is Banach space $C^1[0, 1]$ equipped with the norm $\|x\|_* = \max\{\|x\|, \|x'\|\}$, here $\|x\| = \max_{t \in [0,1]} |x(t)|$, $\|x'\| = \max_{t \in [0,1]} |x'(t)|$. We say that a monotone increasing function $u \in C^1[0, 1]$ is a solution of BVP (1.1) if u satisfies

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the boundary conditions in (1.1), $u(\xi) = 0$ for some $\xi \in (0,1)$, ${}^{C}D^{\alpha}_{0^+}u(t)$ is continuous on $(0,1] \setminus \{\xi\}$ and satisfies the equation in (1.1) for $t \in (0,1] \setminus \{\xi\}$.

In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives due to their wide range of applications in varied fields of sciences and engineering. Many research papers have appeared recently concerning the existence of positive solutions for fractional boundary value problems with singularities on time and/or space variables, see, for example, the papers [8, 10–12, 14, 21, 23] and the references therein. In [1–4, 6, 7, 17–20, 22], using techniques of nonlinear analysis such as fixed point theorems on cones and nonlinear alternatives combined with the methods of regularization and sequential approximation, the authors proved the existence of positive solutions for singular fractional boundary value problems in which the singularities are with respect to space variables.

The singular boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) + f(t, u(t), u'(t), {}^{C}D^{\mu}_{0^+}u(t)) = 0, \\ u'(0) = 0,\ u(1) = 0, \end{cases}$$

is studied in [2], where $1 < \alpha < 2, 0 < \mu < 1$, $f(t,x,y,z)$ is positive and may be singular at the value 0 of its space variables x, y and z. $f(t,x,y,z)$ is a L^q-Carathéodory function on $[0,1] \times \mathcal{B}$ with $q > \frac{1}{\alpha-1}$, $\mathcal{B} = (0,\infty) \times (-\infty,0) \times (-\infty,0)$. An existence result of positive solutions in space $C^1[0,1]$ is proved by the combination of regularization and sequential techniques with the Guo–Krasnosel'skii fixed point theorem on cones.

In [17] the author discussed the existence of positive solutions for the singular fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0^+}u(t) + f(t, u(t), u'(t), D^{\mu}_{0^+}u(t)) = 0, \\ u(0) = 0,\ u'(0) = u'(1) = 0, \end{cases}$$

where $2 < \alpha < 3, 0 < \mu < 1$, $D^{\alpha}_{0^+}$ and $D^{\mu}_{0^+}$ are the standard Riemann–Liouville fractional derivatives of order α and μ respectively. The function $f(t,x,y,z)$ is positive and may be singular at the value 0 of its arguments x, y and z, moreover, $f(t,x,y,z)$ satisfies the local Carathéodory conditions on $[0,1] \times (0,\infty) \times (0,\infty) \times (0,\infty)$. By regularization and sequential techniques and by the Guo–Krasnosel'skii fixed point theorem on cones, positive solutions in $C^1[0,1]$ are obtained.

Although the singular fractional boundary value problems have been investigated widely, the solutions allowing negative values of fractional boundary value problems with singularities on space variables are seldom considered. By Schaefer's fixed point theorem and Vitali convergence theorem, O'Regan and Staněk in [13] investigated monotone solutions in space $C[0,1]$ of the fractional boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = f(t, u(t)), \\ u(0) + u(1) = 0,\ u'(0) = 0, \end{cases}$$

where $1 < \alpha < 2$, $f(t,x) \in C([0,1] \times (\mathbb{R}\setminus\{0\}))$. $f(t,x)$ is nonnegative and may be singular at $x = 0$.

Inspired by above works, we prove the existence of monotone increasing solutions for BVP (1.1). The main tool used in this paper is Schaefer's fixed point theorem. Our proofs are based on regularization and sequential techniques. Compared with the existing literature, this paper presents the following new features. Firstly, as far as we know, the existence results of solutions allowing negative values are even less for fractional boundary value problems with singularities on space variables. Our result compensates for this deficiency to some extent. Secondly, the significant difference with the problem discussed in [13] lies in that the nonlinear term f in BVP (1.1) is related to fractional derivatives and permits singularities on all its space variables. That is to say the problem considered in this paper performs a more general form. Moreover, the conditions on f in our paper are more general than those in [13].

2 Preliminaries

In this section, we introduce some notations and preliminary facts which are used throughout this paper.

The Riemann–Liouville fractional integral of order $\delta > 0$ of a function $f(t) \in L^1(a,b)$ is defined by (see [9, p. 69])

$$I^{\delta}_{a^+} f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} f(s)ds, \qquad t > a.$$

The Riemann–Liouville fractional derivative of order $\delta > 0$ of a continuous function f on $(a,b]$ is given by (see [9, p. 70])

$$D^{\delta}_{a^+} f(t) = \left(\frac{d}{dt}\right)^n I^{n-\delta}_{a^+} f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\delta-1} f(s)ds,$$

provided that the right-hand side is pointwise defined on $(a,b]$, where n is the smallest integer greater than or equal to δ. In particular, for $\delta = n$, $D^n_{a^+} f(t) = f^{(n)}(t)$.

The Caputo fractional derivative of order $\delta > 0$ of a function $f(t) \in C(a,b]$ is defined by (see [9, p. 91])

$${}^C D^{\delta}_{a^+} f(t) = D^{\delta}_{a^+} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right],$$

provided that the right-hand side is pointwise defined on $(a,b]$, where n is the smallest integer greater than or equal to δ. In particular, for $\delta = n$, ${}^C D^n_{a^+} f(t) = f^{(n)}(t)$.

Remark 2.1. For a function $f(t) \in L^1(a,b)$, a sufficient condition for the existence of Riemann–Liouville fractional derivative almost everywhere is that $I^{n-\delta}_{a^+} f(t) \in AC^{n-1}[a,b]$. In this case, the function f is said to have a summable fractional derivative of order δ ([15, Definition 2.4]). In view of the definition of Caputo fractional derivative, ${}^C D^{\delta}_{a^+} f(t) = D^{\delta}_{a^+} f(t)$ for $\delta \in \mathbb{N}$ and ${}^C D^{\delta}_{a^+} f(t) = D^{\delta}_{a^+} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\delta+1)} (t-a)^{k-\delta}$ for $\delta \notin \mathbb{N}$ (see (2.4.6) in [9]), thus this is also a sufficient condition for the existence of Caputo fractional derivative. It is worth mentioning that the solution $u(t)$ in our main result not only has summable fractional derivative ${}^C D^{\alpha}_{0^+} u(t)$ on $[0,1]$ but also has continuous fractional derivative ${}^C D^{\alpha}_{0^+} u(t)$ on $(0,1] \setminus \{\xi\}$. For more details, see Step 3 in the proof of Theorem 4.1 and Remark 4.2 in Section 4.

Remark 2.2. The following properties are useful for our discussion.

(i) ([9, Lemma 2.8]) $I^{\delta}_{a^+} : C[a,b] \to C[a,b]$ for $\delta > 0$.

(ii) ([9, Lemma 2.3]) If $\delta > 0$, $\gamma > 0$, $\delta + \gamma > 1$ and $f \in L^p(a,b)$ $(1 \le p \le \infty)$, then $I^{\delta}_{a^+} I^{\gamma}_{a^+} f(t) = I^{\delta+\gamma}_{a^+} f(t)$, $t \in [a,b]$.

(iii) ([9, Theorem 2.2]) If $n-1<\delta\leq n$ and $f(t)\in C^n[a,b]$, then ${}^C D^{\delta}_{a^+}f(t)=I^{n-\delta}_{a^+}f^{(n)}(t)$, $t\in[a,b]$.

(iv) ([9, Lemma 2.21]) If $\delta>0$ and $f\in C[a,b]$, then ${}^C D^{\delta}_{a^+}I^{\delta}_{a^+}f(t)=f(t)$, $t\in[a,b]$.

(v) ([17, Lemma 2.1]) $I^{\delta}_{a^+}:L^1[a,b]\to L^1[a,b]$ for $\delta\in(0,1)$ and $I^{\delta}_{a^+}:L^1[a,b]\to AC^{[\delta]-1}[a,b]$ for $\delta\geq 1$, where $[\delta]$ means the integral part of δ.

For convenience, in the following discussion we use I^{α}, ${}^C D^{\alpha}$ and D^{α} to denote $I^{\alpha}_{0^+}$, ${}^C D^{\alpha}_{0^+}$ and $D^{\alpha}_{0^+}$, respectively.

A sequence $\{\phi_n\}\subset L^1[0,1]$ is said to have uniformly absolutely continuous integrals on $[0,1]$ if for any $\epsilon>0$, there exists $\delta>0$ such that if $E\subset[0,1]$ and $\mathrm{meas}(E)<\delta$, then $\int_E|\phi_n(t)|dt<\epsilon$ for all $n\in\mathbb{N}$ (see [5, p. 178]). To prove the main result, we need the following Vitali convergence theorem and nonlinear alternative.

Lemma 2.3 ([5, pp. 178–179] Vitali convergence theorem). *Let $\{\phi_n\}\subset L^1[0,1]$, $\lim_{n\to+\infty}\phi_n(t)=\phi(t)$ for a.e. $t\in[0,1]$ and $|\phi(t)|<\infty$ for a.e. $t\in[0,1]$. Then the following statements are equivalent.*

(1) $\phi\in L^1[0,1]$ and $\lim_{n\to+\infty}\|\phi_n-\phi\|_1=0$.

(2) The sequence $\{\phi_n\}$ has uniformly absolutely continuous integrals on $[0,1]$.

Lemma 2.4 ([16, p. 29] Schaefer's fixed point theorem). *Let X be a Banach space and $T:X\to X$ be completely continuous. Then the following alternative holds. Either the equation $x=\lambda T(x)$ has a solution for every $\lambda\in[0,1]$ or the set $A=\{x\in X: x=\lambda Tx$ for some $\lambda\in(0,1)\}$ is unbounded.*

Denote $\mathbb{R}_0=\mathbb{R}\setminus\{0\}$, $\mathbb{R}^+=[0,+\infty)$ and $\mathbb{R}^+_0=(0,+\infty)$. We work with the following conditions on the function f in (1.1).

(H1) $f\in C([0,1]\times\mathbb{R}_0\times\mathbb{R}^+_0\times\mathbb{R}^+_0)$, $\lim_{x\to 0}f(t,x,y,z)=\lim_{y\to 0^+}f(t,x,y,z)=\lim_{z\to 0^+}f(t,x,y,z)=+\infty$ and $f(t,x,y,z)\geq mt^{2-\alpha}$ for $(t,x,y,z)\in[0,1]\times\mathbb{R}_0\times\mathbb{R}^+_0\times\mathbb{R}^+_0$.

(H2) $f(t,x,y,z)\leq\rho(t)g(|x|,y,z)+p(|x|)+q(y,z)$ for $(t,x,y,z)\in[0,1]\times\mathbb{R}_0\times\mathbb{R}^+_0\times\mathbb{R}^+_0$, where $\rho(t)$ is nonnegative on $[0,1]$, $g(x,y,z)\in C(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+)$ is nonnegative and nondecreasing in all its arguments, $p(x)\in C(\mathbb{R}^+_0)$ is nonnegative and nonincreasing, $q(y,z)\in C(\mathbb{R}^+_0\times\mathbb{R}^+_0)$ is nonnegative and nonincreasing in all its arguments.

(H3) $\lim_{x\to+\infty}\frac{g(x,x,x)}{x}=0$. $p(\lambda x)\leq\lambda^{-\sigma}p(x)$ for some $\sigma\in(0,\frac{\alpha-1}{2})$ and for any $\lambda\in(0,1]$, $x\in\mathbb{R}^+_0$. $\rho(t)$, $p(t^2)$ and $q(m\Gamma(3-\alpha)t,\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta})\in L^{\nu}[0,1]$ for some $\nu\in(\frac{1}{\alpha-1},\frac{1}{2\sigma})$.

Remark 2.5. In [13], the nonlinear term satisfies $f(t,x)\leq g(|x|)+\frac{A}{|x|^{\nu}}$, where $A>0$ is a constant and $\nu>0$ is a suitable small number. It is easy to verify that the simple function $p(x)=\frac{1}{x^{\omega}}$ for $0<\omega<\frac{\alpha-1}{2}$ fulfils the conditions (H2) and (H3) with $\omega\leq\sigma<\frac{\alpha-1}{2}$ and $\nu\in(\frac{1}{\alpha-1},\frac{1}{2\sigma})$.

Remark 2.6. By Lemma 2.1 and 2.2 in [2], for any $f(t)\in L^{\nu}[0,1]$ with $\nu>\frac{1}{\alpha-1}$, $I^{\alpha-1}f(t)\in C[0,1]$ and $\left|\int_0^t(t-s)^{\alpha-2}f(s)ds\right|\leq\left(\frac{t^d}{d}\right)^{\frac{1}{\mu}}\|f\|_{\nu}$, where $d=(\alpha-2)\mu+1$ and $\mu=\frac{\nu}{\nu-1}$. Thus we can know easily $\lim_{t\to 0^+}I^{\alpha-1}f(t)=0$. Similarly, $I^{\alpha}f(t)\in C[0,1]$ and $\lim_{t\to 0^+}I^{\alpha}f(t)=0$. The continuity of $I^{\alpha}f(t)$ on $[0,1]$ can also be derived from the continuity of $I^{\alpha-1}f(t)$, Remark 2.2 (i) and (ii).

3 Auxiliary regular problem

This section deals with an auxiliary regular problem. We prove its solvability and give the properties of its solutions. We also state a necessary lemma and its useful corollary.

Consider the integral equation defined by

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_n(s, u(s), u'(s), {}^C D^\beta u(s)) ds \\ & - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f_n(s, u(s), u'(s), {}^C D^\beta u(s)) ds, \end{aligned} \tag{3.1}$$

where

$$f_n(t,x,y,z) = \begin{cases} f(t,x,\chi_n(y),\chi_n(z)), & x \geq \frac{1}{n}, \\ \frac{n}{2} \left[f(t,\frac{1}{n},\chi_n(y),\chi_n(z))(\frac{1}{n}+x) + f(t,-\frac{1}{n},\chi_n(y),\chi_n(z))(\frac{1}{n}-x) \right], & |x| \leq \frac{1}{n}, \\ f(t,x,\chi_n(y),\chi_n(z)), & x \leq -\frac{1}{n}, \end{cases}$$

and

$$\chi_n(\tau) = \begin{cases} \tau, & \tau \geq \frac{1}{n}, \\ \frac{1}{n}, & \tau \leq \frac{1}{n}. \end{cases}$$

Then the conditions (H1) and (H2) give

(K1) $f_n \in C([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ and $f_n(t,x,y,z) \geq mt^{2-\alpha}$ for $(t,x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

(K2) $f_n(t,x,y,z) \leq \rho(t)g(|x|+1, y+1, z+1) + p(\frac{1}{n}) + q(\frac{1}{n}, \frac{1}{n})$ for $(t,x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, $f_n(t,x,y,z) \leq \rho(t)g(|x|+1, y+1, z+1) + p(|x|) + q(y,z)$ for $(t,x,y,z) \in [0,1] \times \mathbb{R}_0 \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$.

Define an operator T_n by the formula

$$\begin{aligned} T_n u(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_n(s, u(s), u'(s), {}^C D^\beta u(s)) ds \\ & - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f_n(s, u(s), u'(s), {}^C D^\beta u(s)) ds. \end{aligned} \tag{3.2}$$

Obviously, the fixed points of T_n are exactly the solutions of integral equation (3.1).

Lemma 3.1. *Suppose that (H1) holds. Then $T_n : C^1[0,1] \to C^1[0,1]$ is completely continuous.*

Proof. Let $u \in C^1[0,1]$. Using Remark 2.2 (i) and (iii) we have ${}^C D^\beta u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} u'(s) ds$ and ${}^C D^\beta u(t) \in C[0,1]$. Thus, in view of (3.2), Remark 2.2 (i) and (K1) ensure $T_n u(t) \in C[0,1]$. Moreover, according to (K1), Remark 2.2 (i), (ii) and (iv), we know $(T_n u)'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f_n(s, u(s), u'(s), {}^C D^\beta u(s)) ds$ and $(T_n u)'(t) \in C[0,1]$. So we have $T_n : C^1[0,1] \to C^1[0,1]$.

T_n is a continuous operator. In fact, let $\{u_k\} \subset C^1[0,1]$ be such that $\lim_{k \to +\infty} \|u_k - u\|_* = 0$, then $u(t) \in C^1[0,1]$. Since

$$\begin{aligned} |{}^C D^\beta u_k(t) - {}^C D^\beta u(t)| & \leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} |u_k'(s) - u'(s)| ds \\ & \leq \frac{\|u_k' - u'\|}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} ds \leq \frac{\|u_k' - u'\|}{\Gamma(2-\beta)}, \end{aligned}$$

we get $\|{}^CD^\beta u_k - {}^C D^\beta u\| \to 0$ and thus $\|f_n(t,u_k,u_k',{}^CD^\beta u_k) - f_n(t,u,u',{}^CD^\beta u)\| \to 0$ as $k \to +\infty$. Note that

$$
\begin{aligned}
&|T_n u_k(t) - T_n u(t)| \\
&\quad = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f_n(s,u_k(s),u_k'(s),{}^CD^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^CD^\beta u(s)) \right] ds \right. \\
&\qquad \left. - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[f_n(s,u_k(s),u_k'(s),{}^CD^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^CD^\beta u(s)) \right] ds \right| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f_n(s,u_k(s),u_k'(s),{}^CD^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^CD^\beta u(s)) \right| ds \\
&\qquad + \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left| f_n(s,u_k(s),u_k'(s),{}^CD^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^CD^\beta u(s)) \right| ds \\
&\quad \le \frac{\|f_n(t,u_k,u_k',{}^CD^\beta u_k) - f_n(t,u,u',{}^CD^\beta u)\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&\qquad + \frac{\|f_n(t,u_k,u_k',{}^CD^\beta u_k) - f_n(t,u,u',{}^CD^\beta u)\|}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\
&\quad = \frac{\|f_n(t,u_k,u_k',{}^CD^\beta u_k) - f_n(t,u,u',{}^CD^\beta u)\|}{\Gamma(\alpha+1)} \left(t^\alpha + \frac{1}{2} \right) \\
&\quad \le \frac{3\|f_n(t,u_k,u_k',{}^CD^\beta u_k) - f_n(t,u,u',{}^CD^\beta u)\|}{2\Gamma(\alpha+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
&|(T_n u_k)'(t) - (T_n u)'(t)| \\
&\quad = \left| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left[f_n(s,u_k(s),u_k'(s),{}^CD^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^CD^\beta u(s)) \right] ds \right| \\
&\quad \le \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left| f_n(s,u_k(s),u_k'(s),{}^CD^\beta u_k(s)) - f_n(s,u(s),u'(s),{}^CD^\beta u(s)) \right| ds \\
&\quad \le \frac{\|f_n(t,u_k,u_k',{}^CD^\beta u_k) - f_n(t,u,u',{}^CD^\beta u)\|}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \\
&\quad = \frac{\|f_n(t,u_k,u_k',{}^CD^\beta u_k) - f_n(t,u,u',{}^CD^\beta u)\|}{\Gamma(\alpha)} t^{\alpha-1} \\
&\quad \le \frac{\|f_n(t,u_k,u_k',{}^CD^\beta u_k) - f_n(t,u,u',{}^CD^\beta u)\|}{\Gamma(\alpha)}.
\end{aligned}
$$

So we obtain $\lim_{k\to+\infty} \|T_n u_k - T_n u\|_* = 0$. Therefore, T_n is a continuous operator.

Furthermore, T_n is completely continuous. Suppose that $\Omega \subset C^1[0,1]$ is bounded and let $M_n = \sup\{\|f_n(t,u,u',{}^CD^\beta u)\|, u \in \Omega\}$, here M_n is well defined because ${}^CD^\beta u(t) \le \frac{\|u'\|}{\Gamma(2-\beta)}$. Then we have

$$
\begin{aligned}
|T_n u(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f_n(s,u(s),u'(s),{}^CD^\beta u(s)) \right| ds \\
&\quad + \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left| f_n(s,u(s),u'(s),{}^CD^\beta u(s)) \right| ds \\
&\le \frac{M_n}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{M_n}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \le \frac{3M_n}{2\Gamma(\alpha+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
|(T_n u)'(t)| &\le \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \big| f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \big| ds \\
&\le \frac{M_n}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \le \frac{M_n}{\Gamma(\alpha)}.
\end{aligned}
$$

Therefore, $T_n(\Omega)$ is bounded. Now we are in the position to prove $T_n(\Omega) \subset C^1[0,1]$ is an equicontinuous set. Let $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, then $|T_n u(t_2) - T_n u(t_1)| \le \frac{M_n}{\Gamma(\alpha)}(t_2 - t_1)$ by the mean value theorem and $|(T_n u)'(t)| \le \frac{M_n}{\Gamma(\alpha)}$. Moreover,

$$
\begin{aligned}
&|(T_n u)'(t_2) - (T_n u)'(t_1)| \\
&= \frac{1}{\Gamma(\alpha-1)} \Big| \int_0^{t_2} (t_2-s)^{\alpha-2} f_n(s,u(s),u'(s),{}^C D^\beta u(s)) ds \\
&\qquad - \int_0^{t_1} (t_1-s)^{\alpha-2} f_n(s,u(s),u'(s),{}^C D^\beta u(s)) ds \Big| \\
&\le \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \left[(t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2} \right] \big| f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \big| ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} \big| f_n(s,u(s),u'(s),{}^C D^\beta u(s)) \big| ds \\
&\le \frac{M_n}{\Gamma(\alpha-1)} \int_0^{t_1} \left[(t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2} \right] ds + \frac{M_n}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} ds \\
&= \frac{M_n}{\Gamma(\alpha)} \big(t_1^{\alpha-1} - t_2^{\alpha-1} + 2(t_2-t_1)^{\alpha-1} \big).
\end{aligned}
$$

Keeping in mind that the function $t^{\alpha-1}$ is uniformly continuous on $[0,1]$, we have $T_n(\Omega)$ is equicontinuous. Consequently, the Arzelà–Ascoli theorem guarantees that T_n is a completely continuous operator. The proof of Lemma 3.1 is finished. □

The next lemma presents the existence of fixed points for the operator T_n.

Lemma 3.2. *Assume that the conditions (H1), (H2) and (H3) are satisfied. Then T_n has a fixed point in $C^1[0,1]$ for any $n \in \mathbb{N}$.*

Proof. In view of Lemma 2.4 and Lemma 3.1, it is sufficient to prove the set $A_n = \{u \in C^1[0,1] : u = \lambda T_n u$ for some $\lambda \in (0,1)\}$ is bounded. For any $u \in A_n$, we have

$$
\begin{aligned}
u(t) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_n(s,u(s),u'(s),{}^C D^\beta u(s)) ds \\
&\quad - \frac{\lambda}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f_n(s,u(s),u'(s),{}^C D^\beta u(s)) ds,
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
u'(t) &= \frac{\lambda}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f_n(s,u(s),u'(s),{}^C D^\beta u(s)) ds \\
&\ge \frac{m\lambda}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} s^{2-\alpha} ds \\
&= \frac{m\lambda t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} s^{2-\alpha} ds = m\lambda\Gamma(3-\alpha) t \ge 0
\end{aligned} \tag{3.4}
$$

by (K1). According to (3.3) and (3.4), one has $u(0)+u(1)=0$, $u'(0)>0$ on $(0,1]$. Thus there exists $\xi\in(0,1)$ such that $u(\xi)=0$. It follows that $|u(t)|=|u(t)-u(\xi)|\le\|u'\||t-\xi|$ and hence $\|u\|\le\|u'\|$. Since ${}^{C}D^{\beta}u(t)\ge 0$ by (3.4) and ${}^{C}D^{\beta}u(t)\le\frac{\|u'\|}{\Gamma(2-\beta)}$, applying the conditions (H2), (H3) and (K2) we can derive

$$\begin{aligned}
u'(t)&\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\left[\rho(s)g(|u(s)|+1,u'(s)+1,{}^{C}D^{\beta}u(s)+1)+p\left(\frac{1}{n}\right)+q\left(\frac{1}{n},\frac{1}{n}\right)\right]ds\\
&\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\left[\rho(s)g(\|u'\|+1,\|u'\|+1,\frac{\|u'\|}{\Gamma(2-\beta)}+1)+p\left(\frac{1}{n}\right)+q\left(\frac{1}{n},\frac{1}{n}\right)\right]ds\\
&\le\frac{g(\|u'\|+1,\|u'\|+1,\frac{\|u'\|}{\Gamma(2-\beta)}+1)}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\rho(s)ds+\frac{p(\frac{1}{n})+q(\frac{1}{n},\frac{1}{n})}{\Gamma(\alpha)}\\
&\le Cg\left(\|u'\|+1,\|u'\|+1,\frac{\|u'\|}{\Gamma(2-\beta)}+1\right)+\frac{p(\frac{1}{n})+q(\frac{1}{n},\frac{1}{n})}{\Gamma(\alpha)},
\end{aligned}$$

here $C=\max_{t\in[0,1]}\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\rho(s)ds$, C is well defined by Remark 2.6 and (H3). In particular, the inequality

$$1\le\frac{Cg(\|u'\|+1,\|u'\|+1,\frac{\|u'\|}{\Gamma(2-\beta)}+1)}{\|u'\|}+\frac{p(\frac{1}{n})+q(\frac{1}{n},\frac{1}{n})}{\|u'\|\Gamma(\alpha)}$$

is fulfilled. The condition $\lim_{x\to+\infty}\frac{g(x,x,x)}{x}=0$ in (H3) guarantees that there exists $L>0$ such that

$$\frac{Cg(\|u'\|+1,\|u'\|+1,\frac{\|u'\|}{\Gamma(2-\beta)}+1)}{\|u'\|}+\frac{p(\frac{1}{n})+q(\frac{1}{n},\frac{1}{n})}{\|u'\|\Gamma(\alpha)}<1$$

for $\|u'\|>L$. Consequently, we obtain $\|u\|\le\|u'\|\le L$ for $u\in A_n$. Therefore, A_n is bounded and we complete the proof. □

Lemma 3.2 shows that the integral equation (3.1) admits a solution u_n in $C^1[0,1]$ for any $n\in\mathbb{N}$. The properties of solutions to (3.1) are collected in the following lemma.

Lemma 3.3. *Let the conditions (H1), (H2) and (H3) be valid and u_n be solution of (3.1). Then*

(1) $u_n(0)+u_n(1)=0$, $u_n'(0)=0$, $u_n'(t)\ge m\Gamma(3-\alpha)t$ and there exists $\xi_n\in(0,1)$ such that $u_n(\xi_n)=0$.

(2) $|u_n(t)|\ge\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|$.

(3) $\{u_n(t),n\in\mathbb{N}\}$ is a compact subset of $C^1[0,1]$.

(4) There exists a constant $l\in(0,1)$ such that $l\le\xi_n<1$ for any $n\in\mathbb{N}$.

Proof. **Proof of (1).** Similar to (3.4), the condition (K1) ensures $u_n'(t)\ge m\Gamma(3-\alpha)t$. Other assertions in (1) are obvious so we omit their proofs.
Proof of (2). Using (1), one has easily $|u_n(t)|=\left|\int_{\xi_n}^t u_n'(s)ds\right|\ge m\Gamma(3-\alpha)\left|\int_{\xi_n}^t sds\right|=\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|$.
Proof of (3). In order to apply the Arzelà–Ascoli theorem, we need to prove $\{u_n(t)\}$ is bounded in $C^1[0,1]$ and $\{u_n'(t)\}$ is equicontinuous. Firstly, we prove $\{u_n(t)\}$ is bounded. In view of (1), we get

$$\|u_n\|\le\|u_n'\|,\quad {}^{C}D^{\beta}u_n(t)\ge m\frac{\Gamma(3-\alpha)}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}sds=\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta}.$$

We also know ${}^CD^{\beta}u_n(t) \leq \frac{\|u_n'\|}{\Gamma(2-\beta)}$. Thus, for $t \in (0,1] \setminus \{\xi_n\}$, by (H2), (K2), (1) and (2) we derive

$$\begin{aligned} f_n(t, u_n(t), u_n'(t), {}^CD^{\beta}u_n(t)) &\leq \rho(t)g(|u_n(t)|+1, u_n'(t)+1, {}^CD^{\beta}u_n(t)+1) \\ &\quad + p(|u_n(t)|) + q(u_n'(t), {}^CD^{\beta}u_n(t)) \\ &\leq \rho(t)g\left(\|u_n'\|+1, \|u_n'\|+1, \frac{\|u_n'\|}{\Gamma(2-\beta)}+1\right) \\ &\quad + p\left(\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|\right) + q\left(m\Gamma(3-\alpha)t, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta}\right). \end{aligned}$$

Hence,

$$\begin{aligned} u_n'(t) \leq &\frac{g(\|u_n'\|+1, \|u_n'\|+1, \frac{\|u_n'\|}{\Gamma(2-\beta)}+1)}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\rho(s)ds \\ &+ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} p\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right) ds \\ &+ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} q\left(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta}\right) ds. \end{aligned} \tag{3.5}$$

Furthermore, by (H3) and Remark 2.6, we can let

$$C_1 = \max_{t\in[0,1]} \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\rho(s)ds, \tag{3.6}$$

$$C_2 = \max_{t\in[0,1]} \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} q\left(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta}\right) ds \tag{3.7}$$

and by the Hölder inequality one has

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-2} p\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right) ds \\ &\qquad \leq [(\alpha-2)\mu+1]^{-\frac{1}{\mu}} \left(\int_0^t p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right) ds\right)^{\frac{1}{\nu}}, \end{aligned} \tag{3.8}$$

here $(\alpha-2)\mu+1 > 0$ and $\frac{1}{\mu}+\frac{1}{\nu} = 1$, μ is well defined by the choice of ν in condition (H3). Next we estimate the integral on the right side of (3.8).

$$\begin{aligned} &\int_0^t p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right) ds \\ &\quad \leq \int_0^1 p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right) ds \\ &\quad = \int_0^{\xi_n} p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right) ds + \int_{\xi_n}^1 p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right) ds \\ &\quad = I_1 + I_2. \end{aligned} \tag{3.9}$$

In view of the monotone property of p and (H3), we get

$$\begin{aligned} I_1 &\leq \int_0^{\xi_n} p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}\xi_n(\xi_n-s)\right) ds = \int_0^1 \xi_n p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}\xi_n^2(1-s)\right) ds \\ &\leq A\xi_n^{1-2\sigma\nu} \int_0^1 (p(1-s))^{\nu} ds \leq A \int_0^1 (p(s))^{\nu} ds \leq A \int_0^1 (p(s^2))^{\nu} ds = C_3 < +\infty, \end{aligned} \tag{3.10}$$

where $A = 1$ if $\frac{m\Gamma(3-\alpha)}{2} \geq 1$, otherwise $A = \left(\frac{m\Gamma(3-\alpha)}{2}\right)^{-\sigma\nu}$.

$$\begin{aligned}
I_2 &= \int_{\xi_n}^{1} p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}(s+\xi_n)(s-\xi_n)\right)ds \\
&= \int_0^1 (1-\xi_n)p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}(1-\xi_n)s((1-\xi_n)s+2\xi_n)\right)ds \\
&\leq \int_0^1 (1-\xi_n)p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}(1-\xi_n)^2 s^2\right)ds \leq A(1-\xi_n)^{1-2\sigma\nu}\int_0^1 (p(s^2))^{\nu}ds \\
&\leq A\int_0^1 (p(s^2))^{\nu}ds = C_3 < +\infty.
\end{aligned} \tag{3.11}$$

As a result, the inequalities from (3.5) to (3.11) show that for any $n \in \mathbb{N}$ and $t \in [0,1]$,

$$u_n'(t) \leq C_1 g\left(\|u_n'\|+1, \|u_n'\|+1, \frac{\|u_n'\|}{\Gamma(2-\beta)}+1\right) + \frac{[(\alpha-2)\mu+1]^{-\frac{1}{\mu}}}{\Gamma(\alpha-1)}(2C_3)^{\frac{1}{\nu}} + C_2.$$

Consequently, similar to the proof in Lemma 3.2, we can conclude that $\{u_n(t)\}$ is bounded.

Now it remains to prove that $\{u_n'(t)\}$ is equicontinuous. Let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$ and $L = \sup\{\|u_n\|_*, n \in \mathbb{N}\}$. Then

$$\begin{aligned}
|u_n'(t_2)-u_n'(t_1)| &\leq \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}\left((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}\right)f_n(s,u_n(s),u_n'(s),{}^C D^{\beta}u_n(s))ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}f_n(s,u_n(s),u_n'(s),{}^C D^{\beta}u_n(s))ds \\
&\leq \frac{\|\rho\|_{\nu} g(L+1, L+1, \frac{L}{\Gamma(2-\beta)}+1)}{\Gamma(\alpha-1)}\left(\int_0^{t_1}\left((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}\right)^{\mu}ds\right)^{\frac{1}{\mu}} \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\left(\int_0^{t_1} p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right)ds\right)^{\frac{1}{\nu}}\left(\int_0^{t_1}\left((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}\right)^{\mu}ds\right)^{\frac{1}{\mu}} \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\left(\int_0^{t_1} q^{\nu}\left(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta}\right)ds\right)^{\frac{1}{\nu}} \\
&\quad \cdot \left(\int_0^{t_1}\left((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}\right)^{\mu}ds\right)^{\frac{1}{\mu}} \\
&\quad + \frac{\|\rho\|_{\nu} g(L+1, L+1, \frac{L}{\Gamma(2-\beta)}+1)}{\Gamma(\alpha-1)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-2)\mu}ds\right)^{\frac{1}{\mu}} \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\left(\int_{t_1}^{t_2} p^{\nu}\left(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|\right)ds\right)^{\frac{1}{\nu}}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-2)\mu}ds\right)^{\frac{1}{\mu}} \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\left(\int_{t_1}^{t_2} q^{\nu}\left(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta}\right)ds\right)^{\frac{1}{\nu}}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-2)\mu}ds\right)^{\frac{1}{\mu}}.
\end{aligned}$$

According to (3.9), (3.10), (3.11) and the condition (H3) we know $\int_0^{t_1} p^{\nu}(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|)ds$, $\int_0^{t_1} q^{\nu}(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta})ds$, $\int_{t_1}^{t_2} p^{\nu}(\frac{m\Gamma(3-\alpha)}{2}|s^2-\xi_n^2|)ds$, $\int_{t_1}^{t_2} q^{\nu}(m\Gamma(3-\alpha)s, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}s^{2-\beta})ds$ are bounded. Furthermore, the relation $(x-y)^{\eta} \leq x^{\eta} - y^{\eta}$ for $x \geq y \geq 0, \eta > 1$ ensures that

$$\begin{aligned}
\int_0^{t_1}\left((t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}\right)^{\mu}ds &\leq \int_0^{t_1}\left((t_1-s)^{(\alpha-2)\mu}-(t_2-s)^{(\alpha-2)\mu}\right)ds \\
&= \frac{t_1^{(\alpha-2)\mu+1}-t_2^{(\alpha-2)\mu+1}+(t_2-t_1)^{(\alpha-2)\mu+1}}{(\alpha-2)\mu+1}.
\end{aligned}$$

In addition,

$$\int_{t_1}^{t_2}(t_2-s)^{(\alpha-2)\mu}ds=\frac{(t_2-t_1)^{(\alpha-2)\mu+1}}{(\alpha-2)\mu+1}.$$

Hence we can obtain that $\{u_n'(t)\}$ is equicontinuous. Consequently, the Arzelà–Ascoli theorem implies that $\{u_n(t)\}$ is a compact subset of $C^1[0,1]$.

Proof of (4). Suppose that there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\lim_{k\to+\infty}\xi_{n_k}=0$. Since $|u_{n_k}(0)|=|u_{n_k}(0)-u_{n_k}(\xi_{n_k})|\le\|u'_{n_k}\|\xi_{n_k}$, we have $\lim_{k\to+\infty}u_{n_k}(0)=0$. Thus, $\lim_{k\to+\infty}u_{n_k}(1)=0$ because $u_{n_k}(0)+u_{n_k}(1)=0$, which contradicts $u_{n_k}(1)-u_{n_k}(0)=\int_0^1u'_{n_k}(s)ds\ge\frac{m\Gamma(3-\alpha)}{2}$. Hence, $\inf\{\xi_n:n\in\mathbb{N}\}>0$. As a result, we arrive at $\xi_n\in[l,1)$ for $n\in\mathbb{N}$ with some $l>0$.

We complete the proof of Lemma 3.3. □

In order to apply the Vitali convergence theorem in the proof of our main theorem, we need the following result.

Lemma 3.4. *Let the conditions (H1), (H2) and (H3) be satisfied and u_n be solution of (3.1). Then $\{f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t)),n\in\mathbb{N}\}\subset C[0,1]$ has uniformly absolutely continuous integrals on $[0,1]$.*

Proof. Let $E\subset[0,1]$ be measurable and $L=\sup\{\|u_n\|_*,n\in\mathbb{N}\}$. Then

$$\begin{aligned}\int_E f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t))dt\le g\left(L+1,L+1,\frac{L}{\Gamma(2-\beta)}+1\right)\int_E\rho(t)dt\\+\int_E p\left(\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|\right)dt\\+\int_E q\left(m\Gamma(3-\alpha)t,\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta}\right)dt.\end{aligned}$$

Applying the Hölder inequality, we have

$$\int_E\rho(t)dt\le(\mathrm{meas}(E))^{\frac{1}{\mu}}\left(\int_E(\rho(t))^\nu dt\right)^{\frac{1}{\nu}},$$

$$\int_E p\left(\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|\right)dt\le(\mathrm{meas}(E))^{\frac{1}{\mu}}\left(\int_E p^\nu\left(\frac{m\Gamma(3-\alpha)}{2}|t^2-\xi_n^2|\right)dt\right)^{\frac{1}{\nu}},$$

$$\int_E q\left(m\Gamma(3-\alpha)t,\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta}\right)dt\le(\mathrm{meas}(E))^{\frac{1}{\mu}}\left(\int_E q^\nu\left(m\Gamma(3-\alpha)t,\frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)}t^{2-\beta}\right)dt\right)^{\frac{1}{\nu}}.$$

Noticing the condition (H3), (3.9), (3.10) and (3.11), we conclude that the sequence $\{f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t))\}$ has uniformly absolutely continuous integrals on $[0,1]$. □

Corollary 3.5. *Let the conditions (H1), (H2) and (H3) hold and u_n be solution of (3.1). Then $\{(t_0-t)^{\alpha-1}f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t)),n\in\mathbb{N}\}\subset C[0,t_0]$ has uniformly absolutely continuous integrals on $[0,t_0]$ for any $t_0\in[0,1]$.*

The assertion in Corollary 3.5 follows from Lemma 3.4 and the fact

$$(t_0-t)^{\alpha-1}f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t))\le f_n(t,u_n(t),u_n'(t),{}^CD^\beta u_n(t)),\qquad t\in[0,t_0].$$

4 Main result

Now we can give the existence result for the singular BVP (1.1).

Theorem 4.1. *Assume that the conditions (H1), (H2) and (H3) are valid. Then there exists at least one increasing function $u(t) \in C^1[0,1]$ solving the BVP (1.1).*

Proof. For clarity, we divide the proof into several steps.
Step 1: Firstly, Lemma 3.3 and the Bolzano–Weierstrass theorem guarantee that there exist subsequences $\{u_{n_k}\} \subset \{u_n\}$, $\{\xi_{n_k}\} \subset \{\xi_n\}$ and $u \in C^1[0,1]$, $\xi \in [l,1]$ such that $\lim_{k\to+\infty} \xi_{n_k} = \xi$ and $\lim_{k\to+\infty} \|u_{n_k} - u\|_* = 0$. Then again by Lemma 3.3, $u(\xi) = 0$, $u(0) + u(1) = 0$, $u'(0) = 0$, $u'(t) > 0$ for $t \in (0,1]$ and $|u(t)| \geq \frac{m\Gamma(3-\alpha)}{2}|t^2 - \xi^2|$. The last inequality together with $u(0) + u(1) = 0$ implies $u(1) \neq 0$, that is, $\xi \in [l,1)$.

Furthermore, since ${}^C D^\beta u_{n_k}(t) \geq \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} t^{2-\beta}$ and $\lim_{k\to+\infty} \|{}^C D^\beta u_{n_k} - {}^C D^\beta u\| = 0$, we get ${}^C D^\beta u(t) \geq \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} t^{2-\beta}$ and thus ${}^C D^\beta u(t) > 0$ on $(0,1]$. Hence, $f(t, u(t), u'(t), {}^C D^\beta u(t)) \in C((0,1] \setminus \{\xi\})$ and

$$\lim_{k\to+\infty} f_{n_k}(t, u_{n_k}(t), u'_{n_k}(t), {}^C D^\beta u_{n_k}(t)) = f(t, u(t), u'(t), {}^C D^\beta u(t)), \qquad t \in (0,1] \setminus \{\xi\}.$$

Also, we can know $f(t, u(t), u'(t), {}^C D^\beta u(t)) \in L^1[0,1]$ by Lemma 2.3 and Lemma 3.4. Moreover, according to Lemma 2.3 and Corollary 3.5, passing to the limit as $k \to +\infty$ on both sides of the equality

$$\begin{aligned} u_{n_k}(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_{n_k}(s, u_{n_k}(s), u'_{n_k}(s), {}^C D^\beta u_{n_k}(s)) ds \\ & - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f_{n_k}(s, u_{n_k}(s), u'_{n_k}(s), {}^C D^\beta u_{n_k}(s)) ds, \end{aligned}$$

we obtain

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u'(s), {}^C D^\beta u(s)) ds \\ & - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s), u'(s), {}^C D^\beta u(s)) ds, \qquad t \in [0,1]. \end{aligned} \tag{4.1}$$

Therefore, $u(t)$ is a solution of integral equation (4.1). Next we prove $u(t)$ is a solution of (1.1).
Step 2: In this step we prove that the right side integral in (4.1) belongs to $C^1[0,1]$ and satisfies the boundary value conditions in (1.1).

Let $L = \|u\|_*$. In view of for any $t \in (0,1] \setminus \{\xi\}$,

$$\begin{aligned} f(t, u(t), u'(t), {}^C D^\beta u(t)) \leq {} & g\left(L+1, L+1, \frac{L}{\Gamma(2-\beta)} + 1\right)\rho(t) + p\left(\frac{m\Gamma(3-\alpha)}{2}|t^2 - \xi^2|\right) \\ & + q\left(m\Gamma(3-\alpha)t, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} t^{2-\beta}\right), \end{aligned}$$

this together with (3.9), (3.10), (3.11) and (H3) guarantees that $f(t, u(t), u'(t), {}^C D^\beta u(t)) \in L^\nu[0,1]$. Hence $I^\alpha f \in C[0,1]$ and $I^{\alpha-1} f \in C[0,1]$ by Remark 2.6. Furthermore, by Remark 2.2 (ii) one has for any $t \in [0,1]$,

$$\begin{aligned} D^1 I^\alpha f(t, u(t), u'(t), {}^C D^\beta u(t)) &= D^1 I^1 I^{\alpha-1} f(t, u(t), u'(t), {}^C D^\beta u(t)) \\ &= I^{\alpha-1} f(t, u(t), u'(t), {}^C D^\beta u(t)). \end{aligned}$$

Thus we obtain that the right side integral in (4.1) belongs to $C^1[0,1]$. Since by Remark 2.6

$$\lim_{t\to 0^+} I^\alpha f(t,u(t),u'(t),{}^CD^\beta u(t)) = \lim_{t\to 0^+} I^{\alpha-1} f(t,u(t),u'(t),{}^CD^\beta u(t)) = 0,$$

we can know easily the right side integral in (4.1) satisfies the boundary value conditions in BVP (1.1).

Step 3: Now it remains to prove that the Caputo derivative of order α of the right side integral in (4.1) exists and is continuous on $(0,1]\setminus\{\xi\}$ and satisfies the differential equation in (1.1) for $t\in(0,1]\setminus\{\xi\}$.

In fact, using the definitions of Caputo fractional derivative and Riemann–Liouville fractional derivative, we have

$$\begin{aligned}
{}^CD^\alpha u(t) &= D^\alpha[u(t)-u(0)-u'(0)t]\\
&= D^\alpha\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),u'(s),{}^CD^\beta u(s))ds\right)\\
&= D^\alpha I^\alpha f(t,u(t),u'(t),{}^CD^\beta u(t)) = \left(\frac{d}{dt}\right)^2 I^{2-\alpha} I^\alpha f(t,u(t),u'(t),{}^CD^\beta u(t)).
\end{aligned}$$

Thus we need to prove that $\left(\frac{d}{dt}\right)^2 I^{2-\alpha} I^\alpha f(t,u(t),u'(t),{}^CD^\beta u(t))$ exists and is continuous on $(0,1]\setminus\{\xi\}$ and is equal to $f(t,u(t),u'(t),{}^CD^\beta u(t))$ for $t\in(0,1]\setminus\{\xi\}$.

Firstly, by $f\in L^1[0,1]$ and Remark 2.2 (ii), for any $t\in[0,1]$, we have

$$\begin{aligned}
I^{2-\alpha} I^\alpha f(t,u(t),u'(t),{}^CD^\beta u(t)) &= I^2 f(t,u(t),u'(t),{}^CD^\beta u(t))\\
&= \int_0^t (t-s) f(s,u(s),u'(s),{}^CD^\beta u(s))ds.
\end{aligned}$$

Secondly, by $f\in C((0,1]\setminus\{\xi\})$, for any $t\in(0,1]\setminus\{\xi\}$, let Δt be small enough so that f is continuous on $[t-|\Delta t|, t+|\Delta t|]$ (for $t=1$, f is continuous on $[t-|\Delta t|, t]$), then applying mean value theorem for integrals, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^t f(s,u(s),u'(s),{}^CD^\beta u(s))ds\right)\\
&= \lim_{\Delta t\to 0} \frac{\int_0^{t+\Delta t} f(s,u(s),u'(s),{}^CD^\beta u(s))ds - \int_0^t f(s,u(s),u'(s),{}^CD^\beta u(s))ds}{\Delta t}\\
&= \lim_{\Delta t\to 0} \frac{\int_t^{t+\Delta t} f(s,u(s),u'(s),{}^CD^\beta u(s))ds}{\Delta t} = f(t,u(t),u'(t),{}^CD^\beta u(t)).
\end{aligned}$$

Similarly, $\frac{d}{dt}\left(\int_0^t s f(s,u(s),u'(s),{}^CD^\beta u(s))ds\right) = t f(t,u(t),u'(t),{}^CD^\beta u(t))$. As a result we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^t (t-s) f(s,u(s),u'(s),{}^CD^\beta u(s))ds\right)\\
&= \frac{d}{dt}\left(t\int_0^t f(s,u(s),u'(s),{}^CD^\beta u(s))ds\right) - \frac{d}{dt}\left(\int_0^t s f(s,u(s),u'(s),{}^CD^\beta u(s))ds\right)\\
&= \int_0^t f(s,u(s),u'(s),{}^CD^\beta u(s))ds,
\end{aligned}$$

and hence, $\left(\frac{d}{dt}\right)^2\left(\int_0^t (t-s) f(s,u(s),u'(s),{}^CD^\beta u(s))ds\right) = f(t,u(t),u'(t),{}^CD^\beta u(t))$ for any $t\in(0,1]\setminus\{\xi\}$.

We complete the proof of our main result. □

Remark 4.2. In Theorem 4.1, by $f \in L^1[0,1]$ and Remark 2.2 (v) we can know $I^2 f(t,u(t),u'(t),{}^C D^\beta u(t)) \in AC^1[0,1]$. Thus the function $u(t)$ defined by (4.1) has summable fractional derivative and ${}^C D^\alpha u(t) = f(t,u(t),u'(t),{}^C D^\beta u(t))$ for a.e. $t \in [0,1]$. Furthermore, $f \in C((0,1]\setminus\{\xi\})$ and this ensures ${}^C D^\alpha u(t) = f(t,u(t),u'(t),{}^C D^\beta u(t))$ for any $t \in (0,1]\setminus\{\xi\}$.

5 An example

In this section we give an example to illustrate our result.

Example 5.1. Consider the boundary value problem

$$\begin{cases} {}^C D^{\frac{3}{2}} x(t) = (t+1)^2 + |\cos t| \left[\ln(1+|x(t)|) + \arctan x'(t) + ({}^C D^{\frac{1}{2}} x(t))^{\frac{1}{2}}\right] \\ \qquad\qquad + \frac{e^t}{|x(t)|^{\frac{1}{8}}} + \frac{1}{\left(x'(t){}^C D^{\frac{1}{2}} x(t)\right)^{\frac{1}{10}}}, \\ x(0) + x(1) = 0, \ x'(0) = 0. \end{cases} \tag{5.1}$$

Clearly $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$ and the nonlinear term is

$$f(t,x,y,z) = (t+1)^2 + |\cos t| \left[\ln(1+|x|) + \arctan y + z^{\frac{1}{2}}\right] + \frac{e^t}{|x|^{\frac{1}{8}}} + \frac{1}{(yz)^{\frac{1}{10}}}, \qquad (t,x,y,z) \in [0,1] \times \mathbb{R}_0 \times \mathbb{R}_0^+ \times \mathbb{R}_0^+.$$

The conditions (H1), (H2) and (H3) are satisfied with $m = \min_{t\in[0,1]}(t+1)^2 = 1$, $\sigma \in [\frac{1}{8}, \frac{1}{4})$, $\nu \in (2, \frac{1}{2\sigma})$, $\rho(t) = (t+1)^2 + |\cos t|$, $g(x,y,z) = 1 + \ln(1+x) + \arctan y + z^{\frac{1}{2}}$, $p(x) = \frac{e}{x^{1/8}}$ and $q(y,z) = \frac{1}{(yz)^{1/10}}$. We only verify that $p(x)$ and $q(y,z)$ satisfy the conditions in (H3). Other conditions are easy to verify and we omit here. First of all, we have $p(\lambda x) = \lambda^{-\frac{1}{8}} \frac{e}{x^{1/8}} \le \lambda^{-\sigma} \frac{e}{x^{1/8}} = \lambda^{-\sigma} p(x)$ for $\lambda \in (0,1]$ and $x \in \mathbb{R}_0^+$. Moreover, $\frac{\nu}{4} < 1$ and this ensures $p(x^2) \in L^\nu[0,1]$ and $q(m\Gamma(3-\alpha)x, \frac{m\Gamma(3-\alpha)}{\Gamma(3-\beta)} x^{2-\beta}) = (\frac{\sqrt{\pi}}{3})^{-\frac{1}{10}} \frac{1}{x^{1/4}} \in L^\nu[0,1]$. As a result, Theorem 4.1 guarantees that the problem (5.1) has an increasing solution.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (11671181). The authors would like to thank the reviewers for their valuable suggestions and useful comments that have significantly improved the original manuscript.

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