On the impulsive Dirichlet problem
for second-order differential inclusions

Martina Pavlačková[1] and **Valentina Taddei**✉[2]

[1]Dept. of Computer Science and Appl. Mathematics, Moravian Business College Olomouc,
tř. Kosmonautů 1288/1, 77900 Olomouc, Czech Republic
[2]Dept. of Sciences and Methods for Engineering, University of Modena and Reggio Emilia,
Via G. Amendola 2 - pad. Morselli, I-42122 Reggio Emilia, Italy

Abstract. Solutions in a given set of an impulsive Dirichlet boundary value problem are investigated for second-order differential inclusions. The method used for obtaining the existence and the localization of a solution is based on the combination of a fixed point index technique developed by ourselves earlier with a bound sets approach and Scorza-Dragoni type result. Since the related bounding (Liapunov-like) functions are strictly localized on the boundaries of parameter sets of candidate solutions, some trajectories are allowed to escape from these sets.

Keywords: impulsive Dirichlet problem, differential inclusions, topological methods, bounding functions, Scorza-Dragoni technique.

2020 Mathematics Subject Classification: 34A60, 34B15, 47H04.

1 Introduction

Let us consider the Dirichlet boundary value problem

$$\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0, T], \\ x(T) = x(0) = 0, \end{cases} \tag{1.1}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping.

Moreover, let a finite number of points $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, and real $n \times n$ matrices A_i, B_i, $i = 1, \ldots, p$, be given. In the paper, the solvability of the Dirichlet b.v.p. (1.1) will be investigated in the presence of the following impulse conditions

$$x(t_i^+) = A_i x(t_i), \qquad i = 1, \ldots, p, \tag{1.2}$$
$$\dot{x}(t_i^+) = B_i \dot{x}(t_i), \qquad i = 1, \ldots, p, \tag{1.3}$$

where the notation $\lim_{t \to a^+} x(t) = x(a^+)$ is used.

✉Corresponding author. Email: valentina.taddei@unimore.it

By a *solution* of problem (1.1)–(1.3) we shall mean a function $x \in PAC^1([0, T], \mathbb{R}^n)$ (see Section 2 for the definition) satisfying (1.1), for almost all $t \in [0, T]$, and fulfilling the conditions (1.2) and (1.3).

Boundary value problems with impulses have been widely studied because of their applications in areas, where the parameters are subject to certain perturbations in time. For instance, in the treatment of some diseases, impulses may correspond to administration of a drug treatment or in environmental sciences, they can describe the seasonal changes or harvesting.

While the theory of single valued impulsive problems is deeply examined (see, e.g. [9, 10, 22]), the theory dealing with multivalued impulsive problems has not been studied so much yet (for the overview of known results see, e.g., the monographs [11, 19] and the references therein). However, it is worth to study also the multivalued case, since the multivalued problems come e.g. from single valued problems with discontinuous right-hand sides, or from control theory.

The most of mentioned results dealing with impulsive problems have been obtained using fixed point theorems, upper and lower-solutions methods, or using topological and variational approaches.

In this paper, the existence and the localization of a solution for the impulsive Dirichlet b.v.p. (1.1)–(1.3) will be studied using a continuation principle. On this purpose, it will be necessary to embed the original problem into a family of problems and to ensure that the boundary of a prescribed set of candidate solutions is fixed point free, i.e. to verify so called transversality condition. This condition can be guaranteed by a bound sets technique that was described by Gaines and Mawhin in [17] for single valued problems without impulses. Recently, in [25], a bound sets technique for the multivalued impulsive b.v.p. using non strictly localized bounding (Liapunov-like) functions has been developed. Such a non-strict localization of bounding functions makes parameter sets of candidate solutions "only" positively invariant.

In this paper, the conditions imposed on the bounding function will be strictly localized on the boundary of the set of candidate solutions, which eliminates this unpleasant handicap. Both the possible cases will be discussed – problems with an upper semicontinuous r.h.s. and also problems with an upper-Carathéodory r.h.s. More concretely, in Theorem 4.3 below, the upper semicontinuous case is considered and the transversality condition is obtained reasoning pointwise via a C^1-bounding function with a locally Lipschitzian gradient. In Theorem 5.2, the upper-Carathéodory case and a C^2-bounding function will be considered and the reasoning will be based on a Scorza-Dragoni approximation technique. In fact, even if the first kind of regularity of the r.h.s. is a special case of the second one, in the first case the stronger regularity will allow to use C^1-bounding functions, while in the second case, C^2-bounding functions will be needed. Moreover, even when using C^2-bounding functions, the more regularity of the r.h.s. allows to obtain the result under weaker conditions. Let us note that a similar approach was employed for problems with upper semicontinous r.h.s. without impulses e.g. in [3, 6] and for problems with upper-Carathéodory r.h.s. without impulses e.g. in [4, 24].

This paper is organized as follows. In the second section, we recall suitable definitions and statements which will be used in the sequel. Section 3 is devoted to the study of bound sets and Liapunov-like bounding functions for impulsive Dirichlet problems with an upper semicontinuous r.h.s. At first, C^1-bounding functions with locally Lipschitzian gradients are considered. Consequently, it is shown how conditions ensuring the existence of bound set

change in case of C^2-bounding functions. In Section 4, the bound sets approach is combined with a continuation principle and the existence and localization result is obtained in this way for the impulsive Dirichlet problem (1.1)–(1.3). Section 5 deals with the existence and localization of a solution of the Dirichlet impulsive problem in case when the r.h.s. is an upper-Carathéodory mapping. In Section 6, the obtained result is applied to an illustrative example.

2 Some preliminaries

Let us recall at first some geometric notions of subsets of metric spaces. If (X, d) is an arbitrary metric space and $A \subset X$, by $\text{Int}(A), \overline{A}$ and ∂A we mean the interior, the closure and the boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x, a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X.

For a given compact real interval J, we denote by $C(J, \mathbb{R}^n)$ (by $C^1(J, \mathbb{R}^n)$) the set of all functions $x : J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $AC^1(J, \mathbb{R}^n)$, we shall mean the set of all functions $x : J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J.

Let $PAC^1([0, T], \mathbb{R}^n)$ be the space of all functions $x : [0, T] \to \mathbb{R}^n$ such that

$$
x(t) = \begin{cases}
x_{[0]}(t), & \text{for } t \in [0, t_1], \\
x_{[1]}(t), & \text{for } t \in (t_1, t_2], \\
\quad \vdots \\
x_{[p]}(t), & \text{for } t \in (t_p, T],
\end{cases}
$$

where $x_{[0]} \in AC^1([0, t_1], \mathbb{R}^n), x_{[i]} \in AC^1((t_i, t_{i+1}], \mathbb{R}^n), x(t_i^+) = \lim_{t \to t_i^+} x(t) \in \mathbb{R}$ and $\dot{x}(t_i^+) = \lim_{t \to t_i^+} \dot{x}(t) \in \mathbb{R}$, for every $i = 1, \ldots, p$. The space $PAC^1([0, T], \mathbb{R}^n)$ is a normed space with the norm

$$
\|x\| := \sup_{t \in [0,T]} |x(t)| + \sup_{t \in [0,T]} |\dot{x}(t)|. \tag{2.1}
$$

In a similar way, we can define the spaces $PC([0, T], \mathbb{R}^n)$ and $PC^1([0, T], \mathbb{R}^n)$ as the spaces of functions $x : [0, T] \to \mathbb{R}^n$ satisfying the previous definition with $x_{[0]} \in C([0, t_1], \mathbb{R}^n), x_{[i]} \in C((t_i, t_{i+1}], \mathbb{R}^n)$ or with $x_{[0]} \in C^1([0, t_1], \mathbb{R}^n), x_{[i]} \in C^1((t_i, t_{i+1}], \mathbb{R}^n)$, for every $i = 1, \ldots, p$, respectively. The space $PC^1([0, T], \mathbb{R}^n)$ with the norm defined by (2.1) is a Banach space (see [23, page 128]).

A subset $A \subset X$ is called a *retract* of a metric space X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$. We say that a space X is an *absolute retract* (AR-space) if, for each space Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y. If f is extendable only over some neighborhood of A, for each closed $A \subset Y$ and each continuous mapping $f : A \to X$, then X is called an *absolute neighborhood retract* (ANR-space). Let us note that X is an *ANR*-space if and only if it is a retract of an open subset of a normed space and that X is an *AR*-space if and only if it is a retract of some normed space (see, e.g. [2]). Conversely, if X is a retract (of an open subset) of a convex set in a Banach space, then it is an *AR*-space (*ANR*-space). So, the space $C^1(J, \mathbb{R}^n)$, where $J \subset \mathbb{R}$ is a compact interval, is an *AR*-space as well as its convex subsets or retracts, while its open subsets are *ANR*-spaces.

A nonempty set $A \subset X$ is called an R_δ-*set* if there exists a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of compact *AR*-spaces such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

The following hierarchy holds for nonempty subsets of a metric space:

$$\text{compact+convex} \subset \text{compact } AR\text{-space} \subset R_\delta\text{-set,} \tag{2.2}$$

and all the above inclusions are proper. For more details concerning the theory of retracts, see [14].

We also employ the following definitions from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that φ is a multivalued mapping from X to Y (written $\varphi : X \multimap Y$) if, for every $x \in X$, a nonempty subset $\varphi(x)$ of Y is prescribed. We associate with F its *graph* Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

Let us mention also some basic notions concerning multivalued mappings. A multivalued mapping $\varphi : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X.

Let $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper semicontinuous multimap and let, for all $r > 0$, exist an integrable function $\mu_r : J \to [0, \infty)$ such that $|y| \leq \mu_r(t)$, for every $(t, x) \in J \times \mathbb{R}^m$, with $|x| \leq r$, and every $y \in F(t, x)$. Then if we consider the composition of F with a function $q \in PC^1([0, T], \mathbb{R}^n)$, the corresponding superposition multioperator $\mathcal{P}_F(q)$ given by

$$\mathcal{P}_F(q) = \{f \in L^1([0, T]; \mathbb{R}^m) \ : \ f(t) \in F(t, q(t)) \text{ a.a. } t \in [0, T]\},$$

is well defined and nonempty (see [12, Proposition 6]).

Let Y be a metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space*, i.e. a nonempty set Ω equipped with a σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $\varphi : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$. Obviously, every u.s.c. mapping is measurable.

We say that mapping $\varphi : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $\varphi(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $\varphi(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all $t \in J$, and the set $\varphi(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

If $X \cap Y \neq \emptyset$ and $\varphi : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed point* of φ if $x \in \varphi(x)$. The set of all fixed points of φ is denoted by $\text{Fix}(\varphi)$, i.e.

$$\text{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}.$$

For more information and details concerning multivalued analysis, see, e.g., [2,8,18,21].

The continuation principle which will be applied in the paper requires in particular the transformation of the studied problem into a suitable family of associated problems which does not have solutions tangent to the boundary of a given set Q of candidate solutions. This will be ensured by means of Hartman-type conditions (see Section 3) and by means of the following result based on Nagumo conditions (see [27, Lemma 2.1] and [20, Lemma 5.1]).

Proposition 2.1. *Let $\psi : [0, +\infty) \to [0, +\infty)$ be a continuous and increasing function, with*

$$\lim_{s \to \infty} \frac{s^2}{\psi(s)} ds = \infty, \tag{2.3}$$

and let R be a positive constant. Then there exists a positive constant

$$B = \psi^{-1}(\psi(2R) + 2R) \tag{2.4}$$

such that if $x \in PC^1([0, T], \mathbb{R}^n)$ is such that $|\ddot{x}(t)| \leq \psi(|\dot{x}(t)|)$, for a.a. $t \in [0, T]$, and $|x(t)| \leq R$, for every $t \in [0, T]$, then it holds that $|\dot{x}(t)| \leq B$, for every $t \in [0, T]$.

Let us note that the previous result is classically given for C^2-functions. However, it is easy to prove (see, e.g., [5]) that the statement holds also for piecewise continuously differentiable functions.

For obtaining the existence and localization result for the case of upper-Carathéodory r.h.s., we will need the following Scorza-Dragoni type result for multivalued maps (see [15, Proposition 5.1]).

Proposition 2.2. *Let $X \subset \mathbb{R}^m$ be compact and let $F : [a, b] \times X \multimap \mathbb{R}^n$ be an upper-Carathéodory map. Then there exists a multivalued mapping $F_0 : [a, b] \times X \multimap \mathbb{R}^n \cup \{\emptyset\}$ with compact, convex values and $F_0(t, x) \subset F(t, x)$, for all $(t, x) \in [a, b] \times X$, having the following properties:*

  (i) *if $u : [a, b] \to \mathbb{R}^m$, $v : [a, b] \to \mathbb{R}^n$ are measurable functions with $v(t) \in F(t, u(t))$, on $[a, b]$, then $v(t) \in F_0(t, u(t))$, a.e. on $[a, b]$;*

  (ii) *for every $\epsilon > 0$, there exists a closed $I_\epsilon \subset [a, b]$ such that $\nu([a, b] \setminus I_\epsilon) < \epsilon$, $F_0(t, x) \neq \emptyset$, for all $(t, x) \in I_\epsilon \times X$ and F_0 is u.s.c. on $I_\epsilon \times X$.*

3   Bound sets for Dirichlet problems with upper semicontinuous r.h.s.

In this section, we consider an u.s.c. multimap F and we are interested in introducing a Liapunov-like function V, usually called a *bounding function*, verifying suitable transversality conditions which assure that there does not exist a solution of the b.v.p. lying in a closed set $\overline{K}$ and touching the boundary ∂K of K at some point.

Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$ and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$(H1)$ $V|_{\partial K} = 0$,

$(H2)$ $V(x) \leq 0$, for all $x \in \overline{K}$.

Definition 3.1. A nonempty open set $K \subset \mathbb{R}^n$ is called a *bound set* for problem (1.1)–(1.3) if there does not exist a solution x of (1.1)–(1.3) such that $x(t) \in \overline{K}$, for each $t \in [0, T]$, and $x(t_0) \in \partial K$, for some $t_0 \in [0, T]$.

Firstly, we show sufficient conditions for the existence of a bound set for the second-order impulsive Dirichlet problem (1.1)–(1.3) in the case of a smooth bounding function V with a locally Lipschitzian gradient.

Proposition 3.2. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$, $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions (H1) and (H2). Suppose moreover that, for all $x \in \partial K$, $t \in (0, T) \setminus \{t_1, \dots, t_p\}$ and $v \in \mathbb{R}^n$ with*

$$\langle \nabla V(x), v \rangle = 0, \tag{3.1}$$

the following condition holds

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} > 0, \tag{3.2}$$

for all $w \in F(t, x, v)$. Then all solutions $x : [0, T] \to \overline{K}$ of problem (1.1) satisfy $x(t) \in K$, for every $t \in [0, T] \setminus \{t_1, \dots, t_p\}$.

Proof. Let $x : [0, T] \to \overline{K}$ be a solution of problem (1.1). We assume by a contradiction that there exists $\bar{t} \in [0, T] \setminus \{t_1, \dots, t_p\}$ such that $x(\bar{t}) \in \partial K$. Since $x(0) = x(T) = 0 \in K$, it must be $\bar{t} \in (0, T)$.

Let us define the function g in the following way $g(h) := V(x(\bar{t} + h))$. Then $g(0) = 0$ and there exists $\alpha > 0$ such that $g(h) \le 0$, for all $h \in [-\alpha, \alpha]$, i.e., there is a local maximum for g at the point 0, and $g \in C^1([-\alpha, \alpha], \mathbb{R}^n)$, so $\dot{g}(0) = \langle \nabla V(x(\bar{t})), \dot{x}(\bar{t}) \rangle = 0$. Consequently, $x := x(\bar{t}), v := \dot{x}(\bar{t})$ satisfy condition (3.1).

Since ∇V is locally Lipschitzian, there exist an open set $U \subset \mathbb{R}^n$, with $x(\bar{t}) \in U$, and a constant $L > 0$ such that $\nabla V|_U$ is Lipschitzian with constant L. We can assume, without loss of generality, that $x(\bar{t} + h) \in U$ for all $h \in [-\alpha, \alpha]$.

Since $g(0) = 0$ and $g(h) \le 0$, for all $h \in [-\alpha, 0)$, there exists an increasing sequence of negative numbers $\{h_k\}_{k=1}^{\infty}$ such that $h_1 > -\alpha, h_k \to 0^-$ as $k \to \infty$, and $\dot{g}(h_k) \ge 0$, for each $k \in \mathbb{N}$. Since $x \in C^1([-\alpha, 0], \mathbb{R}^n)$, it holds, for each $k \in \mathbb{N}$, that

$$x(\bar{t} + h_k) = x(\bar{t}) + h_k[\dot{x}(\bar{t}) + b_k], \tag{3.3}$$

where $b_k \to 0$ as $k \to \infty$.

Since $x([-\alpha, 0])$ and $\dot{x}([-\alpha, 0])$ are compact sets and F is globally upper semicontinuous with compact values, $F(\cdot, x(\cdot), \dot{x}(\cdot))$ must be bounded on $[-\alpha, 0]$, by which $\dot{x}$ is Lipschitzian on $[-\alpha, 0]$. Thus, there exists a constant λ such that, for all $k \in \mathbb{N}$,

$$\left| \frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} \right| \le \lambda,$$

i.e. the sequence $\left\{ \frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} \right\}_{k=1}^{\infty}$ is bounded. Therefore, there exist a subsequence, for the sake of simplicity denoted as the sequence, of $\left\{ \frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} \right\}$ and $w \in \mathbb{R}^n$ such that

$$\frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} \to w \tag{3.4}$$

as $k \to \infty$.

Let $\varepsilon > 0$ be given. Then, as a consequence of the regularity assumptions on F and of the continuity of both x and $\dot{x}$ on $[-\alpha, 0]$, there exists $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$ such that, for each $h \in [-\alpha, 0]$, $h \ge -\bar{\delta}$, it follows that

$$F(\bar{t} + h, x(\bar{t} + h), \dot{x}(\bar{t} + h)) \subset F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})) + \varepsilon \overline{B}_n,$$

where B_n denotes the unit open ball in $\mathbb{R}^n$ centered at the origin. Subsequently, since F is convex valued, according to the Mean-Value Theorem (See [8], Theorem 0.5.3), there exists $k_\varepsilon \in \mathbb{N}$ such that, for each $k \geq k_\varepsilon$,

$$\frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} = \frac{1}{-h_k} \int_{\bar{t}+h_k}^{\bar{t}} \ddot{x}(s) \, ds \in F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})) + \varepsilon \overline{B}_n.$$

Since F has compact values and $\varepsilon > 0$ is arbitrary,

$$w \in F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})).$$

As a consequence of property (3.4), there exists a sequence $\{a_k\}_{k=1}^\infty$, $a_k \to 0$ as $k \to \infty$, such that

$$\dot{x}(\bar{t} + h_k) = \dot{x}(\bar{t}) + h_k[w + a_k], \tag{3.5}$$

for each $k \in \mathbb{N}$. Since $h_k < 0$ and $\dot{g}(h_k) \geq 0$, in view of (3.3) and (3.5),

$$0 \geq \frac{\dot{g}(h_k)}{h_k} = \frac{\langle \nabla V(x(\bar{t} + h_k)), \dot{x}(\bar{t} + h_k) \rangle}{h_k}$$
$$= \frac{\langle \nabla V(x(\bar{t}) + h_k[\dot{x}(\bar{t}) + b_k]), \dot{x}(\bar{t}) + h_k[w + a_k] \rangle}{h_k}.$$

Since $b_k \to 0$ when $k \to +\infty$, it is possible to find $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, it holds that $x(\bar{t}) + \dot{x}(\bar{t})h_k \in U$, because U is open. By means of the local Lipschitzianity of ∇V, for all $k \geq k_0$,

$$0 \geq \frac{\dot{g}(h_k)}{h_k} \geq \frac{\langle \nabla V(x(\bar{t}) + h_k \dot{x}(\bar{t})), \dot{x}(\bar{t}) + h_k[w + a_k] \rangle}{h_k} - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w + a_k]|$$
$$= \frac{\langle \nabla V(x(\bar{t}) + h_k \dot{x}(\bar{t})), \dot{x}(\bar{t}) + h_k w \rangle}{h_k} - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w + a_k]| + \langle \nabla V(x(\bar{t}) + h_k \dot{x}(\bar{t})), a_k \rangle.$$

Since $\langle \nabla V(x(\bar{t}) + h_k \dot{x}(\bar{t})), a_k \rangle - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w + a_k]| \to 0$ as $k \to \infty$,

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x(\bar{t}) + h\dot{x}(\bar{t})), \dot{x}(\bar{t}) + hw \rangle}{h} \leq 0 \tag{3.6}$$

in contradiction with (3.2). Thus $x(t) \in K$ for every $t \in [0, T] \setminus \{t_1, \dots, t_p\}$. □

Remark 3.3. It is obvious that condition (3.2) in Proposition 3.2 can be replaced by the following assumption: suppose that, for all $x \in \partial K$, $t \in (0, T) \setminus \{t_1, \dots, t_p\}$ and $v \in \mathbb{R}^n$ satisfying (3.1) the following condition holds

$$\liminf_{h \to 0^+} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} > 0, \tag{3.7}$$

for all $w \in F(t, x, v)$.

Now, let us focus our attention also to the impulsive points $t_1, \dots, t_p$.

Theorem 3.4. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$, $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies*

conditions (H1) *and* (H2). *Furthermore, assume that* A_i, B_i, $i = 1, \ldots, p$, *are real* $n \times n$ *matrices such that* A_i, $i = 1, \ldots, p$, *satisfy*

$$A_i(\partial K) = \partial K, \quad \text{for all } i = 1, \ldots, p. \tag{3.8}$$

Moreover, let, for all $x \in \partial K$, $t \in (0, T) \setminus \{t_1, \ldots t_p\}$ *and* $v \in \mathbb{R}^n$ *satisfying* (3.1), *condition* (3.2) *holds, for all* $w \in F(t, x, v)$.

At last, suppose that, for all $x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with*

$$\langle \nabla V(A_i x), B_i v \rangle \le 0 \le \langle \nabla V(x), v \rangle, \quad \text{for some } i = 1, \ldots, p, \tag{3.9}$$

the following condition

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} > 0 \tag{3.10}$$

holds, for all $w \in F(t_i, x, v)$. *Then* K *is a bound set for the impulsive Dirichlet problem* (1.1)–(1.3).

Proof. Applying Proposition 3.2, we only need to show that if $x : [0, T] \to \overline{K}$ is a solution of problem (1.1), then $x(t_i) \in K$, for all $i = 1, \ldots, p$. As in the proof of Proposition 3.2, we argue by a contradiction, i.e. we assume that there exists $i \in \{1, \ldots, p\}$ such that $x(t_i) \in \partial K$. Following the same reasoning as in the proof of Proposition 3.2, for $\bar{t} = t_i$, we obtain

$$\langle \nabla V(x(t_i)), \dot{x}(t_i) \rangle \ge 0,$$

because $V(x(t_i)) = 0$ and $V(x(t)) \le 0$, for all $t \in [0, T]$.

Moreover, according to the condition (3.8), $V(A_i(x(t_i))) = 0$ as well, and so we can apply the same reasoning to the function $\tilde{g}(h) = V(x(t_i + h))$ for $h > 0$ and $\tilde{g}(0) = V(x(t_i^+))$. Since $x \in PC^1([0, T], \mathbb{R}^n)$, also $\tilde{g} \in C^1([0, \alpha], \mathbb{R})$ and $\tilde{g}(h) \le 0$ for $h > 0$ and $\tilde{g}(0) = 0$ imply $\dot{\tilde{g}}(0) \le 0$, i.e.

$$0 \ge \langle \nabla V(A_i(x(t_i))), B_i \dot{x}(t_i) \rangle.$$

Therefore, $x := x(t_i)$, $v := \dot{x}(t_i)$ satisfy condition (3.9).

Using the same procedure as in the proof of Proposition 3.2, for $\bar{t} = t_i$, we obtain the existence of a sequence of negative numbers $\{h_k\}_{k=1}^{\infty}$ and of point $w \in F(t_i, x(t_i), \dot{x}(t_i))$ such that

$$\frac{\dot{x}(t_i + h_k) - \dot{x}(t_i)}{h_k} \to w \quad \text{as } k \to \infty.$$

By the same arguments as in the previous proof, we get

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x(t_i) + h \dot{x}(t_i)), \dot{x}(t_i) + hw \rangle}{h} \le 0. \tag{3.11}$$

Inequality (3.11) is in a contradiction with condition (3.10), which completes the proof. $\square$

Remark 3.5. If condition (3.10) holds, for some $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (3.9) and $w \in F(t_i, x, v)$, then, according to the continuity of ∇V,

$$\langle \nabla V(x), v \rangle = 0. \tag{3.12}$$

Indeed

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} = \liminf_{h \to 0^-} \left[\frac{\langle \nabla V(x + hv), v \rangle}{h} + \langle \nabla V(x + hv), w \rangle \right]$$

which, since $\langle \nabla V(x), v \rangle \ge 0$, can be positive only if (3.12) holds.

Definition 3.6. A function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying all assumptions of Theorem 3.4 is called a *bounding function* for the set K relative to (1.1)–(1.3).

For our main result concerning the existence and localization of a solution of the Dirichlet b.v.p., we need to ensure that no solution of given b.v.p lies on the boundary ∂Q of a parameter set Q of candidate solutions. In the following section, it will be shown that if the set Q is defined as follows

$$Q := \{ q \in PC^1([0,T], \mathbb{R}^n) \mid q(t) \in \overline{K}, \text{ for all } t \in [0,T] \} \tag{3.13}$$

and if all assumptions of Theorem 3.4 are satisfied, then solutions of the b.v.p. (1.1)–(1.3) behave as indicated.

Proposition 3.7. *Let $K \subset \mathbb{R}^n$ be a nonempty open bounded set with $0 \in K$, let $Q \subset PC^1([0,T], \mathbb{R}^n)$ be defined by the formula (3.13) and let $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions $(H1)$ and $(H2)$. Moreover, assume that A_i, B_i, $i = 1, \ldots, p$, are real $n \times n$ matrices such that A_i, $i = 1, \ldots, p$, satisfy (3.8).*

Furthermore, suppose that, for all $x \in \partial K$, $t \in (0,T) \setminus \{t_1, \ldots, t_p\}$ and $v \in \mathbb{R}^n$ satisfying (3.1), condition (3.2) holds, for all $w \in F(t, x, v)$, and that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ satisfying (3.9), the condition (3.10) holds, for all $w \in F(t_i, x, v)$. Then problem (1.1)–(1.3) has no solution on ∂Q.

Proof. One can readily check that if $x \in \partial Q$, then there exists a point $t_x \in [0,T]$ such that $x(t_x) \in \partial K$. But then, according to Theorem 3.4, x cannot be a solution of (1.1)–(1.3). □

Let us now consider the particular case when the bounding function V is of class C^2. Then conditions (3.2) and (3.10) can be rewritten in terms of gradients and Hessian matrices and the following result can be directly obtained.

Corollary 3.8. *Let $K \subset \mathbb{R}^n$ be a nonempty open bounded set with $0 \in K$, let $Q \subset PC^1([0,T], \mathbb{R}^n)$ be defined by the formula (3.13) and let $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ which satisfies conditions $(H1)$ and $(H2)$. Moreover, assume that A_i, B_i, $i = 1, \ldots, p$, are real $n \times n$ matrices such that A_i, $i = 1, \ldots, p$, satisfy (3.8).*

Furthermore, suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ the following holds:

$$\text{if } \langle \nabla V(x), v \rangle = 0, \quad \text{then } \langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0, \tag{3.14}$$

for all $t \in (0,T) \setminus \{t_1, \ldots, t_p\}$ and $w \in F(t, x, v)$, and fixed $i = 1, \ldots, n$

$$\text{if } \langle \nabla V(A_i x), B_i v \rangle \leq 0 \leq \langle \nabla V(x), v \rangle \quad \text{then } \langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0, \tag{3.15}$$

for all $w \in F(t_i, x, v)$. Then problem (1.1)–(1.3) has no solution on ∂Q.

Proof. The statement of Corollary 3.8 follows immediately from Remark 3.5 and the fact that if $V \in C^2(\mathbb{R}^n, \mathbb{R})$, then, for all $x \in \partial K$, $t \in (0,T)$, $v \in \mathbb{R}^n$ and $w \in F(t, x, v)$, there exists

$$\lim_{h \to 0} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} = \lim_{h \to 0} \frac{\langle \nabla V(x + hv), v + hw \rangle - \langle \nabla V(x), v \rangle}{h}$$
$$= \langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle. \qquad \square$$

Remark 3.9. In conditions (3.2), (3.10), (3.14) and (3.15), the element v plays the role of the first derivative of the solution x. If x is a solution of (1.1)–(1.3) such that $x(t) \in \overline{K}$, for every $t \in [0, T]$, and there exists a continuous increasing function $\psi : [0, \infty) \to [0, \infty)$ satisfying condition (2.3) and such that

$$|F(t, c, d)| \leq \psi(|d|), \tag{3.16}$$

for a.a. $t \in [0, T]$ and every $c, d \in \mathbb{R}^n$ with $|c| \leq R := \max\{|x| : x \in \overline{K}\}$, then, according to Proposition 2.1, it holds that $|\dot{x}(t)| \leq B$, for every $t \in [0, T]$, where B is defined by (2.4). Hence, it is sufficient to require conditions (3.2), (3.10), (3.14) and (3.15) in Proposition 3.2, Theorem 3.4 and Corollary 3.8 only for all $v \in \mathbb{R}^n$ with $|v| \leq B$ and not for all $v \in \mathbb{R}^n$.

4 Existence and localization result for the impulsive Dirichlet problem with upper semi-continuous r.h.s.

In order to obtain the main existence theorem, the bound sets technique described in the previous section will be combined with the topological method which was developed by ourselves in [25] for the impulsive boundary value problems. The version of the continuation principle for problems without impulses can be found e.g. in [7].

Proposition 4.1 ([25, Proposition 2.4]). *Let us consider the b.v.p.*

$$\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0, T], \\ x \in S, \end{cases} \tag{4.1}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of $PC^1([0, T], \mathbb{R}^n)$. Let $H : [0, T] \times \mathbb{R}^{4n} \times [0, 1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$H(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in [0, T] \times \mathbb{R}^{2n}. \tag{4.2}$$

Assume that

(i) *there exists a retract Q of $PC^1([0, T], \mathbb{R}^n)$, with $Q \setminus \partial Q \neq \emptyset$, and a closed subset S_1 of S such that the associated problem*

$$\begin{cases} \ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), & \text{for a.a. } t \in [0, T], \\ x \in S_1 \end{cases} \tag{4.3}$$

has, for each $(q, \lambda) \in Q \times [0, 1]$, a non-empty and convex set of solutions $\mathfrak{T}(q, \lambda)$;

(ii) *there exists a nonnegative, integrable function $\alpha : [0, T] \to \mathbb{R}$ such that*

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \leq \alpha(t)(1 + |x(t)| + |\dot{x}(t)|), \quad \text{for a.a. } t \in [0, T],$$

for any $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$;

(iii) *$\mathfrak{T}(Q \times \{0\}) \subset Q$;*

(iv) *there exist constants $M_0 \geq 0$, $M_1 \geq 0$ such that $|x(0)| \leq M_0$ and $|\dot{x}(0)| \leq M_1$, for all $x \in \mathfrak{T}(Q \times [0, 1])$;*

(v) *the solution map $\mathfrak{T}(\cdot, \lambda)$ has no fixed points on the boundary ∂Q of Q, for every $\lambda \in [0, 1)$.*

Then the b.v.p. (4.1) *has a solution in* $S_1 \cap Q$.

Remark 4.2. The condition that Q is a retract of $PC^1([0,T],\mathbb{R}^n)$ in Proposition 4.1 can be replaced by the assumption that Q is an absolute neighborhood retract and $\text{ind}(\mathfrak{T}(\cdot,0),Q,Q) \neq 0$ (see, e.g., [2]). It is therefore possible to assume alternatively that Q is a retract of a convex subset of $PC^1([0,T],\mathbb{R}^n)$ or of an open subset of $PC^1([0,T],\mathbb{R}^n)$ together with $\text{ind}(\mathfrak{T}(\cdot,0),Q,Q) \neq 0$.

The solvability of (1.1) will be now proved, on the basis of Proposition 4.1. Defining namely the set Q of candidate solutions by the formula (3.13), we are able to verify, for each $(q,\lambda) \in Q \times [0,1)$, the transversality condition (v) in Proposition 4.1.

Theorem 4.3. *Let* $K \subset \mathbb{R}^n$ *be a nonempty, open, bounded and convex set with* $0 \in K$ *and let us consider the impulsive Dirichlet problem* (1.1)–(1.3), *where* $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper semicontinuous multivalued mapping,* $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, *and* $A_i, B_i, i = 1,\dots,p$, *are real* $n \times n$ *matrices with* $A_i \partial K = \partial K$, *for all* $i = 1,\dots,p$. *Moreover, assume that*

(i) *there exists a function* $\beta : [0,\infty) \to [0,\infty)$ *continuous and increasing satisfying*

$$\lim_{s\to\infty} \frac{s^2}{\beta(s)} ds = \infty$$

such that

$$|F(t,c,d)| \leq \beta(|d|),$$

for a.a. $t \in [0,T]$ *and every* $c,d \in \mathbb{R}^n$ *with* $|c| \leq R := \max\{|x| : x \in \overline{K}\}$;

(ii) *the problem*

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p, \end{cases} \tag{4.4}$$

has only the trivial solution;

(iii) *there exists a function* $V \in C^1(\mathbb{R}^n,\mathbb{R})$, *with* ∇V *locally Lipschitzian, satisfying conditions* $(H1)$ *and* $(H2)$;

(iv) *for all* $x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with* $|v| \leq \beta^{-1}(\beta(2R) + 2R)$, *the inequality*

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+h\lambda w \rangle}{h} > 0$$

holds, for all $t \in (0,T) \setminus \{t_1,\dots,t_p\}, \lambda \in (0,1)$ *and* $w \in F(t,x,v)$ *if* $\langle \nabla V(x),v \rangle = 0$ *and for all* $\lambda \in (0,1), w \in F(t_i,x,v)$ *if* $\langle \nabla V(A_i x), B_i v \rangle \leq 0 \leq \langle \nabla V(x),v \rangle$.

Then the Dirichlet problem (1.1)–(1.3) *has a solution* $x(\cdot)$ *such that* $x(t) \in \overline{K}$, *for all* $t \in [0,T]$.

Proof. Define

$$B = \beta^{-1}(\beta(2R) + 2R),$$

$$S = S_1 = Q := \{q \in PC^1([0,T],\mathbb{R}^n) \mid q(t) \in \overline{K}, |\dot{q}(t)| \leq 2B, \text{ for all } t \in [0,T]\}$$

and $H(t, c, d, e, f, \lambda) = \lambda F(t, e, f)$. Thus the associated problem (4.3) is the fully linearized problem

$$
\begin{cases}
\ddot{x}(t) \in \lambda F(t, q(t), \dot{q}(t)), & \text{for a.a. } t \in [0, T], \\
x(T) = x(0) = 0, \\
x(t_i^+) = A_i x(t_i), & i = 1, \dots, p, \\
\dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1, \dots, p.
\end{cases}
\tag{4.5}
$$

For each $(q, \lambda) \in Q \times [0, 1]$, let $\mathfrak{T}(q, \lambda)$ be the solution set of (4.5). We will check now that all the assumptions of Proposition 4.1 are satisfied.

Since the closure of a convex set is still a convex set, it follows that Q is convex, and hence a retract of $PC^1([0, T], \mathbb{R}^n)$. Moreover,

$$
\text{Int } Q = \{q \in PC^1([0, T], \mathbb{R}^n) \mid q(t) \in K, \ |\dot{q}(t)| < 2B, \ \text{for all } t \in [0, T]\} \neq \varnothing,
$$

since K is nonempty.

Notice now that, for every $t \in [0, T], c, d \in \mathbb{R}^n$, the inequality

$$
|H(t, e, f, c, d, \lambda)| = \lambda |F(t, e, f)| \leq \beta(|f|)
\tag{4.6}
$$

holds. Hence, denoting $z = (c, d, e, f, \lambda) \in \mathbb{R}^{4n+1}$, since $|f| \leq |z|$, when $|z| \leq r$, the monotonicity of β implies that $|H(t, c, d, e, f, \lambda)| \leq \beta(r)$, which ensures, for every $q \in Q$, the existence of $f_q \in \mathcal{P}_F(q)$. Given $q \in Q$, $\lambda \in [0, 1]$, and a L^1-selection $f_q(\cdot)$ of $F(\cdot, q(\cdot), \dot{q}(\cdot))$, let us consider the corresponding single valued linear problem with linear impulses

$$
\begin{cases}
\ddot{x}(t) = \lambda f_q(t), & \text{for a.a. } t \in [0, T], \\
x(T) = x(0) = 0, \\
x(t_i^+) = A_i x(t_i), & i = 1, \dots, p, \\
\dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1, \dots, p.
\end{cases}
\tag{4.7}
$$

Clearly, for all $q \in Q$ and $\lambda \in [0, 1]$,

$$
\mathfrak{T}(q, \lambda) = \{x_{\lambda f_q} \in PC^1([0, T], \mathbb{R}^n) : x_{\lambda f_q} \text{ is a solution of (4.7), for some } f_q \in \mathcal{P}_F(q)\}.
$$

Using the notation

$$
C := \begin{cases}
B_1(T - t_1) + A_1 t_1 & \text{if } p = 1 \\
\displaystyle\prod_{l=1}^{p} B_l(T - t_p) + \prod_{k=1}^{p} A_k t_1 + \sum_{j=2}^{p} \prod_{k=j}^{p} A_k \prod_{l=1}^{j-1} B_l(t_j - t_{j-1}) & \text{if } p \geq 2,
\end{cases}
\tag{4.8}
$$

it is easy to prove that the initial problem

$$
\begin{cases}
\ddot{x}(t) = 0, & \text{for a.a. } t \in [0, T], \\
x(0) = 0, \\
x(t_i^+) = A_i x(t_i), & i = 1, \dots, p, \\
\dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1, \dots, p
\end{cases}
$$

has infinitely many solutions given by

$$
x_0(t) = \begin{cases}
\dot{x}_0(0) t & \text{if } t \in [0, t_1], \\
B_1 \dot{x}_0(0)(t - t_1) + A_1 \dot{x}_0(0) t_1 & \text{if } t \in (t_1, t_2] \\
\left[\displaystyle\prod_{l=1}^{i} B_l(t - t_i) + \prod_{k=1}^{i} A_k t_1 + \sum_{j=2}^{i} \prod_{k=j}^{i} A_k \prod_{l=1}^{j-1} B_l(t_j - t_{j-1}) \right] \dot{x}_0(0) & \text{if } t \in (t_i, t_{i+1}], \ 2 \leq i \leq p
\end{cases}
$$

with $\dot{x}_0(0) \in \mathbb{R}^n$. Since $x_0(T) = 0$ if and only if $C\dot{x}_0(0) = 0$, assumption (ii) holds if and only if C is regular. Then (4.7) has a unique solution given by

$$
x_{\lambda f_q}(t) = \begin{cases}
\dot{x}_{\lambda f_q}(0)t + \lambda \displaystyle\int_0^t (t-\tau)f_q(\tau)d\tau & \text{if } t \in [0, t_1], \\[2mm]
B_1 \dot{x}_{\lambda f_q}(0)(t - t_1) + \lambda \displaystyle\int_{t_1}^t (t-\tau)f_q(\tau)d\tau + B_1(t - t_1)\lambda \displaystyle\int_0^{t_1} f_q(\tau)d\tau + A_1 \dot{x}_{\lambda f_q}(0)t_1 \\[2mm]
\qquad + A_1 \lambda \displaystyle\int_0^{t_1}(t_1 - \tau)f_q(\tau)d\tau \quad \text{if } t \in (t_1, t_2], \\[2mm]
\displaystyle\prod_{l=1}^i B_l \dot{x}_{\lambda f_q}(0)(t - t_i) + \lambda \displaystyle\int_{t_i}^t (t-\tau)f_q(\tau)d\tau + \sum_{r=1}^i \prod_{l=r}^i B_l(t - t_i)\lambda \displaystyle\int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \\[2mm]
\qquad + \displaystyle\prod_{k=1}^i A_k \dot{x}_{\lambda f_q}(0)t_1 + \prod_{k=1}^i A_k \lambda \displaystyle\int_0^{t_1}(t_1 - \tau)f_q(\tau)d\tau \\[2mm]
\qquad + \displaystyle\sum_{j=2}^i \prod_{k=j}^i A_k \left[\prod_{l=1}^{j-1} B_l \dot{x}_{\lambda f_q}(0)(t_j - t_{j-1}) + \lambda \int_{t_{j-1}}^{t_j}(t_j - \tau)f_q(\tau)d\tau \right. \\[2mm]
\qquad \left. + \displaystyle\sum_{r=1}^{k-1} \prod_{l=r}^{k-1} B_l(t_j - t_{j-1})\lambda \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \right] \quad \text{if } t \in (t_i, t_{i+1}], \ 2 \le i \le p
\end{cases}
$$

with

$$
\dot{x}_{\lambda f_q}(0) = -C^{-1}\left(\lambda \int_{t_1}^T (T-\tau)f_q(\tau)d\tau + B_1(T - t_1)\lambda \int_0^{t_1} f_q(\tau)d\tau + A_1\lambda \int_0^{t_1}(t_1 - \tau)f_q(\tau)d\tau \right) \tag{4.9}
$$

if $p = 1$ and

$$
\dot{x}_{\lambda f_q}(0) = -C^{-1}\left(\lambda \int_{t_p}^T (T-\tau)f_q(\tau)d\tau + \sum_{r=1}^p \prod_{l=r}^p B_l(T - t_p)\lambda \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \right.
$$
$$
\qquad + \prod_{k=1}^p A_k \lambda \int_0^{t_1}(t_1 - \tau)f_q(\tau)d\tau \tag{4.10}
$$
$$
\qquad \left. + \sum_{j=2}^p \prod_{k=j}^p A_k \left[\lambda \int_{t_{j-1}}^{t_j}(t_j - \tau)f_q(\tau)d\tau + \sum_{r=1}^{k-1} \prod_{l=r}^{k-1} B_l(t_j - t_{j-1})\lambda \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \right] \right)
$$

if $p \ge 2$. Therefore $\mathfrak{T}(q, \lambda) \ne \varnothing$. Moreover, given $x_1, x_2 \in \mathfrak{T}(q, \lambda)$, there exist f_q^1, f_q^2 such that $x_1 = x_{\lambda f_q^1}$ and $x_2 = x_{\lambda f_q^2}$. Since the right-hand side F has convex values, it holds that, for any $c \in [0,1]$ and $t \in [0, T]$, $c f_q^1(t) + (1 - c)f_q^2(t) \in F(t, q(t), \dot{q}(t))$ as well. The linearity of both the equation and of the impulses yields that $c x_1 + (1 - c)x_2 = x_{cf_q^1 + (1-c)f_q^2}$, i.e. that the set of solutions of problem (4.5) is, for each $(q, \lambda) \in Q \times [0,1]$, convex. Hence assumption (i) of Proposition 4.1 is satisfied.

Moreover, from (4.6), we obtain that, for every $\lambda \in [0,1], q \in Q, x \in \mathfrak{T}(q, \lambda)$,

$$
|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda| \le \beta(|\dot{q}(t)|) \le \beta(2B) \le \beta(2B)(1 + |x(t)| + |\dot{x}(t)|), \tag{4.11}
$$

thus also assumption (ii) of the same proposition holds.

The fulfillment of condition (iii) in Proposition 4.1 follows from the fact that, for $\lambda = 0$, problems (4.7) and (4.4) coincide and the latter one has only the trivial solution. Hence, $\mathfrak{T}(q, 0) = 0 \in \text{Int } Q$, because $0 \in K$.

For every $\lambda \in [0,1]$, $q \in Q$ and every solution $x_{\lambda f_q}$ of (4.7), $|x_{\lambda f_q}(0)| = 0$. Moreover, according to assumption (i) and formulas (4.9) and (4.10),

$$|\dot{x}_{\lambda f_q}(0)| \leq \|C^{-1}\| \left[\beta(2B) \frac{1}{2} T^2 + T^2 \|B_1\| \beta(2B) + \frac{1}{2} T^2 \|A_1\| \beta(2B) \right]$$

$$= T^2 \|C^{-1}\| \cdot \beta(2B) \left[\frac{1}{2} + \|B_1\| + \frac{1}{2} \|A_1\| \right]$$

if $p = 1$ and

$$|\dot{x}_{\lambda f_q}(0)| \leq \|C^{-1}\| \left[\frac{1}{2} T^2 \beta(2B) + T^2 \prod_{l=1}^{p} \|B_l\| \cdot \beta(2B) \right.$$

$$\left. + T^2 \prod_{k=1}^{p} \|A_k\| \beta(2B) + T^2 \prod_{l=1}^{p} \|B_l\| \prod_{k=1}^{p} \|A_k\| \cdot \beta(2B) \right]$$

$$= T^2 \|C^{-1}\| \cdot \beta(2B) \left[\frac{1}{2} + \prod_{l=1}^{p} \|B_l\| + \prod_{k=1}^{p} \|A_k\| + \prod_{l=1}^{p} \|B_l\| \prod_{k=1}^{p} \|A_k\| \right]$$

if $p \geq 2$. Therefore there exists a constant M_1 such that $|\dot{x}(0)| \leq M_1$, for all solutions x of (4.5). Hence, condition (iv) in Proposition 4.1 is satisfied as well.

At last, let us assume that $q_* \in Q$ is, for some $\lambda \in [0,1)$, a fixed point of the solution mapping $\mathfrak{T}(\cdot, \lambda)$. We will show now that q_* can not lay in ∂Q. We already proved this property if $\lambda = 0$, thus we can assume that $\lambda \in (0,1)$. From (4.11), we have, for a.a. $t \in [0,T]$, that

$$|\ddot{q}_*(t)| = \lambda |F(t, q_*(t), \dot{q}_*(t))| \leq \beta(|\dot{q}_*(t)|).$$

Therefore, since $|q_*(t)| \leq R$, for every $t \in [0,T]$, Proposition 2.1 implies that $|\dot{q}_*(t)| \leq B < 2B$, for every $t \in [0,T]$. Moreover, according to Theorem 3.4 and Remark 3.9, hypotheses (iii) and (iv) guarantee that $q_*(t) \in K$, for all $t \in [0,T]$. Thus $q_* \in$ Int Q, which implies that condition (v) from Proposition 4.1 is satisfied, for all $\lambda \in [0,1)$, and the proof is completed. $\quad\square$

Remark 4.4. An easy example of impulses conditions guaranteeing assumption (ii) in Theorem 4.3 are the antiperiodic impulses, i.e. $A_i = B_i = -I$, for every $i = 1, \ldots, p$. In this case, the matrix $C = (-1)^p T I$ (see [25]) and it is clearly regular. If $p = 1$ condition (ii) holds also e.g. for $A_1 = -I$ and $B_1 = I$ provided $T \neq 2t_1$.

5 Existence and localization result for the impulsive Dirichlet problem with upper-Carathéodory r.h.s.

In this section, we will study the impulsive Dirichlet b.v.p. (1.1)–(1.3) with an upper-Carathéodory r.h.s. and we will develop the bounding functions method with the strictly localized bounding functions also in this more general case. The technique which will be applied for obtaining the final result consists in replacing the original problem by the sequence of problems with non-strict localized bounding functions which satisfy all the assumptions of the following result developed by ourselves recently in [25].

Proposition 5.1 ([25, Theorem 4.1 and Remark 4.3]). *Let $K \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with $0 \in K$ and let us consider the impulsive Dirichlet problem (1.1)–(1.3), where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, and $A_i, B_i, i = 1, \ldots, p$, are real $n \times n$ matrices with $A_i \partial K = \partial K$, for all $i = 1, \ldots, p$. Moreover, assume that*

(i) *there exists a function $\varphi : [0, \infty) \to [0, \infty)$ continuous and increasing satisfying*

$$\lim_{s \to \infty} \frac{s^2}{\varphi(s)} ds = \infty \qquad (5.1)$$

such that

$$|F(t, c, d)| \leq \varphi(|d|), \qquad (5.2)$$

for a.a. $t \in [0, T]$ and every $c, d \in \mathbb{R}^n$ with $|c| \leq R := \max\{|x| : x \in \overline{K}\}$;

(ii) *the problem*

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0, T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), & i = 1, \ldots, p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1, \ldots, p, \end{cases} \qquad (5.3)$$

has only the trivial solution;

(iii) *there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$, with ∇V locally Lipschitzian, satisfying conditions $(H1)$ and $(H2)$;*

(iv) *there exists $\varepsilon > 0$ such that, for all $\lambda \in (0, 1), x \in \overline{K} \cap N_\varepsilon(\partial K), t \in (0, T)$, and $v \in \mathbb{R}^n$, with $|v| \leq \varphi^{-1}(\varphi(2R) + 2R)$, the following condition*

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0 \qquad (5.4)$$

holds, for all $w \in \lambda F(t, x, v)$;

(v) *for all $i = 1, \ldots, p, x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \leq \varphi^{-1}(\varphi(2R) + 2R)$ and $\langle \nabla V(x), v \rangle \neq 0$, it holds that*

$$\langle \nabla V(A_i x), B_i v \rangle \cdot \langle \nabla V(x), v \rangle > 0.$$

Then the Dirichlet problem (1.1)–(1.3) has a solution $x(\cdot)$ such that $x(t) \in \overline{K}$, for all $t \in [0, T]$.

Approximating the original problem by a sequence of problems satisfying conditions of Proposition 5.1 and applying the Scorza-Dragoni type result (Proposition 2.2), we are already able to state the second main result of the paper. The transversality condition is now required only on the boundary ∂K of the set K and not on the whole neighborhood $\overline{K} \cap N_\varepsilon(\partial K)$, as in Proposition 5.1.

Theorem 5.2. *Let $K \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with $0 \in K$ and let us consider the impulsive Dirichlet problem (1.1)–(1.3), where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper Carathéodory multivalued mapping, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, and $A_i, B_i, i = 1, \ldots, p$, are real $n \times n$ matrices with $A_i \partial K = \partial K$, for all $i = 1, \ldots, p$. Moreover, assume that*

(i) *there exists a function $\beta : [0, \infty) \to [0, \infty)$ continuous and increasing satisfying*

$$\lim_{s \to \infty} \frac{s^2}{\beta(s)} ds = \infty$$

such that

$$|F(t, c, d)| \leq \beta(|d|),$$

for a.a. $t \in [0, T]$ and every $c, d \in \mathbb{R}^n$ with $|c| \leq R := \max\{|x| : x \in \overline{K}\}$;

(ii) the problem

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), & i = 1, \ldots, p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1, \ldots, p, \end{cases} \tag{5.5}$$

has only the trivial solution;

(iii) there exists $h > 0$ and a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$, with $HV(x)$ positive semidefinite in $N_h(\partial K)$, satisfying conditions $(H1), (H2)$;

(iv) for all $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \leq \beta^{-1}(\beta(2R) + 2R)$, the inequality

$$\langle \nabla V(x), w \rangle > 0$$

holds for all $t \in (0, T)$ and $w \in F(t, x, v)$;

(v) for all $i = 1, \ldots, p$, $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \leq \beta^{-1}(\beta(2R) + 2R)$ and $\langle \nabla V(x), v \rangle \neq 0$, it holds that

$$\langle \nabla V(A_i x), B_i v \rangle \cdot \langle \nabla V(x), v \rangle > 0.$$

Then the Dirichlet problem (1.1)–(1.3) *has a solution $x(\cdot)$ such that $x(t) \in \overline{K}$, for all $t \in [0, T]$.*

Proof. Since $V \in C^2(\mathbb{R}^n, \mathbb{R})$, the function $x \to |\nabla V(x)|$ is continuous on the compact set ∂K, and hence there exists $k > 0$ such that $|\nabla V(x)| > 0$ for every $x \in N_k(\partial K)$. Define $\delta = \min\{h, k\}$. According to Urysohn's Lemma, there exists a function $\mu \in C(\mathbb{R}^n, [0,1])$ such that $\mu \equiv 1 \in N_{\frac{\delta}{2}}(\partial K)$ and $\mu \equiv 0 \in \mathbb{R}^n \setminus N_\delta(\partial K)$. Take a sequence of positive numbers $\{\epsilon_m\}$ decreasing to zero, an open and bounded set G, with $\overline{K} \subset G$, and $L > \beta^{-1}(\beta(2R) + 2R)$. According to Proposition 2.2 there exist a monotone decreasing sequence $\{\theta_m\}$ of open subsets of $[0,T]$ and a measurable multimap $F_0 : [0,T] \times \overline{G} \times \{v \in \mathbb{R}^n : |v| \leq L\} \multimap \mathbb{R}^n$ such that $\nu(\theta_m) \leq \epsilon_m$, $F_0(t, x, v) \subset F(t, x, v)$ and F_0 is u.s.c. on $([0,T] \setminus \theta_m) \times \overline{G} \times \{v \in \mathbb{R}^n : |v| \leq L\}$ for every $m \in \mathbb{N}$. Trivially $\nu(\cap_{m=1}^\infty \theta_m) = 0$ and $\lim_{m \to \infty} \chi_{\theta_m}(t) = 0$ for every $t \notin \cap_{m=1}^\infty \theta_m$.

Define, for each $m \in \mathbb{N}$, $(t, x, v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$F_m(t,x,v) = \begin{cases} F_0(t,x,v) + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)\frac{\nabla V(x)}{|\nabla V(x)|} & \text{if } x \in G \text{ and } |v| < L \\ F(t,x,v) + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)\frac{\nabla V(x)}{|\nabla V(x)|}, & \text{otherwise.} \end{cases}$$

Since $\delta \leq k$, we have that $\mu(x) = 0$ for $x \in \mathbb{R}^n \setminus N_\delta(\partial K)$ and $\nabla V(x) \neq 0$ in $N_\delta(\partial K)$, hence it follows that F_m is well defined. Since μ and β are continuous, V is of class C^2, G is open, $F_0(t,x,v) \subset F(t,x,v)$, and F is an upper-Carathéodory map, F_m is a Carathéodory map as well.

Let us now prove that problem

$$\begin{cases} \ddot{x}(t) \in F_m(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), & i = 1, \ldots, p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1, \ldots, p. \end{cases} \tag{5.6}$$

satisfies the assumptions of Proposition 5.1.

First of all notice that, since $0 \leq \mu(x) \leq 1, 0 \leq \chi_{\theta_m}(t) \leq 1$, for every $x \in \mathbb{R}^n, t \in [0, T]$, it holds, according to (i),

$$|F_m(t, c, d)| \leq |F(t, c, d)| + 2\beta(|d|) \leq 3\beta(|d|),$$

for every $(t, c, d) \in t \times \mathbb{R}^n \times \mathbb{R}^n$ with $|c| \leq R$. Thus condition (i) of Proposition 5.1 is satisfied by the continuous increasing function $\varphi = 3\beta$, since it clearly holds that

$$\lim_{s \to \infty} \frac{s^2}{\varphi(s)} = \frac{1}{3} \lim_{s \to \infty} \frac{s^2}{\beta(s)} = \infty.$$

Moreover, conditions (ii) and (iii) imply the analogous conditions in Proposition 5.1.

Let us now observe that, since $\varphi(s) = 3\beta(s)$, then $\varphi^{-1}(s) = \beta^{-1}(\frac{s}{3})$, which is an increasing function, as inverse of an increasing function. Hence

$$\varphi^{-1}(\varphi(2R) + 2R) = \beta^{-1}\left(\frac{3\beta(2R) + 2R}{3}\right) = \beta^{-1}\left(\beta(2R) + \frac{2}{3}R\right) \leq \beta^{-1}(\beta(2R) + 2R).$$

Therefore, condition (v) implies the analogous condition of Proposition 5.1. Moreover, for every $\lambda \in (0, 1), x \in \overline{K} \cap N_{\frac{\delta}{2}}(\partial K), t \in (0, T)$, and $v \in \mathbb{R}^n$, with $|v| \leq \varphi^{-1}(\varphi(2R) + 2R), w_1 \in \lambda F_m(t, x, v)$,

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w_1 \rangle = \langle HV(x)v, v \rangle + \lambda[\langle \nabla V(x), w \rangle + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)|\nabla V(x)|]$$
$$= \langle HV(x)v, v \rangle + \lambda[\langle \nabla V(x), w \rangle + 2\beta(|v|)\chi_{\theta_m}(t)|\nabla V(x)|].$$

Then, if $t \in [0, T] \setminus \theta_m$

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w_1 \rangle = \langle HV(x)v, v \rangle + \lambda \langle \nabla V(x), w \rangle \geq \lambda \langle \nabla V(x), w \rangle,$$

with $w \in F_0(t, x, v)$, because $\overline{K} \cap N_{\frac{\delta}{2}}(\partial K) \subset \overline{K} \subset G$ and $\varphi^{-1}(\varphi(2R) + 2R) \leq \beta^{-1}(\beta(2R) + 2R) < L$. Since V is of class C^2, F_0 is u.s.c. on the compact set $([0, T] \setminus \theta_m) \times \partial K \times \{v \in \mathbb{R}^n : |v| \leq \varphi^{-1}(\varphi(2R) + 2R)\}$, and F_0 is compact valued, condition (iv) implies that there exists $k_1 > 0$ such that

$$\langle \nabla V(x), w \rangle > 0$$

for every $t \in [0, T] \setminus \theta_m, x \in \overline{K} \cap N_{k_1}(\partial K), v \in \mathbb{R}^n : |v| \leq \varphi^{-1}(\varphi(2R) + 2R), w \in F_0(t, x, v)$. Hence,

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w_1 \rangle \geq \lambda \langle \nabla V(x), w \rangle > 0,$$

for all $\lambda \in (0, 1), t \in [0, T] \setminus \theta_m, x \in \overline{K} \cap N_{k_1}(\partial K), v \in \mathbb{R}^n : |v| \leq \varphi^{-1}(\varphi(2R) + 2R), w_1 \in \lambda F_m(t, x, v)$.

On the other hand, if $t \in \theta_m$, since $x \in N_{\frac{\delta}{2}}(\partial K)$ and $h \geq \delta$,

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w_1 \rangle \geq \lambda[\langle \nabla V(x), w \rangle + 2\beta(|v|)|\nabla V(x)|]$$
$$\geq \lambda[-|w| + 2\beta(|v|)]|\nabla V(x)| \geq \lambda\beta(|v|)|\nabla V(x)| > 0.$$

Condition (iv) in Proposition 5.1 follows taking $\epsilon = \min\{k_1, \frac{\delta}{2}\}$.

Applying Proposition 5.1 we obtain that, for every $m \in \mathbb{N}$, there exists a solution x_m of (5.6) such that $x_m(t) \in \overline{K}$ and $|\dot{x}_m(t)| \leq \varphi^{-1}(\varphi(2R) + 2R)$, for every $t \in [0, T]$. Hence $|\ddot{x}_m(t)| \leq \varphi(2R) + 2R$ for every $t \in [0, T]$. The Ascoli–Arzelà theorem implies that $\{x_m\} \to x$ uniformly

in $C^1([0,T],\mathbb{R}^n)$ and $\ddot{x}_m \to \ddot{x}$ weakly in $L^1([0,T],\mathbb{R}^n)$. Thus $x(t) \in \overline{K}, |\dot{x}(t)| \le \varphi^{-1}(\varphi(2R)+2R)$ for every $t \in [0,T]$, and x satisfies (1.2)–(1.3). Moreover, since $\nu(\cap_{n=1}^\infty \theta_m) = 0$,

$$\lim_{m\to\infty} 2\mu(x_m(t))\beta(|\dot{x}_m(t)|)\chi_{\theta_m}(t)\frac{\nabla V(x_m(t))}{|\nabla V(x_m(t))|} = 0,$$

for a.a. $t \in [0,T]$. Consequently, a standard limiting argument (see e.g. [28, Theorem 3.1.2]) implies that x is a solution of

$$\ddot{x}(t) = F_0(t,x(t),\dot{x}(t))$$

and, since $F_0(t,x(t),\dot{x}(t)) \subset F(t,x(t),\dot{x}(t))$, a solution of the problem (1.1)–(1.3). □

Remark 5.3. Both Theorems 4.3 and 5.2 give an existence result for an impulsive Dirichlet boundary value problem with a strictly localized bounding function respectively for u.s.c. and upper-Carathéodory multimap. However Theorem 5.2 does not represent an extension of Theorem 4.3, since the first one deals with a C^2-bounding function, while the second one is related to a C^1-bounding function and can not be easily extended to the Carathéodory case.

In the case when the multivalued mapping F is u.s.c. and the bounding function V is of class C^2, i.e. when it is possible to apply both theorems, conditions of Theorem 4.3 are weaker than assumptions of Theorem 5.2. In fact, in this case, according to Corollary 3.8, condition (iv) of the first theorem reads as

$$\langle HV(x)v,v\rangle + \lambda\langle \nabla V(x),w\rangle > 0$$

for every $x \in \partial K, \lambda \in (0,1), v \in \mathbb{R}^n$, with $|v| \le \beta^{-1}(\beta(2R)+2R)$, and for every $t \in [0,T] \setminus \{t_1,\dots,t_p\}, w \in F(t,x,v)$ if $\langle \nabla V(x),v\rangle \ne 0$, or for every $w \in F(t_i,x,v)$ if $\langle \nabla V(A_ix),B_iv\rangle \le 0 \le \langle \nabla V(x),v\rangle$, which are implied by assumptions (iii) and (iv) of the second theorem.

6   An application of the main result

As an application of Theorem 5.2, let us consider the second-order inclusion

$$\ddot{x}(t) \in a(t)\dot{x}(t) + h(t,x(t))), \quad \text{for a.a. } t \in [0,T], \tag{6.1}$$

together with antiperiodic impulses and Dirichlet boundary conditions

$$x(t_i^+) = -x(t_i), \qquad i = 1,\dots,p, \tag{6.2}$$
$$\dot{x}(t_i^+) = -\dot{x}(t_i), \qquad i = 1,\dots,p, \tag{6.3}$$
$$x(0) = x(T) = 0, \tag{6.4}$$

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$. Assume that $a \in L^\infty([0,T],\mathbb{R})$, with $\|a\|_\infty > 0$, and $h : [0,T] \times \mathbb{R} \multimap \mathbb{R}$ is an upper-Carathéodory multivalued mapping with

$$|h(t,y)| \le \alpha(t)g(y)$$

for some $\alpha \in L^\infty([0,T],\mathbb{R})$, $g \in C(\mathbb{R},\mathbb{R})$.

When h is a function, the impulsive Dirichlet boundary value problem associated to the single valued equation $\ddot{x}(t) = a(t)\dot{x}(t) + h(t,x(t))$ represents a generalization of a wide class of equations which are widely studied in literature (see, e.g., [1,13,16,26,29]) for its several applications (including biological phenomena involving thresholds, models describing population dynamics or inspection processes in operations research). Much more rare are the

results concerning the multivalued case which can be e.g. used for modelling optimal control problems in economics.

We will show now that, under very general conditions, the Dirichlet multivalued problem (6.1), (6.4) together with impulse conditions (6.2), (6.3) satisfies all the assumptions of Theorem 5.2. On this purpose, let us consider the nonempty, open, bounded, convex and symmetric neighbourhood of the origin $K = (-k, k)$, with k to be specified later, and the C^2-function $V(x) = \frac{1}{2}(x^2 - k^2)$ that trivially satisfies conditions $(H1)$ and $(H2)$.

In order to verify condition (i), let us define the continuous and increasing function

$$\beta(d) = \|a\|_\infty d + \|\alpha\|_\infty \overline{g}, \quad \text{for all } d \in [0, +\infty),$$

where $\overline{g} = \max_{|x| \leq k} |g(x)|$. The function β obviously satisfies (5.1) and $F(t, c, d) := a(t)d + h(t, c)$ satisfies (3.16), for all $t \in [0, T]$ and all $c, d \in \mathbb{R}$, with $|c| \leq k$.

Assumption (ii) holds as well since, according to Remark 4.4, the associated homogeneous problem has only the trivial solution.

Condition (iii) follows from the fact that $\dot{V}(x) = x$ and $\ddot{V}(x) = 1$, for every $x \in \mathbb{R}$.

Notice moreover that, whenever $xv \neq 0$, then $(-x)(-v)xv = x^2v^2 > 0$, hence also condition (v) holds.

Finally, since $\beta^{-1}(d) = \frac{1}{\|a\|_\infty}(d - \|\alpha\|_\infty \overline{g})$, we easily get that

$$\beta^{-1}(\beta(2k) + 2k) = 2k\left(1 + \frac{1}{\|a\|_\infty}\right).$$

Thus condition (iv) reads as

$$a(t)xv + xw > 0 \tag{6.5}$$

for every $t \in [0, T], x$ with $|x| = k, v$ with $|v| \leq 2k(1 + \frac{1}{\|a\|_\infty})$ and $w \in h(t, x)$. Taking $x = k$ we then get $w > -a(t)v$, for every $w \in h(t, k)$. Since the previous condition must hold both for positive and negative values of $v, h(t, k)$ must take only positive values and the transversality condition is satisfied if

$$w > \|a\|_\infty 2k\left(1 + \frac{1}{\|a\|_\infty}\right) = 2k(\|a\|_\infty + 1) \qquad \forall w \in h(t, k).$$

Similarly, taking $x = -k$ we get that (6.5) is equivalent to $w < -a(t)v$, for every $w \in h(t, -k)$ which is satisfied only if w is negative. A sufficient condition then becomes

$$w < -2k(\|a\|_\infty + 1) \qquad \forall w \in h(t, -k).$$

Thus condition (iv) holds if there exists $k > 0$ such that for every $w_1 \in h(t, k), w_2 \in h(t, -k)$,

$$w_1 > 2k(\|a\|_\infty + 1) \quad \text{and} \quad w_2 < -2k(\|a\|_\infty + 1). \tag{6.6}$$

The previous result can be stated in the form of the following theorem.

Theorem 6.1. *Assume that $a \in L^\infty([0, T], \mathbb{R})$, with $\|a\|_\infty > 0$, $h : [0, T] \times \mathbb{R} \multimap \mathbb{R}$ is an upper-Carathédory multivalued mapping with*

$$|h(t, y)| \leq \alpha(t)g(y),$$

for some $\alpha \in L^\infty([0, T], \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$. Moreover, assume that there exists $k > 0$ such that, for every $t \in [0, T]$, and $w \in h(t, k)$,

$$w > 2k(\|a\|_\infty + 1)$$

and that, for every $t \in [0, T]$, and $w \in h(t, -k)$,

$$w < -2k(\|a\|_\infty + 1).$$

Then problem (6.1)–(6.4) *has a solution $x(\cdot)$ such that $|x(t)| \leq k$, for every $t \in [0, T]$.*

Remark 6.2. Suppose that, in (6.1), $h(t, x) = \gamma(t) + \alpha(t)f(x)$, where f is an odd semicontinuous multimap and $\alpha, \gamma \in L^\infty([0, T], \mathbb{R})$. Then (6.6) is equivalent to require the existence of $k > 0$ such that, for every $t \in [0, T]$,

$$\alpha(t)f(k) > 2k(|a\|_\infty + 1) - \gamma(t).$$

If $\alpha(t) \geq \overline{\alpha} > 0$, for every $t \in [0, T]$, then (6.6) is equivalent to

$$\overline{\alpha}f(k) > 2k(\|a\|_\infty + 1) - \|\gamma^-\|_\infty,$$

where $\gamma^-(t) = \min\{0, \gamma(t)\}$, which holds, e.g., if f is superlinear at infinity, which is true in many applications. The superlinearity of f at infinity is a sufficient condition also if $\alpha(t) \leq -\overline{\alpha} < 0$, for every $t \in [0, T]$. Notice that the obtained solution is a nonzero function whenever γ is a nonzero function.

Acknowledgment

References

[1] R. Agarwal, D. O'Regan, Multiple nonnegative solutions for second order impulsive differential equations, *Appl. Math. Comp.* **114**(2000), 97–132. https://doi.org/10.1016/S0096-3003(99)00074-0

[2] J. Andres, L. Górniewicz, *Topological fixed point principles for boundary value problems*, Topological Fixed Point Theory and Its Applications, Vol. 1, Kluwer, Dordrecht, 2003. https://doi.org/10.1007/978-94-017-0407-6

[3] J. Andres, M. Kožušníková, L. Malaguti, On the Floquet problem for second-order Marchaud differential systems, *J. Math. Anal. Appl.* **351**(2009), 360–372. https://doi.org/10.1016/j.jmaa.2008.10.028

[4] J. Andres, L. Malaguti, M. Pavlačková, Strictly localized bounding functions for vector second-order boundary value problems, *Nonlin. Anal.* **71**(2009), No. 12, 6019–6028. https://doi.org/10.1016/j.na.2009.05.035

[5] J. Andres, L. Malaguti L., M. Pavlačková, Hartman-type conditions for multivalued Dirichlet problems in abstract spaces, *Discrete Contin. Dyn. Syst.* 10th AIMS Conf. Suppl. (2015), 103–111. https://doi.org/10.3934/proc.2015.0038

[6] J. Andres, L. Malaguti, V. Taddei, Floquet boundary value problems for differential inclusions: a bound sets approach, *Z. Anal. Anwendungen* **20**(2001), No. 3, 709–725. https://doi.org/10.4171/ZAA/1040

[7] J. ANDRES, M. PAVLAČKOVÁ, Asymptotic boundary value problems for second-order differential systems, *Nonlin. Anal.* **71**(2009), No. 5–6, 1462–1473. https://doi.org/10.1016/j.na.2008.12.013

[8] J. AUBIN, A. CELLINA, *Differential inclusions*, Springer, Berlin, 1984. https://doi.org/10.1007/978-3-642-69512-4

[9] D. BAINOV, P. SIMEONOV, *Impulsive differential equations: periodic solutions and applications*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 66, Longman Scientific and Technical, Essex, 1993. MR1266625

[10] G. BALLINGER, X. LIU, Practical stability of impulsive delay differential equations and applications to control problems, in: *Optimization methods and applications*, Appl. Optim., Vol. 52, Kluwer Academic, Dordrecht, 2001, pp. 3–21. https://doi.org/10.1007/978-1-4757-3333-4_1

[11] M. BENCHOHRA, J. HENDERSON, S. NTOUYAS, *Impulsive differential equations and inclusions*, Vol. 2, Hindawi, New York, 2006. https://doi.org/10.1155/9789775945501

[12] I. BENEDETTI, V. OBUKHOVSKII, V. TADDEI, On noncompact fractional order differential inclusions with generalized boundary condition and impulses in a Banach space, *J. Funct. Spaces* **2015**, Art. ID 651359, 10 pp. https://doi.org/10.1155/2015/651359

[13] G. BONANNO, B. DI BELLA, J. HENDERSON, Existence of solutions to second-order boundary-value problems with small perturbations of impulses, *Electronic J. Differential Equations* **2013**, No. 126, 1–14. MR3065079

[14] K. BORSUK, *Theory of retracts*, Monografie Matematyczne, Vol. 44, PWN, Warsaw, 1967. MR0216473

[15] K. DEIMLING, *Multivalued differential equations*, W. de Gruyter, Berlin, 1992. https://doi.org/10.1515/9783110874228

[16] M. FERRARA, S. HEIDARKHANI, P. PIZZIMENTI, Multiple solutions for Dirichlet impulsive equations, *Adv. Nonlinear Anal.* **3**(2014), No. 1, 89–98. https://doi.org/10.1515/anona-2014-0023

[17] R. GAINES, J. MAWHIN, *Coincidence degree and nonlinear differential equations*, Springer, Berlin, 1977. https://doi.org/10.1007/BFb0089537

[18] L. GÓRNIEWICZ, *Topological fixed point theory of multivalued mappings*, 2nd ed., Springer, Berlin, 2006. https://doi.org/10.1007/978-94-015-9195-9

[19] J. GRAEF, J. HENDERSON, A. OUAHAB, *Impulsive differential inclusions: a fixed point approach*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 20, De Gruyter, Berlin, 2013. https://doi.org/10.1515/9783110295313

[20] P. HARTMAN, *Ordinary differential equations*, Wiley-Interscience, New York, 1969. https://doi.org/10.1137/1.9780898719222

[21] S. HU, N. PAPAGEORGIOU, *Handbook of multivalued analysis. Vol. I. Theory*, Kluwer, Dordrecht, 1997. https://doi.org/10.1007/978-1-4615-6359-4

[22] V. Lakshmikantham, D. Bainov, P. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989. https://doi.org/10.1142/0906

[23] J. Meneses, R. Naulin, Ascoli–Arzelà theorem for a class of right continuous functions, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **38**(1995), 127–135.

[24] M. Pavlačková, A Scorza-Dragoni approach to Dirichlet problem with an upper-Carathéodory right-hand side, *Topol. Methods Nonlinear Anal.* **44**(2014), No. 1, 239–247. https://doi.org/10.12775/TMNA.2014.045

[25] M. Pavlačková, V. Taddei, A bounding function approach to impulsive Dirichlet problem with an upper-Carathéodory right-hand side, *Electronic J. Differential Equations* **2019**, No. 12, 1–18. MR3904853

[26] I. Rachůnková, J. Tomeček, Second order BVPs with state dependent impulses via lower and upper functions, *Cent. Eur. J. Math.* **12**(2014), No. 1, 128–140. https://doi.org/10.2478/s11533-013-0324-7

[27] K. Schmitt, R. Thompson, Boundary value problems for infinite systems of second-order differential equations, *J. Differential Equations* **18**(1975), 277–295. https://doi.org/10.1016/0022-0396(75)90063-7

[28] I. Vrabie, *Compactness methods for nonlinear evolutions*, 2nd Edition, Longman, Harlow, 1990.

[29] Z. Zhang, R. Yuan, An application of variational methods to Dirichlet boundary value problem with impulses, *Nonlinear Anal. Real World Appl.* **11**(2010), No. 1, 155–162. https://doi.org/10.1016/j.nonrwa.2008.10.044