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3D incompressible flows with small viscosity around distant obstacles

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Abstract. In this paper, we analyze the behavior of three-dimensional incompressible flows, with small viscosities $\nu > 0$, in the exterior of material obstacles $\Omega_R = \Omega_0 + (R, 0, 0)$, where Ω_0 belongs to a class of smooth bounded domains and $R > 0$ is sufficiently large. Applying techniques developed by Kato, we prove an explicit energy estimate which, in particular, indicates the limiting flow, when both $\nu \to 0$ and $R \to \infty$, as that one governed by the Euler equations in the whole space. According to this approach, it is natural to contrast our main result to that one already known in the literature for families of viscous flows in expanding domains.

Keywords: singular perturbation in context of PDEs, vanishing viscosity limit, Navier–Stokes equations, Euler equations.

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1 Introduction

Let $\Omega_0 \subset \mathbb{R}^3$ be a smooth bounded domain, such that $\mathbb{R}^3 \setminus \overline{\Omega}_0$ is connected and simply connected. We also assume that $\mathbf{0} = (0, 0, 0)$ lies inside Ω_0. For each $R \geq 0$, let us set

$$\mathbf{R} = (R, 0, 0), \qquad \Omega_R = \Omega_0 + \mathbf{R}, \qquad \Pi_R = \mathbb{R}^3 \setminus \overline{\Omega}_R \quad \text{and} \quad \Gamma_R = \partial\Omega_R = \partial\Pi_R.$$

Under these notation, we recall the definition of some usual spaces related to incompressible fluids:

$$V(\Pi_R) = \{v \in (H^1(\Pi_R))^3 : \operatorname{div} v = 0 \text{ in } \Pi_R \text{ and } v = 0 \text{ on } \Gamma_R\} \tag{1.1}$$

and

$$H(\Pi_R) = \{v \in (L^2(\Pi_R))^3 : \operatorname{div} v = 0 \text{ in } \Pi_R \text{ and } v \cdot \mathbf{n} = 0 \; on \; \Gamma_R\}, \tag{1.2}$$

where $\mathbf{n}$ is the outward directed unit normal vector field to Γ_R.

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We fix an initial vorticity ω_0, which is a smooth, divergence-free and compactly supported vector field in $\mathbb{R}^3$. Since Π_R is simply connected, there exists a unique $v_{0,R} \in H(\Pi_R)$ such that $\operatorname{curl} v_{0,R} = \omega_0|_{\Pi_R}$ (see Proposition 3.4). In addition, let us denote by u_0 the velocity defined on $\mathbb{R}^3$ which is associated to the vorticity ω_0, as follows:

$$u_0(x) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \omega_0(y)dy, \tag{1.3}$$

for each $x \in \mathbb{R}^3$, where $\times$ represents the cross product of vectors in $\mathbb{R}^3$. In this context, there exists $T^* > 0$ with the following property: for all $T \in (0, T^*)$, we can find a smooth solution $u = u(x,t)$ to the three-dimensional Euler equations in the whole space

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x), \\ |u| \to 0 \text{ as } |x| \to +\infty, \end{cases} \tag{1.4}$$

defined on $\mathbb{R}^3 \times [0,T]$. In (1.4), u is understood as the velocity of an ideal incompressible fluid, while p denotes its pressure.

Taking $T \in (0, T^*)$ and a small viscosity $\nu > 0$, let us also consider the incompressible Navier–Stokes equations in Π_R, with initial data $v_{0,R}$, given by

$$\begin{cases} v_t^{\nu,R} + (v^{\nu,R} \cdot \nabla)v^{\nu,R} - \nu\Delta v^{\nu,R} + \nabla P^{\nu,R} = 0, & (x,t) \in \Pi_R \times (0,T), \\ \operatorname{div} v^{\nu,R} = 0, & (x,t) \in \Pi_R \times [0,T), \\ v^{\nu,R}(x,t) = 0, & (x,t) \in \partial\Pi_R \times (0,T), \\ v^{\nu,R}(x,0) = v_{0,R}(x), & x \in \Pi_R. \end{cases} \tag{1.5}$$

Above, $v^{\nu,R}$ represents the velocity of the particles of a viscous fluid and $P^{\nu,R}$ is its pressure. It is well-known that there exists a Leray–Hopf weak solution $v^{\nu,R} = v^{\nu,R}(x,t)$ to (1.5) (see Definition 4.1 and Theorem 4.2). We emphasize that, since we consider weak solutions to (1.5), there is no dependence of solution's existence time on the viscosity. Under all these notations we have just described, we are ready to state the main result of this paper.

Theorem 1.1. *As mentioned previously, let $\omega_0 \in (C_c^\infty(\mathbb{R}^3))^3$ be a divergence-free vector field in $\mathbb{R}^3$, and consider the smooth solution $u = u(x,t)$ of (1.4), defined on $\mathbb{R}^3 \times [0,T]$, with initial data given in (1.3). For $\nu > 0$ and $R > 0$, let $v^{\nu,R}$ be a weak solution of (1.5) in $\Pi_R \times [0,T)$, with initial data $v_{0,R}$, where $v_{0,R}$ is the L^2-orthogonal projection of $u_0|_{\Pi_R}$ on $H(\Pi_R)$. Then, there exist $C = C(T, \Omega_0, \omega_0) > 0$ and $R_0 > 0$ such that, for all $R > R_0$, we have*

$$\|v^{\nu,R} - u\|_{L^\infty([0,T];[L^2(\Pi_R)]^3)} \leq C\left(\frac{1}{R} + \sqrt{\nu}\right). \tag{1.6}$$

At this moment, we would like to list some papers where asymptotic behavior of incompressible flows under singular domain perturbation has been considered. Initially, we recall the study of incompressible flows in the presence of small obstacles, presented in [7] and [6]. In [7], it was investigated the asymptotic behavior of 2D incompressible ideal flows in the exterior of a single smooth obstacle that shrinks homothetically to a point. The work developed in [7] allowed to identify the equation satisfied by the limit flow. In fact, if γ is the circulation around the obstacle and $\gamma = 0$, then the limit velocity verifies the Euler equations in the full-plane, with the same initial vorticity. On the other hand, when $\gamma \neq 0$, the limit equation involves a new forcing term, with an initial vorticity that acquires a pointwise Dirac mass. In a similar analysis for the 2D Navier–Stokes equations, considered in [6], it was proved that, if the circulation is sufficiently small, then the limit equation is the Navier–Stokes equations in the whole space, but an additional pointwise Dirac mass still appears in the vorticity of the limit equation. In [4], the corresponding problem was considered in the three-dimensional case, where it was established that the limit velocity is a solution of the Navier–Stokes equations in the full-space. Later, in [1], the research proceeded with the asymptotic behavior of solutions of the incompressible 2D Euler equations on a bounded domain with a finite number of holes, assuming that the size of one of them vanishes. In that situation, the limit flow was identified as a modified Euler system in the domain without its small hole.

In [5], incompressible flows around a small obstacle, with small viscosity, are considered. Under specific assumptions, it can be seen that solutions of the Navier–Stokes system in exterior domains converge to solutions of the Euler system in the full space when both viscosity and the size of the obstacle vanish. In the proof of this result, it is presented a rate of convergence in terms of the viscosity and the size of the obstacle. In addition, the complementary situation was treated in [9], where 2D Euler and Navier–Stokes systems were analyzed in expanding domains. To be more precise, such asymptotic analysis also pointed out that solutions in large domains converge to the corresponding solution in the full plane.

As we can see, in the context of fluid dynamics, limits of singularly perturbed domain have been extensively studied over the last years. Last but not least, we would like to highlight [10], where Kelliher, Lopes Filho and Nussenzveig Lopes examined, in dimensions 2 and 3, the limiting behavior of incompressible flows with small viscosity inside expanding domains. Based on energy estimates developed by Kato in [8], these three authors identified conditions under which the limit velocity satisfies the Euler system in the whole space when both viscosity vanishes and the domain becomes large. We are supposed to remark that their analysis also exhibits a rate of convergence which takes into account the small viscosity of the fluid and the enlarged boundary domain. The current work intends to be part of the list of papers we have just mentioned. However, our purpose here is closer to [10]. In fact, we study, in dimension 3, the limiting behavior of incompressible flows, with small viscosity, around far obstacles. In this sense, here the boundary domain becomes distant through the translation of $\Gamma_0 = \partial\Pi_0$ by $\mathbf{R} = (R, 0, 0)$, while, in [10], the boundary goes far by dilatation. In both cases, there is some effect of distant boundaries in the vanishing viscosity limit. Thus, Theorem 1.1 should be contrasted with the corresponding three-dimensional main result of [10].

The remainder of this paper is organized as follows: in Section 2, we deal with the behavior of smooth solutions of (1.4) at infinity. In Section 3, we set some suitable approximate solutions to the Euler equations and, applying the decay results obtained in Section 2, some indispensable estimates are achieved in Propositions 3.3 and 3.4. Section 4 is devoted to a brief discussion about the Navier–Stokes in exterior domains. At this point, we must emphasize Proposition 4.5, where a well-known relation involving weak solutions to (4.1) is extended to a larger class of test vector fields. In Section 5 we prove Theorem 1.1, our main result. In Section 6, we make further comments about some aspects related to this work. At the end, for the sake of clarity, there is an appendix, where we list domains, differential operators, function spaces and notations related to the PDEs mentioned throughout the development of this work.

2 Incompressible inviscid flow in the whole space

At the beginning of this section, we would like to have a few words on the local well-posedness for the 3D Euler system in the whole space. As said before, we fix an initial vorticity $\omega_0 \in (C_c^\infty(\mathbb{R}^3))^3$, which is divergence-free, and take the associated velocity u_0, expressed in (1.3). Under these assumptions, it was proved in [12] that, for sufficiently small times, there exists a smooth solution of (1.4), with u_0 as the initial data. It means that there exists $T^* > 0$, depending on u_0, such that, for all $T \in (0, T^*)$, there exists a unique smooth solution (u, p) of (1.4), defined on $\mathbb{R}^3 \times [0, T]$.

Additionally, for each $t \in [0, T]$, the vector field $\omega = \operatorname{curl}(u(\cdot, t))$ is compactly supported and there exists $r > 0$ such that

$$\operatorname{supp}(\omega(u)) \subset B_r(\mathbf{0}) \times [0, T] \tag{2.1}$$

what can be seen in [10], for example. It is important to notice that ω and u solve the system

$$\begin{cases} \omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, & (x,t) \in \mathbb{R}^3 \times (0,T), \\ \operatorname{div} \omega = 0, & (t,x) \in \mathbb{R}^3 \times [0,T], \\ \omega = \operatorname{curl} u, & (x,t) \in \mathbb{R}^3 \times [0,T], \end{cases} \tag{2.2}$$

and, due to the second PDE in (2.2), it is true that $u = \operatorname{curl} \Psi$, where Ψ is the vector-valued stream function given by

$$\Psi(x,t) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(y,t)}{|x-y|} dy. \tag{2.3}$$

As a consequence, for all $(x, t) \in \mathbb{R}^3 \times [0, T]$, u can be recovered from ω throughout the *Biot–Savart* law

$$u(x,t) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \omega(y,t) dy, \tag{2.4}$$

which we had already stated in (1.3), for $t = 0$.

In the rest of this section, we will focus our attention on the behavior of the smooth solution (u, p) of (1.4) at infinity.

Lemma 2.1. *Let $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in (C^\infty(\mathbb{R}^3))^3$ be a compactly supported vector field and consider $M > 0$ such that $\operatorname{supp} \Phi \subset \bar{B}_M(\mathbf{0})$. Then, there exists $C > 0$ such that, for any $x \in \mathbb{R}^3 \setminus B_{2M}(\mathbf{0})$, we have*

$$\left| \int_{\mathbb{R}^3} \frac{\Phi(y)}{|x-y|} dy - \frac{1}{|x|} \int_{\mathbb{R}^3} \Phi(y) dy \right| \leq \frac{C}{|x|^2}. \tag{2.5}$$

Additionally, if $\int_{\mathbb{R}^3} \Phi(y)dy = \mathbf{0}$, then the inequality

$$\left| \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \Phi(y) dy \right| \leq \frac{C}{|x|^3} \tag{2.6}$$

also holds.

Proof. We start proving (2.6). Consider the vector field $g = (g_1, g_2, g_3) \in (C^\infty(\mathbb{R}^3 \setminus \{\mathbf{0}\}))^3$, given by $g(x) = \frac{x}{|x|^3}$ for each $x \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Let us take $x \in \mathbb{R}^3 \setminus B_{2M}(\mathbf{0})$ and $y \in \bar{B}_M(\mathbf{0})$. Since

$$\{(1-t)x + t(x-y) : t \in [0,1]\} \subset \mathbb{R}^3 \setminus \{\mathbf{0}\},$$

applying the mean value theorem, we obtain $\theta_i \in (0,1)$ such that

$$g_i(x-y) = g_i(x) + Dg_i(x-\theta_i y)(-y),$$

where $i \in \{1,2,3\}$. Using that $\int_{\mathbb{R}^3} \Phi(y)dy = \mathbf{0}$ and $|x-\theta_i y| \geq |x| - \theta_i|y| \geq |x| - \frac{|x|}{2} = \frac{|x|}{2}$, we easily check that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\left[\frac{(x_i-y_i)}{|x-y|^3}\Phi_j(y) - \frac{(x_j-y_j)}{|x-y|^3}\Phi_i(y)\right]dy\right| \\
&\quad\leq \left|\int_{\bar{B}_M(\mathbf{0})} Dg_i(x-\theta_i y)(-y)\Phi_j(y)dy\right| + \left|\int_{\bar{B}_M(\mathbf{0})} Dg_j(x-\theta_j y)(-y)\Phi_i(y)dy\right| \\
&\quad\leq C\left(\int_{\bar{B}_M(\mathbf{0})}\frac{|\Phi_j(y)|}{|x-\theta_i y|^3}dy + \int_{\bar{B}_M(\mathbf{0})}\frac{|\Phi_i(y)|}{|x-\theta_j y|^3}dy\right) \leq \frac{C}{|x|^3},
\end{aligned}$$

for all $i,j \in \{1,2,3\}$. From this, (2.6) follows.

The proof of (2.5) is analogous, but the condition $\int_{\mathbb{R}^3} \Phi(y)dy = \mathbf{0}$ is not required. In fact, we can find $\lambda \in (0,1)$ such that

$$\left|\int_{\mathbb{R}^3}\frac{\Phi(y)}{|x-y|}dy - \frac{1}{|x|}\int_{\mathbb{R}^3}\Phi(y)dy\right| = \left|\int_{\bar{B}_M(\mathbf{0})}\frac{(x-\lambda y)\cdot y}{|x-\lambda y|^3}\Phi(y)dy\right| \leq \int_{\bar{B}_M(\mathbf{0})}\frac{M|\Phi(y)|}{|x-\lambda y|^2}dy \leq \frac{C}{|x|^2},$$

following the desired conclusion. ☐

Next, we apply Lemma 2.1 in order to state the decay of u and its derivatives, as $|x| \to \infty$.

Proposition 2.2. *Consider u and ω as mentioned above and take $M > 0$ such that* $\operatorname{supp}(\omega(\cdot,t)) \subset \bar{B}_M(\mathbf{0})$ *for all $t \in [0,T]$. Then, there exists $C > 0$ such that*

$$|u(x,t)| \leq \frac{C}{|x|^2}, \qquad |u_t(x,t)| \leq \frac{C}{|x|^3} \quad \text{and} \quad |\nabla u(x,t)| \leq \frac{C}{|x|^3}, \tag{2.7}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \bar{B}_{2M}(\mathbf{0})) \times [0,T]$.

Proof. During this proof, suppose that $(x,t) \in (\mathbb{R}^3 \setminus \bar{B}_{2M}(\mathbf{0})) \times [0,T]$ is fixed. Thus, we easily get

$$|u(x,t)| = \left|\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)}{|x-y|^3}\times\omega(y,t)dy\right| \leq \left(\frac{1}{\pi}\int_{\bar{B}_M(\mathbf{0})}|\omega(y,t)|dy\right)\frac{1}{|x|^2} \leq \frac{C}{|x|^2}.$$

For the second desired estimate, we use (2.2) and (2.4) in order to obtain

$$u_t(x,t) = \frac{-1}{4\pi}\int_{\bar{B}_M(\mathbf{0})}\frac{(x-y)}{|x-y|^3}\times[(\omega\cdot\nabla)u - (u\cdot\nabla)\omega](y,t)dy.$$

Recalling that u and ω are two divergence-free vector fields, we take

$$\int_{\mathbb{R}^3}[(\omega\cdot\nabla)u - (u\cdot\nabla)\omega](y,t) = \mathbf{0}.$$

Hence, applying the inequality (2.6) from Lemma 2.1, we have

$$|u_t(x,t)| \leq \frac{C}{|x|^3}.$$

In the last part of this proof, we will estimate $\nabla u = [\partial_i u_j]_{i,j=1}^3$. Notice that, if $0 < \varepsilon < M$ and $y \in \overline{B}_M(\mathbf{0})$, we clearly get $|x-y| \geq \frac{|x|}{2} > M > \varepsilon$ for all $y \in \bar{B}_M(0)$. Consequently, for any 3×1 matrix B, we take

$$\begin{aligned}
|[\nabla u(x,t)]B| &= \lim_{\varepsilon \to 0} \left| \int_{|y-x| \geq \varepsilon} \left(\frac{\omega(y,t) \times B}{4\pi |x-y|^3} + \frac{3\{[(x-y) \times \omega(y,t)] \otimes (x-y)\}B}{4\pi |x-y|^5} \right) dy \right| \\
&\leq \int_{\bar{B}_M(\mathbf{0})} \left| \frac{\omega(y,t) \times B}{4\pi |x-y|^3} + \frac{3\{[(x-y) \times \omega(y,t)] \otimes (x-y)\}B}{4\pi |x-y|^5} \right| dy \\
&\leq \left(\frac{1}{4\pi} \int_{\bar{B}_M(\mathbf{0})} \frac{|\omega(y,t)|}{|x-y|^3} dy \right) |B| \\
&\leq \frac{C}{|x|^3} |B|,
\end{aligned}$$

where $h \otimes k$ denotes the 3×3 matrix $[h_i k_j]_{i,j=1}^3$ for each $h, k \in \mathbb{R}^3$. It completes the proof. $\square$

In the next two results, we will specify the decay of the scalar pressure p, given in (1.4), as $|x| \to \infty$.

Lemma 2.3. *Let (u,p) be the solution of (1.4) and consider $\bar{y} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. The following properties hold:*

(a) There exists $C > 0$ such that

$$|\nabla p(x,t)| \leq \frac{C}{|x|^3}$$

for any $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$.

(b) For each $t \in [0,T]$, there exists

$$L(t) = \lim_{\theta \to \infty} p(\theta \bar{y}, t).$$

Proof. The pointwise estimate for ∇p comes immediately from Proposition 2.2.

Let us prove the second part of the result. Let $(\theta_n)_{n=1}^\infty$ be a sequence of positive real numbers which tends to infinity. Since

$$|p(\theta_m \bar{y}, t) - p(\theta_n \bar{y}, t)| = \left| \int_{\theta_n}^{\theta_m} \nabla p(s\bar{y}, t) \cdot \bar{y} ds \right| \leq \frac{C}{2|\bar{y}|^2} \left| \frac{1}{\theta_n^2} - \frac{1}{\theta_m^2} \right| \tag{2.8}$$

for all positive integers m and n, we conclude that, for each $t \in [0,T]$, the sequence $(p(\theta_n \bar{y}, t))_{n=1}^\infty$ converges as $n \to \infty$. Analogously, if $(\lambda_n)_{n=1}^\infty$ is another sequence of positive real numbers which tends to infinity, we take

$$|p(\theta_n \bar{y}, t) - p(\lambda_n \bar{y}, t)| \leq \frac{C}{2|\bar{y}|^2} \left| \frac{1}{\theta_n^2} - \frac{1}{\lambda_n^2} \right|$$

for all positive integers m and n, and $t \in [0,T]$. It means that there exists $\lim_{\theta \to \infty} p(\theta \bar{y}, t)$, as desired. $\square$

Next, we will see that Lemma 2.3 allows us to collect some properties of the pressure p.

Proposition 2.4. *Let (u,p) be the solution (1.4) and consider $\bar{y}, \bar{z} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Then*

(a) $\lim_{\theta \to \infty} p(\theta \bar{y}, t) = \lim_{\theta \to \infty} p(\theta \bar{z}, t)$ for each $t \in [0,T]$;

(b) *there exists a continuous function* $p_\infty : [0,T] \longrightarrow \mathbb{R}$ *and* $C > 0$ *such that*

$$|p(x,t) - p_\infty(t)| \leq \frac{C}{|x|^2} \tag{2.9}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$.

Proof. Firstly, let us focus on the proof of (a). Without loss of generality, we can assume that $\bar{z} \notin \mathbb{R}\bar{y}$ and $|\bar{y}| \geq |\bar{z}|$. Take $\theta > 0$ and consider the sphere

$$\mathcal{S} = \{x \in \mathbb{R}^3 : |x| = \theta|\bar{z}|\}.$$

Let $\sigma : [s_1, s_2] \subset [0, 2\pi] \longrightarrow \mathcal{S}$ be the geodesic on $\mathcal{S}$ from $\theta\bar{z}$ to $q = \frac{\theta|\bar{z}|}{|\bar{y}|}\bar{y}$, given by

$$\sigma(s) = (\sin s)q + (\cos s)\theta|\bar{z}|v,$$

where v belongs to the tangent plane to $\mathcal{S}$ at q. Thus, from Lemma 2.3, we obtain

$$\begin{aligned} |p(\theta\bar{y},t) - p(\theta\bar{z},t)| &\leq |p(\theta\bar{y},t) - p(q)| + |p(q) - p(\theta\bar{z},t)| \\ &= \left|\int_{s_1}^{s_2} \nabla p(\sigma(s)) \cdot \sigma'(s) ds\right| + \left|\int_{\frac{\theta|\bar{z}|}{|\bar{y}|}}^{\theta} \nabla p(s\bar{y}) \cdot \bar{y} ds\right| \leq \left[\frac{2\pi C}{|\bar{z}|^2} + \frac{C}{2|\bar{y}|}\left(\frac{|\bar{y}|^2}{|\bar{z}|^2} - 1\right)\right]\frac{1}{\theta^2}. \end{aligned}$$

Therefore, $\lim_{\theta\to\infty} p(\theta\bar{y},t) = \lim_{\theta\to\infty} p(\theta\bar{z},t)$ for each $t \in [0,T]$.

Secondly, we must prove the part (b). Let us set the scalar function $p_\infty : [0,T] \longrightarrow \mathbb{R}$ by $p_\infty(t) = \lim_{\theta\to\infty} p(\theta\bar{y},t)$. Arguing as in (2.8), we easily check that,

$$|p(x,t) - p_\infty(t)| = \lim_{\theta\to\infty} |p(x,t) - p(\theta x,t)| \leq \lim_{\theta\to\infty} \frac{C}{|x|^2}\left(1 - \frac{1}{\theta^2}\right) = \frac{C}{|x|^2}.$$

This completes the proof of Proposition 2.4. □

In the last result of this section, we will make use of Propositions 2.2 and 2.4 in order to describe how the stream vector field Ψ behaves at infinity.

Proposition 2.5. *Let* (u,p) *be the solution of* (1.4) *and* Ψ *be the associated stream vector field given in* (2.3). *Consider* $M > 0$ *such that* $\operatorname{supp}(\omega(\cdot,t)) \subset \bar{B}_M(\mathbf{0})$ *for all* $t \in [0,T]$. *Then, there exists* $C > 0$ *such that*

$$|\Psi(x,t)| \leq \frac{C}{|x|}, \qquad |\Psi_t(x,t)| \leq \frac{C}{|x|^2} \quad \text{and} \quad |\nabla\Psi(x,t)| \leq \frac{C}{|x|^2}, \tag{2.10}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \bar{B}_{2M}(\mathbf{0})) \times [0,T]$.

Proof. The first inequality in (2.10) is straightforward. Now, take $(x,t) \in (\mathbb{R}^3 \setminus \bar{B}_{2M}(\mathbf{0})) \times [0,T]$. Since

$$\Psi_t(x,t) = \frac{1}{4\pi}\int_{\bar{B}_M(\mathbf{0})} \frac{(\omega\cdot\nabla)u - (u\cdot\nabla)\omega}{|x-y|}(y,t)dy$$

and $\int_{\mathbb{R}^3}[(\omega\cdot\nabla)u - (u\cdot\nabla)\omega](y,t) = \mathbf{0}$, the inequality (2.5) gives us the second estimate in (2.10). Finally, for each $i \in \{1,2,3\}$, we notice that

$$|\partial_i\Psi(x,t)| = \left|\frac{1}{4\pi}\int_{\bar{B}_M(\mathbf{0})} \frac{x_i - y_i}{|x-y|^3}\omega(y,t)dy\right| \leq \int_{\bar{B}_M(\mathbf{0})} \frac{|\omega(y,t)|}{|x-y|^2}dy,$$

and thus the third estimate in (2.10) also holds. □

3 Approximate inviscid solutions

Firstly, let us recall some notations given in Section 2. Consider $\omega_0 \in (C_c^\infty(\mathbb{R}^3))^3$ and let (u, p) be the smooth solution of (1.4) in $\mathbb{R}^3 \times [0, T]$, with initial data

$$u_0 = u_0(x) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega_0(y)dy. \tag{3.1}$$

Also, let Ψ be the stream vector field associated to u, given in (2.3).

In this section, we intend to approximate the solution u by an appropriate net $(u^R)_{R>0}$ of divergence-free vector fields. Suppose that $\bar{\Omega}_0 \cup \operatorname{supp} \omega(\cdot, t) \subset B_{M_0}(\mathbf{0})$ for all $t \in [0, T]$, where $M_0 > 0$, and let us take $\chi \in C^\infty(\mathbb{R})$ satisfying $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $\mathbb{R} \setminus (-2M_0, 2M_0)$, and $\chi \equiv 0$ in $[-M_0, M_0]$. For each $R > 0$, let us set $\chi^R(x) = \chi(|x - \mathbf{R}|)$ and

$$u^R(x,t) := \operatorname{curl}(\chi^R(\Psi + C_R)) = \nabla\chi^R \times (\Psi + C_R) + \chi^R u, \tag{3.2}$$

where $(x, t) \in \mathbb{R}^3 \times [0, T]$ and $C_R = \frac{-1}{4\pi R} \int_{\mathbb{R}^3} \omega_0(y)dy$. Clearly, each u^R is a smooth and divergence-free vector field in $\mathbb{R}^3$, which vanishes in the neighborhood of $\Gamma_R = \partial(\Omega_R)$, where $\Omega_R = \Omega_0 + \mathbf{R}$. Besides, taking

$$\overline{p} = p - p_\infty, \tag{3.3}$$

where the function $p_\infty : [0, T] \longrightarrow \mathbb{R}$ was obtained in Proposition 2.4, we also have

$$u_t^R = \nabla\chi^R \times \Psi_t - \chi^R(u \cdot \nabla)u - \chi^R \nabla\bar{p} \tag{3.4}$$

in $\Pi_R \times (0, T)$, recalling that $\Pi_R = \mathbb{R}^3 \setminus \overline{\Omega}_R$.

Next, we will prove some important estimates involving $(u^R)_{R>0}$, which will allow us to obtain Theorem 1.1.

Lemma 3.1. *Let us consider Ψ, C_R and $M_0 > 0$ as mentioned at the beginning of this section. Then there exist $C > 0$ and $R_0 > 0$ such that*

$$\sup_{|y| \in [M_0, 2M_0]} |\Psi(y + \mathbf{R}, t) + C_R| \leq \frac{C}{R^2} \tag{3.5}$$

for all $R > R_0$ and $t \in [0, T]$.

Proof. Firstly, we observe that, for all $x_1, x_2 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ satisfying $|x_1| \geq 2|x_2|$, we can apply the mean value theorem in order to obtain

$$\left| \frac{1}{|x_1 - x_2|} - \frac{1}{|x_1|} \right| \leq \frac{4|x_2|}{|x_1|^2}. \tag{3.6}$$

Let us fix $y \in \mathbb{R}^3$ such that $|y| \in [M_0, 2M_0]$. Since $\int_{\mathbb{R}^3} \omega(z,t)dz = \int_{\mathbb{R}^3} \omega_0(z)dz$, for any $t \in [0, T]$, we have

$$\begin{aligned}
|\Psi(y + \mathbf{R}, t) + C_R| &= \left| \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(z,t)}{|(y + \mathbf{R}) - z|} dz - \frac{1}{4\pi R} \int_{\mathbb{R}^3} \omega_0(z)dz \right| \\
&\leq \left| \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(z,t)}{|(y + \mathbf{R}) - z|} dz - \frac{1}{4\pi|y + \mathbf{R}|} \int_{\mathbb{R}^3} \omega(z,t)dz \right| \\
&\quad + \frac{1}{4\pi} \left| \frac{1}{|y + \mathbf{R}|} - \frac{1}{|\mathbf{R}|} \right| \int_{\mathbb{R}^3} |\omega_0(z)|dz \\
&=: A + B.
\end{aligned}$$

Taking $R_0 = 4M_0$, it is clear that, if $z \in B_{M_0}(\mathbf{0})$ and $R > R_0$, then $|(y+\mathbf{R}) - z| \geq R - 3M_0 > 0$ and $|y+\mathbf{R}| \geq R - 2M_0 > 0$. As a consequence,

$$
\begin{aligned}
A &\leq \frac{1}{4\pi}\int_{B_{M_0}(\mathbf{0})} \left| \frac{1}{|(y+\mathbf{R})-z|} - \frac{1}{|y+\mathbf{R}|} \right| |\omega(z,t)| dz \\
&\leq \frac{1}{4\pi}\int_{B_{M_0}(\mathbf{0})} \frac{|z|}{|(y+\mathbf{R})-z||y+\mathbf{R}|} |\omega(z,t)| dz \\
&\leq \frac{C}{(R-3M_0)^2} \\
&\leq \frac{C}{R^2}.
\end{aligned}
$$

for all $R > R_0$. Finally, applying (3.6) with $x_1 = -\mathbf{R}$ and $x_2 = y$, we also obtain

$$
B \leq \frac{|y|}{\pi R^2}\int_{B_{M_0}(\mathbf{0})} |\omega_0(z)| dz \leq \frac{C}{R^2}
$$

for $R > R_0$. Hence, (3.5) follows. □

Remark 3.2. The estimates proved in Propositions 2.2 and 2.5 are valid for any $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$.

Proposition 3.3. *Under the previous notation, there exist two constants $C = C(\Omega_0, T) > 0$ and $R_0 > 0$ such that, for all $R > R_0$, we have:*

(a) $\|u^R - u\|_{L^\infty([0,T];L^2(\Pi_R))} + \|u^R - \chi^R u\|_{L^\infty([0,T];L^2(\Pi_R))} \leq C/R^2$;

(b) $\|\nabla u^R - \nabla u\|_{L^\infty([0,T];L^2(\Pi_R))} \leq C/R^2$;

(c) $\|\bar{p}\nabla\chi^R\|_{L^\infty([0,T];L^2(\Pi_R))} + \|\nabla\chi^R \times \Psi_t\|_{L^\infty([0,T];L^2(\Pi_R))} \leq C/R^2$;

(d) $\|u^R\|_{L^\infty([0,T];L^\infty(\Pi_R))} + \|\nabla u^R\|_{L^\infty([0,T];L^\infty(\Pi_R))} + \|\nabla u^R\|_{L^\infty([0,T];L^2(\Pi_R))} \leq C$.

Proof. **ESTIMATE (a)**: In the first place, using (3.2) and (3.5), we get

$$
\begin{aligned}
\|u^R(\cdot,t) - \chi^R u(\cdot,t)\|^2_{L^2(\Pi_R)} &= \|\nabla\chi^R \times (\Psi + C_R)\|^2_{L^2(\Pi_R)} \\
&= \int_{|x-\mathbf{R}|\in[M_0,2M_0]} |\chi'(|x-\mathbf{R}|)|^2 |\Psi(x,t) + C_R|^2 dx \\
&= \int_{|y|\in[M_0,2M_0]} |\chi'(|y|)|^2 |\Psi(y+\mathbf{R},t) + C_R|^2 dy \\
&\leq C \sup_{|y|\in[M_0,2M_0]} |\Psi(y+\mathbf{R},t) + C_R|^2 \\
&\leq \frac{C}{R^4}
\end{aligned} \tag{3.7}
$$

for all $t \in [0,T]$ and $R > 0$ sufficiently large. On the other hand, recalling Proposition 2.2, we also obtain

$$
\begin{aligned}
\|(\chi^R-1)u(\cdot,t)\|^2_{L^2(\Pi_R)} &= \int_{B_{2M_0}(\mathbf{R})\cap\Pi_R} |\chi(|x-\mathbf{R}|)-1|^2|u(x,t)|^2dx \\
&= \int_{B_{2M_0}(\mathbf{0})\cap\Pi_0} |\chi(|y|)-1|^2|u(y+\mathbf{R},t)|^2dy \\
&\le C\int_{B_{2M_0}(\mathbf{0})\cap\Pi_0} \frac{|\chi(|y|)-1|^2}{|y+\mathbf{R}|^4}dy \\
&\le \frac{C}{(R-2M_0)^4} \\
&\le \frac{C}{R^4}
\end{aligned} \tag{3.8}
$$

for all $t\in[0,T]$ and $R>0$ sufficiently large. As a result, desired estimate holds.

ESTIMATE (b): From (3.2), we know that

$$\partial_i u^R - \partial_i u = \partial_i(\nabla\chi^R)\times(\Psi+C_R)+\nabla\chi^R\times\partial_i\Psi+(\partial_i\chi^R)u+(\chi^R-1)\partial_i u.$$

Hence, using Propositions 2.2 and 2.5, and Lemma 3.1, we can argue as in (3.7) and (3.8) in order to prove that

$$\|\nabla u^R-\nabla u\|_{L^\infty([0,T];L^2(\Pi_R))}\le CR^{-2}$$

for all $R>0$ sufficiently large.

ESTIMATE (c): The proof of the third desired estimate is very similar to the last ones. In fact, it is a consequence of (2.9) and (2.10), given in Propositions 2.4 and 2.5, respectively.

ESTIMATE (d): Let us prove the last estimate. Since the inviscid velocity u is a smooth vector field, Proposition 2.2 assures that $\|\nabla u^R\|_{L^\infty([0,T];L^2(\Pi_R))}\le C$ for $R>0$ is sufficiently large. Finally, for all $i\in\{1,2,3\}$, it is clear that

$$|\chi^R(x)|+|\partial_i\chi^R(x)|+|\partial_i\partial_j\chi^R(x)|\le C,$$

for all $x\in\mathbb{R}^3$ and $R>0$. It implies that $\|u^R\|_{L^\infty([0,T];L^\infty(\Pi_R))}$ and $\|\nabla u^R\|_{L^\infty([0,T];L^\infty(\Pi_R))}$ are uniformly bounded with respect to $R>0$. This ends the proof. □

Next, we prove the last result of this section, which yields a suitable convergence related to the initial data.

Proposition 3.4. *As before, let $\omega_0\in(C_c^\infty(\mathbb{R}^3))^3$ be a divergence-free vector field and consider the initial velocity u_0 as in (3.1). Then, for each $R>0$, there exists a unique $v_{0,R}\in H(\Pi_R)$ such that* $\operatorname{curl} v_{0,R}=\omega_0|_{\Pi_R}$. *In addition, there exist $C>0$ and $R_0>0$ such that*

$$\|\tilde{v}_{0,R}-u_0\|_{L^2(\mathbb{R}^3)}\le\frac{C}{R^2} \tag{3.9}$$

for all $R>R_0$, where $\tilde{v}_{0,R}$ vanishes on $\overline{\Omega}_R$ and equals $v_{0,R}$ on Π_R.

Proof. For each $R>0$, the existence of exactly one vector field $v_{0,R}\in H(\Pi_R)$ satisfying $\operatorname{curl} v_{0,R}=\omega_0|_{\Pi_R}$ was given in [4], where the authors have used the Leray–Helmholtz–Weyl orthogonal decomposition as well as the simple connectedness of Π_R.

In order to prove (3.9), we observe that

$$\|u_0|_{\Pi_R}-v_{0,R}\|_{L^2(\Pi_R)}\le\|u_0|_{\Pi_R}-w\|_{L^2(\Pi_R)}$$

for all $w \in H(\Pi_R)$. In particular, taking $w(x) = u^R(x,0)$, where $x \in \mathbb{R}^3$, and applying Proposition 3.3, we obtain

$$\begin{aligned}\|\tilde{v}_{0,R} - u_0\|^2_{L^2(\mathbb{R}^3)} &= \|v_{0,R} - u_0|_{\Pi_R}\|^2_{L^2(\Pi_R)} + \|u_0\|^2_{L^2(\bar{\Omega}_R)} \\ &\leq \|u_0|_{\Pi_R} - u^R(\cdot,0)\|^2_{L^2(\Pi_R)} + \|u_0\|^2_{L^2(\bar{\Omega}_R)} \\ &\leq \frac{C}{R^4} + \|u_0\|^2_{L^2(\bar{\Omega}_R)},\end{aligned}$$

for all $R > 0$ sufficiently large. Besides, taking $R > 2M_0$, we observe that $\bar{\Omega}_R \cap \operatorname{supp}\omega_0 = \emptyset$. As a consequence, for any $x \in \overline{\Omega}_R$ and $y \in \operatorname{supp}\omega_0$, we have

$$|x-y| \geq R - |y| - |x - \mathbf{R}| \geq R - 2M_0 > 0.$$

Therefore,

$$\|u_0\|^2_{L^2(\bar{\Omega}_R)} \leq C \int_{B_{M_0}(\mathbf{0})} \frac{|\omega_0(y)|^2}{|x-y|^4} dydx \leq \frac{C}{(R-2M_0)^4} \leq \frac{C}{R^4},$$

and (3.9) holds. □

4 Leray–Hopf solutions in exterior domains

Throughout this section, let $\Pi = \mathbb{R}^3 \setminus \overline{\Omega} \subset \mathbb{R}^3$ be a smooth exterior domain, which means that Ω is a smooth compact set in $\mathbb{R}^3$. Given $T > 0$ and $v_0 \in H(\Pi)$, we consider the Navier–Stokes system

$$\begin{cases} v_t + (v\cdot\nabla)v - \nu\Delta v + \nabla P = 0, & (x,t) \in \Pi \times (0,T), \\ \operatorname{div} v = 0, & (x,t) \in \Pi \times [0,T), \\ v(x,t) = \mathbf{0}, & (x,t) \in \partial\Pi \times (0,T), \\ v(x,0) = v_0(x), & x \in \Pi, \end{cases} \tag{4.1}$$

where $v = v(x,t)$ is the velocity field evaluated at the point $x \in \Pi$ and at the time $t \in [0,T]$, $P = P(x,t)$ is the related scalar pressure field, and $\nu > 0$ is the kinematic viscosity.

Definition 4.1. Under the notation above, a measurable vector field

$$v \in L^2(0,T;V(\Pi)) \cap L^\infty(0,T;H(\Pi))$$

is said to be a weak solution of (4.1) in $\Pi \times [0,T)$ if, for any $\Phi \in \mathcal{D}_T(\Pi)$, we have

$$\int_0^T \int_\Pi [v\cdot\Phi_t - \nu(\nabla v\cdot\nabla\Phi) - (v\cdot\nabla)v\cdot\Phi](x,t)dxdt = -\int_\Pi v_0\cdot\Phi(x,0)dx. \tag{4.2}$$

The next result assures the existence of a weak solution to (4.1) satisfying a very important additional estimate, which is called a *Leray–Hopf solution* of (4.1).

Theorem 4.2. *Given $T > 0$ and $v_0 \in H(\Pi)$, the system (4.1) there exists a weak solution $v : \Pi \times [0,T) \longrightarrow \mathbb{R}^3$ which satisfies the energy estimate*

$$\|v(\cdot,t)\|^2_{L^2(\Pi)} + 2\nu \int_0^t \|\nabla v(\cdot,\tau)\|^2_{L^2(\Pi)} d\tau \leq \|v_0\|^2_{L^2(\Pi)} \tag{4.3}$$

for all $t \in [0,T]$

Proof. The proof of this result can be found in [2]. □

Remark 4.3. Taking $T > 0$ and $v_0 \in H(\Pi)$, let us consider a weak solution $v : \Pi \times [0, T) \longrightarrow \mathbb{R}^3$ of (4.1). It is known that, for any $\Phi \in \mathcal{D}(\Pi \times [0, T))$, v satisfies

$$\int_\Pi (v \cdot \Phi)(x, t)dx - \int_\Pi (v \cdot \Phi)(x, 0)dx = \int_0^t \int_\Pi [v \cdot \Phi_t - \nu(\nabla v \cdot \nabla \Phi) - (v \cdot \nabla)v \cdot \Phi](x, \tau)d\tau \tag{4.4}$$

for all $t \in [0, T)$ (see [3], for instance).

Later, we will apply the relation (4.4) replacing Φ by each approximate inviscid solutions u^R, where $R > 0$. For this reason, we are supposed to prove that (4.4) remains valid when Φ decays sufficiently fast at infinity, but is not compactly supported.

Let $\eta : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function which satisfies $0 \le \eta \le 1$ in $\mathbb{R}^3$, $\eta \equiv 1$ in $B_1(\mathbf{0})$, and $\eta \equiv 0$ in $\mathbb{R}^3 \setminus B_2(\mathbf{0})$. For each $s > 0$, we set $\eta_s(x) = \eta(s^{-1}x)$, where $x \in \mathbb{R}$. Under these notations, we are ready to present the next two results.

Lemma 4.4. *Let $F : \Pi \longrightarrow \mathbb{R}^3$ and $G : \Pi \times [0, T] \longrightarrow \mathbb{R}^3$ be two smooth vector fields, with $F \in H^1(\Pi)$. Also, suppose that there exist $C > 0$ and $\alpha > 0$ such that*

$$|G(x, t)| \le \frac{C}{|x|^\alpha} \tag{4.5}$$

for all $(x, t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0, T]$. The following properties hold:

(a) $\|\eta_s F - F\|_{H^1(\Pi)} \to 0$ *as* $s \to \infty$;

(b) If $a \in (3, \infty]$, *then* $\|\nabla \eta_s\|_{L^a(\Pi)} \to 0$ *as* $s \to \infty$;

(c) $\|\partial_i \eta_s G(\cdot, t)\|_{L^2(\Pi)} \le \frac{C}{s^{\alpha-1/2}}$ *for all* $t \in [0, T]$ *and* $i \in \{1, 2, 3\}$;

(d) $\|\partial^2_{ij} \eta_s G(\cdot, t)\|_{L^2(\Pi)} \le \frac{C}{s^{\alpha+1/2}}$ *for all* $t \in [0, T]$ *and* $i, j \in \{1, 2, 3\}$;

(e) If $\alpha > \frac{3}{2}$, *then* $\|\eta_s G(\cdot, t) - G(\cdot, t)\|_{L^2(\Pi)} \le \frac{C}{s^{\alpha-3/2}}$ *for all* $t \in [0, T]$.

Proof. Take $d > 0$ such that $\Omega \subset B_d(\mathbf{0})$, where $\Pi = \mathbb{R}^3 \setminus \overline{\Omega}$. In this proof, we will only consider the functions $\eta_s : \mathbb{R}^3 \longrightarrow \mathbb{R}$, with $s \ge d$.

PART (a): It follows immediately from Lebesgue's dominated convergence theorem.

PART (b): Let $i \in \{1, 2, 3\}$. If $a \in (3, +\infty)$, the desired convergence is a consequence of

$$\int_\Pi |\partial_i \eta_s(x)|^a dx = \int_{\mathbb{R}^3} \left|\partial_i \eta\left(\frac{x}{s}\right)\right|^a \frac{1}{s^a} dx = \frac{1}{s^{a-3}} \|\partial_i \eta\|^a_{L^a}.$$

On the other hand, the estimate $\|\partial_i \eta_s\|_{L^\infty(\Pi)} \le \frac{C}{s}$ gives us the complete conclusion.

PARTS (c) and (d): Let us take $t \in [0, T]$ and $i, j \in \{1, 2, 3\}$. Thus, using (4.5), we take

$$\begin{aligned}\int_\Pi |\partial_i \eta_s(x) G(x.t)|^2 dx &= \int_{|x| \in [s, 2s]} \frac{1}{s^2} \left|\partial_i \eta\left(\frac{x}{s}\right)\right|^2 |G(x, t)|^2 dx \\ &= \int_{|y| \in [1, 2]} s\, |\partial_i \eta(y)|^2 \, |G(sy, t)|^2 dy \\ &\le \frac{C}{s^{2\alpha-1}} \left(\int_{|y| \in [1, 2]} |\partial_i \eta(y)|^2 dy\right)\end{aligned}$$

and

$$\int_{\Pi} |\partial^2_{ij}\eta_s(x)G(x,t)|^2 dx = \int_{|x|\in[s,2s]} \frac{1}{s^4}\left|\partial^2_{ij}\eta\left(\frac{x}{s}\right)\right|^2 |G(x,t)|^2 dx$$
$$= \int_{|y|\in[1,2]} \frac{1}{s}\left|\partial^2_{ij}\eta(y)\right|^2 |G(sy,t)|^2 dy$$
$$\leq \frac{C}{s^{2\alpha+1}}\left(\int_{|y|\in[1,2]} |\partial^2_{ij}\eta(y)|^2 dy\right).$$

PART (e): For the last estimate, we assume that $\alpha > 3/2$. Once again, applying (4.5), we have

$$\int_{\Pi} |\eta_s(x)G(x,t) - G(x,t)|^2 dx = \int_{|x|\geq s} \left|\left[\eta\left(\frac{x}{s}\right) - 1\right] G(x,t)\right|^2 dx$$
$$= \int_{|y|\geq 1} |\eta(y) - 1|^2 |G(sy,t)|^2 s^3 dy$$
$$\leq \frac{C}{(2\alpha - 3)s^{2\alpha-3}}$$

for each $t \in [0,T]$. It concludes the proof. □

The following result is the last one of this section. We emphasize that its content brings the information that (4.4) holds for a larger class of test functions.

Proposition 4.5. *Let $\tilde{\Psi} : \mathbb{R}^3 \times [0,T] \longrightarrow \mathbb{R}^3$ and $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be two smooth vector fields satisfying:*

(a) $\operatorname{supp}(\tilde{\Psi}) \subset (\mathbb{R}^3 \setminus \bar{B}) \times [0,T]$, *where B is an open ball containing Ω;*

(b) F is divergence-free and $\operatorname{supp}(F)$ *is a compact subset of Π;*

(c) There exists $C_1 > 0$ such that

$$|\tilde{\Psi}(x,t)| \leq \frac{C_1}{|x|}, \qquad |\tilde{\Psi}_t(x,t)| \leq \frac{C_1}{|x|^2} \quad \text{and} \quad |\nabla\tilde{\Psi}(x,t)| \leq \frac{C_1}{|x|^2}, \tag{4.6}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$.

Additionally, consider $\tilde{\Phi} := \operatorname{curl}(\tilde{\Psi}) + F$ and suppose that there exists $C_2 > 0$ such that

$$|\tilde{\Phi}(x,t)| \leq \frac{C_2}{|x|^2}, \qquad |\tilde{\Phi}_t(x,t)| \leq \frac{C_2}{|x|^3} \quad \text{and} \quad |\nabla\tilde{\Phi}(x,t)| \leq \frac{C_2}{|x|^3}, \tag{4.7}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$. Then, a weak solution v of (4.1), with initial data $v_0 \in H(\Pi)$, satisfies

$$\int_{\Pi} (v \cdot \tilde{\Phi})(x,t)dx - \int_{\Pi} v_0(x) \cdot \tilde{\Phi}(x,0)dx$$
$$= \int_0^t \int_{\Pi} [v \cdot \tilde{\Phi}_t - \nu(\nabla v \cdot \nabla\tilde{\Phi}) + (v \cdot \nabla)\tilde{\Phi} \cdot v](x,\tau)d\tau. \tag{4.8}$$

Proof. As in the proof of Lemma 2.10, consider $\Pi = \mathbb{R}^3 \setminus \overline{\Omega}$ and take $d > 0$ such that $\Omega \subset B_d(\mathbf{0})$. For each $s \geq d$, define $\Phi^s \in \mathcal{D}(\Pi \times [0,T))$ given by

$$\Phi^s := \operatorname{curl}(\eta_s \tilde{\Psi}) + F = \nabla \eta_s \times \tilde{\Psi} + \eta_s \operatorname{curl} \tilde{\Psi} + F.$$

From (4.4), we obtain

$$\begin{aligned}\int_\Pi (v \cdot \Phi^s)(x,t)dx - \int_\Pi (v \cdot \Phi^s)(x,0)dx \\ = \int_0^t \int_\Pi [v \cdot (\Phi^s)_t - \nu(\nabla v \cdot \nabla \Phi^s) - (v \cdot \nabla) v \cdot \Phi^s](x,\tau) d\tau\end{aligned} \tag{4.9}$$

for all $t \in [0,T)$.

Next, we fix $t \in [0,T)$, in order to pass to the limit in (4.9) as $s \to \infty$. Firstly, using (4.3) and Lemma 4.4, we take

$$\begin{aligned}\left| \int_\Pi (v \cdot \Phi^s)(x,t)dx - \int_\Pi (v \cdot \tilde{\Phi})(x,t)dx \right| &\leq \|v(\cdot,t)\|_{L^2(\Pi)} \|\nabla \eta_s \times \tilde{\Psi} + (\eta_s - 1) \operatorname{curl} \tilde{\Psi}\|_{L^2(\Pi)} \\ &\leq \frac{C\|v_0\|_{L^2(\Pi)}}{s^{1/2}}\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\left| \int_0^t \int_\Pi (v \cdot (\Phi^s)_t) dx d\tau - \int_0^t \int_\Pi (v \cdot \tilde{\Phi}_t) dx d\tau \right| &\leq \int_0^t \int_\Pi |v| |\nabla \eta_s \times \tilde{\Psi}_t + (\eta_s - 1)(\operatorname{curl} \tilde{\Psi})_t|(x,\tau) dx d\tau \\ &\leq \|v_0\|_{L^2(\Pi)} \int_0^t (\|\nabla \eta_s \times \tilde{\Psi}_t\|_{L^2(\Pi)} + \|(\eta_s - 1)(\operatorname{curl} \tilde{\Psi})_t\|_{L^2(\Pi)})(\tau) d\tau \\ &\leq \frac{CT\|v_0\|_{L^2(\Pi)}}{s^{3/2}}.\end{aligned} \tag{4.11}$$

Likewise, using Lemma 4.4, (4.3) and

$$\partial_i \Phi^s - \partial_i \tilde{\Phi} = \partial_i(\nabla \eta_s) \times \tilde{\Psi} + \nabla \eta_s \times \partial_i \tilde{\Psi} + (\partial_i \eta_s) \operatorname{curl} \tilde{\Psi} + (\eta_s - 1) \partial_i(\operatorname{curl} \tilde{\Psi})$$

where $i \in \{1,2,3\}$, we obtain the estimate

$$\begin{aligned}&\left| \int_0^t \int_\Pi (\nabla v \cdot \nabla \Phi^s)(x,\tau) dx d\tau - \int_0^t \int_\Pi (\nabla v \cdot \nabla \tilde{\Phi})(x,\tau) dx d\tau \right| \\ &\leq \sum_{i=1}^3 \int_0^t \|\partial_i v(\cdot,\tau)\|_{L^2(\Pi)} \|(\partial_i \Phi^s - \partial_i \tilde{\Phi})(\cdot,\tau)\|_{L^2(\Pi)} d\tau \\ &\leq \frac{\|v_0\|_{L^2(\Pi)}}{\sqrt{2\nu}} \sum_{i=1}^3 \left(\int_0^t \|(\partial_i \Phi^s - \partial_i \tilde{\Phi})(\cdot,\tau)\|^2_{L^2(\Pi)} d\tau \right)^{1/2} \\ &\leq \frac{C}{s^{1/2}}.\end{aligned} \tag{4.12}$$

To conclude the proof, we use

$$\int_0^t \int_\Pi [(v \cdot \nabla) v] \cdot \Phi^s(x.\tau) dx d\tau = -\int_0^t \int_\Pi [(v \cdot \nabla) \Phi^s] \cdot v(x.\tau) dx d\tau,$$

as well as Lemma 4.4 and (4.3) in order to get

$$\begin{aligned}
&\left|\int_0^t \int_\Pi [(v\cdot\nabla)v]\cdot\Phi^s(x.\tau)dxd\tau + \int_0^t\int_\Pi [(v\cdot\nabla)\tilde{\Phi}]\cdot v(x.\tau)dxd\tau\right| \\
&\quad\le \int_0^t\int_\Pi |v|^2|\nabla\Phi^s - \nabla\tilde{\Phi}|^2(x,\tau)dxd\tau \\
&\quad\le \int_0^t \|v(\cdot,\tau)\|^2_{L^4(\Pi)}\|(\nabla\Phi^s - \nabla\tilde{\Phi})(\cdot,\tau)\|_{L^2(\Pi)}d\tau \\
&\quad\le \frac{C}{s^{1/2}}\int_0^t \|v(\cdot,\tau)\|^2_{H^1(\Pi)}d\tau \\
&\quad\le \frac{C}{s^{1/2}}.
\end{aligned} \tag{4.13}$$

Therefore, from (4.10), (4.11), (4.12) and (4.13), the relation (4.8) holds. □

5 Proof of Theorem 1.1

This section is devoted to the main result of this paper. Let us recall its hypotheses:

- ω_0 is a smooth, compactly supported and divergence-free vector field in $\mathbb{R}^3$;
- (u,p) is the smooth solution of (1.4), defined in $\mathbb{R}^3\times(0,T)$, with initial data u_0, as in (1.3).

Taken $T\in(0,T^*)$, $R>0$ and $\nu>0$, we also consider:

- u^R as defined in (3.2);
- $v_{0,R}$ is the L^2-orthogonal projection of $u_0|_{\Pi_R}$ on $H(\Pi_R)$, mentioned in Proposition 3.4;
- $v^{\nu,R}$ is a weak solution of (4.1) in $\Pi_R\times[0,T)$, with initial data $v_{0,R}$, given by Theorem 4.2.

Proof of Theorem 1.1. **CLAIM**: There exist $C = C(T,\Omega_0,\omega_0) > 0$ and $R_0 > 0$ such that, if $R > R_0$, then

$$\|v^{\nu,R}(\cdot,t) - u^R(\cdot,t)\|^2_{L^2(\Pi_R)} + \nu\int_0^t \|\nabla v^{\nu,R}(\cdot,\tau) - \nabla u^R(\cdot,\tau)\|^2_{L^2(\Pi_R)}d\tau \le C\left(\frac{1}{R^2}+\nu\right) \tag{5.1}$$

for almost every $t\in[0,T]$. As a consequence, Theorem 1.1 holds.

From now on, we will focus on the verification of (5.1). To do so, the main arguments used below take into account those presented in [8]. Let us fix $t\in[0,T]$. From (4.3),

$$\frac{1}{2}\|v^{\nu,R}(\cdot,t)\|^2_{L^2(\Pi_R)} + \nu\int_0^t \|\nabla v^{\nu,R}(\cdot,\tau)\|^2_{L^2(\Pi_R)}d\tau \le \frac{1}{2}\|v_{0,R}\|^2_{L^2(\Pi_R)} = \frac{1}{2}\|\tilde{v}_{0,R}\|^2_{L^2(\mathbb{R}^3)}, \tag{5.2}$$

where $\tilde{v}_{0,R}$ equals $v_{0,R}$ on Π_R and vanishes otherwise.

Next, due to (2.7) and (2.10), we can apply Proposition 4.5, with $v = v^{\nu,R}$ and $\tilde{\Phi} = u^R$, in order to obtain

$$
\begin{aligned}
&-\int_{\Pi_R}(v^{\nu,R}\cdot u^R)(x,t)dx+\int_{\Pi_R}(v_{0,R}\cdot u^R)(x,0)dx\\
&\quad=-\int_0^t\int_{\Pi_R}[v^{\nu,R}\cdot u^R_t-\nu\nabla v^{\nu,R}\cdot\nabla u^R](x,\tau)dxd\tau-\int_0^t\int_{\Pi_R}[(v^{\nu,R}\cdot\nabla)u^R]\cdot v^{\nu,R}dxd\tau\\
&\quad=-\int_0^t\int_{\Pi_R}v^{\nu,R}\cdot[\nabla\chi^R\times\Psi_t-\chi^R(u\cdot\nabla)u-\chi^R\nabla\overline{p}](x,\tau)dxd\tau\\
&\qquad+\nu\int_0^t\int_{\Pi_R}(\nabla v^{\nu,R}\cdot\nabla u^R)(x,\tau)dxd\tau-\int_0^t\int_{\Pi_R}[(v^{\nu,R}\cdot\nabla)u^R]\cdot v^{\nu,R}(x,\tau)dxd\tau\\
&\quad=-\int_0^t\int_{\Pi_R}v^{\nu,R}\cdot[\nabla\chi^R\times\Psi_t-\chi^R\nabla\overline{p}](x,\tau)dxd\tau+\nu\int_0^t\int_{\Pi_R}(\nabla v^{\nu,R}\cdot\nabla u^R)(x,\tau)dxd\tau\\
&\qquad+\int_0^t\int_{\Pi_R}[(\chi^R u-u^R)\nabla u]\cdot v^{\nu,R}(x,\tau)dxd\tau+\int_0^t\int_{\Pi_R}[(u^R\cdot\nabla)(u-u^R)]\cdot v^{\nu,R}(x,\tau)dxd\tau\\
&\qquad+\int_0^t\int_{\Pi_R}[(v^{\nu,R}-u^R)\nabla u^R]\cdot(v^{\nu,R}-u^R)(x,\tau)dxd\tau. \qquad (5.3)
\end{aligned}
$$

Besides, recalling that the kinetic energy $E(t)=\frac{1}{2}\int_{\mathbb{R}^3}|u(x,t)|^2dx$ is a conserved quantity in time, we also take

$$
\begin{aligned}
\frac{1}{2}\|u^R(\cdot,t)\|^2_{L^2(\Pi_R)}&\le\frac{1}{2}\|(\nabla\chi^R\times(\Psi+C_R))(\cdot,t)\|^2_{L^2(\Pi_R)}\\
&\quad+\|(\nabla\chi^R\times(\Psi+C_R))(\cdot,t)\|_{L^2(\Pi_R)}\|\chi^R u\|_{L^2(\Pi_R)}+\frac{1}{2}\|\chi^R u(\cdot,t)\|^2_{L^2(\Pi_R)}\\
&\le\frac{1}{2}\|(u^R-\chi^R u)(\cdot,t)\|^2_{L^2(\Pi_R)}\\
&\quad+\|(u^R-\chi^R u)(\cdot,t)\|_{L^2(\Pi_R)}\|\chi^R u\|_{L^2(\Pi_R)}+\frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^3)}. \qquad (5.4)
\end{aligned}
$$

As a result, from (5.2), (5.3) and (5.4) , we conclude that

$$
\begin{aligned}
&\frac{1}{2}\|v^{\nu,R}-u^R\|^2_{L^2(\Pi_R)}+\nu\int_0^t\|(\nabla v^{\nu,R}-\nabla u^R)(\cdot,\tau)\|^2_{L^2(\Pi_R)}d\tau\\
&\quad\le\left[\frac{1}{2}\|\tilde{v}_{0,R}-u_0\|^2_{L^2(\mathbb{R}^3)}+\int_{\Pi_R}v_{0,R}\cdot(u_0-u^R)(x,0)dx\right]\\
&\quad-\left[\int_0^t\int_{\Pi_R}v^{\nu,R}\cdot(\nabla\chi^R\times\Psi_t-\chi^R\nabla\overline{p})(x,\tau)dxd\tau\right]\\
&\quad+\nu\left[\int_0^t\|\nabla u^R\|^2_{L^2(\Pi_R)}-\int_0^t\int_{\Pi_R}(\nabla v^{\nu,R}\cdot\nabla u^R)(x,\tau)dxd\tau\right]\\
&\quad+\left[\int_0^t\int_{\Pi_R}[(\chi^R u-u^R)\nabla u]\cdot v^{\nu,R}(x,\tau)dxd\tau+\int_0^t\int_{\Pi_R}[(u^R\cdot\nabla)(u-u^R)]\cdot v^{\nu,R}(x,\tau)dxd\tau\right]\\
&\quad+\left[\int_0^t\int_{\Pi_R}[(v^{\nu,R}-u^R)\nabla u^R]\cdot(v^{\nu,R}-u^R)(x,\tau)dxd\tau\right]\\
&\quad+\left[\frac{1}{2}\|(u^R-\chi^R u)(\cdot,t)\|^2_{L^2(\Pi_R)}+\|(u^R-\chi^R u)(\cdot,t)\|_{L^2(\Pi_R)}\|\chi^R u\|_{L^2(\Pi_R)}\right]\\
&\quad=:A_1+A_2+A_3+A_4+A_5+A_6. \qquad (5.5)
\end{aligned}
$$

Firstly, using Proposition 3.3(d), we easily get

$$|A_5| \le C\int_0^t \|v^{\nu,R} - u^R\|^2_{L^2(\Pi_R)}(x,\tau)d\tau.$$

Likewise, A_6 was completely analyzed in Proposition 3.3(a). Thus, in the rest of this proof, we will estimate each A_i, with $i \in \{1,2,3,4\}$, in terms of $R > 0$ and $\nu > 0$.

To see that

$$|A_1| \le \frac{C}{R^2},$$

we just apply Propositions 3.3(a) and 3.4. In fact, observe that

$$\begin{aligned}\int_{\Pi_R} v_{0,R}\cdot(u_0 - u^R)(x,0)dx &\le \|v_{0,R}\|_{L^2(\Pi_R)}\|(u_0-u^R)(\cdot,0)\|_{L^2(\Pi_R)}\\ &\le \|u_0\|_{L^2(\mathbb{R}^3)}\|(u_0-u^R)(\cdot,0)\|_{L^2(\Pi_R)}\\ &\le \frac{C}{R^2},\end{aligned}$$

and recall (3.9).

Next, using the relation

$$-\int_0^t\int_{\Pi_R} v^{\nu,R}\cdot(\chi^R\nabla\overline{p})dxd\tau = \int_0^t\int_{\Pi_R} v^{\nu,R}\cdot(\overline{p}\nabla\chi^R)dxd\tau,$$

and Proposition 3.3(c), we obtain

$$\begin{aligned}|A_2| &= \left|\int_0^t\int_{\Pi_R} v^{\nu,R}\cdot(\nabla\chi^R\times\Psi_t + \overline{p}\nabla\chi^R)(x,\tau)dxd\tau\right|\\ &\le \int_0^t \|v^{\nu,R}\|_{L^2(\Pi_R)}(\|\nabla\chi^R\times\Psi_t\|_{L^2(\Pi_R)} + \|\overline{p}\nabla\chi^R\|_{L^2(\Pi_R)})(\cdot,\tau)dxd\tau\\ &\le T\|v_{0,R}\|_{L^2(\Pi_R)}(\|\nabla\chi^R\times\Psi_t\|_{L^\infty([0,T];L^2(\Pi_R))} + \|\overline{p}\nabla\chi^R\|_{L^\infty([0,T];L^2(\Pi_R))})\\ &\le \frac{CT\|u_0\|_{L^2(\mathbb{R}^3)}}{R^2}.\end{aligned}$$

Besides, using Young's inequality and Proposition 3.3(d), we can easily check that

$$\begin{aligned}|A_3| &= \left|\nu\int_0^t\int_{\Pi_R}\nabla u^R\cdot(\nabla u^R - v^{\nu,R})(x,\tau)dxd\tau\right|\\ &\le \frac{\nu}{2}\int_0^t\|\nabla u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau + \frac{\nu}{2}\int_0^t\|\nabla v^{\nu,R} - \nabla u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau\\ &\le C\nu + \frac{\nu}{2}\int_0^t\|\nabla v^{\nu,R} - \nabla u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau.\end{aligned}$$

At last, the estimate

$$\begin{aligned}|A_4| \le &\int_0^t \|\chi^R u - u^R\|_{L^2(\Pi_R)}\|\nabla u\|_{L^\infty(\Pi_R)}\|v^{\nu,R}\|_{L^2(\Pi_R)}d\tau\\ &+\int_0^t\|u^R\|_{L^\infty(\Pi_R)}\|\nabla u^R - \nabla u\|_{L^2(\Pi_R)}\|v^{\nu,R}\|_{L^2(\Pi_R)}d\tau\\ &\le \frac{CT}{R^2}\end{aligned}$$

comes from Theorem 4.2 and Proposition 3.3(a),(b),(d).

Therefore, there exist $K > 0$ and $L > 0$, independent of time t, such that

$$\frac{1}{2}\|v^{\nu,R}(\cdot,t) - u^R(\cdot,t)\|^2_{L^2(\Pi_R)} + \frac{\nu}{2}\int_0^t \|\nabla v^{\nu,R} - \nabla u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau$$
$$\leq K\left(\frac{1}{R^2} + \nu\right) + L\int_0^t \|v^{\nu,R} - u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau$$

for any $t \in [0,T]$. Thus, the integral form of Gronwall's inequality allows us to achieve the estimate (5.1) . □

6 Some additional comments

1. In order to obtain Proposition 3.4, the circulation of the velocity on the boundary Γ_R is not required, since each Π_R is a 3D simply connected domain.

2. In this paper, we deal with three-dimensional incompressible flows, with small viscosity, around distant obstacles. Perhaps, the analogous two-dimensional case can also be studied. To be more precise, let $\mathcal{U}_0 \subset \mathbb{R}^2$ be a smooth bounded domain, which is also connected and simply connected. For each $R \geq 0$, let us set

$$\mathcal{U}_R = \mathcal{U}_0 + (R,0), \qquad \mathcal{V}_R = \mathbb{R}^2 \setminus \overline{\mathcal{U}}_R \quad \text{and} \quad \mathcal{C}_R = \partial\mathcal{U}_R = \partial\Pi_R.$$

Let us consider $\omega^0 \in C_c^\infty(\mathbb{R}^2)$ and $\gamma \in \mathbb{R}$, which are both independent of $R > 0$. Set

$$y_{0,R} = K_R[\omega^0] + (\gamma + m)H_R,$$

where $m = \int_{\mathbb{R}^2} \omega^0 dx$, $K_R[\omega^0] = K_R[\omega^0](x)$ is the Biot–Savart operator in $\mathcal{V}_R$ and $H_R = H_R(x)$ is the generator of the harmonic vector fields in $\mathcal{V}_R$. Thanks to Lemma 2.2 and Proposition 2.1 of [7], for each $R > 0$, we have

$$\begin{cases} \operatorname{div} y_{0,R} = 0, & \text{in } \mathcal{V}_R, \\ \operatorname{curl} y_{0,R} = \omega^0, & \text{in } \mathcal{V}_R, \\ y_{0,R} \cdot \hat{n} = 0, & \text{on } \mathcal{C}_R, \\ |y_{0,R}(x)| \to 0, & \text{as } |x| \to +\infty, \\ \int_{\mathcal{C}_R} u_{0,R} \cdot ds = \gamma. \end{cases}$$

Thus, it would be very nice to understand the asymptotic behavior of the family of 2D incompressible flows, with small viscosity, around distant obstacles, governed by

$$\begin{cases} y_t^{\nu,R} + (y^{\nu,R} \cdot \nabla)y^{\nu,R} - \nu\Delta y^{\nu,R} + \nabla\pi^{\nu,R} = 0, & (x,t) \in \mathcal{V}_R \times (0,T), \\ \operatorname{div} y^{\nu,R} = 0, & (x,t) \in \mathcal{V}_R \times [0,T), \\ y^{\nu,R}(x,t) = 0, & (x,t) \in \mathcal{C}_R \times (0,T), \\ y^{\nu,R}(x,0) = y_{0,R}(x), & x \in \mathcal{V}_R. \end{cases} \tag{6.1}$$

It means that we have fixed ω^0 as an initial vorticity and γ as the circulation of the initial flow around each obstacle $\mathcal{U}_R$. We should notice that $y_{0,R} \in L^{2,\infty}(\Pi_R)$, but $y_{0,R} \notin L^2(\Pi_R)$. In this case, it seems to us that Kato's argument can not be applied following the same structure of the proof of Theorem 1.1. However, another approach may work if we take mild solutions of (6.1), obtained [11].

3. In order to simplify calculations and estimates, we have considered material obstacles of the form $\Omega_R = \Omega_0 + \mathbf{R} = \Omega_0 + R\mathbf{1}$, where $\mathbf{1} = (1,0,0)$. However, our main result remains valid if we replace Ω_R by $\Omega_0 + R\vec{z}$, for each $R > 0$, where $\vec{z}$ is another unit vector in $\mathbb{R}^3$.

4. We believe that Theorem 1.1 is related to that one obtained in [10], in the three-dimensional case. Let $\Omega^0 \subset \mathbb{R}^3$ be a smooth and simply connected bounded domain $\Omega^0 \subset \mathbb{R}^3$. In that work, the authors started with a smooth initial vorticity $\omega_0 \in C_c^\infty(\mathbb{R}^3)$, which is divergence-free and, for each $L > 0$, they took u_0^L as the unique divergence-free vector field in $\Omega^L = L\Omega^0$ that is tangent to $\partial\Omega^L$ and has a curl equal to ω_0 in Ω^L. At this point, they studied the limiting behavior of a family

$$\{u^{\nu,L} : \nu > 0 \text{ and } L > 0\},$$

consisting of weak solutions to the Navier–Stokes system

$$\begin{cases} u_t^{\nu,L} + (u^{\nu,L}\cdot\nabla)u^{\nu,L} - \nu\Delta u^{\nu,L} + \nabla\pi^{\nu,L} = 0, & (x,t) \in \Omega^L \times (0,T), \\ \operatorname{div} u^{\nu,L} = 0, & (x,t) \in \Omega^L \times [0,T), \\ u^{\nu,L}(x,t) = 0, & (x,t) \in \partial\Omega^L \times (0,T), \\ u^{\nu,L}(x,0) = u_0^L(x), & x \in \Omega^L. \end{cases} \tag{6.2}$$

Summarizing, it is proved in Theorem 1.2 of [10] that

$$\|u^{\nu,L} - u\|_{L^\infty([0,T];L^2(\Omega^L))} \le \left[C\left(\nu + \frac{1}{\sqrt{L}}\right) + \|u_0^L - u_0\|_{L^2(\Omega^L)}\right] e^{CT}, \tag{6.3}$$

where $u = u(x,t)$ is a smooth solution to (1.4), with initial data u_0, given in (1.3). Above, we observe that

$$\|u_0^L - u_0\|_{L^2(\Omega^L)} \le CL^{-1/2},$$

which allows us to contrast the current main result of this paper with (6.3). In both cases, there are results for 3D incompressible flows, with small viscosity, in domains with distant boundaries. Thus, it seems to us that the rates of convergence

$$R^{-1} \quad \text{and} \quad L^{-1/2}$$

on the right side of (1.6) and (6.3), respectively, suggest that the boundary $\Gamma_R = \partial\Pi_R$, in (1.5), produces a bigger effect in the vanishing viscosity limit, when compared to the same effect produced by $\partial\Omega^L$, in (6.2).

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Appendix: List of notations

This appendix reunites some notations and definitions that we have used in the previous sections. Our purpose here is to become the reading of this article easier.

Domains

- $\Omega_0 \subset \mathbb{R}^3$ be a smooth bounded domain, such that $\mathbb{R}^3 \setminus \overline{\Omega}_0$ is connected and simply connected;
- For each $R \geq 0$,
$$\mathbf{R} := (R, 0, 0), \qquad \Omega_R := \Omega_0 + \mathbf{R}, \qquad \Pi_R := \mathbb{R}^3 \setminus \overline{\Omega}_R \quad \text{and} \quad \Gamma_R := \partial\Omega_R = \partial\Pi_R.$$

Operators

For a smooth vector field $F = (F_1, F_2, F_3)$ in $\mathbb{R}^3$, we denote:

- $\operatorname{div} F := \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$;
- $\operatorname{curl} F := (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$;
- $(F \cdot \nabla)F := F_1 \partial_1 F + F_2 \partial_2 F + F_3 \partial_3 F$.

Function spaces

Given an open set $\mathcal{O} \subset \mathbb{R}^3$ and a function $f : \mathcal{O} \longrightarrow \mathbb{R}$, the *support* of f is the closed set
$$\operatorname{supp} f := \overline{\{x \in \mathcal{O} : f(x) \neq 0\}}.$$

Under these notations,

- $L^{2,\infty}(\mathcal{O})$ denotes the Lorentz space of f satisfying $\sup_{s>0} A_s < +\infty$, where
$$A_s := s^2 \mu(\{x \in \mathcal{O} : |f(x)| > s\})$$
and μ is the Lebesgue measure in $\mathbb{R}^3$.
- $C_c^\infty(\mathcal{O})$ denotes the space of all infinitely differentiable real functions with compact support in $\mathcal{O}$;
- $\mathcal{D}(\mathcal{O}) := \{\Psi \in (C_c^\infty(\mathcal{O}))^3 : \operatorname{div} \Psi = 0 \text{ in } \mathcal{O}\}$;
- $V(\mathcal{O}) := \{\Psi \in (H^1(\mathcal{O}))^3 : \operatorname{div} \Psi = 0 \text{ in } \mathcal{O} \text{ and } u = 0 \text{ on } \partial\mathcal{O}\}$;
- $H(\mathcal{O}) := \{\Psi \in (L^2(\mathcal{O}))^3 : \operatorname{div} \Psi = 0 \text{ in } \mathcal{O} \text{ and } \Psi \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}$, where $\mathbf{n}$ is the outward directed unit normal vector field to $\partial\mathcal{O}$;
- Given $T > 0$, $\mathcal{D}_T(\mathcal{O})$ denotes the set of all $\varphi \in (C_c^\infty(\mathcal{O} \times [0, T)))^3$ such that
$$\operatorname{div}_x \varphi(x,t) := \partial_1 \varphi_1 + \partial_2 \varphi_2 + \partial_3 \varphi_3 = 0 \quad \text{in } \Pi \times [0, T).$$

Solutions and initial data

- $\omega_0 \in \mathcal{D}(\mathbb{R}^3)$ denotes a fixed initial vorticity;
- u_0 represents the velocity defined on $\mathbb{R}^3$, associated to the vorticity ω_0, as in (1.3);
- (u, p) denotes a smooth solution of (1.4), with initial data u_0;
- (v, P) denotes a Leray–Hopf solution of (4.1);
- Given $R > 0$ and $\nu > 0$, $(v^{\nu,R}, P^{\nu,R})$ denotes a Leray–Hopf solution of (1.5).

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