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Existence of a positive bound state solution for the nonlinear Schrödinger–Bopp–Podolsky system

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Abstract. In this paper, we study a class of Schrödinger–Bopp–Podolsky system. Under some suitable assumptions for the potentials, by developing some calculations of sharp energy estimates and using a topological argument involving the barycenter function, we establish the existence of positive bound state solution.

Keywords: Schrödinger–Bopp–Podolsky system, variational approach, competing potentials, bound state solution.

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1 Introduction

In this paper, we consider the following Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = Q(x)|u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p \in (3,5)$, $V(x)$, $K(x)$ and $Q(x)$ are positive functions such that

$$\lim_{|x|\to\infty} V(x) = V_\infty > 0, \qquad \lim_{|x|\to\infty} Q(x) = Q_\infty > 0, \qquad \lim_{|x|\to\infty} K(x) = 0.$$

This system appears when a Schrödinger field $\psi = \psi(t,x)$ couple with its electromagnetic field in the Bopp–Podolsky electromagnetic theory. The Bopp–Podolsky theory, developed by Bopp [8], and independently by Podolsky [20], is a second order gauge theory for the electromagnetic field. As the Mie theory [19] and its generalizations given by Born and Infeld [7, 9], it was introduced to solve the so called infinity problem that appears in the classical Maxwell theory. From the well known Gauss law (or Poisson equation), the electrostatic potential ϕ for a given charge distribution whose density is ρ satisfies the equation

$$-\Delta\phi = \rho \ \text{ in } \mathbb{R}^3. \tag{1.2}$$

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If $\rho = 4\pi\delta_{x_0}$, with $x_0 \in \mathbb{R}^3$, the fundamental solution of (1.2) is $E(x - x_0)$, where $E(x) = \frac{1}{|x|}$, and the electrostatic energy is $\varepsilon(E) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla E(x)|^2 \mathrm{d}x = +\infty$. Thus, equation (1.2) was replaced by

$$-\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1 - |\nabla\phi|^2}}\right) = \rho \ \text{ in } \mathbb{R}^3$$

in the Born–Infeld theory (see [2]) or replaced by

$$-\Delta\phi + a^2\Delta^2\phi = \rho \ \text{ in } \mathbb{R}^3$$

in the Bopp–Podolsky one, from the reason that, in both case if $\rho = 4\pi\delta_{x_0}$, their energy is finite. In particular, when we consider the operator $-\Delta + \Delta^2$, by [3], we know that $\mathcal{K}(x - x_0)$, where $\mathcal{K}(x) := \frac{1-e^{-|x|}}{|x|}$, is the fundamental solution of the equation

$$-\Delta\phi + \Delta^2\phi = 4\pi\delta_{x_0},$$

it has no singularity in x_0 since it satisfies $\lim_{x\to x_0} \mathcal{K}(x - x_0) = 1$, and its energy satisfies

$$\varepsilon_{BP}(\mathcal{K}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\mathcal{K}|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |\Delta\mathcal{K}|^2 \,\mathrm{d}x < +\infty.$$

In addition, the Bopp–Podolsky theory may be interpreted as an effective theory for short distances and for large distances it is indistinguishable from the Maxwell one. For more physical details we refer the reader to the recent paper[10,11,14] and their references therein. Indeed the operator $-\Delta + \Delta^2$ appears also in other different interesting mathematical and physical situations [5,15].

Recently, P. d'Avenia and G. Siciliano in [3] introduced the following Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where a, $\omega > 0$, $q \neq 0$, they developed the variational framework for system (1.3) and proved that when $p \in (2, 6)$, there exists $q^* > 0$, for every $q \in (-q^*, q^*) \setminus \{0\}$, problem (1.3) admits a nontrivial solution, when $p \in (3, 6)$, for $q \neq 0$, problem (1.3) admits a nontrivial solution. In [22], G. Siciliano and K. Silva proved that the multiplicity and nonexistence of solutions for problem (1.3) by using the fibering method.

If $a = 0$ in problem (1.3), it reduces to the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

In the last decades, there are lots of results about the existence and multiplicity of solutions for system (1.4), we do not review the huge documents, just list some of them for interesting readers to see [1,6,12,21] and the references therein.

The purpose of this paper is to describe some phenomena that can occur when the coefficients $V(x)$, $K(x)$ and $Q(x)$ are competing. In order to describe our main results, we first rewrite problem (1.1) in a more appropriate way to our aim. Let

$$V(x) = V_\infty + a(x), \qquad Q(x) = Q_\infty - b(x),$$

where $a(x)$ and $b(x)$ satisfies the following assumptions:

(H_1) $a(x) \in L^{\frac{3}{2}}(\mathbb{R}^3)$, $a(x) \geq 0$, $a(x) \not\equiv 0$, and $\lim_{|x|\to\infty} a(x) = 0$;

(H_2) $b(x) \in L^\infty(\mathbb{R}^3)$, $0 \leq b(x) < Q_\infty$, $b(x) \not\equiv 0$, and $\lim_{|x|\to\infty} b(x) = 0$;

(H_3) $K \in L^2(\mathbb{R}^3)$ and there exists $R_0 > 0$ such that $K(x) \leq Ce^{-2\sqrt{V_\infty}|x|}$ for all $|x| \geq R_0$.

Clearly, (1.1) becomes the following form

$$\begin{cases} -\Delta u + (V_\infty + a(x))u + K(x)\phi u = (Q_\infty - b(x))|u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = K(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

From the variational framework described in Section 2, we find that the difference between problem (1.5) and system (1.4) is the nonlocal kernel $\mathcal{K}(x) = \frac{1-e^{-|x|}}{|x|}$, comparing the Poisson kernel $\mathcal{P}(x) = \frac{1}{|x|}$, $\mathcal{K}(x)$ is nonhomogeneous and not singular at $x = 0$ because $\lim_{|x|\to 0} \mathcal{K}(x) = 1$, which implies that $\mathcal{K} \in L^\infty(\mathbb{R}^3)$. The non-homogeneity of $\mathcal{K}$ makes difficult the use of rescaling of type $t \to u(t^\gamma\cdot)$ and the non-singularity maybe weak some conditions in the estimates.

To the best of knowledge, the system (1.5) is a new one in the field of variational methods, there are only few works about the existence and multiplicity of solutions, such as the ground state. The purpose of this paper is to study the existence of bound state solution for system (1.5). The approach is inspired by the ideas in [4, 12], we explore some calculations of sharp energy estimates and apply a topological argument involving the barycenter function to show that there exists a critical value of the energy functional, in a higher level of energy which can yield a solution of the problem (1.5). The main difficulties of this work are that the problem is given in the whole space, leading to the loss of compactness, and some sharp energy estimates. For dealing with these obstacles, we borrows a global compactness lemma to recover the compactness and use some careful computations to get the energy estimates.

Now we are ready to give the main results of the paper.

Theorem 1.1. *Suppose that* (H_1)–(H_3) *hold and*

(H_4) $\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)|x|^2 e^{2\sqrt{V_\infty}|x|}\,\mathrm{d}x < +\infty$ *and* $\int_{\mathbb{R}^3} b(x)|x|^2 e^{2\sqrt{V_\infty}|x|}\,\mathrm{d}x < +\infty$.

Then (1.5) *admits a positive bound state solution.*

The paper is organised as follows. In Section 2, we give general preliminaries in order to attack our problem. In Section 3, we prove Theorem 1.1.

2 Preliminaries

In what follows, we will use the following notations:

- Let $H^1(\mathbb{R}^3)$ be the Sobolev space endowed with the inner product and norm

$$(u, v) := \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)dx, \qquad \|u\|^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx.$$

- $\mathcal{D}$ is the completion of $\mathbf{C}_c^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta u|^2)\,\mathrm{d}x\right)^{\frac{1}{2}}$$

- $\mathcal{L}^q(\mathcal{O})$, $1 \le q \le \infty$, $\mathcal{O} \subseteq \mathbb{R}^3$ a measurable set, denotes the Lebesgue space, the norm in $\mathcal{L}^q(\mathcal{O})$ is denoted by $|\cdot|_{\mathcal{L}^q(\mathcal{O})}$ when $\mathcal{O}$ is a proper measurable subset of $\mathbb{R}^3$ and by $|\cdot|_q$ when $\mathcal{O} = \mathbb{R}^3$.
- $B_R(y)$ denotes the ball of radius R centered at y, if $y = 0$, we denote it by B_R.
- $c, c_i, C, C_i, \ldots$ denote a number of positive constants.

In what follows, without any loss of generality we assume $V_\infty = Q_\infty = 1$.

From [3], we know that for $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi \in \mathcal{D}$, denoted by ϕ_u^K, such that $-\Delta\phi + \Delta^2\phi = K(x)u^2$ and it possesses the explicit formula

$$\phi_u^K(x) := \phi(x) = \int_{\mathbb{R}^3} \frac{(1 - e^{-|x-z|})}{4\pi|x-z|} K(z)u^2(z)\mathrm{d}z, \qquad x \in \mathbb{R}^3. \tag{2.1}$$

Replacing ϕ by ϕ_u^K in the first equation in system (1.5), then this system reduces to a class of Schrödinger equation

$$-\Delta u + (V_\infty + a(x))u + K(x)\phi_u^K u = (Q_\infty - b(x))|u|^{p-1}u \quad \text{in } \mathbb{R}^3. \tag{2.2}$$

The energy functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ corresponding to equation (2.2) is defined as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (V_\infty + a(x))u^2 \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^K u^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^3} (Q_\infty - b(x))|u|^{p+1} \,\mathrm{d}x,$$

clearly, $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and its critical points are weak solutions of problem (2.2). Therefore, in order to find the solutions of system (1.5), we only need to seek the critical points of functional I. In other words, if $u \in H^1(\mathbb{R}^3)$ is a critical point of I, then (u, ϕ_u) is a weak solution for system (1.5).

Now, by Lemma 3.4 in [3] and applying a similar argument as in Proposition 2.2 of [12], we can show some properties of ϕ_u.

Proposition 2.1. *For each $u \in H^1(\mathbb{R}^3)$, the following statements hold:*

(i) $\phi_u^K \in \mathcal{D} \hookrightarrow L^\infty(\mathbb{R}^3)$;

(ii) $\phi_u^K \ge 0$;

(iii) *For every* $s \in (3, +\infty]$, $\phi_u^K \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;

(iv) *For every* $s \in (\frac{3}{2}, +\infty]$, $\nabla\phi_u^K \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;

(v) $\phi_{tu}^K = t^2\phi_u^K$;

(vi) $|\phi_u^K|_6 \le c\|u\|^2$;

(vii) *For every* $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)}^K = \phi_u^{K(\cdot-y)}(\cdot + y)$;

$(viii)$ *If* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, *then*

$$\phi_{u_n}^K \to \phi_u^K \quad \text{in } \mathcal{D}, \qquad \int_{\mathbb{R}^3} K(x)\phi_{u_n}^K u_n^2 \,\mathrm{d}x \to \int_{\mathbb{R}^3} K(x)\phi_u^K u^2 \,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}^K u_n \varphi \,\mathrm{d}x \to \int_{\mathbb{R}^3} K(x)\phi_u^K u\varphi \,\mathrm{d}x \qquad \forall\, \varphi \in H^1(\mathbb{R}^3).$$

Proof. We only need to verify that (iii), (iv) and $(viii)$ hold true.

Observe that $Ku^2 \in L^r(\mathbb{R}^3)$ for $r \in [1, \frac{3}{2})$ owing to $K \in L^2(\mathbb{R}^3)$ and $u \in H^1(\mathbb{R}^3)$. By (ii) of Lemma 3.3 in [3], we know that $\phi_u^K \in L^q(\mathbb{R}^3)$ for $q \in (\frac{3r}{3-2r}, +\infty]$. From $\frac{3r}{3-2r} \in [3, +\infty)$ and using Lemma 2.20 in [18], we see that $\phi_u^K \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$. Similarly, we can get (iv).

$(viii)$ For all $u \in H^1(\mathbb{R}^3)$, consider the linear functional $\mathcal{L}_u : \mathcal{D} \to \mathbb{R}^3$ defined by

$$\mathcal{L}_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v \,\mathrm{d}x,$$

by the definition of ϕ_u^K, we have $\|\phi_u^K\|_{\mathcal{D}} = \|\mathcal{L}_u\|_{\mathcal{L}(\mathcal{D},\mathbb{R})}$. Therefore we intend to show that as $n \to \infty$

$$\|\mathcal{L}_{u_n} - \mathcal{L}_u\|_{\mathcal{L}(\mathcal{D},\mathbb{R})} \to 0.$$

Let $\varepsilon > 0$ be fixed, there exists a positive number R_ε so large that $|K|_{L^2(\mathbb{R}^3\setminus B(0,R_\varepsilon))} < \varepsilon$. Therefore for $v \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\begin{aligned}
|\mathcal{L}_{u_n}(v) - \mathcal{L}_u(v)| &= \int_{\mathbb{R}^3} K(u_n^2 - u^2)v \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^3\setminus B(0,R_\varepsilon)} K|u_n^2 - u^2||v| \,\mathrm{d}x + \int_{B(0,R_\varepsilon)} K|u_n^2 - u^2||v| \,\mathrm{d}x \\
&\le |K|_{L^2(\mathbb{R}^3\setminus B(0,R_\varepsilon))}|u_n^2 - u^2|_3|v|_6 + \left[\int_{B(0,R_\varepsilon)} K^{\frac{6}{5}}|u_n^2 - u^2|^{\frac{6}{5}} \,\mathrm{d}x\right]^{\frac{5}{6}} |v|_6 \\
&\le \left[\varepsilon c + \left[\int_{B(0,R_\varepsilon)} K^{\frac{6}{5}}|u_n + u|^{\frac{6}{5}}|u_n - u|^{\frac{6}{5}} \,\mathrm{d}x\right]^{\frac{5}{6}}\right] \|v\|_{\mathcal{D}}
\end{aligned}$$

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, we know that $u_n \to u$ in $L_{loc}^{\frac{12}{5}}(\mathbb{R}^3)$. Furthermore, set $B_M = \{x \in B(0,R_\varepsilon) : K(x) > M\}$ and remark being $K \in L^2(\mathbb{R}^3)$, $|B_M| \to 0$ as $M \to \infty$. Therefore, for large M, $\left(\int_{B_M} K^2\right)^{\frac{3}{5}} < \varepsilon$. So we have

$$\begin{aligned}
&\int_{B(0,R_\varepsilon)} K^{\frac{6}{5}}|u_n + u|^{\frac{6}{5}}|u_n - u|^{\frac{6}{5}} \,\mathrm{d}x \\
&\quad= \int_{B_M} K^{\frac{6}{5}}|u_n + u|^{\frac{6}{5}}|u_n - u|^{\frac{6}{5}} \,\mathrm{d}x + \int_{B(0,R_\varepsilon)\setminus B_M} K^{\frac{6}{5}}|u_n + u|^{\frac{6}{5}}|u_n - u|^{\frac{6}{5}} \,\mathrm{d}x \\
&\quad\le \int_{B_M} |K^2|^{\frac{3}{5}} \left(\int_{\mathbb{R}^3} |u_n + u|^6\right)^{\frac{1}{5}} \left(\int_{\mathbb{R}^3} |u_n - u|^6\right)^{\frac{1}{5}} \,\mathrm{d}x \\
&\quad+ M^{\frac{6}{5}} \left(\int_{B(0,R_\varepsilon)} |u_n + u|^{\frac{12}{5}} \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B(0,R_\varepsilon)} |u_n - u|^{\frac{12}{5}} \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad\le c\varepsilon + o(1).
\end{aligned}$$

Therefore $\phi_{u_n}^K \to \phi_u^K$ in $\mathcal{D}$. And

$$\int_{\mathbb{R}^3} (K(x)\phi_{u_n}^K u_n^2 - K(x)\phi_u^K u^2) \,\mathrm{d}x = \int_{\mathbb{R}^3} K(x)(u_n^2 - u^2))\phi_{u_n}^K \,\mathrm{d}x + \int_{\mathbb{R}^3} K(x)u^2(\phi_{u_n}^K - \phi_u^K) \,\mathrm{d}x.$$

Similar to the proof of the above, we can show that

$$\int_{\mathbb{R}^3} K(x)(u_n^2 - u^2))\phi_{u_n}^K = o(1).$$

Because when $n \to \infty$, $\phi_{u_n}^K \to \phi_u^K$, and by (H_3) we know that

$$\int_{\mathbb{R}^3} K(x)u^2(\phi_{u_n}^K - \phi_u^K) \,\mathrm{d}x \le |K|_2|u^2|_3|\phi_{u_n}^K - \phi_u^K|_6 = o(1).$$

Finally,

$$\int_{\mathbb{R}^3} (K(x)\phi_{u_n}^K u_n\varphi - K(x)\phi_u^K u\varphi) \,\mathrm{d}x = \int_{\mathbb{R}^3} (K(x)\varphi u_n)(\phi_{u_n}^K - \phi_u^K) \,\mathrm{d}x + \int_{\mathbb{R}^3} (K(x)\varphi\phi_u^K)(u_n - u) \,\mathrm{d}x.$$

This can be easily proved similar to the above. □

It is not difficult to verify that the functional I is bounded neither from above nor from below in $H^1(\mathbb{R}^3)$. Indeed, there exists $t \in \mathbb{R}^+$ such that $tu \in H^1(\mathbb{R}^3)$ satisfies

$$I(tu) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (V_\infty + a(x))u^2 \,\mathrm{d}x + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u^K u^2 \,\mathrm{d}x - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3} (Q_\infty - b(x))|u|^{p+1} \,\mathrm{d}x.$$

Since $p>3$, $\lim_{t\to+\infty} I(tu)=-\infty$. On the other hand, for some $\alpha,\beta\in\mathbb{R}$ and for any $t>0$, we have

$$\begin{aligned} I(t^\alpha u(t^\beta x)) &= \frac{1}{2}\int_{\mathbb{R}^3} t^{2\alpha+2\beta}|\nabla u(t^\beta x)|^2+(V_\infty+a(x))t^{2\alpha}|u(t^\beta x)|^2\,\mathrm{d}x \\ &\quad+\frac{t^{4\alpha}}{4}\int_{\mathbb{R}^3} K(x)\phi_u^K u(t^\beta x)^2\,\mathrm{d}x-\frac{t^{(p+1)\alpha}}{p+1}\int_{\mathbb{R}^3}(Q_\infty-b(x))|u(t^\beta x)|^{p+1}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^3} t^{2\alpha+2\beta-3\beta}|\nabla u|^2+\Big(V_\infty+a\Big(\frac{x}{t^\beta}\Big)\Big)t^{2\alpha-3\beta}|u|^2\,\mathrm{d}x \\ &\quad+\frac{t^{4\alpha-3\beta}}{4}\int_{\mathbb{R}^3} K\Big(\frac{x}{t^\beta}\Big)\phi_u^{K(\frac{\cdot}{t^\beta})}u^2\,\mathrm{d}x-\frac{t^{(p+1)\alpha-3\beta}}{p+1}\int_{\mathbb{R}^3}\Big(Q_\infty-b\Big(\frac{x}{t^\beta}\Big)\Big)u^{p+1}\,\mathrm{d}x. \end{aligned}$$

By (H_1)–(H_3), choosing α,β such that $2\alpha-\beta>(p+1)\alpha-3\beta$, that is $2\beta>(p-1)\alpha$. Particularly, we chose that $\alpha=1,\beta=p$, then $\lim_{t\to+\infty} I(tu)=+\infty$.

Naturally, we consider that the functional I restricted in the Nehari manifold $\mathcal{N}$, that contains all the critical points of I, is bounded from below, where

$$\mathcal{N}:=\Big\{u\in H^1(\mathbb{R}^3)\backslash\{0\}:\ I'(u)[u]=0\Big\}.$$

By using a standard argument, we can show the following lemma.

Lemma 2.2. *Suppose that (H_1)–(H_3) hold, the following statements are true:*

(i) *There exists a positive constant $c>0$ such that for all $u\in\mathcal{N}$, there holds*

$$|u|_{p+1}\geq c>0.$$

(ii) *$\mathcal{N}$ is a C^1 regular manifold diffeomorphic to the sphere of $H^1(\mathbb{R}^3)$.*

(iii) *I is bounded from below on $\mathcal{N}$ by a positive constant.*

(iv) *u is a free critical point of I if and only if u is a critical point of I constrained on $\mathcal{N}$.*

Proof. (i) Let $u\in\mathcal{N}$, by (H_1)–(H_3) and Sobolev's embedding theorem, we have that

$$\begin{aligned} c_1|u|_{p+1}^2\leq\|u\|^2 &\leq\|u\|^2+\int_{\mathbb{R}^3}a(x)u^2\,\mathrm{d}x+\int_{\mathbb{R}^3}K(x)\phi_u^K u^2\,\mathrm{d}x \\ &=\int_{\mathbb{R}^3}(Q_\infty-b(x))|u|^{p+1}\,\mathrm{d}x\leq c_2|u|_{p+1}^{p+1}, \end{aligned} \tag{2.3}$$

where $c_1,c_2>0$ independent of u, and owing to $p>3$, this estimate implies that

$$|u|_{p+1}\geq c>0,\qquad \|u\|\geq c>0,\qquad \forall u\in\mathcal{N}, \tag{2.4}$$

where $c>0$ independent of u.

(ii) It suffices to show that $G'(u)[u]<0$ for $u\in\mathcal{N}$, where $G(u)=I'(u)[u]$. Clearly, $G\in C^1(H^1(\mathbb{R}^3),\mathbb{R})$. By (2.4), for any $u\in\mathcal{N}$, we deduce that

$$\begin{aligned} G'(u)[u] &= 2\int_{\mathbb{R}^3}|\nabla u|^2+(V_\infty+a(x))u^2\,\mathrm{d}x+4\int_{\mathbb{R}^3}K(x)\phi_u^K u^2\,\mathrm{d}x \\ &\quad-(p+1)\int_{\mathbb{R}^3}(Q_\infty-b(x))|u|^{p+1}\,\mathrm{d}x \\ &= -(p-1)\int_{\mathbb{R}^3}|\nabla u|^2+(V_\infty+a(x))u^2\,\mathrm{d}x-(p-3)\int_{\mathbb{R}^3}K(x)\phi_u^K u^2\,\mathrm{d}x \\ &\leq -(p-3)C<0, \end{aligned} \tag{2.5}$$

where $C>0$ is dependent of u. By applying the implicit function theorem, we see that $\mathcal{N}$ is of C^1 manifold.

The remaining proofs of (ii), (iii) and (iv) are standard, we omit them. □

Now, the limit equation corresponding to problem (2.2) is defined as

$$-\Delta u + V_\infty u = Q_\infty |u|^{p-1}u \quad \text{in } \mathbb{R}^3. \tag{2.6}$$

The energy functional $I_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ associated to problem (2.6) given by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty |u|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^3} Q_\infty |u|^{p+1}\,\mathrm{d}x$$

and it is easy to verify that $I_\infty \in C^2(H^1(\mathbb{R}^3), \mathbb{R})$. Denote the Nehari manifold of functional I_∞ by

$$\mathcal{N}_\infty := \Big\{u \in H^1(\mathbb{R}^3)\backslash(0) : I'_\infty(u)[u] = 0\Big\}.$$

Set

$$m_\infty := \inf_{u\in\mathcal{N}_\infty} I_\infty(u).$$

$m_\infty > 0$ is achieved by a positive radially symmetric function ω, that is unique (up to translations) positive solution of (2.6) (see [17]), decreasing when the radial coordinate increases and such that

$$\lim_{|x|\to\infty} |D^j\omega(x)||x|e^{\sqrt{V_\infty}|x|} = d_j > 0, \qquad j = 0, 1,\ d_j \in \mathbb{R}. \tag{2.7}$$

Moreover, for any sign-changing critical point u of I_∞, by standard argument, the following inequality holds true

$$I_\infty(u) \geq 2m_\infty. \tag{2.8}$$

Now we are ready to consider the constrained minimization problem $m := \inf\{I(u),\ u \in \mathcal{N}\}$, we find that the relation between least energy m and m_∞ holds and it is not achieved, thus we can not look for the ground state.

Proposition 2.3. *Suppose that (H_1)–(H_3) hold. Then there holds*

$$m = m_\infty \tag{2.9}$$

and the infimum is not achieved.

Proof. Let $u \in \mathcal{N}$, then there exists $t_u > 0$ such that $t_u u \in \mathcal{N}_\infty$. Thus, we deduce that

$$I(u) \geq I(t_u u) \geq \frac{t_u^2}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V_\infty u^2)\,\mathrm{d}x - \frac{t_u^{p+1}}{p+1}\int_{\mathbb{R}^3} Q_\infty |u|^{p+1}\,\mathrm{d}x = I_\infty(t_u u) \geq m_\infty$$

from which, we get $m \geq m_\infty$.

Next, it suffices to find a sequence $(u_n)_n$, $u_n \in \mathcal{N}$, such that $\lim_{n\to\infty} I(u_n) = m$. For this purpose, let us consider $(y_n)_n$, with $y_n \in \mathbb{R}^3$, $|y_n| \to +\infty$ as $n \to \infty$ and set $u_n = t_n\omega_{y_n} = t_n\omega(x - y_n)$, where $t_n = t(\omega_{y_n})$ is such that $u_n = t_n\omega_{y_n} \in \mathcal{N}$. Since

$$\begin{aligned} I(u_n) &= \frac{t_n^2}{2}\int_{\mathbb{R}^3} |\nabla\omega_{y_n}|^2 + (V_\infty + a(x))\omega_{y_n}^2\,\mathrm{d}x + \frac{t_n^4}{4}\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{y_n}}\omega_{y_n}^2\,\mathrm{d}x \\ &\quad - \frac{t_n^{p+1}}{p+1}\int_{\mathbb{R}^3}(Q_\infty - b(x))|\omega_{y_n}|^{p+1}\,\mathrm{d}x \\ &= \frac{t_n^2}{2}\int_{\mathbb{R}^3} |\nabla\omega|^2 + (V_\infty + a(x+y_n))\omega^2\,\mathrm{d}x + \frac{t_n^4}{4}\int_{\mathbb{R}^3} K(x+y_n)\phi_\omega^{K(\cdot+y_n)}\omega^2\,\mathrm{d}x \\ &\quad - \frac{t_n^{p+1}}{p+1}\int_{\mathbb{R}^3}(Q_\infty - b(x+y_n))|\omega|^{p+1}\,\mathrm{d}x, \end{aligned}$$

and from (H_1), (H_2) and (H_3), by Lebesgue dominated theorem, we can deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} a(x+y_n)\omega^2 = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{R}^3} b(x+y_n))|\omega|^{p+1} = 0,$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} K(x+y_n)\phi_\omega^{K(\cdot+y_n)}\omega^2\,\mathrm{d}x = 0.$$

Combining with $t_n\omega_{y_n} \in \mathcal{N}$, we have that $1 \le t_n \le C$, where $C > 0$ is a positive constant. Therefore, from $\omega \in \mathcal{N}_\infty$ and $p \in (3,5)$, it follows that $t_n \to 1$ and thus $I(u_n) \to m_\infty$.

To complete the proof, we argue by contradiction and we assume that $v \in \mathcal{N}$ exists such that $I(v) = m = m_\infty$. Obviously, there exists $t_v > 0$ such that $t_v v \in \mathcal{N}_\infty$, we have that

$$\begin{aligned} m_\infty \le I_\infty(t_v v) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|t_v v\|^2 \\ &\le \left(\frac{1}{2} - \frac{1}{p+1}\right)\|t_v v\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} K(x)\phi_{t_v v}^K (t_v v)^2\,\mathrm{d}x \\ &\le I(t_v v) \le I(v) = m = m_\infty \end{aligned}$$

which implies that $t_v = 1$ and

$$\int_{\mathbb{R}^3} K(x)\phi_v^K v^2\,\mathrm{d}x = 0. \tag{2.10}$$

Hence, $v \in \mathcal{N}_\infty$ and $I_\infty(v) = m_\infty$. By the uniqueness of solution of problem (2.6), there exists $y \in \mathbb{R}^3$ such that $v(x) = \omega(x-y) > 0$, for every $x \in \mathbb{R}^3$, which leads to $\int_{\mathbb{R}^3} K(x)\phi_v^K v^2\,\mathrm{d}x > 0$, contradicts with (2.10). □

In order to find a bound state in higher energy level in $(m_\infty, 2m_\infty)$, the next results help us to recover the compactness of the bounded (PS) sequence in $(m_\infty, 2m_\infty)$. Following the proof of Lemma 4.5 in [3], we can show the following splitting lemma.

Lemma 2.4 (Splitting lemma)**.** *Suppose that* (H_1)–(H_3) *hold. Let* $(u_n)_n$ *be a* (PS) *sequence of* I *constrained on* $\mathcal{N}$*, i.e.* $u_n \in \mathcal{N}$*, and* $I(u_n)$ *is bounded,* $\nabla I|_{\mathcal{N}}(u_n) \to 0$ *strongly in* $H^1(\mathbb{R}^3)$*. Then, up to a subsequence, there exist a solution* $\bar{u}$ *of* (2.2)*, a number* $k \in \mathbb{N} \cup \{0\}$*,* k *functions* $u^1, \ldots, u^k$ *of* $H^1(\mathbb{R}^3)$ *and* k *sequences of points* $(y_n^j)_n$*,* $y_n^j \in \mathbb{R}^3$*,* $0 \le j \le k$ *such that, as* $n \to +\infty$*,*

- (i) $u_n - \sum_{j=1}^k u^j(\cdot - y_n^j) \to \bar{u}$ *in* $H^1(\mathbb{R}^3)$;
- (ii) $I(u_n) \to I(\bar{u}) + \sum_{j=1}^k I_\infty(u^j)$;
- (iii) $|y_n^j| \to +\infty$, $|y_n^i - y_n^j| \to +\infty$ *if* $i \ne j$;
- (iv) u^j *are weak solution of* (2.6).

Moreover, in the case $k = 0$*, the above holds without* u^j*.*

In the end of this section, we recall a technical result for some estimates in the next section, its proof is found in [4,12].

Lemma 2.5. *If* $g \in L^\infty(\mathbb{R}^3)$ *and* $h \in L^1(\mathbb{R}^3)$ *are such that, for some* $\alpha \ge 0$, $b \ge 0$, $\gamma \in \mathbb{R}$

$$\lim_{|x|\to+\infty} g(x)e^{\alpha|x|}|x|^b = \gamma \quad \text{and} \quad \int_{\mathbb{R}^3} |h(x)|e^{\alpha|x|}|x|^b\,\mathrm{d}x < +\infty,$$

then, for every $z \in \mathbb{R}^3 \setminus \{0\}$*,*

$$\lim_{\rho\to+\infty}\left(\int_{\mathbb{R}^3} g(x+\rho z)h(x)\,\mathrm{d}x\right)e^{\alpha|\rho z|}|\rho z|^b = \gamma\int_{\mathbb{R}^3} h(x)e^{-\alpha(x\cdot z)/|z|}\,\mathrm{d}x.$$

3 Proof of Theorem 1.1

Now, we turn to build tools and topological techniques to prove the existence of an higher energy solution when (1.5) has no ground state solution. First we recall the definition of barycenter $\beta : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ of a function $u \in H^1(R^3)$, $u \neq 0$, given in [13], set

$$\mu(u)(x) = \frac{1}{|B_1(0)|} \int_{B_1(x)} |u(y)| dy,$$

then $\mu(u) \in L^\infty(\mathbb{R}^3)$ and is continuous in $H^1(\mathbb{R}^3)$. Let

$$\hat{u}(x) = \left[\mu(u)(x) - \frac{1}{2} \max \mu(u)(x) \right]^+,$$

it is easy to check that $\hat{u} \in C_0(\mathbb{R}^3)$ and then define $\beta : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ as follows

$$\beta(u) = \frac{1}{|\hat{u}|_1} \int_{\mathbb{R}^3} x \hat{u}(x) \, \mathrm{d}x \in \mathbb{R}^3.$$

Since $\hat{u}$ has compact support, β is well defined and it is easy to verify the following properties:

(1) β is continuous in $H^1(\mathbb{R}^3) \setminus \{0\}$;

(2) if u is a radial function, $\beta(u) = 0$;

(3) for all $t \neq 0$ and for all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, $\beta(tu) = \beta(u)$;

(4) given $z \in \mathbb{R}^3$ and setting $u_z(x) = u(x - z)$, $\beta(u_z) = \beta(u) + z$.

By Proposition 2.3, we see that m can not be achieved, with the help of the barycenter mapping β, we can add some refined constraint in the Nehari manifold $\mathcal{N}$. For this purpose, define the following minimization problem

$$\mathcal{B}_0 := \inf \{ I(u) : u \in \mathcal{N}, \beta(u) = 0 \}.$$

Clearly, we have $m = m_\infty \leq \mathcal{B}_0$. Furthermore, the strict inequality holds true.

Lemma 3.1. *Suppose that (H_1)–(H_3) hold. Then*

$$m = m_\infty < \mathcal{B}_0$$

Proof. By contradiction, we assume that $\mathcal{B}_0 = m_\infty$, then there exists $u_n \in \mathcal{N}$ such that $\beta(u_n) = 0$, $I(u_n) \to m_\infty$ and $\nabla I|_{\mathcal{N}}(u_n) \to 0$. By the Ekeland's variational principle (see Theorem 8.15 in [24]), a sequence of $(v_n)_n \in H^1(\mathbb{R}^3)$ exists so that

$$v_n \in \mathcal{N}, \qquad I(v_n) = m_\infty + o_n(1) \quad \text{and} \quad |\beta(v_n) - \beta(u_n)| = o(1). \tag{3.1}$$

By Lemma 2.4, we have that

$$m_\infty = I(u_n) + o(1) = I(\overline{u}) + \Sigma_{j=1}^k I_\infty(u^j) + o(1).$$

Owing to $I(\overline{u}) \geq m = m_\infty$ and $I_\infty(u^j) \geq m_\infty$, we have that $k = 0$. Thus, $v_n \to \overline{u}$. Since $\overline{u}$ is a nontrivial solution of (1.5), we deduce that

$$I(\overline{u}) = m_\infty, \qquad I'(\overline{u}) = 0, \qquad \beta(\overline{u}) = 0,$$

which means that $m = m_\infty$ is achieved, contradicts with Proposition 2.3. The proof is completed. □

Lemma 3.2. *The functional I constrained on $\mathcal{N}$ satisfies the Palais–Smale condition in $(m_\infty, 2m_\infty)$.*

Proof. Let $\{u_n\}$ be a Palais–Smale sequence of $I|_{\mathcal{N}}$ such that $I(u_n) \to c \in (m_\infty, 2m_\infty)$. By Lemma 2.4, we have that

$$c = I(u_n) + o(1) = I(\overline{u}) + \Sigma_{j=1}^{k} I_\infty(u^j) + o(1).$$

The conclusion follows observing that any critical point $\bar{u}$ of I is such that $I(\bar{u}) \geq m = m_\infty$, any solution of (2.6) verifies that $I_\infty(u) \geq m_\infty$ and if it changes sign, $I_\infty(u) \geq 2m_\infty$. Whatever any case, we can obtain the sequence $\{u_n\}$ strongly convergence in $H^1(\mathbb{R}^3)$. The compactness is proved. □

Let $\xi \in \mathbb{R}^3$ with $|\xi| = 1$ and $\Sigma = \{z \in \mathbb{R}^3 : |z - \xi| = 2\}$. For $\rho > 0$ and $(z, s) \in \Sigma \times [0, 1]$, define

$$\overline{\psi}_\rho[z,s](x) := (1-s)\omega_{\rho z}(x) + s\omega_{\rho\xi}(x) = (1-s)\omega(x-\rho z) + s\omega(x-\rho\xi), \qquad x \in \mathbb{R}^3,$$

where w is a unique radically symmetric positive solution of problem (2.6), then by virtue of standard argument, there exist positive numbers $t_{\rho,z,s} := t_{\overline{\psi}_\rho[z,s]}$ and $\tau_{\rho,z,s} := \tau_{\overline{\psi}_\rho[z,s]}$ such that

$$\psi_\rho[z,s] = t_{\rho,z,s}\overline{\psi}_\rho[z,s] \in \mathcal{N}, \qquad \psi_{\infty,\rho}[z,s] = \tau_{\rho,z,s}\overline{\psi}_\rho[z,s] \in \mathcal{N}_\infty. \tag{3.2}$$

Remark 3.3. Note that $\overline{\psi}_\rho[z,s] \to \omega(x-\rho z)$ as $s \to 0$ and $\overline{\psi}_\rho[z,s] \to \omega(x-\rho\xi)$ as $s \to 1$, moreover, $\tau_{\rho,z,s} \to 1$ as $s \to 0$ or $s \to 1$ due to $\omega(x-\rho z) \in \mathcal{N}_\infty$ and $\omega(x-\rho\xi) \in \mathcal{N}_\infty$.

Lemma 3.4. *For all $\rho > 0$, we have*

$$\mathcal{B}_0 \leq \mathcal{T}_\rho := \max_{\Sigma\times[0,1]} I(\psi_\rho[z,s]).$$

Proof. Observing that $\beta(\psi_\rho[z,0]) = \beta(t_{\rho,z,0}\overline{\psi}_\rho[z,0]) = \beta(t_{\rho,z,0}\omega_{\rho z}) = \beta(\omega_{\rho z}) = \rho z$ and $\beta(\psi_\rho[z,1]) = \rho\xi$. Let

$$\mathcal{G}(z,s) = s\rho\xi + (1-s)\rho z, \tag{3.3}$$

then $\mathcal{G}(z,s) \in C(\Sigma \times (0,1])$. Define a mapping by

$$h(t,z,s) = t\mathcal{G}(z,s) + (1-t)\beta(\psi_\rho[z,s]), \qquad \forall t \in [0,1], \tag{3.4}$$

then $h(t,z,s) \in C([0,1] \times \overline{\Sigma \times (0,1]})$ is continuous and

$$h(t,z,0) = t\rho z + (1-t)\beta(\psi_\rho[z,0]) = t\rho z + (1-t)\rho z = \rho z \neq 0, \qquad \forall z \in \Sigma$$

and

$$h(t,z,1) = t\rho\xi + (1-t)\beta(\psi_\rho[z,1]) = t\rho\xi + (1-t)\rho\xi = \rho\xi \neq 0, \qquad \forall z \in \Sigma$$

which implies that $0 \notin h(t, \partial(\Sigma \times (0,1]))$, for every $t \in [0,1]$. Therefore, by the homotopical invariance of Brouwer degree, we get $\deg(h(t), \Sigma \times (0,1], 0) = constant$. Thus

$$\deg(h(0), \Sigma \times (0,1], 0) = \deg(h(1), \Sigma \times (0,1], 0),$$

that is

$$\deg(\beta(\psi_\rho[z,s]), \Sigma \times (0,1], 0) = \deg(\mathcal{G}(s,z), \Sigma \times (0,1], 0).$$

Clearly, $\deg(\mathcal{G}(z,s), \Sigma \times (0,1], 0) \neq 0$. Thus, it follows from the solvable property of Brouwer degree that there exists $(\overline{z}, \overline{s}) \in \Sigma \times (0,1]$ such that $\beta(\psi_\rho[\overline{z}, \overline{s}]) = 0$. Therefore, by the definition of $\mathcal{B}_0$, we have that

$$\mathcal{B}_0 \leq I(\psi_\rho[\overline{z}, \overline{s}]) \leq \mathcal{T}_\rho.$$

□

In order to show $\mathcal{T}_\rho < 2m_\infty$, we have to give some estimates from the decay of ω and coefficients $a(x)$, $K(x)$ and $b(x)$.

Lemma 3.5. *Suppose that (H_3) holds. There exists $c > 0$ such that for all $\rho > 3R_0$, the following holds*

$$\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho\zeta}}\omega^2_{\rho\zeta}\, dx \leq c e^{-\frac{4}{3}\sqrt{V_\infty}\rho}, \quad \textit{for all } \zeta \in \mathbb{R}^3 \textit{ with } |\zeta| \geq 1. \tag{3.5}$$

Proof. Let $\rho > 3R_0$, then $\frac{1}{3}\rho > R_0$ and if $|y| \le \frac{1}{3}\rho$ and $|\zeta| \ge 1$, we have that $|y-\rho\zeta| \ge \rho|\zeta| - |y| \ge \rho - \frac{1}{3}\rho = \frac{2}{3}\rho$. Thus, by the exponential decay (2.7) of ω, $\frac{1-e^{-|x|}}{|x|} \in L^s(\mathbb{R}^3)$ for all $s \in (3,+\infty]$ (see Lemma 3.3 in [3]), Hölder's inequality and (H_3), we deduce that

$$
\begin{aligned}
\phi^K_{\omega_{\rho\zeta}}(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1-e^{-|x-y|}}{|x-y|}K(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y\\
&= \frac{1}{4\pi}\left(\int_{|y|\le\frac{1}{3}\rho}\frac{1-e^{-|x-y|}}{|x-y|}K(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y + \int_{|y|>\frac{1}{3}\rho}\frac{1-e^{-|x-y|}}{|x-y|}K(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y\right)\\
&\le \frac{1}{4\pi}\left[\left(\int_{|y|\le\frac{1}{3}\rho}K^2(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{|y|\le\frac{1}{3}\rho}\left(\frac{1-e^{-|x-y|}}{|x-y|}\right)^4\mathrm{d}y\right)^{\frac{1}{4}}\left(\int_{|y|\le\frac{1}{3}\rho}\omega^4(y-\rho\zeta)\,\mathrm{d}y\right)^{\frac{1}{4}}\right.\\
&\qquad\left.+\int_{|y|>\frac{1}{3}\rho}\frac{1-e^{-|x-y|}}{|x-y|}K(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y\right]\\
&\le c\left(e^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(\int_{|y|\le(\frac{1}{2}-q)\rho}K^2(y)\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\left(\frac{1-e^{-|x-y|}}{|x-y|}\right)^4\mathrm{d}y\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}^3}\omega^4(y-\rho\zeta)\,\mathrm{d}y\right)^{\frac{1}{4}}\right.\\
&\qquad\left.+e^{-\frac{2}{3}\sqrt{V_\infty}\rho}\int_{|y|>\frac{1}{3}\rho}\frac{1-e^{-|x-y|}}{|x-y|}\omega^2_{\rho\zeta}\,\mathrm{d}y\right)\\
&\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(C+\int_{|y|>\frac{1}{3}\rho}\frac{1-e^{-|x-y|}}{|x-y|}\omega^2_{\rho\zeta}\,\mathrm{d}y\right)\\
&\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}.
\end{aligned}
$$

Thus, a similar computation gives

$$
\begin{aligned}
\int_{\mathbb{R}^3}K(x)\phi^K_{\omega_{\rho\zeta}}(x)\omega^2_{\rho\zeta}\,\mathrm{d}x &\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\int_{\mathbb{R}^3}K(x)\omega^2_{\rho\zeta}\,\mathrm{d}x\\
&= ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(\int_{|x|\le\frac{1}{3}\rho}K(x)\omega^2_{\rho\zeta}\,\mathrm{d}x + \int_{|x|>\frac{1}{3}\rho}K(x)\omega^2_{\rho\zeta}\,\mathrm{d}x\right)\\
&\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left[\left(\int_{|x|\le\frac{1}{3}\rho}K^2(x)\omega^2_{\rho\zeta}\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{|x|\le\frac{1}{3}\rho}\omega^2_{\rho\zeta}\,\mathrm{d}x\right)^{\frac{1}{2}} + \int_{|x|>\frac{1}{3}\rho}K(x)\omega^2_{\rho\zeta}\,\mathrm{d}x\right)\\
&\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}e^{-\frac{2}{3}\sqrt{V_\infty}\rho} = ce^{-\frac{4}{3}\sqrt{V_\infty}\rho}.
\end{aligned}
$$

□

From (3.5), it is easy to very that

$$
\int_{\mathbb{R}^3}K(x)\phi^K_{\omega_{\rho\xi}}\omega^2_{\rho\xi}\,\mathrm{d}x = o(\rho^{-1}e^{-\sqrt{V_\infty}\rho}), \qquad \int_{\mathbb{R}^3}K(x)\phi^K_{\omega_{\rho z}}\omega^2_{\rho z}\,\mathrm{d}x = o(\rho^{-1}e^{-\sqrt{V_\infty}\rho}) \tag{3.6}
$$

and we denote $\varepsilon_\rho = \rho^{-1}e^{-\sqrt{V_\infty}\rho}$ for convenience. Moreover, we can obtain the following estimate:

Lemma 3.6.

$$
\int_{\mathbb{R}^3}K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}^2_\rho[z,s]\,\mathrm{d}x = o(\varepsilon_\rho), \qquad \forall s\in[0,1] \text{ and } z\in\Sigma. \tag{3.7}
$$

Proof. Since

$$
\phi^K_{\overline{\psi}_\rho[z,s]} = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1-e^{-|x-y|}}{|x-y|}K(y)\overline{\psi}^2_\rho[z,s](y)\,\mathrm{d}y \le 2\phi^K_{\omega_{\rho z}} + 2\phi^K_{\omega_{\rho\xi}}
$$

and then

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}^2_\rho[z,s]\,\mathrm{d}x &\le 2\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}(\omega^2_{\rho\xi}+\omega^2_{\rho z})\,\mathrm{d}x\\ &\le 4\int_{\mathbb{R}^3} K(x)(\phi^K_{\omega_{\rho z}}+\phi^K_{\omega_{\rho\xi}})(\omega^2_{\rho\xi}+\omega^2_{\rho z})\,\mathrm{d}x\\ &= 4\int_{\mathbb{R}^3} K(x)(\phi^K_{\omega_{\rho z}}\omega^2_{\rho z}+\phi^K_{\omega_{\rho\xi}}\omega^2_{\rho z}+\phi^K_{\omega_{\rho z}}\omega^2_{\rho\xi}+\phi^K_{\omega_{\rho\xi}}\omega^2_{\rho z})\,\mathrm{d}x.\end{aligned}$$

Now, similar to the proof of Lemma 3.5, we have that

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho z}}\omega^2_{\rho\xi}(x)\,\mathrm{d}x \le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\int_{\mathbb{R}^3} K(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x\\ &= ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(\int_{|x|\le\frac{1}{3}\rho} K(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x+\int_{|x|>\frac{1}{3}\rho} K(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x\right)\\ &\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(\left(\int_{|x|\le\frac{1}{3}\rho} K^2(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{|x|\le\frac{1}{3}\rho}\omega^2_{\rho\xi}(x)\,\mathrm{d}x\right)^{\frac{1}{2}}+\int_{|x|>\frac{1}{3}\rho} K(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x\right)\\ &\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}e^{-\frac{2}{3}\sqrt{V_\infty}\rho}=ce^{-\frac{4}{3}\sqrt{V_\infty}\rho}.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho z}}\omega^2_{\rho\xi}(x)\,\mathrm{d}x=o(\varepsilon_\rho),$$

and the same argument leads to

$$\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho\xi}}\omega^2_{\rho z}(x)\,\mathrm{d}x=o(\varepsilon_\rho).$$

Therefore, from (3.6), the estimate (3.7) follows. □

Next, we give some estimates which are used in the sequel.

Lemma 3.7. *The following estimates hold:*

$$\int_{\mathbb{R}^3}\omega^p_{\rho z}\omega_{\rho\xi}\,\mathrm{d}x=O(\widetilde{\varepsilon_\rho})=o(\varepsilon_\rho),\qquad \int_{\mathbb{R}^3}\omega^p_{\rho\xi}\omega_{\rho z}\,\mathrm{d}x=O(\widetilde{\varepsilon_\rho})=o(\varepsilon_\rho),\tag{3.8}$$

$$\int_{\mathbb{R}^3} a(x)\omega^2_{\rho z}\,\mathrm{d}x=o(\varepsilon_\rho),\qquad \int_{\mathbb{R}^3} a(x)\omega^2_{\rho\xi}\,\mathrm{d}x=o(\varepsilon_\rho),\qquad \int_{\mathbb{R}^3} a(x)\overline{\psi}^2_\rho[z,s]\,\mathrm{d}x=o(\varepsilon_\rho),\tag{3.9}$$

$$\int_{\mathbb{R}^3} b(x)|\overline{\psi}_\rho[z,s]|^{p+1}\,\mathrm{d}x=o(\varepsilon_\rho),\tag{3.10}$$

where $\widetilde{\varepsilon_\rho}=\rho^{-2}e^{-2\sqrt{V_\infty}\rho}$.

Proof. (i) By (2.7), we can deduce that

$$\begin{aligned}\int_{\mathbb{R}^3}\omega^p_{\rho\xi}\omega_{\rho z}\,\mathrm{d}x &= \int_{\mathbb{R}^3}\omega^p(x-\rho\xi)\omega(x-\rho z)\,\mathrm{d}x=\int_{\mathbb{R}^3}\omega^p(y)\omega(y+\rho(\xi-z))\,\mathrm{d}y\\ &\sim c\int_{\mathbb{R}^3}|y+\rho(\xi-z)|^{-1}e^{-\sqrt{V_\infty}|y+\rho(\xi-z)|}\omega^p(y)\,dy\end{aligned}$$

In order to apply Lemma 2.5, let us set $h(x)=\omega^p(x)\in L^1(\mathbb{R}^3)$, $g(x)=|x|^{-1}e^{-\sqrt{V_\infty}|x|}$, taking $\alpha=\sqrt{V_\infty}$ and $b=1$, clearly,

$$\lim_{|x|\to+\infty} g(x)|x|e^{\sqrt{V_\infty}|x|}=1,$$

$$\int_{\mathbb{R}^3}\omega^p(x)e^{\sqrt{V_\infty}|x|}|x|\,dx\le c\int_{\mathbb{R}^3}e^{-(p-1)\sqrt{V_\infty}|x|}|x|^{-(p-1)}\,dx<+\infty.$$

By using Lemma 2.5 and $z \in \Sigma$ ($|\xi - z| = 2$), we get that

$$\begin{aligned}\lim_{\rho\to+\infty} e^{\sqrt{V_\infty}|\rho(\xi-z)|}|\rho(\xi-z)| \int_{\mathbb{R}^3} g(x+\rho(\xi-z))h(x)\,\mathrm{d}x \\ = 2\lim_{\rho\to+\infty} e^{2\sqrt{V_\infty}\rho}\rho \int_{\mathbb{R}^3} g(x+\rho(\xi-z))h(x)\,\mathrm{d}x \\ = \int_{\mathbb{R}^3} \omega^p(x) e^{-\sqrt{V_\infty}\frac{x\cdot(\xi-z)}{|\xi-z|}}\,\mathrm{d}x = c_1,\end{aligned}$$

which means that

$$\lim_{\rho\to+\infty} \rho^2 e^{2\sqrt{V_\infty}\rho} \int_{\mathbb{R}^3} g(x+\rho(\xi-z))h(x)\,\mathrm{d}x = c_1. \tag{3.11}$$

This means that

$$\int_{\mathbb{R}^3} \omega^p_{\rho\xi}\omega_{\rho z}\,\mathrm{d}x = O(\tilde{\varepsilon_\rho}) = o(\varepsilon_\rho).$$

Similar argument as above we can show that

$$\int_{\mathbb{R}^3} \omega^p_{\rho z}\omega_{\rho\xi}\,\mathrm{d}x = O(\tilde{\varepsilon_\rho}) = o(\varepsilon_\rho).$$

(ii) By Hölder's inequality, we have that

$$\begin{aligned}\int_{\mathbb{R}^3} a(x)\omega^2_{\rho\xi}\,\mathrm{d}x = \int_{\mathbb{R}^3} a(x)\omega^2(x-\rho\xi)\,\mathrm{d}x \le \left(\int_{\mathbb{R}^3}\omega^2(x-\rho\xi)\,\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}\omega^2(x-\rho\xi)\,\mathrm{d}x\right)^{\frac{2}{3}} \\ \le C\left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}\omega^2(x-\rho\xi)\,\mathrm{d}x\right)^{\frac{2}{3}} \\ \sim \left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}|x-\rho\xi|^{-2}e^{-2\sqrt{V_\infty}|x-\rho\xi|}\,\mathrm{d}x\right)^{\frac{2}{3}}.\end{aligned}$$

Taking $\alpha = 2\sqrt{V_\infty}$, $b = 2$, $h(x) = a^{\frac{3}{2}}(x)$, $g(x) = |x|^{-2}e^{-2\sqrt{V_\infty}|x|}$ in Lemma 2.5, by (H_4), it is easy to see that

$$\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)|x|^2 e^{2\sqrt{V_\infty}|x|}\,\mathrm{d}x < +\infty \quad \text{and} \quad \lim_{|x|\to+\infty} g(x)|x|^2 e^{2\sqrt{V_\infty}|x|} = 1.$$

Thus

$$\begin{aligned}\lim_{\rho\to+\infty} e^{2\sqrt{V_\infty}|-\rho\xi|}|-\rho\xi|^2\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho\xi)\,\mathrm{d}x = \lim_{\rho\to+\infty}\rho^2 e^{2\sqrt{V_\infty}\rho}\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho\xi)\,\mathrm{d}x \\ = \int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)e^{-2\sqrt{V_\infty}\frac{x\cdot(-\xi)}{|\xi|}}\,\mathrm{d}x = c_2,\end{aligned}$$

which yields that

$$\lim_{\rho\to+\infty}\left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho\xi)\,\mathrm{d}x\right)^{\frac{2}{3}} = O(\rho^{-\frac{4}{3}}e^{-\frac{4}{3}\sqrt{V_\infty}\rho}) = o(\rho^{-1}e^{-\sqrt{V_\infty}\rho}) = o(\varepsilon_\rho).$$

Thus, we conclude that

$$\int_{\mathbb{R}^3} a(x)\omega^2_{\rho\xi}\,\mathrm{d}x = o(\varepsilon_\rho).$$

Note that $z \in \Sigma$ implies that $1 \le |z| \le 3$, parallel to the above argument, we can get that

$$\lim_{\rho\to+\infty}\rho^2|z|^2 e^{2\sqrt{V_\infty}\rho|z|}\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho z)\,\mathrm{d}x = \int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)e^{-2\sqrt{V_\infty}\frac{x\cdot(-z)}{|z|}}\,\mathrm{d}x = c_3,$$

which leads to

$$\lim_{\rho\to+\infty}\left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho z)\,\mathrm{d}x\right)^{\frac{2}{3}} = O(\rho^{-\frac{4}{3}}e^{-\frac{4}{3}\sqrt{V_\infty}\rho}) = o(\rho^{-1}e^{-\sqrt{V_\infty}\rho}) = o(\varepsilon_\rho).$$

Thus, it follows that

$$\int_{\mathbb{R}^3} a(x)\omega_{\rho z}^2 \,\mathrm{d}x = o(\varepsilon_\rho).$$

Since

$$\int_{\mathbb{R}^3} a(x)\overline{\psi}_\rho^2[z,s] \,\mathrm{d}x = \int_{\mathbb{R}^3} a(x)[(1-s)\omega_{\rho z} + s\omega_{\rho\xi}]^2 \,\mathrm{d}x \le 2\int_{\mathbb{R}^3} a(x)[\omega_{\rho z}^2 + \omega_{\rho\xi}^2] \,\mathrm{d}x,$$

according to the two estimates which have been proved, it is easily to get

$$\int_{\mathbb{R}^3} a(x)\overline{\psi}_\rho^2[z,s] \,\mathrm{d}x = o(\varepsilon_\rho).$$

(iii) By (H_2), we can easily check that $b(x)\omega_{\rho\xi}^{p-1} \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $b(x)\omega_{\rho z}^{p-1} \in L^{\frac{3}{2}}(\mathbb{R}^3)$. In view of (H_4), $b(x) \in L^\infty(\mathbb{R}^3)$ and $\omega(x) \in L^\infty(\mathbb{R}^3)$, we have that

$$\int_{\mathbb{R}^3} b^{\frac{3}{2}}(x)\omega_{\rho z}^{\frac{3(p-1)}{2}} |x|^2 e^{2\sqrt{V_\infty}|x|} \,\mathrm{d}x \le C\int_{\mathbb{R}^3} b(x)|x|^2 e^{2\sqrt{V_\infty}|x|} \,\mathrm{d}x < +\infty.$$

Similar argument as the proof of (ii), we can show that

$$\int_{\mathbb{R}^3} b(x)\omega_{\rho\xi}^{p+1} \,\mathrm{d}x = o(\varepsilon_\rho), \qquad \int_{\mathbb{R}^3} b(x)\omega_{\rho z}^{p+1} \,\mathrm{d}x = o(\varepsilon_\rho).$$

Owing to

$$\begin{aligned}\int_{\mathbb{R}^3} b(x)|\overline{\psi}_\rho[z,s]|^{p+1} \,\mathrm{d}x &= \int_{\mathbb{R}^3} b(x)[(1-s)\omega_{\rho z} + s\omega_{\rho\xi}]^{p+1} \,\mathrm{d}x \\ &\le 2^p \int_{\mathbb{R}^3} b(x)(\omega_{\rho z}^{p+1} + \omega_{\rho\xi}^{p+1}) \,\mathrm{d}x,\end{aligned}$$

we conclude that (3.10) follows. □

Lemma 3.8. *Suppose that (H_1)–(H_4) hold. Let $t_{\rho,z,s}$ and $\tau_{\rho,z,s}$ be the number defined in (3.2). Then there exists a constant $C > 0$ such that*

$$0 < t_{\rho,z,s} \le C, \qquad \forall \rho > 0,\ \forall (z,s) \in \Sigma \times [0,1]. \tag{3.12}$$

Moreover,

$$t_{\rho,z,s} = \tau_{\rho,z,s} + o(\varepsilon_\rho). \tag{3.13}$$

Proof. Observing that

$$0 < \frac{1}{2}\left(\int_{B_1(0)} |\nabla\omega|^2 + \omega^2\right)^{\frac{1}{2}} \le \|\overline{\psi}_\rho[z,s]\| \le \|\omega_{\rho z}\| + \|\omega_{\rho\xi}\| = 2\|\omega\| \tag{3.14}$$

and

$$0 < \frac{1}{2}\left(\int_{B_1(0)} \omega^{p+1}\right)^{\frac{1}{p+1}} \le |\overline{\psi}_\rho[z,s]|_{p+1} \le 2^p(|\omega_{\rho z}|_{p+1} + |\omega_{\rho\xi}|_{p+1}) = 2^{p+1}|\omega|_{p+1}. \tag{3.15}$$

Since $t_{\rho,z,s}\overline{\psi}_\rho[z,s] \in \mathcal{N}$, we have that

$$\begin{aligned}t_{\rho,z,s}^2 \left(\|\overline{\psi}_\rho[z,s]\|^2 + \int_{\mathbb{R}^3} a(x)\overline{\psi}_\rho^2[z,s] \,\mathrm{d}x\right) + t_{\rho,z,s}^4 \int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}_\rho^2[z,s] \,\mathrm{d}x \\ - t_{\rho,z,s}^{p+1}\int_{\mathbb{R}^3} (Q_\infty - b(x))|\overline{\psi}_\rho[z,s]|^{p+1} \,\mathrm{d}x = 0.\end{aligned} \tag{3.16}$$

By (3.14), (3.15) and (3.16), it is easy to check that (3.12) holds. Thus, by (3.6), we have that

$$t_{\rho,z,s}^2 \int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}_\rho^2[z,s] \,\mathrm{d}x = o(\varepsilon_\rho). \tag{3.17}$$

By (3.16), we have that

$$t_{\rho,z,s}^{p-1} = \frac{\|\overline{\psi}_\rho[z,s]\|^2}{\int_{\mathbb{R}^3}(Q_\infty - b(x))|\overline{\psi}_\rho[z,s]|^{p+1}\,\mathrm{d}x} + \frac{\int_{\mathbb{R}^3} a(x)\overline{\psi}_\rho^2[z,s]\,\mathrm{d}x + t_{\rho,z,s}^2\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}_\rho^2[z,s]\,\mathrm{d}x}{\int_{\mathbb{R}^3}(Q_\infty - b(x))|\overline{\psi}_\rho[z,s]|^{p+1}\,\mathrm{d}x}.$$

Therefore, by (3.17), (3.9), (3.10) and $\tau_{\rho,z,s}\overline{\psi}_\rho[z,s] \in \mathcal{N}_\infty$, we deduce that

$$t_{\rho,z,s}^{p-1} = \frac{\|\overline{\psi}_\rho[z,s]\|^2}{\int_{\mathbb{R}^3} Q_\infty|\overline{\psi}_\rho[z,s]|^{p+1}\,\mathrm{d}x} + o(\varepsilon_\rho) = \tau_{\rho,z,s}^{p-1} + o(\varepsilon_\rho)$$

which yields the conclusion. The proof is completed. □

Lemma 3.9. *Suppose that (H_1)–(H_4) hold. Then there exists ρ_0 such that, for all $\rho > \rho_0$,*

$$\mathcal{T}_\rho = \max_{\Sigma\times[0,1]} I(\psi_\rho[z,s]) < 2m_\infty.$$

Proof. By (3.2), (3.9), (3.13), (3.14) and (3.17), for any $(z,s) \in \Sigma\times[0,1]$, we deduce that

$$\begin{aligned}
I(\psi_\rho[z,s]) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|t_{\rho,z,s}\overline{\psi}_\rho[z,s]\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)t_{\rho,z,s}^2\int_{\mathbb{R}^3} a(x)|\overline{\psi}_\rho[z,s]|^2\,\mathrm{d}x \\
&\quad + \left(\frac{1}{4} - \frac{1}{p+1}\right)t_{\rho,z,s}^4\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}|\overline{\psi}_\rho[z,s]|^2\,\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|t_{\rho,z,s}\overline{\psi}_\rho[z,s]\|^2 + o(\varepsilon_\rho) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\tau_{\rho,z,s}\overline{\psi}_\rho[z,s]\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)(t_{\rho,z,s}^2 - \tau_{\rho,z,s}^2)\|\overline{\psi}_\rho[z,s]\|^2 + o(\varepsilon_\rho) \qquad (3.18)\\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\tau_{\rho,z,s}\overline{\psi}_\rho[z,s]\|^2 + o(\varepsilon_\rho) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|\overline{\psi}_\rho[z,s]\|^2}{|\overline{\psi}_\rho[z,s]|_{p+1}^2}\right)^{\frac{p+1}{p-1}} + o(\varepsilon_\rho) \\
&= I_\infty(\psi_{\infty,\rho}[z,s]) + o(\varepsilon_\rho)
\end{aligned}$$

By direction computation, we have that

$$\|\overline{\psi}_\rho[z,s]\|^2 = (\overline{\psi}_\rho[z,s],\overline{\psi}_\rho[z,s])_{H^1(\mathbb{R}^3)} = [(1-s)^2 + s^2]\|\omega\|^2 + 2s(1-s)(\omega_{\rho\xi},\omega_{\rho z})_{H^1(\mathbb{R}^3)}. \qquad (3.19)$$

Since $\omega_{\rho\xi}$ is a positive solution of problem (2.6), it follows that

$$(\omega_{\rho\xi},\omega_{\rho z})_{H^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3}\omega_{\rho\xi}^p\omega_{\rho z}\,\mathrm{d}x := A_\rho.$$

By (3.19), we see that

$$\|\overline{\psi}_\rho[z,s]\|^2 = (\overline{\psi}_\rho[z,s],\overline{\psi}_\rho[z,s])_{H^1(\mathbb{R}^3)} = [(1-s)^2 + s^2]\|\omega\|^2 + 2s(1-s)A_\rho. \qquad (3.20)$$

According to the following equality:

$$(a+b)^{p+1} \geq a^{p+1} + b^{p+1} + (p+1)(a^pb + ab^p), \quad \text{for all } a,b \in \mathbb{R}^+ \text{ and } p \geq 2,$$

we have that

$$\begin{aligned}
|\overline{\psi}_\rho[z,s]|_{p+1}^{p+1} &= \int_{\mathbb{R}^3}[(1-s)\omega_{\rho z} + s\omega_{\rho\xi}]^{p+1} \\
&\geq [(1-s)^{p+1} + s^{p+1}]|\omega|_{p+1}^{p+1} + (p+1)[(1-s)^ps + (1-s)s^p]A_\rho.
\end{aligned} \qquad (3.21)$$

When s or $(1-s)$ is small enough, $\psi_{\infty,\rho}[z,s]$ tends to $\omega_{\rho z}$ or $\omega_{\rho\xi}$. Then

$$I_\infty(\psi_{\infty,\rho}[z,s]) \to m_\infty.$$

Therefore there exists $\delta > 0$ such that for $\min\{s, 1-s\} \le \delta$,

$$I_\infty(\psi_{\infty,\rho}[z,s]) < 2m_\infty.$$

In what follows we assume that $\min\{s, 1-s\} \ge \delta$, by virtue of (3.20) and (3.21), we get that

$$\begin{aligned}
\frac{\|\overline{\psi}_\rho[z,s]\|^2}{|\overline{\psi}_\rho[z,s]|_{p+1}^2} &\le \frac{[(1-s)^2+s^2]\|\omega\|^2 + 2s(1-s)A_\rho}{([(1-s)^{p+1}+s^{p+1}]|\omega|_{p+1}^{p+1} + (p+1)[(1-s)^p s + (1-s)s^p]A_\rho)^{\frac{2}{p+1}}} \\
&= \frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}} \frac{\|\omega\|^2}{|\omega|_{p+1}^2} \times \frac{1 + \frac{2s(1-s)}{((1-s)^2+s^2)\|\omega\|^2} A_\rho}{\left(1 + \frac{(p+1)((1-s)^p s + (1-s)s^p)}{(1-s)^{p+1}+s^{p+1}} \frac{A_\rho}{|\omega|_{p+1}^{p+1}}\right)^{\frac{2}{p+1}}} \\
&\le \frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}} \frac{\|\omega\|^2}{|\omega|_{p+1}^2} \frac{1 + \frac{2s(1-s)}{(1-s)^2+s^2} \frac{A_\rho}{\|\omega\|^2}}{1 + \frac{2((1-s)^p s + (1-s)s^p)}{(1-s)^{p+1}+s^{p+1}} \frac{A_\rho}{|\omega|_{p+1}^{p+1}}}.
\end{aligned} \tag{3.22}$$

Notice that we have the following inequalities:

$$\frac{s(1-s)}{(1-s)^2+s^2} < \frac{(1-s)^p s + (1-s)s^p}{(1-s)^{p+1}+s^{p+1}} \quad \text{for } 0 < s < 1,$$

$$\frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}} < 2^{\frac{p-1}{p+1}} \quad \text{for } 0 \le s < 1.$$

Then

$$\begin{aligned}
&I_\infty(\psi_{\infty,\rho}[z,s]) \\
&\le \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}} \frac{\|\omega\|^2}{|\omega|_{p+1}^2}\right)^{\frac{p+1}{p-1}} \left(\frac{1 + \frac{2s(1-s)}{(1-s)^2+s^2} \frac{A_\rho}{\|\omega\|^2}}{1 + \frac{2((1-s)^p s + (1-s)s^p)}{(1-s)^{p+1}+s^{p+1}} \frac{A_\rho}{|\omega|_{p+1}^{p+1}}}\right)^{\frac{p+1}{p-1}} \\
&\le \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}} \frac{\|\omega\|^2}{|\omega|_{p+1}^2}\right)^{\frac{p+1}{p-1}} \\
&< 2\left(\frac{1}{2} - \frac{1}{p+1}\right)|\omega|_{p+1}^{p+1} \\
&= 2m_\infty.
\end{aligned}$$

Therefore we see that the conclusion follows. □

Lemma 3.10. *There exists $\rho_1 > 0$ such that*

$$\mathcal{A}_\rho := \max\left\{I(\psi_\rho[z,0]) : z \in \Sigma\right\} < \mathcal{B}_0, \qquad \forall \rho > \rho_1. \tag{3.23}$$

Proof. Observing that $\psi_\rho[z,0] = t_{\rho,z,0}\overline{\psi}_\rho[z,0]$ and $\overline{\psi}_\rho[z,0] = \omega_{\rho z}$. We claim that

$$\lim_{\rho\to+\infty} I(t_{\rho,z,0}\omega_{\rho z}) = m_\infty.$$

Indeed, by (3.6), (3.9) and (3.10), we deduce that

$$\begin{aligned} I(t_{\rho,z,0}\omega_{\rho z}) &= \frac{t_{\rho,z,0}^2}{2}\int_{\mathbb{R}^3}|\nabla\omega_{\rho z}|^2+(V_\infty+a(x))\omega_{\rho z}^2\,\mathrm{d}x+\frac{t_{\rho,z,0}^4}{4}\int_{\mathbb{R}^3}K(x)\phi_{\omega_{\rho z}}^K\omega_{\rho z}^2\,\mathrm{d}x \\ &\quad -\frac{t_{\rho,z,0}^{p+1}}{p+1}\int_{\mathbb{R}^3}(Q_\infty-b(x))|\omega_{\rho z}|^{p+1}\,\mathrm{d}x \\ &= \frac{t_{\rho,z,0}^2}{2}\int_{\mathbb{R}^3}|\nabla\omega_{\rho z}|^2+V_\infty\omega_{\rho z}^2\,\mathrm{d}x-\frac{t_{\rho,z,0}^{p+1}}{p+1}\int_{\mathbb{R}^3}Q_\infty|\omega_{\rho z}|^{p+1}\,\mathrm{d}x+o(\varepsilon_\rho) \\ &= \frac{\tau_{\rho,z,0}^2}{2}\int_{\mathbb{R}^3}|\nabla\omega_{\rho z}|^2+V_\infty\omega_{\rho z}^2\,\mathrm{d}x-\frac{\tau_{\rho,z,0}^{p+1}}{p+1}\int_{\mathbb{R}^3}Q_\infty|\omega_{\rho z}|^{p+1}\,\mathrm{d}x+o(\varepsilon_\rho) \\ &= I_\infty(\tau_{\rho,z,0}\omega_{\rho z})+o(\varepsilon_\rho). \end{aligned}$$

Owing to $\tau_{\rho,z,0}\omega_{\rho z}\in\mathcal{N}_\infty$, thus $I_\infty(\tau_{\rho,z,0}\omega_{\rho z})\to m_\infty$ as $\rho\to+\infty$. By Lemma 3.1, (3.23) follows. □

Proof of Theorem 1.1. From Proposition 2.3, we see that $m = m_\infty$ and m is not achieved and the problem cannot be solved by minimization. However we are now going to prove the existence of a positive solution of (1.3) having energy greater than m_∞, through using the deformation argument. For this purpose, we denote $I^c=\{u\in\mathcal{N}: I(u)\le c\}$, $c\in\mathbb{R}$.

By Lemma 3.1, Lemma 3.4, Lemma 3.9 and Lemma 3.10, the following chain of inequality holds

$$m_\infty<\mathcal{A}_\rho<\mathcal{B}_0\le\mathcal{T}_\rho<2m_\infty,\quad \text{for all } \rho>\max\{3R_0,\rho_0,\rho_1\}.$$

We aim at showing that there exists a Palais–Smale sequence of the functional I constrained on N at level $c^*\in[\mathcal{B}_0,\mathcal{T}_\rho]$. If this is done, the existence of a nontrivial critical point u with $I(u)<2m_\infty$ follows from Lemma 3.2.

Assume by contradiction, that no Palais–Smale sequence exists in $[\mathcal{B}_0,\mathcal{T}_\rho]$. By using the usual deformation arguments ([23]), there exist a number $\delta>0$ and a continuous function $\eta: I^{\mathcal{T}_\rho}\to I^{\mathcal{B}_0-\delta}$ such that $\mathcal{B}_0-\delta>\mathcal{A}_\rho$ and $\eta(u)=u$ for all $u\in I^{\mathcal{B}_0-\delta}$. Let us define the map $\mathcal{H}:\Sigma\times[0,1]\to\mathbb{R}^3$ by $\mathcal{H}(z,s)=\beta\circ\eta\circ\psi_\rho[z,s]$. Lemma 3.10 tells us that $\psi_\rho[z,0]\subset I^{\mathcal{A}_\rho}\subset I^{\mathcal{B}_0-\delta}$, thus $\eta(\psi[z,0])=\psi[z,0]$ and then $\beta\circ\eta\circ\psi_\rho[z,0]=\beta(\psi[z,0])=\rho z$. Define $h(t,z,s)=t\mathcal{G}(z,s)+(1-t)\mathcal{H}(z,s):[0,1]\times\Sigma\times(0,1]\to\mathbb{R}^3$,, where $\mathcal{G}$ is defined in Lemma 3.4. Clearly, $h\in C([0,1]\times\Sigma\times(0,1])$ and for all $t\in[0,1]$, $z\in\Sigma$, we have that $h(t,z,0)=\rho z\ne0$, that is $0\notin h(t,\partial(\Sigma\times(0,1]))$. Similar to the proof of Lemma 3.4, we get that there exists $(\bar z,\bar s)\in\Sigma\times(0,1]$ such that

$$\beta\circ\eta\circ\psi_\rho[\bar z,\bar s]=0. \tag{3.24}$$

According to Lemma 3.4, we know that $\Psi_\rho[z,s]\in I^{\mathcal{T}_\rho}$ and then by the properties of η, we have that

$$\eta\circ\psi[z,s]\in I^{\mathcal{B}_0-\delta},\qquad \forall(z,s)\in\Sigma\times[0,1]. \tag{3.25}$$

Clearly, $\eta\circ\psi[z,s]\in\mathcal{N}$, $\forall(z,s)\in\Sigma\times[0,1]$, in particularly, $\eta\circ\psi[\bar z,\bar s]\in\mathcal{N}$, combining with (3.24) and by the definition of $\mathcal{B}_0$, we see that $I(\eta\circ\psi_\rho[\bar z,\bar s])\ge\mathcal{B}_0$, contradicts with (3.25).

Let $u\in I^{\mathcal{T}_\rho}$ be a critical point we have found. To show that u is a constant sign function, we assume by contradiction that $u=u^++u^-$ with $u^\pm\ne0$. Similar to the proof of Lemma 2.1, Lemma 2.2 and Lemma 2.3 in [16], we conclude that there exists $0<t_{u^+}<1$ and $0<t_{u^-}<1$ such that $t_{u^\pm}u^\pm\in\mathcal{N}$. Thus, by Proposition 2.3, we obtain that

$$2m_\infty=2m\le I(t_{u^+}u^+)+I(t_{u^-}u^-)\le I(t_{u^+}u^++t_{u^-}u^-)<I(u^++u^-)=I(u).$$

which is contrary with $I(u)<2m_\infty$. □

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