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Existence and Ulam type stability for nonlinear Riemann–Liouville fractional differential equations with constant delay

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

Ravi Agarwal ✉ [1, 2], **Snezhana Hristova**[3] and **Donal O'Regan**[2]

[1]Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX 78363, USA
[2]Distinguished University Professor of Mathematics, Florida Institute of Technology, Melbourne, FL 32901, USA
[3]Department of Applied Mathematics and Modeling, University of Plovdiv Paisii Hilendarski, Plovdiv, Bulgaria
[4]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

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Abstract. A nonlinear Riemann–Liouville fractional differential equation with constant delay is studied. Initially, some existence results are proved. Three Ulam type stability concepts are defined and studied. Several sufficient conditions are obtained. Some of the obtained results are illustrated on fractional biological models.

Keywords: Riemann–Liouville fractional derivative, constant delay, initial value problem, existence, Ulam type stability.

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1 Introduction

Ulam type stability concept is quite significant in realistic problems in many applications in numerical analysis, optimization, biology and economics etc. This type of stability guarantees that there is a close exact solution. Recently, several authors extended and discussed Ulam type stability to fractional differential equations. The Ulam type stability is well studied recently for Caputo fractional differential equations. For example, about Caputo fractional differential equations we refer [8, 12], for Caputo fractional differential equations with impulses [14], about Caputo fractional differential equations with delays see, for example, [2, 13], for ψ-Hilfer fractional derivative and a constant delay [11]. Note that in the case of

✉Corresponding author. Email: agarwal@tamuk.edu

the Riemann–Louisville (RL) fractional derivative only the case without any delays is studied (see, for example, [3,7,13,16]).

In addition, many real world processes and phenomena are characterized by the influence of past values of the state variable on the recent one and this leads to the inclusion of delays in the models. The analysis of RL delay fractional differential equations is rather complex (analytical solution computation, controllability analysis, etc.) and a very small class of equations could be solved in explicit form. It requires theoretical proofs of methods guarantee existence of enough close function to the unknown solution. One of these types of method is Ulam type stability. According to our knowledge this type of stability is not studied for nonlinear RL fractional differential equations with delays.

The main goal of the paper is to study scalar nonlinear RL fractional differential equations with a constant delay, to obtain some sufficient conditions for uniqueness and existence and to study Ulam type stability. The present paper is organized as follows. In Section 2, some notations and results about fractional calculus are given. In Section 3, an existence result, based on the Banach contraction principle, for the studied problem is presented. In Section 4, we prove three types of Ulam–Hyers stability results for the given RL fractional differential equation with a constant delay. Finally, in the last section, we illustrate the application of some of th obtained results on two fractional biological models: fractional generalization of Lasota–Ważewska model and fractional generalization of the logistic equation with a biological delay depending on the mechanistic details of the model.

2 Preliminary notes on fractional derivatives and equations

In this section, we introduce notations, definitions, and preliminary facts which are used throughout the paper. Let $T: \ 0 < T < \infty$, $\bar{J} = [0, T]$, $J = (0, T]$, $\tau > 0$ be given (the delay).

There exists a natural number N such that $N\tau < T \leq (N+1)\tau$ holds, i.e. $J = \cup_{k=0}^{N-1}(k\tau, (k+1)\tau] \cup (N\tau, T]$. To be easier for the notation without lose of generalization we could assume that $T = (N+1)\tau$ and then $J = \cup_{k=0}^{N}(k\tau, (k+1)\tau]$.

By $C(J, \mathbb{R})$ we denote the set of all continuous function with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$.

By C_0 we denote the set of all functions $x \in C([-\tau, 0], \mathbb{R})$ with the norm $\|x\|_0 = \sup\{|x(t)| : t \in [-\tau, 0]\}$.

We consider the weighted space of functions $C_\gamma(J) = \{y \in J \to \mathbb{R} : \ t^\gamma y(t) \in C(J, \mathbb{R})\}$ with the norm $\|y\|_{C_\gamma} = \sup_{t \in J} |t^\gamma y(t)|$.

Note $C(J, \mathbb{R})$, $C_\gamma(J)$ are Banach spaces.

In this paper we will use the following definitions for fractional derivatives and integrals:

- *Riemann–Liouville fractional integral* of order $q \in (0,1)$ [9]

$$ {}_0I_t^q m(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{m(s)}{(t-s)^{1-q}} ds, \qquad t \in \bar{J}, $$

  where $\Gamma(\cdot)$ is the Gamma function.

- *Riemann–Liouville fractional derivative* of order $q \in (0,1)$ [9]

$$ {}_0^{RL}D_t^q m(t) = \frac{d}{dt}\big({}_0I_t^{1-q} m(t)\big) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} m(s) ds, \qquad t \in \bar{J}. $$

We will give fractional integrals and RL fractional derivatives of some elementary functions which will be used later:

Proposition 2.1 ([5]). *The following equalities are true:*

$$ {}_{t_0}^{RL}D_t^q (t-t_0)^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-q)}(t-t_0)^{\beta-q}, $$

$$ {}_{t_0}I_t^q (t-t_0)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(1+\beta+q)}(t-t_0)^{\beta+q}, $$

$$ {}_{t_0}I_t^{1-q}(t-t_0)^{q-1} = \Gamma(q), $$

$$ {}_{t_0}^{RL}D_t^q (t-t_0)^{q-1} = 0. $$

The definitions of the initial condition for fractional differential equations with RL derivatives are based on the following result:

Proposition 2.2 ([9, Lemma 3.2]). *Let $q \in (0,1)$, $t_0, T \geq 0: \ t_0 < T \leq \infty$ and $m \in L_1^{loc}([t_0,T],\mathbb{R})$.*

(a) If there exists a.e. a limit $\lim_{t\to t_0+}[(t-t_0)^{q-1}m(t)] = c$, then there also exists a limit

$$ {}_{t_0}I_t^{1-q}m(t)|_{t=t_0} := \lim_{t\to t_0+} {}_{t_0}I_t^{1-q}m(t) = c\Gamma(q). $$

(b) If there exists a.e. a limit ${}_{t_0}I_t^{1-q}m(t)|_{t=t_0} = b$ and if the limit $\lim_{t\to t_0+}[(t-t_0)^{1-q}m(t)]$ exists, then

$$ \lim_{t\to t_0+}[(t-t_0)^{1-q}m(t)] = \frac{b}{\Gamma(q)}. $$

In the case of a scalar linear RL fractional differential equation we have the following result:

Proposition 2.3 ([9, Example 4.1]). *The solution of the Cauchy type problem*

$$ {}_a^{RL}D_t^q x(t) = \lambda x(t) + f(t), \qquad {}_aI_t^{1-q}x(t)|_{t=a} = b $$

has the following form [9, formula (4.1.14)]

$$ x(t) = \frac{b}{(t-a)^{1-q}}E_{q,q}(\lambda(t-a)^q) + \int_a^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds \tag{2.1} $$

where $E_{p,q}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jp+q)}$ is the Mittag-Leffler function with two parameters (see, for example, [9]).

Proposition 2.4. *The inequality*

$$ \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)ds \leq \frac{1}{|a|}(E_q(|a|t^q)-1) \tag{2.2} $$

holds.

Proof.

$$\begin{aligned}
&\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)ds = \int_0^t s^{q-1}E_{q,q}(as^q)ds = \int_0^t s^{q-1}\sum_{n=0}^{\infty}\frac{(as^q)^n}{\Gamma((n+1)q)}ds \\
&= \sum_{n=0}^{\infty}\frac{a^n\int_0^t s^{(n+1)q-1}ds}{\Gamma((n+1)q)} = \left|\sum_{n=0}^{\infty}\frac{a^n t^{(n+1)q}}{(n+1)q\Gamma((n+1)q)}\right| \le \sum_{n=1}^{\infty}\frac{|a^{n-1}|t^{nq}}{\Gamma(nq+1)} \\
&\le \frac{1}{|a|}\sum_{n=0}^{\infty}\frac{(|a|t^q)^n}{\Gamma(nq+1)} - \frac{1}{|a|} = \frac{1}{|a|}(E_q(|a|t^q)-1).
\end{aligned}$$

□

From Proposition 2.3 and Proposition 2.2 (a) we obtain the following result for the weighted form of the initial condition:

Proposition 2.5. *The solution of the Cauchy type problem*

$$ {}_a^{RL}D_t^q x(t) = \lambda x(t) + f(t), \qquad \lim_{t\to a+}\Big((t-a)^{1-q}x(t)\Big) = C$$

has the following form

$$x(t) = \frac{C\,\Gamma(q)}{(t-a)^{1-q}}E_{q,q}(\lambda(t-a)^q) + \int_a^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds. \tag{2.3}$$

Proposition 2.6 ([9]). *For $q\in(0,1)$ the following properties*

$$0 \le E_{q,q}(-\lambda t^q) \le \frac{1}{\Gamma(q)}, \qquad t\ge 0,\ \lambda\ge 0,$$

$$\lim_{t\to 0+} E_{q,q}(-\lambda t^q) = E_{q,q}(0) = \frac{1}{\Gamma(q)}$$

hold.

Proposition 2.7 ([17, Corollary 2]). *Let $a(t)$ be a nondecreasing function on J, $g(t)$ be a nonegative, nondecreasing continuous function on J, and*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\beta-1}u(s)ds, \qquad t\in J.$$

Then $u(t)\le a(t)E_\beta(g(t)\Gamma(\beta)t^\beta)$, $t\in J$.

3 Statement of the problem

Consider the initial value problem (IVP) for a nonlinear system of fractional differential equations with constant delay and $q\in(0,1)$

$$\begin{aligned}
&{}_0^{RL}D_t^q x(t) = ax(t) + bx(t-\tau) + f(t,x(t),x(t-\tau)) \quad \text{for } t\in J,\\
&x(t) = g(t) \quad \text{for } t\in[-\tau,0],\\
&\lim_{t\to 0+}\Big(t^{1-q}x(t)\Big) = g(0)
\end{aligned} \tag{3.1}$$

where ${}_0^{RL}D_t^q$ denotes the RL fractional derivative, $a,b\in\mathbb{R}$ are constants, $\tau>0$ is the constant delay, the functions $f: J\times\mathbb{R}\to\mathbb{R}\to\mathbb{R}$, $g\in C_0$.

First, we will consider the partial linear case of (3.1) without a delay, i.e.

$$\begin{aligned} &{}_0^{RL}D_t^q x(t) = ax(t) + \sigma(t, x(t)) \quad \text{for } t \in J, \\ &\lim_{t\to 0+} \left(t^{1-q}x(t)\right) = x_0. \end{aligned} \tag{3.2}$$

where $a \in \mathbb{R}$ is a constant, $\sigma \in C(J, \mathbb{R})$, $q \in (0, 1)$.

Lemma 3.1 ([9])**.** *The linear initial value problem* (3.2) *has the following integral representation for a solution*

$$x(t) = t^{q-1}x_0\Gamma(q)E_{q,q}(at^q) + \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\sigma(s, x(s))ds, \qquad t \in J.$$

We consider also the integral presentation (see [1]) of a special case of (3.1), i.e. we will consider the non-homogeneous scalar linear Riemann–Liouville fractional differential equations with a constant delay

$${}_0^{RL}D_t^q x(t) = Ax(t) + Bx(t-\tau) + \sigma(t) \quad \text{for } t \in J, \tag{3.3}$$

with the initial conditions

$$x(t) = g(t), \qquad t \in [-\tau, 0], \tag{3.4}$$

$$\lim_{t\to 0+} \left(t^{1-q}x(t)\right) = g(0) \tag{3.5}$$

where $\sigma \in C(\mathbb{R}_+, \mathbb{R})$, $g \in C([-\tau, 0], \mathbb{R})$, A, B are real constants.

Lemma 3.2 ([1])**.** *The solution of the IVP* (3.3), (3.4), (3.5) *is given by*

$$x(t) = \begin{cases} g(t) & t \in (-\tau, 0] \\ g(0)\Gamma(q)E_{q,q}(At^q)t^{q-1} + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\big(Bg(s-\tau) + \sigma(s)\big)ds & t \in (0, \tau] \\ g(0)\Gamma(q)E_{q,q}(At^q)t^{q-1} + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\sigma(s)ds & \\ \quad +B\sum_{i=0}^{n-1}\int_{i\tau}^{(i+1)\tau}(t-s)^{q-1}E_{q,q}(A(t-s)^q)x(s-\tau)ds & \\ \quad +B\int_{n\tau}^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)x(s-\tau)ds & \\ \qquad \text{for } t \in (n\tau, (n+1)\tau], \quad n = 1, 2, \dots, N. & \end{cases}$$

We will consider the assumptions:

(A1) The function $f \in C(\bar{J} \times \mathbb{R}^2, \mathbb{R})$ and there exist constants $K, L > 0$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le K|u_1 - u_2| + L|v_1 - v_2|, \quad t \in \bar{J}, \ \ u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

(A2) The function $g \in C_0$.

4 Existence and integral presentation of the solution

Now we will study the existence of the solution of (3.1) and its presentation, based on Lemma 3.2. We will use the Banach contraction principle.

Lemma 4.1. *Let the assumptions (A1), (A2) be satisfied and $q \in [0.5, 1)$.*
Then the operator $\Omega : C_{1-q}(J) \to C_{1-q}(J)$ where

$$\Omega(y(t)) = \begin{cases} g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} \\ \quad + \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(bg(s-\tau)+f(s,y(s),g(s-\tau))\Big)ds, \quad t \in (0,\tau] \\ g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} \\ \quad + \int_0^\tau (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(bg(s-\tau)+f(s,y(s),g(s-\tau))\Big)ds \\ \quad + \int_\tau^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau)+f(s,y(s),y(s-\tau))\Big)ds, \quad t \in (\tau, T]. \end{cases} \tag{4.1}$$

Proof. Let $y \in C_{1-q}(J)$. We will prove the inclusion

$$\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau)+f(s,y(s),y(s-\tau))\Big)ds \in C_{1-q}(J) \quad \text{for } t \in J.$$

Let $t \in (0,\tau]$ then according to assumption (A1) and Proposition 2.1 with $\beta = q-1$ we get

$$\begin{aligned} &\left| t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(bg(s-\tau)+f(s,y(s),g(s-\tau))\Big)ds \right| \\ &\leq Kt^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|y(s)|ds \\ &\quad + Lt^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|g(s-\tau)|ds \\ &\quad + t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|f(s,0,0)|ds \\ &\quad + |b|\|g\|_0 t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)ds \\ &\leq KSt^{1-q}\int_0^t (t-s)^{q-1}s^{q-1}|s^{1-q}y(s)|ds + \Big(L\|g\|_0 + C\Big)\frac{tS}{q} \\ &\leq \frac{KSt^q\Gamma^2(q)}{\Gamma(2q)}\|y\|_{C_{1-q}} + \Big((L+|b|)\|g\|_0 + C\Big)\frac{St}{q}. \end{aligned} \tag{4.2}$$

where $C = \sup_{t\in J}|f(t,0,0)|$, $S = \sup_{t\in J}E_{q,q}(at^q)$.

Let $t > \tau$. Then according to assumption (A1), equality $\int_\tau^t \frac{(s-\tau)^{q-1}}{(t-s)^{1-q}}ds = \frac{\Gamma^2(q)}{\Gamma(2q)}(t-\tau)^{2q-1}$ (see Proposition 2.1 with $\beta = q-1$, $t_0 = \tau$) and $t^{1-q}(t-\tau)^{2q-1} \leq t^q$ it follows

$$\begin{aligned} &\left| t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau)+f(s,y(s),y(s-\tau))\Big)ds \right| \\ &\leq t^{1-q}S\int_0^t (t-s)^{q-1}(|b|+L)|y(s-\tau)|ds + t^{1-q}KS\int_0^t (t-s)^{q-1}|y(s)|ds + \frac{t}{q}SC \\ &\leq t^{1-q}S\int_0^\tau (t-s)^{q-1}(|b|+L)|g(s-\tau)|ds + t^{1-q}S\int_\tau^t (t-s)^{q-1}(b+L)|y(s-\tau)|ds \\ &\quad + \frac{t}{q}SC + SK\|y(s)\|_{C_{1-q}}\frac{t^q\Gamma^2(q)}{\Gamma(2q)} \leq \frac{t-(t-\tau)^q t^{1-q}}{q}S(|b|+L)\|g\|_0 \\ &\quad + S(|b|+L)\|y(s)\|_{C_{1-q}}\frac{t^{1-q}(t-\tau)^{2q-1}\Gamma^2(q)}{\Gamma(2q)} + \frac{t}{q}SC + SK\|y(s)\|_{C_{1-q}}\frac{t^q\Gamma^2(q)}{\Gamma(2q)} \\ &\leq \frac{t}{q}S((b+L)\|g\|_0 + C) + S(|b|+L+K)\|y(s)\|_{C_{1-q}}\frac{t^q\Gamma^2(q)}{\Gamma(2q)}, \quad t > \tau. \end{aligned} \tag{4.3}$$

From inequalities (4.2) and (4.3) it follows

$$\begin{aligned}&\left|t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau)+f(s,y(s),y(s-\tau))\Big)ds\right|\\&\quad\le \frac{T}{q}S((|b|+L)\|g\|_0+C)+S(|b|+L+K)\|y(s)\|_{C_{1-q}}\frac{T^q\Gamma^2(q)}{\Gamma(2q)}.\end{aligned}\tag{4.4}$$

Therefore, the integrals exist and $\Omega y(t)\in C_{1-q}(J)$. □

Remark 4.2. Note the restriction $q\in[0.5,1)$ is necessary to prove the inequality (4.3) and it is deeply connected with the presence of the delay.

Lemma 4.3. *Suppose (A1) and (A2) hold and* $q\in[0.5,1)$.

(i) If the function $y\in C_{1-q}(J)$ is a solution of IVP (3.1) then it is a fixed-point of the operator Ω defined by (4.1).

(ii) If the function $y\in C_{1-q}(J)$ is a fixed-point of the operator Ω with $y(t)=g(t),\ t\in[-\tau,0]$ then it is a solution of IVP (3.1).

Proof. (i) Let the function $y\in C_{1-q}(J)$ be a solution of IVP (3.1). We will use an induction to prove the function y is a fixed point of the operator Ω.

Let $t\in(0,\tau]$. Then y satisfies the initial value problem (3.2) with $\sigma(t,x)=bg(t-\tau)+f(t,x,g(t-\tau))$ and $x_0=g(0)$. From Lemma 3.1 it follows $\Omega(y(t))=y(t),\ t\in(0,\tau]$.

Let $t\in(\tau,2\tau]$. Then y satisfies the initial value problem (3.2) with $\sigma(t,x)=by(t-\tau)+f(t,x,y(t-\tau))$ and $x_0=g(0)$. From Lemma 3.1 it follows $\Omega(y(t))=y(t),\ t\in(\tau,2\tau]$.

By induction it follows the solution y is a fixed point of the operator Ω.

(ii) Let $y\in C_{1-q}(J)$ be a fixed-point of the operator Ω with $y(t)=g(t),\ t\in[-\tau,0]$.

Then from Lemma 4.1, inequalities (4.2), (4.3) and Proposition 2.6 we obtain that

$$\lim_{t\to 0+}\Big(t^{1-q}\Omega(y(t))\Big)=g(0).\tag{4.5}$$

Therefore, the function y solves the IVP (3.1). □

Remark 4.4. If the conditions of Lemma 4.3 are satisfied, and $y\in C_{1-q}(J)$ is a fixed-point of the operator Ω, then we can spread the definition of y over the entire interval $[-\tau,T]$ by $y(t)=g(t),\ t\in[-\tau,0]$ and then y is a solution of IVP (3.1).

Theorem 4.5 (Existence result)**.** *Let $q\in[0.5,1)$ and the assumption (A1) and (A2) be satisfied and the inequality*

$$\alpha=(K+L+|b|)\frac{T^q\Gamma^2(q)}{\Gamma(2q)}\sup_{t\in J}E_{q,q}(at^q)<1\tag{4.6}$$

holds.

Then the initial value problem (3.1) *has a unique solution* $y \in C_{1-q}(J)$ *satisfying the integral presentation*

$$y(t) = \begin{cases} g(t), \quad t \in [-\tau, 0] \\ g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} + b\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)g(s-\tau)\Big)ds \\ \qquad + \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)f(s,y(s),g(s-\tau))ds, \quad t \in (0,\tau] \\ g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} + \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)f(s,y(s),y(s-\tau))ds \\ \qquad +b\sum_{i=0}^{n-1}\int_{i\tau}^{(i+1)\tau}(t-s)^{q-1}E_{q,q}(a(t-s)^q)y(s-\tau)ds \\ \qquad +b\int_{n\tau}^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)y(s-\tau)ds \\ \qquad\qquad \textit{for} \quad t \in (n\tau,(n+1)\tau], n = 1,2,\dots,N \end{cases} \tag{4.7}$$

Proof. According to Lemma 4.1 the operator $\Omega : C_{1-q}(J) \to C_{1-q}(J)$.

We will prove the operator Ω has an unique fixed point in $C_{1-q}(J)$.

Let $y, y^* \in C_{1-q}(J)$ and $t \in (0,\tau]$. Then we obtain

$$\begin{aligned} &|t^{1-q}[\Omega(y(t)) - \Omega(y^*(t))]| \\ &\quad \le t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|f(s,y(s),g(s-\tau)) - f(s,y^*(s),g(s-\tau))|ds \\ &\quad \le t^{1-q}KS\int_0^t (t-s)^{q-1}s^{q-1}|s^{1-q}(y(s)-y^*(s))|ds \\ &\quad \le t^{1-q}KS\|y-y^*\|_{C_{1-q}}\int_0^t (t-s)^{q-1}s^{q-1}ds \\ &\quad \le \frac{KST^q\Gamma^2(q)}{\Gamma(2q)}\|y-y^*\|_{C_{1-q}} \le \alpha\|y-y^*\|_{C_{1-q}}. \end{aligned} \tag{4.8}$$

Let $y, y^* \in C_{1-q}(J)$ and $t > \tau$. Then we obtain

$$\begin{aligned} &\Big|t^{1-q}[\Omega(y(t)) - \Omega(y^*(t)]\Big| \\ &\quad \le t^{1-q}\int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|f(s,y(s),y(s-\tau)) - f(s,y^*(s),y^*(s-\tau))|ds \\ &\qquad + t^{1-q}|b|\int_\tau^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)|y(s-\tau)-y^*(s-\tau)|ds \\ &\quad \le Kt^{1-q}S\int_0^t (t-s)^{q-1}|y(s)-y^*(s)|ds \\ &\qquad + t^{1-q}(L+|b|)S\int_\tau^t (t-s)^{q-1}(s-\tau)^{q-1}|(s-\tau)^{1-q}(y(s-\tau)-y^*(s-\tau))|ds \\ &\quad \le KS\|y-y^*\|_{C_{1-q}}\frac{t^q\Gamma^2(q)}{\Gamma(2q)} + (L+|b|)S\|y-y^*\|_{C_{1-q}}\frac{t^{1-q}(t-\tau)^{2q-1}\Gamma^2(q)}{\Gamma(2q)} \\ &\quad \le (K+L+|b|)\frac{St^q\Gamma^2(q)}{\Gamma(2q)}\|y-y^*\|_{C_{1-q}} \le \alpha\|y-y^*\|_{C_{1-q}}. \end{aligned} \tag{4.9}$$

Therefore

$$\|\Omega(y) - \Omega(y^*)\|_{C_{1-q}} \le \alpha\|y-y^*\|_{C_{1-q}}. \tag{4.10}$$

According to Lemma 4.3 it follows the claim of Theorem 4.5. □

Corollary 4.6. *Let the assumptions (A1), (A2) are satisfied with* $a \le 0$, $q \in [0.5, 1)$ *and*

$$(K+L+|b|)T^q\Gamma(q) < \Gamma(2q). \tag{4.11}$$

Then the initial value problem (3.1) *has a unique solution* $y \in C_{1-q}(J)$.

The proof follows from Theorem 4.5 and Proposition 2.6.

In the case of an equation without a delay we obtain the following corollary.

Corollary 4.7. *Let $\tau = 0$, $a = b = 0$, $q \in (0,1)$, the function $f \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a constant $K > 0$ such that $|f(t,u_1) - f(t,u_2)| \le K|u_1 - u_2|$, $t \in J$, $u_1, u_2 \in \mathbb{R}$ and*

$$KT^q\Gamma(q) < \Gamma(2q). \tag{4.12}$$

Then the reduced initial value problem

$${}_0^{RL}D_t^q x(t) = f(t, x(t)),\ t \in J, \qquad \lim_{t\to 0+} \left(t^{1-q}x(t)\right) = x_0 \tag{4.13}$$

has a unique solution $y \in C_{1-q}(J)$.

Note in the case without a delay it follows from the proof of Theorem 4.5 that we do not need the restriction $q \in [0.5, 1)$.

Remark 4.8. Note the result of Corollary 4.7 coincides the result of [3, Theorem 3.4] with $L = 0$.

5 Ulam type stability

Let $\varepsilon > 0$ and $\Phi \in C(\bar{J}, [0,\infty))$ be non-decreasing and such that for any $t \in \bar{J}$ the inequality $\int_0^t (t-s)^{q-1}\Phi(s)ds < \infty$ holds.

Definition 5.1 ([10])**.** The equation (3.1) is Ulam–Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C_{1-q}(J)$ of the inequalities

$$\begin{aligned} &\left| {}_0^{RL}D_t^q y(t) - ay(t) - by(t-\tau) - f(t, y(t), y(t-\tau))\right| \le \varepsilon \quad \text{for } t \in J, \\ &y(t) = g(t) \quad \text{for } t \in [-\tau, 0], \\ &\lim_{t\to 0+}\left(t^{1-q}x(t)\right) = g(0) \end{aligned} \tag{5.1}$$

there exists a solution $x \in C_{1-q}(J)$ of the problem (3.1) with

$$|x(t) - y(t)| \le \varepsilon\, c_f \quad \text{for } t \in J. \tag{5.2}$$

Definition 5.2 ([10])**.** The problem (3.1) is Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C_{1-q}(J)$ of the inequality

$$\begin{aligned} &\left| {}_0^{C}D_t^q y(t) - ay(t) - by(t-\tau) - f(t, y(t), y(t-\tau))\right| \le \varepsilon\Phi(t) \quad \text{for } t \in J, \\ &y(t) = g(t) \quad \text{for } t \in [-\tau, 0], \\ &\lim_{t\to 0+}\left(t^{1-q}x(t)\right) = g(0) \end{aligned} \tag{5.3}$$

there exists a solution $x \in C_{1-q}(J)$ of the problem (3.1) with

$$|y(t) - x(t)| \le \varepsilon\, c_f\Phi(t), \qquad t \in J. \tag{5.4}$$

Definition 5.3 ([10])**.** The problem (3.1) is generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_f > 0$ such that for each solution $y \in C_{1-q}(J)$ of the inequality

$$\begin{aligned} &\Big| {}_0^C D_t^q y(t) - ay(t) - by(t-\tau) - f(t, y(t), y(t-\tau))\Big| \leq \Phi(t) \quad \text{for } t \in J, \\ &y(t) = g(t) \quad \text{for } t \in [-\tau, 0], \\ &\lim_{t\to 0+} \Big(t^{1-q}x(t)\Big) = g(0) \end{aligned} \tag{5.5}$$

there exists a solution $x \in C_{1-q}(J)$ of the problem (3.1) with

$$|y(t) - x(t)| \leq \ c_f \Phi(t), \qquad t \in J. \tag{5.6}$$

Remark 5.4. If the function $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ then the function $y \in C_{1-q}(J)$ is a solution of the inequality (5.1) iff there exist a function $G \in C_{1-q}(J)$ which depends on y such that

(i) $\|G(t)\| \leq \varepsilon$;

(ii) ${}_0^C D_t^q y(t) = ay(t) + by(t-\tau) + f(t, y(t), y(t-\tau)) + G(t)$ for $t \in J$
with initial conditions $y(t) = g(t),\ t \in [-\tau, 0],\ \lim_{t\to 0+} \big(t^{1-q}x(t)\big) = g(0)$.

Remark 5.5. If the function $f \in C(\bar{J} \times \mathbb{R}^2, \mathbb{R})$ then the function $y \in C_{1-q}(J)$ is a solution of the inequality (5.5) iff there exist a function $G \in C_{1-q}(J)$ which depends on y such that

(i) $|G(t)| \leq \Phi(t)$ for $t \in J$;

(ii) ${}_0^C D_t^q y(t) = ay(t) + by(t-\tau) + f(t, y(t), y(t-\tau)) + G(t)$ for $t \in J$
with initial conditions $y(t) = g(t),\ t \in [-\tau, 0],\ \lim_{t\to 0+} \big(t^{1-q}x(t)\big) = g(0)$.

Note we have a similar remark for inequality (5.3).
Based on Remark 5.4 and Definition 5.1 we get the following result.

Lemma 5.6. *Let the conditions of Theorem 4.5 be satisfied. If $y \in C_{1-q}(J)$ is a solution of inequalities (5.1) then it satisfies the following integral-algebraic inequalities*

$$\begin{aligned} &\Big| y(t) - g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} \\ &\quad - \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(bg(s-\tau) + f(s, y(s), g(s-\tau))\Big)ds\Big| \\ &\leq \frac{\varepsilon}{|a|}(E_q(|a|t^q) - 1), \qquad t \in (0, \tau] \\ &\Big| y(t) - g(0)\Gamma(q)E_{q,q}(At^q)t^{q-1} \\ &\quad - \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(by(s-\tau) + f(s, y(s), y(s-\tau))\Big)ds\Big| \\ &\leq \frac{\varepsilon}{|a|}(E_q(|a|t^q) - 1), \qquad t \in (\tau, T]. \end{aligned} \tag{5.7}$$

Proof. Let $y \in C_{1-q}(J)$ be a solution of inequalities (5.1). According to Remark 5.5 it satisfies the IVP

$$\begin{aligned} &{}_0^C D_t^q y(t) = ay(t) + by(t-\tau) + f(t, y(t), y(t-\tau)) + G(t) \quad \text{for } t \in J, \\ &y(t) = g(t) \quad \text{for } t \in [-\tau, 0], \\ &\lim_{t \to 0+} \left(t^{1-q} x(t) \right) = g(0). \end{aligned} \tag{5.8}$$

Then according to Lemma 4.3 (i) $y(t)$ is a fixed-point of the operator Ω defined by (4.1), where $f(t, x, y)$ is replaced by $f(t, x, y) + G(t)$.

Let $t \in (0, \tau]$. Apply the inequalities $|G(t)| \leq \varepsilon$ and (2.2) and obtain

$$\begin{aligned} &\left| y(t) - g(0)\Gamma(q)E_{q,q}(at^q)t^{q-1} - \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)\Big(bg(s-\tau) + f(s, y(s), g(s-\tau))\Big)ds \right| \\ &\quad = \left| \int_0^t (t-s)^{q-1}E_{q,q}(a(t-s)^q)G(s)ds \right| \leq \frac{\varepsilon}{|a|}(E_q(|a|t^q) - 1). \end{aligned}$$

The proof for $t \in (\tau, T]$ is similar and we omit it. □

Now we will study Ulam type stability of problem (3.1).

Theorem 5.7 (Stability results)**.** *Assume the conditions of Theorem 4.5 are satisfied.*

(i) Suppose for any $\varepsilon > 0$ the inequality (5.1) has at least one solution. Then problem (3.1) is Ulam–Hyers stable.

(ii) Suppose there exists a nondecreasing function $\Phi \in C(\bar{J}, [0, \infty))$ such that for any $t \in \bar{J}$ the inequality $\int_0^t (t-s)^{q-1}\Phi(s)ds \leq \Lambda_\Phi \Phi(t)$ holds where $\Lambda_\Phi > 0$ is a constant and for any $\varepsilon > 0$ the inequality (5.3) has at least one solution. Then problem (3.1) is Ulam–Hyers–Rassias stable with respect to Φ.

(iii) If there exists a nondecreasing function $\Phi \in C(\bar{J}, [0, \infty))$ such that for any $t \in \bar{J}$ the inequality $\int_0^t (t-s)^{q-1}\Phi(s)ds \leq \Lambda_\Phi \Phi(t)$ holds, $\Lambda_\Phi > 0$ is a constant, and inequality (5.5) has at least one solution then problem (3.1) is generalized Ulam–Hyers–Rassias stable with respect to Φ.

Proof. According to Theorem 4.5 the problem (3.1) has an unique solution $x \in C_{1-q}(J)$ for which the integral presentation (4.7) holds.

(i) Let $\varepsilon > 0$ be an arbitrary number and $y \in C_{1-q}(J)$ be a solution of the inequality (5.1). Therefore, the integral inequalities (5.7) hold.

Denote

$$Q = (K + L + |b|)CE_q(\Gamma(q)KC\tau^q)\frac{\tau^q}{q}, \qquad C = \max_{t \in J} E_{q,q}(at^q),$$

and

$$M_{k+1} = M(1+Q)^k, \qquad k = 0, 1, 2, \ldots, N, \qquad M = \frac{1}{|a|}(E_q(|a|T^q) - 1).$$

Let $t \in (0,\tau]$ be an arbitrary fixed point. According to Lemma 5.6, Theorem 4.5, inequality (5.7) and equality (4.7) we obtain

$$\begin{aligned}
&|x(t)-y(t)| \\
&\quad \le \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q)\Big(f(s,x(s),g(s-\tau)) - f(s,y(s),g(s-\tau))\Big)ds \right| \\
&\qquad + \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) G(s) ds \right| \\
&\quad \le \frac{\varepsilon}{|a|}(E_q(|a|t^q)-1) + KC\int_0^t (t-s)^{q-1}|x(s)-y(s)|ds \\
&\quad \le \varepsilon M + KC \int_0^t (t-s)^{q-1}|x(s)-y(s)|ds.
\end{aligned} \tag{5.9}$$

According to Proposition 2.7 we get

$$|x(t)-y(t)| \le \varepsilon M E_q(\Gamma(q)KCt^q), \qquad t\in[0,\tau], \tag{5.10}$$

and

$$|x(t)-y(t)| \le \varepsilon M E_q(\Gamma(q)KC\tau^q) = \varepsilon M_1, \qquad t\in[0,\tau], \tag{5.11}$$

Let $t \in (\tau, 2\tau]$ be an arbitrary fixed point. According to Lemma 5.6, Theorem 4.5, inequalities (5.7), (5.11) and (2.2) we obtain

$$\begin{aligned}
&|x(t)-y(t)| \\
&\quad \le \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q)\Big(f(s,x(s),x(s-\tau)) - f(s,y(s),y(s-\tau))\Big)ds \right| \\
&\qquad + \left| b\int_\tau^t (t-s)^{q-1} E_{q,q}(a(t-s)^q)\Big(x(s-\tau)-y(s-\tau)\Big)ds \right| \\
&\qquad + \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) G(s) ds \right| \\
&\quad \le \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q)\Big(K|x(s)-y(s)| + L|x(s-\tau)-y(s-\tau)|\Big)ds \\
&\qquad + \varepsilon |b| M_1 C \int_\tau^t (t-s)^{q-1}ds + \varepsilon M \\
&\quad \le KC\int_\tau^t (t-s)^{q-1}|x(s)-y(s)|ds + \varepsilon M E_q(\Gamma(q)KC\tau^q)(L+|b|)C\frac{(t-\tau)^q}{q} + \varepsilon M \\
&\qquad + \varepsilon M E_q(\Gamma(q)KC\tau^q)KC\frac{\tau^q}{q} \\
&\quad \le \varepsilon M(1+Q) + KC\int_\tau^t (t-s)^{q-1}|x(s)-y(s)|ds.
\end{aligned} \tag{5.12}$$

According to Proposition 2.7 we get

$$|x(t)-y(t)| \le \varepsilon M(1+Q)E_q(\Gamma(q)KCt^q) = \varepsilon M_2 E_q(\Gamma(q)KC(t-\tau)^q), \qquad t\in(\tau,2\tau]. \tag{5.13}$$

and

$$|x(t)-y(t)| \le \varepsilon M_2 E_q(\Gamma(q)KC\tau^q), \qquad t\in(\tau,2\tau]. \tag{5.14}$$

Let $t \in (2\tau, 3\tau]$ be an arbitrary fixed point. According to Lemma 5.6, Theorem 4.5, inequalities (5.9), (5.11) and (5.14) we obtain

$$\begin{aligned}
&|x(t) - y(t)| \\
&\quad \leq \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) \Big(K|x(s) - y(s)| + L|x(s-\tau) - y(s-\tau)| \Big) ds \\
&\qquad + |b| \int_\tau^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) |x(s-\tau) - y(s-\tau)| ds \\
&\qquad + \left| \int_0^t (t-s)^{q-1} E_{q,q}(a(t-s)^q) G(s) ds \right| \\
&\quad \leq KC \int_0^t (t-s)^{q-1} |x(s) - y(s)| ds + (L+|b|)C \int_\tau^{2\tau} (t-s)^{q-1} |x(s-\tau) - y(s-\tau)| ds \\
&\qquad + (L+|b|)C \int_{2\tau}^t (t-s)^{q-1} |x(s-\tau) - y(s-\tau)| ds + \varepsilon M \\
&\quad \leq KC \int_{2\tau}^t (t-s)^{q-1} |x(s) - y(s)| ds + \varepsilon M (1+Q)^2.
\end{aligned} \tag{5.15}$$

According to Proposition 2.7 we get

$$|x(t) - y(t)| \leq \varepsilon M (1+Q)^2 E_q(\Gamma(q) K C t^q) = \varepsilon M_3 E_q(\Gamma(q) K C (t-2\tau)^q), \qquad t \in (2\tau, 3\tau] \tag{5.16}$$

and

$$|x(t) - y(t)| \leq \varepsilon M_3 E_q(\Gamma(q) K C \tau^q), \qquad t \in (2\tau, 3\tau]. \tag{5.17}$$

Continuing the induction process we prove

$$|x(t) - y(t)| \leq \varepsilon M_{k+1} E_q(\Gamma(q) K C \tau^q), \qquad t \in (k\tau, (k+1)\tau],\ k = 0, 1, 2, \dots, N. \tag{5.18}$$

Inequality (5.18) proves the claim (i) with $c_f = M(1+Q)^N E_q(\Gamma(q) K C \tau^q)$.

(iii) Let $y \in C_{1-q}(J)$ be a solution of the inequality (5.5) with the function $\Phi(t)$ defined in the condition (iii) of Theorem 5.7.

Denote

$$Q = (K + L + |b|) C \Lambda_\Phi E_q(\Gamma(q) K C \tau^q), \qquad C = \max_{t \in J} E_{q,q}(a t^q),$$

and

$$M_{k+1} = M(1+Q)^k, \qquad k = 0, 1, 2, \dots, N, \qquad M = C \Lambda_\Phi.$$

Similar to the case (i) we use an induction to prove the inequality

$$|x(t) - y(t)| \leq M_{k+1} E_q(\Gamma(q) K C \tau^q) \Phi(t), \qquad t \in (k\tau, (k+1)\tau],\ k = 0, 1, 2, \dots, N. \tag{5.19}$$

Therefore, the problem (3.1) is generalized Ulam–Hyers–Rassias stable with respect to Φ with $c_f = C \Lambda_\Phi (1+Q)^n E_q(\Gamma(q) K C \tau^q)$.

(ii) The proof is similar to the one in (i). □

6 Applications to some biological models

In this section we will apply the obtained results to some biological models and their fractional generalizations.

Model 1. The investigation of blood cell dynamics is connected with formulating and studyig mathematical methods and models, numerical results, schemes to estimate parameters and prognosticate optimal treatments to particular diseases. In order to describe the survival of red blood cells in animals, Ważewska-Czyżewska and Lasota proposed in [15] the following delayed equation $x'(t) = -\gamma x(t) + \beta e^{-\alpha x(t-\tau)}$ where $x(t)$ represents the number of red blood cells at time t, $\gamma > 0$ is the death probability for a red blood cell, a and β are positive constants related to the production of red blood cells per unit time and τ is the time delay between the production of immature red blood cells and their maturation for release in circulating blood stream. The well known Lasota–Ważewska model was extended and generalized by many authors. Now we will consider one fractional generalization.

Consider the following fractional generalization of the mentioned above model:

$$ {}_0^{RL}D_t^q x(t) = \beta e^{-\alpha x(t-\tau)} - \gamma x(t), \qquad t \in J \tag{6.1}$$

with the initial conditions (3.4), (3.5), where $q \in [0.5, 1)$, x is the number of red blood cells, $\beta > 0$ is the demand for oxygen, $\tau > 0$ is the time required for erythrocytes to attain maturity, $\gamma > 0$ is the cell destruction rate.

In this case $a = -\gamma$, $b = 0$, $f(t, x, y) = \beta e^{-\alpha y}$ and $|f(t, x, y) - f(t, u, v)| = |\beta e^{-\alpha y} - \beta e^{-\alpha v}| \leq \beta\alpha|y - v|$, i.e. the condition (A1) is satisfied with $K = 0$, $L = \beta\alpha$.

If $\beta\alpha T\Gamma(q) < \Gamma(2q)$ then according to Theorem 4.5 the initial value problem (6.1), (3.4), (3.5) has an unique solution $x \in C_{1-q}(J)$ satisfying the integral presentation

$$x(t) = g(0)\Gamma(q)E_{q,q}(-\gamma t^q)t^{q-1} + \beta \int_0^t (t-s)^{q-1}E_{q,q}(-\gamma(t-s)^q)e^{-\alpha x(s-\tau)}ds, \quad t \in (0, T]. \tag{6.2}$$

Consider the partial case of $q = 0.8$, $\beta = 0.05$, $\alpha = 0.9$,, $\gamma = 0.01$, and $\tau = 2$, $T = 12$, $g(t) = t^2$. Then the inequality $0.9(0.05)\Gamma(0.8)12 < \Gamma(1.6)$ holds and therefore the model (6.1) has a solution satisfying the integral presentation

$$x(t) = 0.05 \int_0^t (t-s)^{-0.2}E_{0.8,0.8}(-0.5(t-s)^{0.8})e^{-0.9x(s-2)}ds, \qquad t \in (0, 12].$$

Consider the function $y(t) = t^2$. Then ${}_0^{RL}D_t^{0.8} t^2 = \frac{\Gamma(3)}{\Gamma(2.2)} t^{1.2}$ and the inequality

$$\left| \frac{\Gamma(3)}{\Gamma(2.2)} t^{1.2} - 0.05e^{-0.9(t-2)^2} + 0.01t^2 \right| \leq 3.5t + 0.0015 \quad \text{for } t \in [0, 12], \tag{6.3}$$

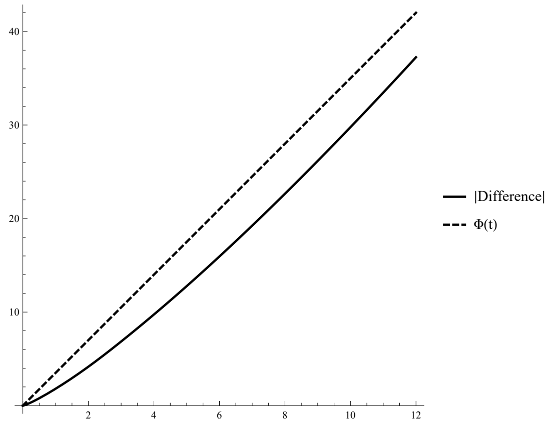
holds (see Figure 6.1a).

Consider $\Phi(t) = 3.5t + 0.0015$. Then $\int_0^t (t-s)^{-0.2}(3.5s + 0.0015)ds \leq \Lambda_\Phi(3.5t + 0.0015)$ with $\Lambda_\Phi = 5.5$ (see Figure 6.1b).

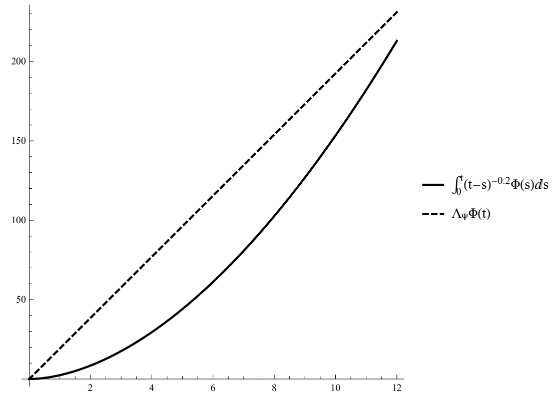
According to Theorem 5.7(iii) the solution $x(t)$ of (6.1) satisfies

$$\left| x(t) - t^2 \right| \leq 39.9618t + 0.0199809, \qquad t \in [0, 12]$$

where $c_f = C\Lambda_\Phi(1 + Q)^4 E_q(0) = 5.5(1.2475)^4 = 13.3206$, $C = \max_{t\in[0,12]} E_{0.8,0.8}(-0.01t^{0.8}) = 1$, $Q = (K + L + |b|)C\Lambda_\Phi E_q(\Gamma(q)KC\tau^q = 0.2475$.



(a) Graph of the difference $\left|\frac{\Gamma(3)}{\Gamma(2.2)}t^{1.2} - 0.05e^{-0.9(t-2)^2} + 0.01t^2\right|$ and the function $\Phi(t) = 3.5t + 0.0015$ on $[0, 12]$.



(b) Graph of the fractional integral of the function $\Phi(t) = 3.5t + 0.0015$ and the function $\Lambda_\Phi \Phi(t)$ on $[0, 12]$.

Figure 6.1: Model 1.

Model 2. Consider the logistic equation where the effect of a biological delay depends on the mechanistic details of the model. For example, suppose that a period of time τ elapses between egg laying and hatching. Let us consider the case of and constant harvesting rate $\mu > 0$ ([6]): $N'(t) = BN(t-\tau)e^{-\delta\tau}(1 - \frac{N(t)}{A}) - \mu N(t)$, where δ is egg mortality and $e^{-\delta\tau}$ could be the birth rate (given as a fraction). Hatchlings were produced by parents alive τ time ago, but complete for sites with individuals alive at their dispersal and recruitment. It has a zero equilibrium.

Now, consider the fractional generalization of the model with $q \in (0, 1)$:

$$ {}_0^{RL}D_t^q N(t) = BN(t-\tau)e^{-\delta\tau}\Big(1 - \frac{N(t)}{A}\Big) - \mu N(t) \tag{6.4} $$

with the initial conditions

$$ N(t) = g(t), \qquad t \in [-\tau, 0], \tag{6.5} $$

$$ \lim_{t \to 0+} \Big(t^{1-q} N(t)\Big) = g(0) \tag{6.6} $$

Note (6.4) has a zero equilibrium.

In this case $f(t, x, y) = -\frac{B}{A}e^{\delta\tau}xy$ and $a = -\mu$, $b = Be^{-\delta\tau}$. Then $|f(t, x, y) - f(t, u, v)| = \frac{B}{A}e^{-\delta\tau}|xy - uv| \le \frac{B}{A}e^{-\delta\tau}|x(y-u) + U(x-v)| \le K|x-v| + L|y-u|$ with $K = \frac{B}{A}e^{-\delta\tau}\max v$ and $L = \frac{B}{A}e^{-\delta\tau}\max x$. We will consider the case $N \le W$, $W \in (0, A]$. Therefore, $K = L = \frac{B}{A}We^{-\delta\tau}$. According to Theorem 4.5 if $\alpha = \big(\frac{2}{A}W + 1\big)Be^{-\delta\tau}\frac{T\Gamma(q)}{\Gamma(2q)} < 1$ then (6.4) has a solution in C_{1-q} satisfying the integral presentation

$$ N(t) = \begin{cases} g(0)\Gamma(q)E_{q,q}(-\mu t^q)t^{q-1} \\ \quad + \int_0^t (t-s)^{q-1}E_{q,q}(-\mu(t-s)^q)\Big(Be^{\delta\tau}g(s-\tau) - \frac{B}{A}e^{-\delta\tau}N(s)N(s-\tau)\Big)ds, \quad t \in (0, \tau] \\ g(0)\Gamma(q)E_{q,q}(-\mu t^q)t^{q-1} - \frac{B}{A}e^{-\delta\tau}\int_0^t (t-s)^{q-1}E_{q,q}(-\mu(t-s)^q)N(s)N(s-\tau)ds \\ \quad + Be^{-\delta\tau}\int_0^t (t-s)^{q-1}E_{q,q}(-\mu(t-s)^q)N(s-\tau)ds, \quad t \in (n\tau, (n+1)\tau],\ n = 1, 2, \ldots \end{cases} $$

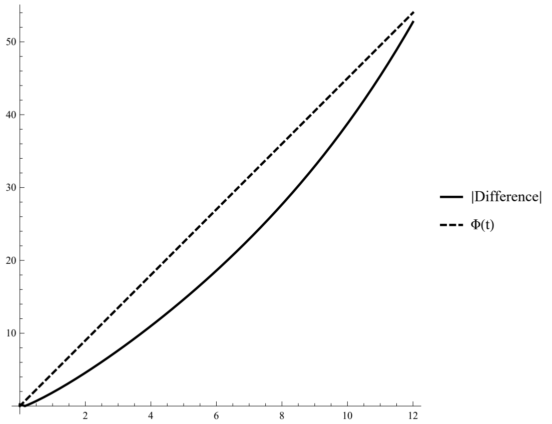
Consider the partial case $B = 0.07$, $\delta = 0.1$, $T = 6$, $q = 0.8$, $\tau = 2$, $\beta = 2$, $A = 100$, $W = 30$, $\mu = 0.1$ and $N \le 1$. Then $\alpha \approx 0.716881 < 1$ and $K = L = \frac{0.07}{100}30e^{-0.02} = 0.0205842$, $a = -0.1$, $b = 0.07e^{-0.2}$. According to Theorem 4.5 the equation (6.4) has a solution.

Consider the function $y(t) = t^2$. Because ${}_0^{RL}D_t^{0.8} t^2 = \frac{\Gamma(3)}{\Gamma(1.5)} t^{1.2}$ the inequality

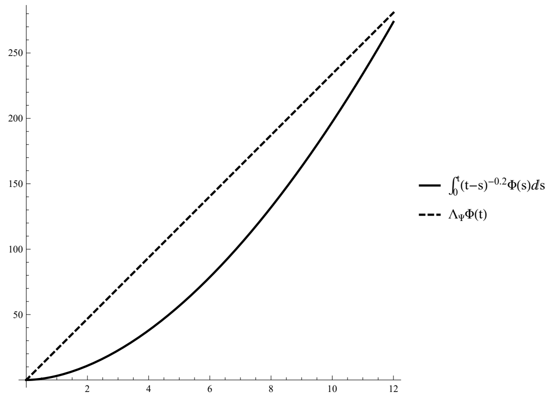
$$\left| \frac{\Gamma(3)}{\Gamma(3-0.8)} t^{2-0.8} - 0.07(t-2)^2 e^{-0.2} \Big(1 - \frac{t^2}{100}\Big) + 0.1t^2 \right| \le 4.5t, \qquad t \in [0, 12].$$

holds (see Figure 6.2a).

Consider $\Phi(t) = 4.5t$. The inequality $\int_0^t (t-s)^{0.8-1} 4.5s ds \le \Lambda_\Phi(4.5t)$ holds with $\Lambda_\Phi = 5.2$ (see Figure 6.2b).



(a) Graph of the difference $|\frac{\Gamma(3)}{\Gamma(3-0.8)} t^{2-0.8} - 0.07(t-2)^2 e^{-0.2}(1 - \frac{t^2}{100}) + 0.1t^2|$ and the function $\Phi(t) = 4.5t$ on $[0, 12]$.



(b) Graph of the fractional integral of the function $\Phi(t) = 4.5t$ and the function $\Lambda_\Phi \Phi(t)$ on $[0, 12]$.

Figure 6.2: Model 2.

According to Theorem 5.7(iii) the solution $N(t)$ of (6.4) satisfies

$$\left| N(t) - t^2 \right| \le 37.5892t, \qquad t \in [0, 12]$$

since $c_f = 5.2(1+Q)^4 1.04604 = 5.5(1.2475)^4 = 8.35315$, $C = \max_{t\in[0,12]} E_{0.8,0.8}(-0.1t^{0.8}) = 1$, $Q = (0.0205842 + 0.0205842 + 0.07e^{-0.2}) 5.2 E_{0.8}(\Gamma(0.8) 0.0205842\ 2^{0.8}) = 0.535671$.

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References

[1] R. Agarwal, S. Hristova, D. O'Regan, Explicit solutions of initial value problems for linear scalar Riemann–Liouville fractional differential equations with a constant delay, *Mathematics* **8**(2020), 32. https://doi.org/10.3390/math8010032

[2] R. Agarwal, S. Hristova, D. O'Regan, Ulam type stability for non-instantaneous impulsive Caputo fractional differential equations with finite state dependent delay, *Georgian Math. J.*, to appear. https://doi.org/10.1515/gmj-2020-2061

[3] M. Benchohra, S. Bouriah, J. J. Nieto, Existence and Ulam stability for nonlinear implicit differential equations with Riemann–Liouville fractional derivative, *Demonstr. Math.*, **52**(2019), 437–450. https://doi.org/10.1515/dema-2019-0032; MR4015245

[4] D. X. Cuong, On the Hyers–Ulam stability of Riemann–Liouville multi-order fractional differential equations, *Afr. Mat.* **30**(2019), 1041–1047. https://doi.org/10.1007/s13370-019-00701-3; MR4022267

[5] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, Vol. 2004, Springer, Berlin, Germany, 2010. https://doi.org/10.1007/978-3-642-14574-2; MR2680847

[6] S. A. H. Geritz, É. Kisdi, Mathematical ecology: why mechanistic models?, *J. Math. Biol* **65**(2012), No. 6–7, 1411–1415. https://doi.org/10.1007/s00285-011-0496-3; MR2993951

[7] S. Hristova, K. Ivanova, Ulam type stability of non-instantaneous impulsive Riemann–Liouville fractional differential equations (changed lower bound of the fractional derivative), *AIP Conference Proceedings* **2159**(2019), 030015. https://doi.org/10.1063/1.5127480

[8] R. W. Ibrahim, Generalized Ulam–Hyers stability for fractional differential equations, *Internat. J. Math.* **23**(2012), 1250056, 9 pp. https://doi.org/10.1142/S0129167X12500565; MR2914657

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V., Amsterdam, 2006. MR2218073

[10] I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.* **26** (2010), No. 1, 103–107. MR2676724

[11] J. Vanterler da C. Sousa, E. Capelas de Oliveira, F. G. Rodrigues, Ulam–Hyers stabilities of fractional functional differential equations, *AIMS Math.* **5**(2020), No. 2, 1346–1358. https://doi.org/10.3934/math.202009; MR4141917

[12] J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 63, 1–10. https://doi.org/10.14232/ejqtde.2011.1.63; MR2832769

[13] J. Wang, Y. Zhang, Ulam–Hyers–Mittag-Leffler stability of fractional-order delay differential equations, *Optimization* **63**(2014), No. 8, 1181–1190, https://doi.org/10.1080/02331934.2014.906597; MR3223606

[14] J. Wang, Y. Zhou, M. Fečkan, Nonlinear impulsive problems for fractional differential equations and Ulam stability, *Comput. Math. Appl.* **64**(2012), No. 10, 3389–3405. https://doi.org/10.1016/j.camwa.2012.02.021; MR2989367

[15] M. Ważewska-Czyżewska, A. Lasota, Mathematical problems of the dynamics of red blood cell population (in Polish), *Math. Stos. (3)* **6**(1976), 23–40. MR682081

[16] L. Xu, Q. Dong, G. Li, Existence and Hyers–Ulam stability for three-point boundary value problems with Riemann–Liouville fractional derivatives and integrals, *Adv. Difference Equ.* **2018**, Paper No. 458, 17 pp. https://doi.org/10.1186/s13662-018-1903-5; MR3887215

[17] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* **328**(2007), 1075–1081. https://doi.org/10.1016/j.jmaa.2006.05.061; MR2290034