



# Topological entropy for impulsive differential equations

*Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday*

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**Abstract.** A positive topological entropy is examined for impulsive differential equations via the associated Poincaré translation operators on compact subsets of Euclidean spaces and, in particular, on tori. We will show the conditions under which the impulsive mapping has the forcing property in the sense that its positive topological entropy implies the same for its composition with the Poincaré translation operator along the trajectories of given systems. It allows us to speak about chaos for impulsive differential equations under consideration. In particular, on tori, there are practically no implicit restrictions for such a forcing property. Moreover, the asymptotic Nielsen number (which is in difference to topological entropy a homotopy invariant) can be used there effectively for the lower estimate of topological entropy. Several illustrative examples are supplied.


**Keywords:** topological entropy, impulsive differential equation, Poincaré's operator, asymptotic Nielsen number, Lefschetz number, Carathéodory periodic solution.

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## 1 Introduction

The main aim of the present paper is to establish a positive topological entropy for impulsive differential equations via the associated Poincaré translation operators along their trajectories. We will present, under natural assumptions, the relationship for the topological entropies of given impulsive maps and their compositions with the Poincaré operators, from which a positive topological entropy of the composition, determining chaos for the impulsive differential equations, is implied by the one of the impulsive map. On tori, the Ivanov theorem (see

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[13, 17]), using effectively the asymptotic Nielsen number (which is in difference to topological entropy a homotopy invariant), is applied for the lower estimate of topological entropy. Moreover, this application can be expressed on tori in terms of the Lefschetz numbers which are significantly easier for calculations.

Although various sorts of chaos have been already investigated for impulsive differential equations (see e.g. [1, 5, 6, 18, 24], and the references therein), as far as we know, a topological entropy has been examined, with only a few exceptions like [3], exclusively for non-impulsive differential equations and dynamical systems (see e.g. [11, 14, 22, 25, 27], and the references therein). That is why we would like, besides other things, to eliminate here this handicap.

For this goal, we will firstly recall Bowen's definition of a topological entropy [7], jointly with its basic properties. We will also recall the Ivanov theorem [13] and its consequences on tori. For the systems of ordinary differential equations on  $\mathbb{R}^n$  and  $\mathbb{R}^n/\mathbb{Z}^n$ , we will define the associated Poincaré translation operators along the trajectories and point out the relationship between Carathéodory periodic solutions and periodic points of the Poincaré operators. Before a separate formulation of the main theorems about a positive topological entropy for impulsive differential equations on Euclidean spaces and tori, we will deduce mentioned crucial relationship for topological entropies of impulsive maps and their compositions with the Poincaré operators. The obtained results will be illustrated by simple examples and commented by concluding remarks.

## 2 Preliminaries

Although the topological entropy, which is a central notion of our paper, was defined by Bowen [7] (cf. also [2, p. 188], [23, pp. 369–370]) for uniformly continuous maps, we will restrict ourselves (from the practical reasons) to a subclass of continuous maps on compact metric spaces. For more details about the topological entropy, see e.g. [19].

**Definition 2.1.** Let  $(X, d)$  be a compact metric space and  $f: X \rightarrow X$  be a continuous map. A set  $S \subset X$  is called  $(n, \varepsilon)$ -separated for  $f$ , for a positive integer  $n$  and  $\varepsilon > 0$ , if for every pair of distinct points  $x, y \in S$ ,  $x \neq y$ , there is at least one  $k$  with  $0 \leq k < n$  such that  $d(f^k(x), f^k(y)) > \varepsilon$ . Then, denoting the number of different orbits of length  $n$  by

$$r(n, \varepsilon, f) := \max \{ \#S : S \subset X \text{ is an } (n, \varepsilon)\text{-separated set for } f \},$$

where  $\#S$  stands for the cardinality (i.e. the number of elements) of  $S$ , the *topological entropy*  $h(f)$  of  $f$  is defined as

$$h(f) := \lim_{\varepsilon \rightarrow 0} \left[ \limsup_{n \rightarrow \infty} \frac{1}{n} \log(r(n, \varepsilon, f)) \right].$$

It will be convenient to recall the following properties of topological entropy. The first lemma justifies Definition 2.1 in the sense that the metric  $d$  in the notation of  $h(f)$  can be omitted.

**Lemma 2.2** (cf. e.g. [19, Proposition 3.1.2], [26, Corollary 7.5.2]). *If  $X$  is a compact metrisable space and  $d'$  is any metric on  $X$ , then  $h(f) = h_{d'}(f)$  holds for any continuous map  $f: X \rightarrow X$ , where  $h_{d'}(f)$  denotes the topological entropy of  $f$  on  $X$  calculated with any specific metric  $d'$ .*

**Lemma 2.3** (cf. e.g. [2, Lemma 4.1.10], [23, Theorem IX.1.3]). *Let  $f$  be a continuous map on  $X$ . Assume  $X = X_1 \cup \dots \cup X_k$  is a decomposition into disjoint closed invariant subsets which are a*

positive distance apart. Then

$$h(f) = \max_{j=1, \dots, k} h\left(f|_{X_j}\right).$$

**Lemma 2.4** (cf. e.g. [2, Lemma 4.1.5], [23, Theorem IX.1.4]). *Let  $f$  be a continuous map on a compact metric space  $X$ . Let  $\Omega \subset X$  be the nonwandering points of  $f$ , i.e. the points  $p \in \Omega$  such that, for every neighbourhood  $U$  of  $p$ , there is an integer  $n > 0$  such that  $f^n(U) \cap U \neq \emptyset$ . Then the entropy  $h(f)$  of  $f$  equals the entropy of  $f$  restricted to its nonwandering set  $\Omega$ , namely  $h(f) = h(f|_{\Omega})$ .*

**Lemma 2.5** (cf. e.g. [23, Theorem IX.1.5]). *Let  $f$  be a continuous map on a compact metric space  $X$  for which the nonwandering set  $\Omega$  consists of a finite number of periodic orbits. Then the topological entropy  $h(f)$  of  $f$  is zero,  $h(f) = 0$ . In particular, the same is true, provided  $\bigcap_{j=0}^{\infty} f^j(X)$  is finite (see e.g. [2, p. 194]).*

Before formulating the following lemma, let us recall that a map  $s: X \rightarrow Y$  is *uniformly finite to one* if  $s^{-1}(y)$  has a finite number of points for each  $y \in Y$ , and there is a bound on the number of elements in  $s^{-1}(y)$  which is independent of  $y \in Y$ .

**Lemma 2.6** (cf. e.g. [23, Theorem IX.1.8]). *Assume that  $f: X \rightarrow X$  and  $g: Y \rightarrow Y$  are continuous maps, where  $(X, d)$  and  $(Y, d')$  are compact metric spaces with metrics  $d$  and  $d'$ , respectively. Assume  $s: X \rightarrow Y$  is a semi-conjugacy from  $f$  to  $g$ , i.e. (i)  $s$  is continuous, (ii)  $s$  is “onto”, (iii)  $s \circ f = g \circ s$ , that is uniformly finite to one. Then  $h(f) = h(g)$ .*

If  $X$  is a compact polyhedron, then we can apply in the form of proposition the following Jiang’s slight generalization (see [17]) of the Ivanov theorem [13], for the lower estimate of the topological entropy. For the definition and properties of the Nielsen number, which is unlike to topological entropy a homotopy invariant, see e.g. [9, 15].

**Proposition 2.7.** *Suppose  $X$  is a compact polyhedron and, in particular (for our needs), the torus  $X = \mathbb{R}^n / \mathbb{Z}^n$ . Let  $f: X \rightarrow X$  be a continuous map. Then for any continuous map  $g: X \rightarrow X$  homotopic to  $f$  (i.e.  $g \sim f$ ), the topological entropy  $h(g)$  satisfies  $h(g) \geq \log N^{\infty}(f)$ , where*

$$N^{\infty}(f) := \max \left\{ 1, \limsup_{m \rightarrow \infty} (N(f^m))^{\frac{1}{m}} \right\}$$

is the asymptotic Nielsen number of  $f$  and  $N(f^m)$  is the standard Nielsen number of the  $m$ -th iterate of  $f$ . Thus, if  $N^{\infty}(f) > 1$ , then

$$h(g) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} \log N(f^m) > 0$$

holds for any  $g \sim f$ .

**Remark 2.8.** For the torus  $X = \mathbb{R}^n / \mathbb{Z}^n$ , we have still (see [8])

$$N(f) = |\lambda(f)|,$$

where  $\lambda(f)$  denotes the *Lefschetz number* of  $f$  (for its definition and properties, see e.g. [9]), by which the inequality

$$h(g) \geq \log N^{\infty}(f) \tag{2.1}$$

can be rewritten into

$$h(g) \geq \log \max \left\{ 1, \limsup_{m \rightarrow \infty} |\lambda(f^m)|^{\frac{1}{m}} \right\}, \tag{2.2}$$

which is significantly easier for verification.

Hence, if

$$\limsup_{m \rightarrow \infty} |\lambda(f^m)|^{\frac{1}{m}} > 1,$$

then

$$h(g) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} \log |\lambda(f^m)| > 0$$

holds for any  $g \sim f$ .

If, in particular,  $f: \mathbb{R}^n/\mathbb{Z}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$  is an endomorphism defined by an integer matrix  $A$ , whose eigenvalues are  $\lambda_1, \dots, \lambda_n$ , then (see e.g. [16, Example, p. 192])

$$N^\infty(f) = \begin{cases} 1, & \text{if } \lambda(f) = 0 \\ \prod_{|\lambda_k| > 1} |\lambda_k|, & \text{otherwise,} \end{cases} \quad (2.3)$$

and  $\lambda(f) = \det(\mathcal{I} - A) = \prod_{k=1}^n (1 - \lambda_k)$ , where  $\lambda(f)$  stands for the Lefschetz number of  $f$ .

Now, consider the vector differential equation

$$x' = F(t, x), \quad (2.4)$$

where  $F: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the *Carathéodory mapping* such that  $F(t, x) \equiv F(t + \omega, x)$ , for some given  $\omega > 0$ , i.e.

- (i)  $F(\cdot, x): [0, \omega] \rightarrow \mathbb{R}^n$  is measurable, for every  $x \in \mathbb{R}^n$ ,
- (ii)  $F(t, \cdot): \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous, for almost all (a.a.)  $t \in [0, \omega]$ .

Let, furthermore (2.4) satisfy a *uniqueness condition* and all solutions of (2.4) entirely exist on the whole line  $(-\infty, \infty)$ .

By a (*Carathéodory*) *solution*  $x(\cdot)$  of (2.4), we understand a locally absolutely continuous function, i.e.  $x \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$ , which satisfies (2.4) for a.a.  $t \in \mathbb{R}$ .

We can associate to (2.4) the *Poincaré translation operator*  $T_\omega: \mathbb{R}^n \rightarrow \mathbb{R}^n$  along its trajectories as follows:

$$T_\omega(x_0) := \{x(\omega): x(\cdot) \text{ is a solution of (2.4) such that } x(0) = x_0\}. \quad (2.5)$$

It is well known (see e.g. [20, Chapter 1.1]) that  $T_\omega$  is a homeomorphism such that  $T_\omega^k = T_{k\omega}$ , for every  $k \in \mathbb{N}$ .

Assuming still that

$$F(t, \dots, x_j, \dots) \equiv F(t, \dots, x_j + 1, \dots), \quad j = 1, \dots, n, \quad (2.6)$$

where  $x = (x_1, \dots, x_n)$ , we can also consider (2.4) on the torus  $\mathbb{R}^n/\mathbb{Z}^n$ , which can be endowed with the metric

$$\hat{d}(x, y) := \min \{d_{Eucl}(a, b): a \in [x], b \in [y]\},$$

for all  $x, y \in \mathbb{R}^n/\mathbb{Z}^n$ , where  $d_{Eucl}(a, b) := \sqrt{\sum_{j=1}^n (a_j - b_j)^2}$ , for all  $a, b \in \mathbb{R}^n$ .

The associated *Poincaré translation operator*  $\hat{T}_\omega: \mathbb{R}^n/\mathbb{Z}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$  along the trajectories of (2.4), considered on  $\mathbb{R}^n/\mathbb{Z}^n$ , takes the form  $\hat{T}_\omega := \tau \circ T_\omega$ , where  $T_\omega$  was defined in (2.5), and  $\tau: \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n, x \rightarrow [x] := \{y \in \mathbb{R}^n: (y - x) \in \mathbb{Z}^n\}$  is the natural (canonical) projection. It is

well known (see e.g. [10, Chapter XVII]) that  $\hat{T}_\omega$  is also a homeomorphism such that  $\hat{T}_\omega^k = \hat{T}_{k\omega}$ , for every  $k \in \mathbb{N}$ . In particular, for  $n = 1$ ,  $\hat{T}_\omega$  is an orientation-preserving homeomorphism.

One can easily detect the one-to-one correspondence between the  $k\omega$ -periodic solutions of (2.4), i.e.  $x(t) \equiv x(t + k\omega)$  but  $x(t) \not\equiv x(t + j\omega)$  for  $j < k$ , and  $k$ -periodic points of  $T_\omega$ , i.e.  $x_0 = T_\omega^k(x_0)$  but  $x_0 \neq T_\omega^j(x_0)$  for  $j < k$ , where  $x_0 = x(0)$  and  $j, k$  are positive integers.

The same correspondence holds between  $k\omega$ -periodic solutions  $\hat{x}(\cdot) := \tau \circ x(\cdot)$  of (2.4), considered on  $\mathbb{R}^n / \mathbb{Z}^n$ , and  $k$ -periodic points  $\hat{x}_0 = \tau \circ x_0$  of  $\hat{T}_\omega := \tau \circ T_\omega$ , where  $\hat{x}_0 = \hat{x}(0)$ .

The impulsive differential equations, i.e. the differential equations (2.4) with impulses at  $t = t_j := j\omega$ ,  $j \in \mathbb{Z}$ , will be considered separately on the spaces  $\mathbb{R}^n$  and  $\mathbb{R}^n / \mathbb{Z}^n$ . Their solutions will be also understood in the same Carathéodory sense, i.e.  $x \in AC[t_j, t_{j+1}]$ ,  $j \in \mathbb{Z}$ .

### 3 Topological entropy for impulsive differential equations on $\mathbb{R}^n$

Consider the vector impulsive differential equation

$$\begin{cases} x' = F(t, x), & t \neq t_j := j\omega, \text{ for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), & j \in \mathbb{Z}, \end{cases} \quad (3.1)$$

where  $F: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the Carathéodory mapping such that  $F(t, x) \equiv F(t + \omega, x)$ , equation (2.4) satisfies a uniqueness condition and a global existence of all its solutions on  $(-\infty, \infty)$ . Let, furthermore,  $I: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a compact continuous impulsive mapping such that  $K_0 := \bar{I}(\mathbb{R}^n)$  and  $I(K_0) = K_0$ .

**Proposition 3.1.** *Let  $T_\omega: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the associated Poincaré translation operator along the trajectories of (2.4), defined in (2.5), such that  $K_1 := T_\omega(K_0)$  and  $K_0 \subset K_1$ . Then the equality*

$$h\left(I|_{K_1} \circ T_\omega|_{K_0}\right) = h\left(I|_{K_0}\right) \quad (3.2)$$

holds for the topological entropies  $h$  of the maps  $I|_{K_1} \circ T_\omega|_{K_0}: K_0 \rightarrow K_0$  and  $I|_{K_0}: K_0 \rightarrow K_0$ .

*Proof.* We have the diagram

$$\begin{array}{ccccc} K_0 & \xrightarrow{T_\omega} & K_1 & \xrightarrow{I} & K_0 \\ T_\omega \downarrow & & T_\omega \uparrow & & \downarrow T_\omega \\ K_1 & \xrightarrow{I} & K_0 & \xrightarrow{T_\omega} & K_1, \end{array}$$

where  $K_0, K_1 \subset \mathbb{R}^n$  are compact subsets, and  $T_\omega|_{K_0}: K_0 \rightarrow K_1$  is (i) continuous, (ii) ‘‘onto’’ and uniformly finite to one, (iii)  $T_\omega|_{K_0} \circ (I|_{K_1} \circ T_\omega|_{K_0}) = (T_\omega|_{K_0} \circ I|_{K_1}) \circ T_\omega|_{K_0}$ , i.e. it is a semi-conjugacy.

Thus, applying Lemma 2.6, we obtain that

$$h\left(I|_{K_1} \circ T_\omega|_{K_0}\right) = h\left(T_\omega|_{K_0} \circ I|_{K_1}\right).$$

Endowing  $K_0, K_1$  with the respective metrics  $d, d'$ , where

$$d(x, y) := d_{\text{Eucl}}(x, y), \text{ for all } x, y \in K_0,$$

$$d'(x, y) := d_{\text{Eucl}}(T_\omega(x), T_\omega(y)), \text{ for all } x, y \in K_0,$$

$$d'(x', y') := d_{\text{Eucl}}(x', y'), \text{ for all } x' (= T_\omega(x)), y' (= T_\omega(y)) \in K_1,$$

we can write in this notation that

$$h\left(T_\omega|_{K_0} \circ I|_{K_1}\right) = h_{d'}\left(I|_{K_1}\right), \quad \text{resp.} \quad h\left(I|_{K_1} \circ T_\omega|_{K_0}\right) = h_{d'}\left(I|_{K_1}\right), \quad (3.3)$$

where the lower index  $d'$  denotes the respective metric.

We can also write that

$$h_{d'}\left(I|_{K_1}\right) = h_{d'}\left(I|_{T_\omega(K_0)}\right) = h_{d'}\left(I|_{K_0}\right). \quad (3.4)$$

Furthermore, since the topological entropy of given continuous maps on compact metric spaces does not depend, according to Lemma 2.2, on the used metrics, we get still that

$$h_{d'}\left(I|_{K_0}\right) = h\left(I|_{K_0}\right). \quad (3.5)$$

Summing up the relations (3.3)–(3.5), we arrive at (3.2), as claimed.  $\square$

**Remark 3.2.** It can be readily seen from (3.2) that a positive topological entropy holds for  $I|_{K_1} \circ T_\omega|_{K_0}$ , when  $h(I|_{K_0}) > 0$  and  $K_0 \subset K_1$ , which is a rather implicit condition. Since  $K_1 \setminus K_0$  is the wandering set for  $I$ , condition (3.2) is in a certain sense sharp (cf. also (3.3)). On the other hand, if  $K_0$  contains only a finite number of periodic orbits for  $I|_{K_0}$ , then according to Lemma 2.5,  $h(I|_{K_0}) = 0$ , by which also  $h(I|_{K_1} \circ T_\omega|_{K_0}) = 0$ .

**Corollary 3.3.** Consider the scalar impulsive differential equation, i.e. (3.1) for  $n = 1$ . If  $[a, b] \subset [T_\omega(a), T_\omega(b)]$  holds for the Poincaré translation operator  $T_\omega$  along the trajectories of (2.4), defined in (2.5), where  $[a, b] = I([a, b])$ , then condition (3.2) takes the form

$$h\left(I|_{[T_\omega(a), T_\omega(b)]} \circ T_\omega|_{[a, b]}\right) = h\left(I|_{[a, b]}\right). \quad (3.6)$$

*Proof.* Since  $T_\omega: \mathbb{R} \rightarrow \mathbb{R}$  must be, under a uniqueness condition, strictly increasing, we have that  $K_1 = [T_\omega(a), T_\omega(b)]$ , where  $K_0 = [a, b]$ . In this notation,  $K_0 \subset K_1$ , and condition (3.2) takes the form (3.6).  $\square$

**Definition 3.4.** We say that the vector impulsive differential equation (3.1) exhibits *chaos in the sense of a positive topological entropy*  $h$  if  $h(I|_{K_1} \circ T_\omega|_{K_0}) > 0$  holds for the composition of the associated Poincaré translation operator  $T_\omega$  along the trajectories of (2.4), defined in (2.5), with the compact impulsive mapping  $I: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , where  $K_0 := \overline{I(\mathbb{R}^n)}$  and  $K_1 := T_\omega(K_0)$ .

**Theorem 3.5.** The vector impulsive differential equation (3.1) exhibits, under the above assumptions, chaos in the sense of Definition 3.4, if  $I(K_0) = K_0$  and  $K_0 \subset K_1$ , where  $K_0 := \overline{I(\mathbb{R}^n)}$  and  $K_1 := T_\omega(K_0)$ , jointly with  $h(I|_{K_0}) > 0$ .

*Proof.* The proof follows directly from the inequality (3.2) in Proposition 3.1.  $\square$

**Corollary 3.6.** The scalar ( $n = 1$ ) impulsive differential equation (3.1) exhibits, under the above assumptions, chaos in the sense of Definition 3.4, provided  $h(I|_{[a, b]}) > 0$  holds, jointly with  $\overline{I(\mathbb{R})} = I([a, b]) = [a, b] \subset [T_\omega(a), T_\omega(b)]$ .

*Proof.* The proof follows directly from the equality (3.6) in Corollary 3.3, where  $K_0 = [a, b]$  and  $K_1 = [T_\omega(a), T_\omega(b)]$ .  $\square$

The following simple illustrative examples demonstrate an application of Corollary 3.6 to scalar ( $n = 1$ ) linear and semi-linear impulsive differential equations.

**Example 3.7.** Consider the linear impulsive equation

$$\begin{cases} x' = p(t)x + q(t), & t \neq t_j := j\omega, \quad \text{for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), & j \in \mathbb{Z}, \end{cases} \quad (3.7)$$

where  $p, q: \mathbb{R} \rightarrow \mathbb{R}$  are measurable functions such that  $p(t) \equiv p(t + \omega)$ ,  $q(t) \equiv q(t + \omega)$ , and the compact (continuous) impulsive function  $I: \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $\overline{I(\mathbb{R})} = [a, b]$  and  $I([a, b]) = [a, b]$ .

Since the general solution of  $x' = p(t)x + q(t)$  reads

$$x(t) = x(0) e^{\int_0^t p(s) ds} + \int_0^t e^{\int_s^t p(r) dr} q(s) ds,$$

the required inclusion  $[a, b] \subset [T_\omega(a), T_\omega(b)]$  in Corollary 3.6 takes the form

$$\begin{aligned} a &\geq a e^{\int_0^\omega p(t) dt} + \int_0^\omega e^{\int_s^\omega p(r) dr} q(s) ds, \\ b &\leq b e^{\int_0^\omega p(t) dt} + \int_0^\omega e^{\int_s^\omega p(r) dr} q(s) ds. \end{aligned}$$

Specially, for  $a = 0$ ,  $b = 1$ :

$$0 \geq \int_0^\omega e^{\int_s^\omega p(r) dr} q(s) ds, \quad 1 \leq e^{\int_0^\omega p(t) dt} + \int_0^\omega e^{\int_s^\omega p(r) dr} q(s) ds.$$

In order to satisfy the first inequality, we can assume that  $q(t) \leq 0$ , for a.a.  $t \in [0, \omega]$ . The second inequality can be then more restrictively rewritten into

$$e^{\int_0^\omega p(t) dt} \geq 1 + \left| \int_0^\omega e^{\int_0^s p(r) dr} q(s) ds \right|.$$

Denoting  $P := \left| \int_0^\omega p(t) dt \right|$  and  $Q := \left| \int_0^\omega q(t) dt \right|$ , we can rewrite it finally as

$$\begin{aligned} e^P(1 - Q) &\geq 1, \quad \text{resp. } Q \leq \frac{e^P - 1}{e^P}, \\ &\text{jointly with } q(t) \leq 0, \quad \text{for a.a. } t \in [0, \omega]. \end{aligned} \quad (3.8)$$

Specially, for  $p(t) \equiv p > 0$ , we can require that

$$Q \leq \frac{e^{p\omega} - 1}{e^{p\omega}} \quad \text{and} \quad q(t) \leq 0,$$

for a.a.  $t \in [0, \omega]$ , or  $-p e^{-p\omega} \leq q(t) \leq 0$ , for a.a.  $t \in [0, \omega]$ .

Thus, the linear impulsive equation (3.7) exhibits chaos in the sense of Definition 3.4, provided (3.8) holds jointly with  $h(I|_{[0,1]}) > 0$ .

The last inequality is satisfied, for instance, for the 1-periodically extended tent map  $I(x) \equiv I(x + 1)$ , where

$$I(x) := \begin{cases} 2x, & \text{for } x \in [0, \frac{1}{2}], \\ 2(1 - x), & \text{for } x \in [\frac{1}{2}, 1], \end{cases}$$

because  $I(\mathbb{R}) = I([0, 1]) = [0, 1]$  and (cf. (3.6))

$$h\left(I|_{[T_\omega(0), T_\omega(1)]} \circ T_\omega|_{[0,1]}\right) = h\left(I|_{[0,1]}\right) = \log 2.$$

For the last inequality, see e.g. [19, Corollary 15.2.14].

**Example 3.8.** Consider the semi-linear impulsive equation

$$\begin{cases} x' = p(t, x)x + q(t, x), & t \neq t_j := j\omega, \quad \text{for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), & j \in \mathbb{Z}, \end{cases} \quad (3.9)$$

where  $p, q: \mathbb{R}^2 \rightarrow \mathbb{R}$  are Carathéodory functions such that  $p(t, x) \equiv p(t + \omega, x)$ ,  $q(t, x) \equiv q(t + \omega, x)$ , and the compact (continuous) impulsive function  $I: \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $I(\mathbb{R}) = [a, b]$  and  $I([a, b]) = [a, b]$ .

Since the solutions  $x_0(\cdot), x_1(\cdot)$  of  $x' = p(t, x)x + q(t, x)$  such that  $x_0(0) = 0$ ,  $x_1(0) = 1$  can be implicitly expressed as

$$\begin{aligned} x_0(t) &= \int_0^t e^{\int_s^t p(r, x_0(r)) dr} q(s, x_0(s)) ds, \\ x_1(t) &= e^{\int_0^t p(s, x_1(s)) ds} + \int_0^t e^{\int_s^t p(r, x_1(r)) dr} q(s, x_1(s)) ds, \end{aligned}$$

one can proceed in a similar way as in Example 3.7.

Hence, the required inclusion  $[0, 1] \subset [T_\omega(0), T_\omega(1)]$  (for  $a = 0, b = 1$ ) in Corollary 3.6 takes this time the form

$$\begin{aligned} 0 &\geq \int_0^\omega e^{\int_s^\omega p(r, x_0(r)) dr} q(s, x_0(s)) ds, \\ 1 &\leq e^{\int_0^\omega p(t, x_1(t)) dt} + \int_0^\omega e^{\int_s^\omega p(r, x_1(r)) dr} q(s, x_1(s)) ds. \end{aligned}$$

In order to satisfy the first inequality, we can assume that  $q(t, x) \leq 0$ , for a.a.  $t \in [0, \omega]$  and all  $x \in \mathbb{R}$ . The second inequality can be then more restrictively rewritten into

$$e^{\int_0^\omega p(t, x_1(t)) dt} \geq 1 + \left| \int_0^\omega e^{\int_s^\omega p(r, x_1(r)) dr} q(s, x_1(s)) ds \right|.$$

Assuming still the existence of real constants  $p_0, p_1, q_1$  such that

$$0 < p_0 \leq p(t, x) \leq p_1 \quad \text{and} \quad |q(t, x)| \leq q_1, \quad \text{for a.a. } t \in [0, \omega] \text{ and all } x \in \mathbb{R},$$

we still require that

$$q_1 \leq \frac{e^{p_0\omega} - 1}{\omega e^{p_1\omega}},$$

i.e. jointly with  $q(t, x) \leq 0$ ,

$$-\frac{e^{p_0\omega} - 1}{\omega e^{p_1\omega}} \leq q(t, x) \leq 0, \quad \text{for a.a. } t \in [0, \omega] \text{ and all } x \in \mathbb{R}, \quad (3.10)$$

where  $0 < p_0 \leq p(t, x)$ , for a.a.  $t \in [0, \omega]$  and all  $x \in \mathbb{R}$ .

Thus, the semi-linear impulsive equation (3.9) exhibits chaos in the sense of Definition 3.4, provided (3.10) holds jointly with  $h(I|_{[0,1]}) > 0$ . This inequality can be satisfied like in Example 3.7, for instance, for the 1-periodically extended tent map.

Now, we would like to apply Theorem 3.5 to the nonlinear vector impulsive differential equation (3.1).



**Example 3.9.** Consider (3.1), where  $F$  and  $I$  are as above, and assume that

$$\begin{cases} f_j(t, \dots, x_j, \dots) > 0 & \text{holds for all } x_j \geq b_j, j = 1, \dots, n, \\ f_j(t, \dots, x_j, \dots) < 0 & \text{holds for all } x_j \leq a_j, j = 1, \dots, n, \end{cases} \quad (3.11)$$

uniformly for a.a.  $t \in [0, \omega]$  and all the remaining components of  $x = (x_1, \dots, x_n)$ , where  $F(t, x) = (f_1(t, x), \dots, f_n(t, x))^T$  and  $(I(\mathbb{R}^n) = ) K_0 := [a_1, b_1] \times \dots \times [a_n, b_n]$ ,  $I(K_0) = K_0$ .

Since, in view of (3.11), the inequalities  $x_j(\omega, a_j) \leq a_j$  and  $x_j(\omega, b_j) \geq b_j$ ,  $j = 1, \dots, n$ , hold for all the components of the solutions  $x(\cdot, a)$  and  $x(\cdot, b)$  such that  $x(0, a) = a$  and  $x(0, b) = b$ , where  $a = (a_1, \dots, a_n)$ ,  $b = (b_1, \dots, b_n)$ , the particular inclusion  $K_0 \subset K_1$  is satisfied, where  $K_0 := [a_1, b_1] \times \dots \times [a_n, b_n]$  and  $K_1 := T_\omega(K_0)$ .

Thus, the vector impulsive equation (3.1) exhibits, according to Theorem 3.5, chaos in the sense of Definition 3.4, provided (3.11) holds jointly with  $h(I|_{K_0}) > 0$ , where  $K_0 := [a_1, b_1] \times \dots \times [a_n, b_n]$ . This inequality can be satisfied, for instance when  $K_0 := [0, 1]^n$  (i.e. for  $[a_j, b_j] = [0, 1]$ ,  $j = 1, \dots, n$ ), for the Cartesian product  $\mathbf{I}$  of 1-periodically extended tent maps, because  $\mathbf{I}([0, 1]^n) = [0, 1]^n$  and (see e.g. [26])

$$h(\mathbf{I}|_{K_1} \circ T_\omega|_{[0, 1]^n}) = h(\mathbf{I}|_{[0, 1]^n}) = n \log 2.$$

**Remark 3.10.** Observe that condition (3.11) imposed on the equations (3.7) and (3.9) takes the simple forms  $p(t) + q(t) > 0$ ,  $q(t) < 0$ , for a.a.  $t \in [0, \omega]$ , resp.  $p(t, 1) + q(t, 1) > 0$ ,  $q(t, 0) < 0$ , for a.a.  $t \in [0, \omega]$ .

## 4 Topological entropy for impulsive differential equations on $\mathbb{R}^n / \mathbb{Z}^n$

Consider (3.1) and assume additionally that (2.6) holds jointly with

$$I(\dots, x_j, \dots) \equiv I(\dots, x_j + 1, \dots) \pmod{1}, \quad j = 1, \dots, n, \quad (4.1)$$

where  $x = (x_1, \dots, x_n)$ .

Because of the commutative diagram

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{T_\omega} & \mathbb{R}^n & \xrightarrow{I} & \mathbb{R}^n \\ \tau \downarrow & & \downarrow \tau & & \downarrow \tau \\ \mathbb{R}^n / \mathbb{Z}^n & \xrightarrow{\hat{T}_\omega} & \mathbb{R}^n / \mathbb{Z}^n & \xrightarrow{\hat{I}} & \mathbb{R}^n / \mathbb{Z}^n, \end{array}$$

where  $\tau$  is the natural (canonical) projection,  $\hat{T}_\omega := \tau \circ T_\omega : \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$ , where  $T_\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the Poincaré translation operator along the trajectories of (2.4), defined in (2.5), and  $\hat{I} = \tau \circ I : \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$ , where  $I : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the impulsive mapping in (3.1), we can advantageously consider (3.1) on the torus  $\mathbb{R}^n / \mathbb{Z}^n$ , in the metric

$$\hat{d} : \mathbb{R}^n / \mathbb{Z}^n \times \mathbb{R}^n / \mathbb{Z}^n \rightarrow \left[0, \frac{\sqrt{n}}{2}\right],$$

where  $\hat{d}(x, y) := \min \{d_{Eucl}(a, b) : a \in [x], b \in [y]\}$ .

Since  $\hat{T}_\omega$  is well known (see e.g. [10, Chapter XVII]) to be a homeomorphism and, in particular for  $n = 1$ , even an orientation-preserving homeomorphism, the composition

$$\widehat{I \circ T_\omega} := \hat{I} \circ \hat{T}_\omega: \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$$

is continuous in  $(\mathbb{R}^n / \mathbb{Z}^n, \hat{d})$ .

We can therefore give the following analogy of Proposition 3.1 on  $\mathbb{R}^n / \mathbb{Z}^n$ .

**Proposition 4.1.** *The equality*

$$h(\widehat{I \circ T_\omega}) = h(\hat{I}) \quad (4.2)$$

holds, under the above assumptions and  $\hat{I}(\mathbb{R}^n / \mathbb{Z}^n) = \mathbb{R}^n / \mathbb{Z}^n$ , for the topological entropies  $h$  of the maps  $\widehat{I \circ T_\omega}: \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$  and  $\hat{I}: \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$  in  $(\mathbb{R}^n / \mathbb{Z}^n, \hat{d})$ .

*Proof.* We can proceed analogously, but (since  $\mathbb{R}^n / \mathbb{Z}^n$  is compact and  $\hat{I}$  is “onto”) in a simpler way, as in the proof of Proposition 3.1.

We have the diagram

$$\begin{array}{ccccc} \mathbb{R}^n / \mathbb{Z}^n & \xrightarrow{\hat{T}_\omega} & \mathbb{R}^n / \mathbb{Z}^n & \xrightarrow{\hat{I}} & \mathbb{R}^n / \mathbb{Z}^n \\ \hat{T}_\omega \downarrow & & \hat{T}_\omega \uparrow & & \downarrow \hat{T}_\omega \\ \mathbb{R}^n / \mathbb{Z}^n & \xrightarrow{\hat{I}} & \mathbb{R}^n / \mathbb{Z}^n & \xrightarrow{\hat{T}_\omega} & \mathbb{R}^n / \mathbb{Z}^n, \end{array}$$

where  $\hat{T}_\omega: \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$  is a homeomorphism and “onto”.

Thus, according to Lemma 2.6, we obtain that

$$h(\hat{I} \circ \hat{T}_\omega) = h(\hat{T}_\omega \circ \hat{I}).$$

Endowing  $\mathbb{R}^n / \mathbb{Z}^n$  with the new metric  $\hat{d}'$ , where

$$\hat{d}'(x, y) := \hat{d}(\hat{T}_\omega(x), \hat{T}_\omega(y)), \quad \text{for all } x, y \in \mathbb{R}^n / \mathbb{Z}^n,$$

we have that

$$h(\widehat{I \circ T_\omega}) = h(\widehat{T_\omega \circ I}) = h_{\hat{d}'}(\hat{I}),$$

where the lower index  $\hat{d}'$  denotes the respective metric. Furthermore, we get still, according to Lemma 2.2,

$$h_{\hat{d}'}(\hat{I}) = h(\hat{I}),$$

and, after all, that

$$h(\widehat{I \circ T_\omega}) = h(\hat{I}),$$

i.e. (4.2), as claimed.  $\square$

**Definition 4.2.** We say that the vector impulsive differential equation (3.1) exhibits on  $\mathbb{R}^n / \mathbb{Z}^n$  (cf. also (2.6), (4.1)) *chaos in the sense of a positive topological entropy*  $h$  if  $h(\widehat{I \circ T_\omega}) > 0$  holds for the map  $\widehat{I \circ T_\omega}: \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$  in  $(\mathbb{R}^n / \mathbb{Z}^n, \hat{d})$ , defined above.

**Theorem 4.3.** *The vector impulsive differential equation (3.1) exhibits on  $\mathbb{R}^n / \mathbb{Z}^n$ , under the above assumptions and additionally (2.6), (4.1), jointly with  $\hat{I}(\mathbb{R}^n / \mathbb{Z}^n) = \mathbb{R}^n / \mathbb{Z}^n$ , chaos in the sense of Definition 4.2, provided  $h(\hat{I}) > 0$  holds for the impulsive mapping  $\hat{I}: \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$  in the metric  $\hat{d}$ .*

*Proof.* The proof follows directly from the equality (4.2) in Proposition 4.1.  $\square$

The following corollary can help us to calculate effectively the topological entropy  $h(\hat{I})$ , and to ensure chaos for (3.1) on  $\mathbb{R}^n/\mathbb{Z}^n$  (cf. [6, Theorem 5.2]).

**Corollary 4.4.** *Let  $\hat{I}: \mathbb{R}^n/\mathbb{Z}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$  be defined by an integer matrix  $A$ , whose eigenvalues are  $\lambda_1, \dots, \lambda_n$ . Then*

$$h(\hat{I}) = \sum_{|\lambda_k| > 1} \log |\lambda_k|$$

*holds for the topological entropy of  $\hat{I}$ , provided  $\prod_{k=1}^n (1 - \lambda_k) \neq 0$ . Therefore, if*

$$\sum_{|\lambda_k| > 1} \log |\lambda_k| > 0 \quad \text{and} \quad \prod_{k=1}^n (1 - \lambda_k) \neq 0,$$

*then (3.1) exhibits on  $\mathbb{R}^n/\mathbb{Z}^n$  under (2.6) chaos in the sense of Definition 4.2.*

*Proof.* The first assertion is well known (see e.g. [26, p. 203] and cf. the preliminaries in Section 2). The second part is, on this basis, an immediate consequence of Theorem 4.3.  $\square$

**Example 4.5.** As an illustrative example of the application of Corollary 4.4, let us consider (3.1) on  $\mathbb{R}^2/\mathbb{Z}^2$  (i.e. for  $n = 2$ ), when assuming (2.6). Let  $\hat{I}: \mathbb{R}^2/\mathbb{Z}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$  be defined by the integer matrix  $A$ , whose real eigenvalues are one, say  $\lambda_1$ , of modulus  $|\lambda_1| > 1$  and the other, say  $\lambda_2$ , with  $|\lambda_2| < 1$ . For instance,  $A$  can take the form,

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},$$

because  $\lambda_1 = 1 + \sqrt{2}$ ,  $\lambda_2 = 1 - \sqrt{2}$ , and so  $(1 - \lambda_1)(1 - \lambda_2) = -2$ , and  $|\lambda_1| = |1 + \sqrt{2}| > 1$ ,  $|\lambda_2| = |1 - \sqrt{2}| < 1$ .

Then  $h(\hat{I}) = \log |\lambda_1| = \log(1 + \sqrt{2}) > 0$ , and (3.1) exhibits on  $\mathbb{R}^2/\mathbb{Z}^2$ , according to Corollary 4.4, chaos in the sense of Definition 4.2.

Observe that since  $\lambda(\hat{I}) = (1 - \lambda_1)(1 - \lambda_2) \neq 0$  holds for the Lefschetz number, we obtain according to (2.3) that  $N^\infty(\hat{I}) = |\lambda_1| = 1 + \sqrt{2}$ , and subsequently (see (2.1))  $h(\hat{I}) \geq \log |\lambda_1| > 0$ , with the same conclusion for (3.1).

Theorem 4.3 can be modified by means of Proposition 2.7 as follows.

**Theorem 4.6.** *Consider, under the above assumptions and (2.6), (4.1), jointly with  $\hat{I}(\mathbb{R}^n/\mathbb{Z}^n) = \mathbb{R}^n/\mathbb{Z}^n$ , the vector impulsive differential equation (3.1) on  $\mathbb{R}^n/\mathbb{Z}^n$ . Assume that the impulsive mapping  $\hat{I}: \mathbb{R}^n/\mathbb{Z}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$  is homotopic to a continuous map  $f: \mathbb{R}^n/\mathbb{Z}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$  such that  $N^\infty(f) > 1$ , i.e. (see (2.2))*

$$\limsup_{m \rightarrow \infty} |\lambda(f^m)|^{\frac{1}{m}} > 1,$$

*where  $\lambda(f^m)$  stands for the Lefschetz number of the  $m$ -th iterate of  $f$ .*

*Then  $h(\hat{I}) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} \log N(f^m) > 0$  holds, where  $N(f^m)$  denotes the Nielsen number of the  $m$ -th iterate of  $f$ , and subsequently equation (3.1) exhibits on  $\mathbb{R}^n/\mathbb{Z}^n$  chaos in the sense of Definition 4.2.*

*Proof.* The proof follows directly from Theorem 4.3, on the basis of Proposition 2.7 and Remark 2.8.  $\square$

**Example 4.7.** Consider the scalar ( $n = 1$ ) impulsive differential equation (3.1) on  $\mathbb{R}/\mathbb{Z}$ , when assuming (2.6). Let  $\hat{I}: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$  be the doubling impulsive mapping, where

$$\hat{I} := \begin{cases} 2x, & \text{for } x \in [0, \frac{1}{2}], \\ 2x - 1, & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

Since one can easily check that (see e.g. [6])

$$N(\hat{I}^k) = |\lambda(\hat{I}^k)| = |1 - 2^k|, \quad k \in \mathbb{N},$$

holds for the Nielsen and Lefschetz numbers, we obtain that

$$N^\infty(\hat{I}) = \limsup_{m \rightarrow \infty} |\lambda(\hat{I}^m)|^{\frac{1}{m}} = \limsup_{m \rightarrow \infty} |1 - 2^m|^{\frac{1}{m}} > 1.$$

Thus, applying Theorem 4.6,  $h(\hat{I}) > 0$  holds, and (3.1) exhibits on  $\mathbb{R}/\mathbb{Z}$  chaos in the sense of Definition 4.2.

According to Corollary 4.4, we have  $h(\hat{I}) = \log 2$ , and the same conclusion.

**Remark 4.8.** Observe that if  $I: [0, 1] \rightarrow [0, 1]$  is the standard tent map defined in Example 3.7, resp. its 1-periodic extension, then  $\hat{I} := \tau \circ I: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$  takes the same form as  $I$ . Thus,  $h(\hat{I}) = \log 2$ , which is sufficient for the application of Theorem 4.3. On the other hand,

$$N(\hat{I}^k) = |\lambda(\hat{I}^k)| = 1, \quad k \in \mathbb{N},$$

holds this time, which excludes the application of Theorem 4.6.

**Example 4.9.** Consider the scalar linear impulsive equation (3.7) with  $p(t) \equiv 0$ , i.e.

$$\begin{cases} x' = q(t), & t \neq t_j := j\omega, \quad \text{for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), & j \in \mathbb{Z}, \end{cases} \quad (4.3)$$

where  $q: \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function such that  $q(t) \equiv q(t + \omega)$  and  $\frac{1}{\omega} \int_0^\omega q(t) dt = 0$ .

- (i) One can easily check that (4.3) exhibits, according to Theorem 3.5, chaos in the sense of Definition 3.4, provided the continuous impulsive function  $I: \mathbb{R} \rightarrow \mathbb{R}$  is compact,  $I(K_0) = K_0$  and such that  $h(I|_{K_0}) > 0$ , where  $K_0 := \bar{I}(\mathbb{R})$ .
- (ii) Furthermore, (4.3) exhibits on  $\mathbb{R}/\mathbb{Z}$ , according to Theorem 4.3, chaos in the sense of Definition 4.2, provided the continuous impulsive function  $I: \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $I(x) \equiv I(x + 1) \pmod{1}$ ,  $\hat{I}(\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$ , and  $h(\hat{I}) > 0$ , where  $\hat{I} := \tau \circ I: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ .
- (iii) At last, (4.3) exhibits on  $\mathbb{R}/\mathbb{Z}$ , according to Theorem 4.6, chaos in the sense of Definition 4.2, provided the continuous impulsive function  $I: \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $I(x) \equiv I(x + 1) \pmod{1}$ ,  $\hat{I}(\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$ , and  $\hat{I}$  is homotopic to  $f: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$  (i.e.  $\hat{I} \sim f$ ) such that

$$\limsup_{m \rightarrow \infty} |\lambda(f^m)|^{\frac{1}{m}} > 1,$$

where  $\lambda(f^m)$  stands for the Lefschetz number of the  $m$ -th iterate of  $f$ .

**Remark 4.10.** One can easily check that since for the 1-periodically extended tent map  $I: \mathbb{R} \rightarrow [0, 1]$ , defined in Example 3.7,  $h(I|_{[0,1]}) = h(\hat{I}) = \log 2 (> 0)$  and  $\limsup_{m \rightarrow \infty} |\lambda(\hat{I}^m)|^{\frac{1}{m}} = 1$ , Theorem 3.5 and 4.3 apply in (i),(ii), while Theorem 4.6 does not apply in (iii). On the other hand, since for the doubling map  $I := 2x: \mathbb{R} \rightarrow \mathbb{R}$ , we have  $I(\mathbb{R}) = \mathbb{R}$ ,  $h(\hat{I}) = \log 2$  and  $\limsup_{m \rightarrow \infty} |\lambda(\hat{I}^m)|^{\frac{1}{m}} = \limsup_{m \rightarrow \infty} |1 - 2^m|^{\frac{1}{m}} > 1$ , Theorems 4.3 and 4.6 apply in (ii), (iii), while Theorem 3.5 does not apply in (i).

## 5 Concluding remarks

It is well known that (see e.g. the main theorem in [21]), for continuous maps on compact intervals, a positive topological entropy is equivalent with Devaney's chaos on a closed invariant subset, i.e. (i) topological transitivity, (ii) density of periodic points, (iii) sensitive dependence on initial conditions. Moreover, transitivity implies period six (see e.g. [12]), and subsequently (in view of the celebrated Sharkovsky cycle coexistence theorem, cf. e.g. [2, Theorem 2.1.1]) the coexistence of  $2k$ -periodic points, for every  $k \in \mathbb{N}$ . Reversely, the existence of a periodic point with period  $k \neq 2^n$ ,  $n \in \mathbb{N} \cup \{0\}$ , implies according to the theorem of Boven and Franks (see e.g. [2, Theorem 4.4.20]), a positive topological entropy, and subsequently Devaney's chaos on a closed invariant subset. The same, except the information about period six, but "only" with period  $k \neq 2^n$ ,  $n \in \mathbb{N} \cup \{0\}$ , is true for continuous maps on a circle, provided they possess a fixed point (see e.g. [2]).

Thus, many results for scalar ( $n = 1$ ) impulsive differential equations about Devaney's chaos and the coexistence of periodic solutions with various periods, including those of the type  $k \neq 2^n$ ,  $n \in \mathbb{N} \cup \{0\}$ , can be also interpreted in terms of a positive topological entropy.

In higher ( $n > 1$ ) dimensions, the situation is more delicate. Nevertheless, the coexistence of infinitely many periodic solutions is also there, in view of Lemma 2.5, a necessary condition for a positive topological entropy.

Under the assumptions of Corollary 4.4, we are able to prove like in [4, Theorem 4.3] the coexistence of  $k\omega$ -periodic (mod 1) solutions of (3.1), for infinitely many  $k \in \mathbb{N}$ , including those for  $k \neq 2^n$ ,  $n \in \mathbb{N} \cup \{0\}$ .

In this light, at least the results about topological entropy for impulsive differential equations, obtained in higher dimensions, seem to be original.

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